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Comparison theorems of isoperimetric type for moments of compact sets

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Abstract

A unified approach to prove isoperimetric inequalities for moments and basic inequalities of interpolation spaces $L(p,q)$ is developed. Instead symmetrization methods we use a monotonicity property of special Stiltjes' means

1. Introduction

We consider q-moments as Lebesgue integral over a compact set $\Omega \subset \mathbb{R}^n (n \geq 1)$

$$I_n(q) = \int_\Omega |x|^{q-n} f(x) \mathrm{d}x \quad (q > 0).$$

We also study Stiltjes integral from the theory of multinomial distributions (see [10])

$$P_n(q) = \int_0^{x_1} \mathrm{d}\psi_1^q(y_1) \int_0^{x_2} \mathrm{d}\psi_2^q(y_2) ... \int_0^{x_n} h(y)\varphi^q(y) \mathrm{d}\psi_n^q(y_n),$$

where $x_k > 0 (k = 1, 2, ..., n)$, $y = (y_1, y_2, ..., y_n)$ and ψ_k are absolutely continuous and strictly increasing functions with the condition $\psi_k(0) = 0$.

The q-moments play an important role in different branches of mathematics and physics (see [1], [3], [7], [9], [14]).

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The aim of this paper is to compare $I_n(q_1)$ with $I_n(q_2)$, and $P_n(q_1)$ with $P_n(q_2)$ if $0 < q_1 < q_2 < \infty$. There are several results of this kind. The first one is the isoperimetric inequality

$$\frac{\left(\int_\Omega \mathrm{d}x\right)^{1+2/n}}{\int_\Omega |x|^2 \mathrm{d}x} \leq \frac{\left(\int_{|x|\leq 1} \mathrm{d}x\right)^{1+2/n}}{\int_{|x|\leq 1} |x|^2 \mathrm{d}x}, \tag{1}$$

which is proved by Pólya and Szegö [13] for $n = 2$. For $n = 3$ Bandle [2] established a generalization of (1) for $I_3(5)$ with bounded $f(x) \geq 0$.

The second result is an inequality by Stein ([15], [16]) for norms in the interpolation spaces $L(p, q)$: *If $\varphi(t) \geq 0$ and $\varphi(t)$ is nonincreasing in $(0, \infty)$, $p > 0$ and $0 < q_1 \leq q_2 \leq \infty$ then the following sharp inequality holds*

$$\left(\frac{q_1}{p}\int_0^\infty [t^{1/p}\varphi(t)]^{q_1}\frac{dt}{t}\right)^{1/q_1} \geq \left(\frac{q_2}{p}\int_0^\infty [t^{1/p}\varphi(t)]^{q_2}\frac{dt}{t}\right)^{1/q_2}. \tag{2}$$

We present here a new unified approach to prove inequalities (1) and (2) and their nontrivial generalizations. Namely, we prove two generalizations of (2) for $P_n(q)$ (see Theorem 1 and Theorem 3, below). In contrast with (2) we obtain

$$\left(\frac{q_1}{p}\int_0^\infty [t^{1/p}\varphi(t)]^{q_1}h(t)\frac{dt}{t}\right)^{1/q_1} \leq \left(\frac{q_2}{p}\int_0^\infty [t^{1/p}\varphi(t)]^{q_2}h(t)\frac{dt}{t}\right)^{1/q_2} \tag{3}$$

provided $0 \leq h(t) \leq 1$, $\varphi(t) \geq 0$, $\varphi(t)$ is nondecreasing in $(0, \infty)$, $p > 0$ and $0 < q_1 \leq q_2 \leq \infty$.

We also show that the integral means

$$m(q) = \left(\frac{q}{\omega_{n-1}}\int_\Omega |x|^{q-n} f(x)\mathrm{d}x\right)^{1/q}$$

is nondecreasing as function in $q \in (0, \infty)$ if $0 \leq f(x) \leq 1$ (see Theorem 2 below). Here $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the $(n-1)$-dimensional area of the unit sphere $|x| = 1$. Note that $m(q)$ does not depend on q if Ω is a ball $B(0, \rho)$ and $f(x) = 1$ in $B(0, \rho)$.

