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Collect. Math. **54**, 3 (2003), 327–340

Curves on a double surface

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Received January 25, 2003

Dedicated to Silvio Greco on the occasion of his sixtieth birthday

Abstract

Let F be a smooth projective surface contained in a smooth threefold T, and let X be the scheme corresponding to the divisor $2F$ on T. A locally Cohen-Macaulay curve $C \subset X$ gives rise to two effective divisors on F, namely the largest divisor P contained in $C \cap F$ and the curve R residual to $C \cap F$ in C. We show that under suitable hypotheses a general deformation of R and P on F lifts to a deformation of C on X, and give applications to the study of Hilbert schemes of locally Cohen-Macaulay space curves.

1. Introduction

The problem of classifying algebraic space curves boils down to understanding the Hilbert schemes $H_{d,g}(\mathbb{P}^3)$ of locally Cohen-Macaulay curves of degree d and arithmetic genus g in $\mathbb{P}^3$ via liaison theory (see [16] for a survey). Since these Hilbert schemes may have many irreducible components and complicated scheme structure, Martin-Deschamps and Perrin stratify $H_{d,g}$ via two invariants γ_C and ρ_C arising from a curve $C \subset \mathbb{P}^3$ [15]: these are finitely supported numerical functions which together determine

Keywords: Hilbert schemes, double surfaces, deformations.

MSC2000: 14C05, 14H50.

all cohomological dimensions $h^i(\mathcal{I}_C(n)) = \dim \mathrm{H}^i(\mathbb{P}^3, \mathcal{I}_C(n))$. Thus the study of $H_{d,g}$ divides into (a) describing the various strata $H_{\gamma,\rho}$ and (b) seeing how these patch together.

The second problem amounts to trying to determine when a component H_0 of a given stratum H_{γ_0,ρ_0} is contained in (or at least meets) the closure of a component H of a possibly different stratum $H_{\gamma,\rho}$: thus one would like to know when there is a family of curves parametrized by the spectrum of a discrete valuation ring with special fibre C_0 in H_0 and general fibre C in H. While various examples of such specializations are known [10, 9, 20, 18, 19], the general question is very difficult: beyond the obvious necessary semicontinuity condition and the more subtle necessary condition that the Rao module $M_C = H^1_* \mathcal{I}_C$ be a flat deformation of a subquotient of M_{C_0} ([11, 5.10]; see also [2]), little is known. Triads were introduced in [11] as a general tool for studying this question.

For curves on a fixed *smooth* surface $F \subset \mathbb{P}^3$, specializations only occur through linear equivalence because $\mathrm{Pic}(F)$ is discrete, so interesting behavior such as jumping cohomology isn't possible. The purpose of the present paper is to study curves on the simplest nonreduced surfaces, namely the divisors $X = 2F$, where $F \subset \mathbb{P}^3$ is a smooth surface: actually, we work in the more general framework of *ribbons* X over F in the sense of Bayer and Eisenbud [1]. Here we can expect plenty of interesting specializations.

Following the successful study of curves in the double plane [12], we define in Section 2 the triple $T(C) = \{Z, R, P\}$ associated to a curve $C \subset X$: P is the largest divisor contained in the scheme-theoretic intersection $C \cap F$, thus we may write $\mathcal{I}_{C\cap F} = \mathcal{I}_P \mathcal{I}_Z$ for a zero dimensional subscheme $Z \subset F$. Taking R to be the residual curve to $C \cap F$ in C, we obtain the triple $T(C) = \{Z, R, P\}$. We show Z is a generalized Gorenstein divisor on R in the sense of [7], while $R \subset P$ are effective divisors on F. In Proposition 2.3 we describe the fibres of the map $C \mapsto T(C) = \{Z, R, P\}$ as an open subset of a vector space of sections of a reflexive sheaf on R, while Proposition 2.5 and Remark 2.7 give cohomological conditions that ensure the fibre over a given triple $\{Z, R, P\}$ is nonempty: in particular, we may produce many families of nonreduced curves in this way. We show by example that the existence of a curve C with $T(C) = \{Z, R, P\}$ becomes delicate when the conditions of 2.5 fail.