The monotonicity of $m(q)$ generates (1) and some new isoperimetric inequalities for the quantity

$$V = \int_\Omega f(x)\mathrm{d}x.$$

In particular, we obtain the following isoperimetric inequalities:

$$\left(\frac{V}{c_n}\right)^{(n+2)/n} \leq \frac{1 + 2/n}{c_n}\int_\Omega |x|^2 f(x)\mathrm{d}x, \tag{4}$$

$$V \log\frac{V}{c_n} \leq V + n\int_\Omega f(x)\log|x|\mathrm{d}x, \tag{5}$$

$$F.p. \int_\Omega |x-a|^{-n} f(x) \mathrm{d}x \le f(a) c_n \log \frac{V}{f(a)c_n} \tag{6}$$

where *F.p.* means "Finite part" of the singular integral [8], and $c_n = \omega_{n-1}/n$ is the volume of the unit ball $|x| \le 1$ in $\mathbb{R}^n$. For (6) we suppose that f is a nonnegative function which satisfies the Hölder condition $|f(x) - f(y)| \le C|x-y|^\alpha$ $(x, y \in \Omega)$ for some $\alpha > 0$ and $a \in \Omega \setminus \partial\Omega$ such that $f(a) = \max_{x\in\Omega} f(x) > 0$.

2. Main results and their applications

Theorem 1

Suppose that $h(y)$ is nondecreasing with respect to $y_k \in [0, x_k]$ for $k = 2, ..., n$ and $\varphi(y)$ is nondecreasing with respect to $y_k \in [0, x_k]$ for $k = 1, 2, ..., n$. If $0 < q_1 < q_2 \le \infty$, $0 \le h(y) \le 1$, $\varphi \ge 0$ then the sharp inequality

$$P_n^{1/q_1}(q_1) \le P_n^{1/q_2}(q_2) \tag{7}$$

is valid.

Proof. First we consider the case $n = 1$. We have to prove that

$$\left(\int_0^x h(t)\varphi^{q_1}(t)\mathrm{d}\psi^{q_1}(t)\right)^{1/q_1} \le \left(\int_0^x h(t)\varphi^{q_2}(t)\mathrm{d}\psi^{q_2}(t)\right)^{1/q_2}, \tag{8}$$

if $0 \le h(t) \le 1$ and $0 \le \varphi(t_1) \le \varphi(t)$ for all t_1 and t such that $0 \le t_1 \le t \le x$.

Let $\Phi(u) = u^{q_2/q_1} (u > 0)$, and let

$$F_i(t) = \int_0^t h(\tau)\varphi^{q_i}(\tau)\mathrm{d}\psi^{q_i}(\tau) \quad (i = 1, 2, \quad 0 \le t \le x).$$

We have

$$F_1(t) \le \varphi^{q_1}(t) \int_0^t \mathrm{d}\psi^{q_1}(\tau) = [\varphi(t)\psi(t)]^{q_1}, \quad 0 \le t \le x. \tag{9}$$

Using (9) and the convexity of $\Phi(u)$ one has

$$\Phi'(F_1(t)) \le \Phi'(\varphi^{q_1}(t)\psi^{q_1}(t)) = \frac{q_2}{q_1}[\varphi(t)\psi(t)]^{q_2-q_1}, \quad 0 \le t \le x.$$

Thus,

$$\Phi'(F_1(t))F_1'(t) \le F_2'(t), \quad 0 \le t \le x. \tag{10}$$

Integrating (10) we get

$$\Phi(F_1(x)) \le F_2(x),$$

which is the inequality (8).

It is clear that equality in (8) holds if and only if equality holds in (10) for almost all $t \in [0,x]$. We now suppose that $F_1(x) > 0$. Therefore there exists $\alpha \in (0,x]$ such that

$$\alpha = \inf \left\{ t \in (0,x] : \int_t^x h(\tau)\varphi^{q_1}(\tau)\mathrm{d}\psi^{q_1}(\tau) = 0 \right\}. \tag{11}$$

On the other hand, using (11) and the equality $\Phi'(F_1(t))F_1'(t) = F_2'(t)$ a.e. in $[0,x]$ we obtain

$$[F_1(t) - \varphi(t)^{q_1}\psi^{q_1}(t)]h(t)\varphi(t) = 0 \text{ a.e. in } [0,\alpha]. \tag{12}$$

Let us remark that $\varphi(t) \not\equiv 0$ in $(0,\alpha)$. Consequently there are $t \in (0,\alpha)$ with $\varphi(t) > 0$. If $0 \le \varphi(t_1) < \varphi(t_2)$ for some $0 < t_1 < t_2 < \alpha$, and $t \in (t_2,\alpha)$, then $\varphi(t_1) < \varphi(t)$ and

$$F_1(t) \le \varphi^{q_1}(t)\psi^{q_1}(t_1) + \varphi^{q_1}(t)[\psi^{q_1}(t) - \psi^{q_1}(t_1)] < [\varphi(t)\psi(t)]^{q_1}. \tag{13}$$

From (12) and (13) one has $h(t) = 0$ a.e. in $[t_2,\alpha]$ in contradiction with (11). Thus, $\varphi(t) = c = \mathit{const} > 0, \quad t \in (0,\alpha)$. Consequently, (12) is equivalent to the equality

$$\left[\int_0^t h(\tau)\mathrm{d}\psi^{q_1}(\tau) - \psi^{q_1}(t)\right] h(t) = 0 \text{ a.e. in } [0,\alpha]. \tag{14}$$

If $h(t) \ne 1$ a.e. in $[0,\alpha]$ then there exists $t_3 \in (0,\alpha)$ such that

$$\int_0^t h(\tau)\mathrm{d}\psi^{q_1}(\tau) < \psi^{q_1}(t), \quad t \in (t_3,\alpha).$$

So, $h(t) = 0$ a.e. in (t_3,α) because (14). This contradicts (11). Thus, $h(t) = 1$ a.e. in $[0,\alpha]$. Therefore equality in (8) holds if and only if $h(t)\varphi(t) = 0$ a.e. in $[0,x]$ or $h(t)\varphi(t) = 0$ a.e. in $[\alpha,x]$, $\varphi(t) = \mathit{const} > 0$ and $h(t) = 1$ a.e. in $(0,\alpha)$.

This completes the proof for $n = 1$.

To prove Theorem 1 for $n \ge 2$ we apply mathematical induction on n. Suppose that Theorem 1 is true for dimensions $1, 2, ..., n-1$. We can write

$$P_n(q) = \int_0^{x_n} P_{n-1}(y_n,q)\mathrm{d}\psi_n^q(y_n),$$

where

$$P_{n-1}(y_n,q) = \int_0^{x_{n-1}} \mathrm{d}\psi_{n-1}^q(y_{n-1}) ... \int_0^{x_1} h(y_1,...,y_n)\varphi^q(y_1,y_2,...,y_n)\mathrm{d}\psi_1^q(y_1).$$

For fixed $y_n \in [0,x_n]$ by induction hypothesis

$$P_{n-1}(y_n,q_1) \le P_{n-1}^{q_1/q_2}(y_n,q_2). \tag{15}$$

The function $\varphi_*(t) := P_{n-1}^{1/q_2}(t,q_2)$ is nondecreasing for $t \in [0,x_n]$, and by (15)

$$P_n(q_1) \le \int_0^{x_n} \varphi_*^{q_1}(y_n)\mathrm{d}\psi_n^{q_1}(y_n). \tag{16}$$