In Section 3 we construct locally closed subschemes $H_{z,r,p}(X) \subset H_{d,g}(X)$ that parametrize curves C whose three associated subschemes Z,R, and P have Hilbert polynomials z, r and p respectively, and thus obtain a stratification of $H_{d,g}(X)$. Of course, the schemes $H_{z,r,p}(X)$ come with natural projection maps t to the Hilbert schemes $D_{z,r,p}(F)$ of flags $Z \subset R \subseteq P$ of subschemes of F with the prescribed Hilbert polynomials.

Our main result (Theorem 3.2) is a relative version of 2.3 which describes the structure of the map t under suitable conditions:

Theorem

Let $V \subset D_{z,r,p}$ be the open subscheme corresponding to triples $\{Z, R, P\}$ satisfying $H^1(\mathcal{O}_R(Z + P - F)) = 0$. Then the map $t^{-1}(V) \to V$ is the composition of an open immersion and an affine bundle projection.

Moreover, the fibres are nonempty if condition (2) of Proposition 2.5 holds. The following immediate consequence (Corollary 3.4) is useful in showing the existence of specializations on a double surface:

Corollary

Let $C_0 \in H_{z,r,p}(X)$ be a curve whose triple $T(C_0) = \{Z, R, P\}$ satisfies $H^1(\mathcal{O}_R(Z + P - F)) = 0$ and belongs to the irreducible component Y of $D_{z,r,p}(F)$. Then C_0 is a specialization of curves $C \in H_{z,r,p}(X)$ whose triples are general in Y.

This result allowed us to produce specializations (Example 3.6) which were crucial in proving the connectedness of the Hilbert schemes of degree four curves [19]. It also recovers deformations from [18] which were originally computed in ad hoc fashion (Example 3.5). We hope that others studying multiple curves can make use of these results.

2. Curves on a ribbon

Let F be a smooth surface over an algebraically closed ground field k. According to [1], a ribbon on F is a scheme X equipped with an isomorphism $F \cong X_{red}$ such that

(1) the ideal sheaf $\mathcal{I}_F$ of F in X has square zero in $\mathcal{O}_X$, and thus may be regarded as an $\mathcal{O}_F$-module;

(2) regarded as an $\mathcal{O}_F$-module, $\mathcal{I}_F$ is locally free of rank one.

If F is contained in a smooth threefold T, the effective divisor $2F$ on T is a ribbon on F, and locally every ribbon on F arises this way: since F is smooth, any ribbon on F is locally split [1, §1]. We use the notation $\mathcal{O}_F(-F) = \mathcal{I}_F$ and $\mathcal{O}_F(F) = \mathcal{H}om_{\mathcal{O}_F}(\mathcal{I}_F, \mathcal{O}_F)$. In this paper we will work with a ribbon X on F, which we will assume projective, although many of our constructions work in the local case as well.

We will use the following conventions: A subscheme $C \subset X$ is a *curve* if all of its associated points have dimension one, thus C is locally Cohen-Macaulay of pure dimension one, or empty. If Y is a subscheme of X, $\mathcal{I}_Y$ denotes the ideal sheaf of Y in X. If R is a Gorenstein scheme and Z a generalized divisor on R [8], then $\mathcal{O}_R(Z) = \mathcal{H}om(\mathcal{I}_{Z,R}, \mathcal{O}_R)$ denotes the reflexive sheaf associated to the divisor Z. If further $R \subset F$, we write $\mathcal{O}_R(Z - F)$ for $\mathcal{O}_R(Z) \otimes \mathcal{O}_F(-F)$. The following proposition generalizes [12, §2]:

Proposition 2.1

To each curve C in X is associated a triple $T(C) = \{Z, R, P\}$ in which $R \subset P$ are effective divisors on F, $Z \subset R$ is Gorenstein and zero-dimensional (possibly empty), and

$$\mathcal{I}_{P,C} \cong \mathcal{O}_R(Z - F).$$

The arithmetic genera are related by

$$p_a(C) = p_a(P) + p_a(R) + \deg_R \mathcal{O}_R(F) - \deg(Z) - 1. \tag{1}$$

Proof. Extracting the possible embedded points from the one dimensional scheme-theoretic intersection $C \cap F \subset F$, we may write

$$\mathcal{I}_{C\cap F,F} = \mathcal{I}_{Z,F}(-P)$$

where P is an effective divisor and Z is zero-dimensional. The inclusion $P \subset C \cap F$ yields a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{I}_{R,F}(-F) & \to & \mathcal{I}_C & \to & \mathcal{I}_{Z,F}(-P) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_F(-F) & \to & \mathcal{I}_P & \to & \mathcal{O}_F(-P) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_R(-F) & \xrightarrow{\sigma} & \mathcal{L} & \to & \mathcal{O}_Z(-P) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2}$$

which defines the residual scheme R to $C \cap F$ in C. The inclusion $\mathcal{O}_R(-F) \hookrightarrow \mathcal{O}_C$ shows that the associated points of R are among those of C, hence R is a curve. By construction, P is the largest curve in $F \cap C$, hence $R \subseteq P$ and $C \subset F$ if and only if R is empty.

We now show that Z is Gorenstein on R and that $\mathcal{L} \cong \mathcal{O}_R(Z - F)$ is a rank one reflexive $\mathcal{O}_R$-module. In view of the bottom row of diagram 2, the submodule $\mathcal{I}_R\mathcal{L} \subset \mathcal{L}$ is supported on Z, but $\mathcal{L} = \mathcal{I}_{P,C} \subset \mathcal{O}_C$ has only associated points of dimension one (because C is purely one-dimensional), hence $\mathcal{I}_R\mathcal{L} = 0$ and $\mathcal{L}$ is an $\mathcal{O}_R$-module. It follows that $\mathcal{O}_Z(-P)$ is an $\mathcal{O}_R$-module as well, hence $Z \subset R$.

Applying the bifunctor $\mathcal{H}om_{\mathcal{O}_R}(\ -\ ,\ -\ )$ to the sequence $0 \to \mathcal{I}_{Z,R} \to \mathcal{O}_R \to \mathcal{O}_Z \to 0$ and the bottom row of diagram (2) we obtain an exact diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \mathcal{H}om_{\mathcal{O}_R}(\mathcal{O}_Z, \mathcal{O}_Z(-P)) \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \to & \mathcal{O}_R(-F) & \xrightarrow{\pi} & \mathcal{O}_R(Z-F) & \to & \mathcal{E}xt^1_{\mathcal{O}_R}(\mathcal{O}_Z, \mathcal{O}_R(-F)) \\ & & \downarrow & & \downarrow \beta & & \downarrow 0 \\ 0 & \to & \mathcal{L} & \xrightarrow{\phi} & \mathcal{H}om_{\mathcal{O}_R}(\mathcal{I}_{Z,R}, \mathcal{L}) & \xrightarrow{\gamma} & \mathcal{E}xt^1_{\mathcal{O}_R}(\mathcal{O}_Z, \mathcal{L}) \\ & & \downarrow & & \downarrow & & \\ \mathcal{O}_Z(-P) & \xrightarrow{\delta} & \mathcal{O}_Z(-P) & \xrightarrow{0} & \mathcal{H}om_{\mathcal{O}_R}(\mathcal{I}_{Z,R}, \mathcal{O}_Z(-P)) & & \end{array}$$

The morphisms π, ϕ and α are injective because $\mathcal{L}$ has no zero dimensional associated points, hence $\mathcal{H}om_{\mathcal{O}_R}(\mathcal{O}_Z, \mathcal{L}) = 0$.