Applying to (16) the inequality (8) with $h(t) = 1$ we obtain

$$P_n(q_1) \le \left(\int_0^{x_n} \varphi_*^{q_2}(y_n) \mathrm{d}\psi_n^{q_2}(y_n) \right)^{q_1/q_2}$$
$$= \left(\int_0^{x_n} P_{n-1}(y_n, q_2) \mathrm{d}\psi_n^{q_2}(y_n) \right)^{q_1/q_2} = P_n^{q_1/q_2}(q_2),$$

which is the desired inequality (7). □

Note that equality in (7) occurs if $\varphi(y) = const > 0$ and $h(y) = 1$ a.e. in $I_\alpha = [0, \alpha_1] \times ... \times [0, \alpha_n] \subset I_x = [0, x_1] \times ... \times [0, x_n]$ and $h(y) = 0$ a.e. in $I_x \setminus I_\alpha$.

Theorem 2

Let Ω be a compact set in $\mathbb{R}^n (n \ge 1)$, $mes(\Omega) > 0$. If $0 < f(x) \le 1$ in Ω and $0 < q_1 < q_2 \le \infty$ then

$$\left(\frac{q_1}{\omega_{n-1}} \int_\Omega |x|^{q_1 - n} f(x) dx \right)^{1/q_1} \le \left(\frac{q_2}{\omega_{n-1}} \int_\Omega |x|^{q_2 - n} f(x) dx \right)^{1/q_2}. \qquad (17)$$

Equality in (17) occurs if and only if $f(x) = 1$ a.e. in Ω and Ω is a ball $B(0, \rho)$ up to a set of Lebesgue measure zero.

Proof. Let $x = |x|\omega, |x| = t, dx = t^{n-1} dt d\omega$. By Fubini theorem

$$m(q) := \left(\frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x \right)^{1/q} = \left(\int_0^\rho h(t) \mathrm{d}t^q \right)^{1/q}, \qquad (18)$$

where $\rho = ||x||_{L^\infty(\Omega)}$ and

$$h(t) = \frac{1}{\omega_{n-1}} \int_{|x|=t} f(x) \chi_\Omega(x) \mathrm{d}\omega.$$

It is clear that (17) is a consequence of (8), and $h(t) = 1$ a.e. in $[0, \rho]$ if and only if $\chi_\Omega(t\omega) = 1$ for almost all $t \in [0, \rho]$ and $f(x) = 1$ a.e. in Ω.

There is another way to investigate the case of equality in (17). If $0 < q_1 < q_2 \le \infty$ and $m(q_1) = m(q_2)$ then $m(q) = m_0 = const$ for $q_1 \le q \le q_2$. Since $m(q)$ is analytic with respect to q in some neighbourhood of $\{q : 0 < q < \infty\}$ then, by the uniqueness theorem for analytic functions, $m(q) = m_0$ for each $q \in (0, \infty)$. Therefore,

$$m(1) = \frac{1}{\omega_{n-1}} \int_\Omega |x|^{1-n} f(x) \mathrm{d}x = \lim_{q \to \infty} m(q) = ||x||_{L^\infty(\Omega)} = \rho.$$

On the other hand, $\mathrm{mes}[\Omega \setminus B(0, \rho)] = 0$ and $m(1) < \rho$, if $f(x) \ne 1$ a.e. in Ω and $\mathrm{mes}(\Omega) > 0$ or $\mathrm{mes}(B(0, \rho) \setminus \Omega) > 0$. This completes the proof of Theorem 2. □

Theorem 3

Suppose that $h(y)$ is nonincreasing with respect to $y_k \in [0, x_k]$ for $k = 2, ..., n$ and $\varphi(y)$ is nonincreasing with respect to $y_k \in [0, x_k]$ for $k = 1, 2, ..., n$. If $0 < q_1 < q_2 \le \infty$, $h(y) \ge 1$, then the sharp inequality

$$P_n^{1/q_1}(q_1) \ge P_n^{1/q_2}(q_2) \qquad (19)$$

is valid.