Smoothness of F implies that R is Gorenstein, hence $\omega_Z \cong \mathcal{E}xt^1_{\mathcal{O}_R}(\mathcal{O}_Z, \mathcal{O}_R(-F))$ and α can be thought as a morphism $\mathcal{O}_Z \to \omega_Z$. Since $\mathcal{O}_Z$ and ω_Z have the same length, α is an isomorphism, explaining the 0 map at the right of the diagram and showing that Z is Gorenstein.

Thinking of $\mathcal{L}$ and $\mathcal{O}_R(Z-F)$ as subsheaves of $\mathcal{H}om_{\mathcal{O}_R}(\mathcal{I}_{Z,R},\mathcal{L})$, the 0 at the bottom of the diagram yields $\mathcal{L}\subset\mathcal{O}_R(Z-F)$ while the 0 at the right gives $\mathcal{O}_R(Z-F)\subset\mathcal{L}$, proving that $\mathcal{L}=\mathcal{O}_R(Z-F)$.

For the arithmetic genus formula, note that $p_a(C)-p_a(P)=-\chi\mathcal{I}_{P,C}=-\chi\mathcal{L}$, which can be read off from the bottom row of diagram (2), keeping in mind that $\deg Z=\chi\,\mathcal{O}_Z$ and $\deg_R\mathcal{E}=\chi\,\mathcal{E}-\chi\,\mathcal{O}_R$ for an invertible sheaf $\mathcal{E}$ on R. $\square$

Example 2.2: (Local construction of triple:) Let A be a local ring with element $x\in A$ such that Ann (x) = (x) and $B=A/(x)$ is regular of dimension two. Given a height one ideal $I\subset A$ (corresponding to the curve C above), we may write the ideal $I+(x)/(x)\subset B$ as $(f)J$ for some $f\in B$ and a height two ideal J, defining $P=\mathrm{Spec}B/(f)$ and $Z=\mathrm{Spec}B/J$. If $K=I\cap(x)$ is the kernel of the natural map $I\to I+(x)/(x)$, then $K\subset(x)\cong B$ can be thought of as an ideal in B. The inclusion $B/K\hookrightarrow A/I$ shows that $R=\mathrm{Spec}B/K$ has no embedded points, hence $K=(g)$ is principal. The image of the composite map $I+(x)\hookrightarrow A\xrightarrow{\cdot x}A$ is contained in I because $x^2=0$, hence $(f)J=I+(x)/(x)\subset K=(g)$ in B. The snake lemma shows that $(f)+(g)/(g)\cong(f)+(g)/((f)J+(g))$, but the latter has finite length (because $(f)/(f)J$ does), hence $(f)+(g)/(g)\subset B/(g)$ is the zero ideal, showing that $(f)\subset(g)$, i.e. $R\subset P$.

Proposition 2.2

Given a triple $\{Z,R,P\}$ of closed subschemes of F as above, the set of curves $C\subset X$ with $T(C)=\{Z,R,P\}$ is in one-to-one correspondence with an open subset of the vector space

$$\mathrm{H}^0(R,\mathcal{O}_R(Z+P-F))\cong\mathrm{Hom}_R(\mathcal{O}_R(-P),\mathcal{O}_R(Z-F)).$$

Proof. We study the fibres of the map $C\mapsto T(C)$. Tensoring the last two rows of diagram (2) with $\mathcal{O}_R$ yields a new commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_R(-F) \arrow[r, "\tau"] \arrow[d, "="'] & \mathcal{I}_P\otimes\mathcal{O}_R \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathcal{O}_R(-P) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_R(-F) \arrow[r, "\sigma"] & \mathcal{O}_R(Z-F) \arrow[r, "\gamma"] & \mathcal{O}_Z(-P) \arrow[r] & 0 \end{tikzcd} \tag{3}$$