Proof. Proof of Theorem 3 follows the proof of Theorem 1 with inequalities opposite to (9), (10), (15) and (16).

It is clear that equality in (19) occurs if $\varphi(y) = const > 0$ and $h(y) = 1$ a.e. in $I_\alpha \subset I_x$ and $\varphi(y) = 0$ in $I_x \setminus I_\alpha$. $\square$

The inequalities (7) and (19) can be used to get some entropy type inequalities. Consider the case related to (19). Suppose that h and φ satisfy the assumptions of Theorem 3. The inequality (19) implies that

$$\frac{\mathrm{d}P_n^{1/q}(q)}{\mathrm{d}q}|_{q=1} \leq 0.$$

Straightforward computations show that this is equivalent to the inequality

$$P_n(1)\log P_n(1) \geq nP_n(1) + \int_0^{x_1} \mathrm{d}\psi_1(y_1) \int_0^{x_2} \mathrm{d}\psi_2(y_2)... \int_0^{x_n} h\varphi \log(\varphi\psi_1...\psi_n)\mathrm{d}\psi_n(y_n),$$

where

$$P_n(1) = \int_0^{x_1} \mathrm{d}\psi_1(y_1) \int_0^{x_2} \mathrm{d}\psi_2(y_2)... \int_0^{x_n} h\varphi \mathrm{d}\psi_n(y_n).$$

The last inequality is well-defined by the entropy theory convention $0\log 0 = 0$.

We now deduce the inequalities (2), (3), (4), (5) and (6). Inequality (2) is a particular case of Theorem 3 for $n = 1$, $h(t) = 1$, $x_1 = \infty$ and $\psi(t) = t^{1/p}$.

We obtain inequality (3) as the case $n = 1$, $x_1 = \infty$, $\psi(t) = t^{1/p}$ of Theorem 1. Letting $q_1 = n$ and $q_2 = n+2$ in Theorem 2 we have inequality (4). By straightforward computations for the function $m(q)$ from (18) we get

$$q^2 m'(q) = -[m(q)]^q \log[m(q)]^q + [m(q)]^q + \frac{q^2}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \log|x| \mathrm{d}x.$$

The inequality (17) implies $m'(q) \geq 0$ and (5) is equivalent to the inequality $m'(q) \geq 0$ at the point $q = n$.

To prove inequality (6) without loss of generality we can assume that $a = 0$ and $f(a) = 1$. First of all, let us remark that Theorem 2 implies that there exists $\lim_{q\to 0+} m(q)$.

Since Ω is a compact set then there exists $\rho > 0$ such that $\Omega \subset B(0,\rho)$. Hence

$$\frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x)\mathrm{d}x \leq \frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n}\mathrm{d}x \leq \frac{q}{\omega_{n-1}} \int_{B(0,\rho)} |x|^{q-n}\mathrm{d}x = \rho^q.$$

Thus,

$$\limsup_{q\to 0} \frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x)\mathrm{d}x \leq 1.$$

Analogously it is easy to show that

$$\liminf_{q\to 0} \frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x \geq 1.$$

Therefore,

$$\frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x \to 1 \text{ as } q \to 0,$$

which implies

$$\lim_{q\to 0} \left(\frac{q}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x \right)^{1/q} = \exp \lim_{q\to 0} \left[\frac{1}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x - \frac{1}{q} \right].$$

Since $0 \in \Omega \setminus \partial\Omega$ then there exists $r > 0$ such that $B(0, r) \subset \Omega$. Further, we have

$$\begin{aligned} \frac{1}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x - \frac{1}{q} &= \frac{1}{\omega_{n-1}} \int_{\Omega \setminus B(0,r)} |x|^{q-n} f(x) \mathrm{d}x \\ &+ \frac{1}{\omega_{n-1}} \int_{B(0,r)} |x|^{q-n} (f(x) - 1) \mathrm{d}x + \frac{r^q - 1}{q}. \end{aligned}$$

Since f is an α-Hölder function then there exists

$$\begin{aligned} &\lim_{q\to 0} \frac{1}{\omega_{n-1}} \int_{B(0,r)} |x|^{q-n} (f(x) - 1) \mathrm{d}x \\ &= \frac{1}{\omega_{n-1}} \int_{B(0,r)} |x|^{-n} (f(x) - 1) \mathrm{d}x \to 0 \text{ as } r \to 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{q\to 0} \left[\frac{1}{\omega_{n-1}} \int_\Omega |x|^{q-n} f(x) \mathrm{d}x - \frac{1}{q} \right] &= \frac{1}{\omega_{n-1}} \int_{\Omega \setminus B(0,r)} |x|^{-n} f(x) \mathrm{d}x \\ &+ \frac{1}{\omega_{n-1}} \int_{B(0,r)} |x|^{-n} (f(x) - 1) \mathrm{d}x - \log \frac{1}{r} \\ &= \lim_{r\to 0} \frac{1}{\omega_{n-1}} \int_{\Omega \setminus B(0,r)} |x|^{-n} f(x) \mathrm{d}x - \log \frac{1}{r} \\ &= F.p. \int_\Omega |x|^{-n} f(x) \mathrm{d}x \,. \end{aligned}$$

The last equality is a definition of *F.p.* (see [8]). Consequently, the inequality (6) follows from Theorem 2 for $q_1 = 0+$ and $q_2 = n$.

If $n \geq 2$ then Theorem 1 implies that the quantity

$$z_n(q) = \left(q^n \int_0^{x_1} \ldots \int_0^{x_n} y_1^{q-1} \ldots y_n^{q-1} h(y_1, \ldots, y_n) \mathrm{d}y_1 \ldots \mathrm{d}y_n \right)^{1/q}$$

is nondecreasing provided $0 \le h \le 1$ and h is nondecreasing as function of y_k for all k except one. It is surprising that the only condition $0 \le h \le 1$ is not sufficient for the monotonicity of $z_n(q)$ in the case $n \ge 2$. For example, the function

$$z_2(q) = \left(q^2 \int_0^1 \int_0^1 (xy)^{q-1} h(x,y) \mathrm{d}x\mathrm{d}y \right)^{1/q}$$

is not monotone for $q > 0$ if $h(x,y) = \frac{\pi}{4}\chi_{\{x^2+y^2\le 1\}}(x,y)$, where χ_Ω denotes the characteristic function of Ω, or $h(x,y) = 1 - xy$.

Nevertheless, (7) is valid without assumptions on monotonicity of h if $h(y) = \Pi h_k(y_k)$. Namely, using twice the one-dimensional case of (7) in mathematical induction on n, we easily obtain:

Proposition 1

Suppose that and φ is nondecreasing with respect to $y_k \in [0, x_k]$ for all $k = 1, 2, ..., n$. If $h(y) = \prod_{k=1}^n h_k(y_k)$ and $0 \le h_k(y_k) \le 1$ for $y_k \in [0, x_k]$ $(k = 1, 2, ..., n)$, then the inequality (7) *holds.*

The following generalization of (2) is a particular case of Theorem 3.

Proposition 2

Suppose that $h(t) \ge 1$, $\varphi(t) \ge 0$ and $\varphi(t)$ is nonincreasing in $(0, \infty)$. If $p > 0$, $0 < q_1 \le q_2 \le \infty$ then the following sharp inequality holds

$$\left(\frac{q_1}{p} \int_0^\infty [t^{1/p}\varphi(t)]^{q_1} h(t) \frac{dt}{t} \right)^{1/q_1} \ge \left(\frac{q_2}{p} \int_0^\infty [t^{1/p}\varphi(t)]^{q_2} h(t) \frac{dt}{t} \right)^{1/q_2}.$$

Other generalizations of (2) were established by Bergh [4], Bergh, Burenkov and Persson [5], [6], Persson and Peĉarić [12], Myasnikov, Persson and Stepanov [11].

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