with exact rows in which ϕ is surjective. The bottom row is obtained by dualizing $0\to\mathcal{I}_{Z,R}\to\mathcal{O}_R\to\mathcal{O}_Z\to0$ and the top row is the restriction to R of the conormal sequence

$$0\to\mathcal{O}_P(-F)\to\mathcal{I}_P\otimes\mathcal{O}_P\to\mathcal{O}_P(-P)\to0$$

which relates the conormal sheaf $\mathcal{I}_P\otimes\mathcal{O}_P$ of P in X to the conormal sheaf $\mathcal{O}_F(-F)$ of F in X and the conormal sheaf $\mathcal{O}_P(-P)$ of P in F. Thus any curve C with $T(C)=\{Z,R,P\}$ gives rise to a surjective morphism $\phi:\mathcal{I}_P\otimes\mathcal{O}_R\to\mathcal{O}_R(Z-F)$ satisfying $\phi\circ\tau=\sigma$.

Conversely, given a surjection ϕ with $\phi \circ \tau = \sigma$, it is easy to check that the kernel of the composite map

$$\mathcal{I}_P \to \mathcal{I}_P \otimes \mathcal{O}_R \xrightarrow{\phi} \mathcal{O}_R(Z-F)$$

is the ideal sheaf $\mathcal{I}$ of a curve C with triple $\{Z, R, P\}$. As in [12], we thus obtain a one-to-one correspondence between curves C in X with triple $\{Z, R, P\}$ and surjections ϕ satisfying $\phi \circ \tau = \sigma$.

Applying the functor $\mathrm{Hom}_R(\ - \ , \mathcal{O}_R(Z-F))$ to the first row of diagram (3) identifies the kernel of the map

$$\mathrm{Hom}_R(\mathcal{I}_P \otimes \mathcal{O}_R, \mathcal{O}_R(Z-F)) \to \mathrm{Hom}_R(\mathcal{O}_R(-F), \mathcal{O}_R(Z-F))$$

with $\mathrm{Hom}_R(\mathcal{O}_R(-P), \mathcal{O}_R(Z-F))$. In particular, the set of ϕ satisfying $\phi \circ \tau = \sigma$ is identified with a coset of this vector space (when nonempty) and the set of surjective such ϕ corresponds to an open subset. $\square$

Remark 2.4. For $R \subset P$ fixed, notice that the top row of diagram (3) splits if and only if $T(C) = \{\emptyset, R, P\}$ for some curve $C \subset X$. Indeed, the map σ along the bottom row becomes the identity map, so the set of surjections ϕ satisfying $\phi \circ \tau = \sigma$ is precisely the set of splittings for τ. For example, if P is the complete intersection of a smooth surface $F \subset \mathbb{P}^3$ (with doubling X) and a hypersurface H of degree d, then the curve $X \cap H$ gives rise to the triple $\{\emptyset, P, P\}$, which agrees with the fact that the normal bundle of P in $\mathbb{P}^3$ splits. If $R = P$ is a general smooth curve, we generally would not expect the triple $\{\emptyset, P, P\}$ to arise from a curve, as this would imply splitting of the normal bundle of P in X.

It is useful to know when a triple actually arises from a curve.

Proposition 2.5

Let $\{Z, R, P\}$ be a triple of subschemes of F as in Proposition 2.1. Suppose that

(1) $\mathrm{H}^1(R, \mathcal{O}_R(Z+P-F)) = 0$*; and*

(2) the map $\mathrm{H}^0(\mathcal{O}_R(Z+P-F)) \otimes \mathcal{O}_R \to \mathcal{O}_Z$ *induced by* γ *is surjective.*

Then the set of curves $C \subset X$ with $T(C) = \{Z, R, P\}$ is parametrized by a non-empty open subset $U \subset \mathrm{H}^0(R, \mathcal{O}_R(Z+P-F))$ of dimension $\deg Z + \chi\mathcal{O}_R(P-F)$.

Proof. The triple $\{Z, R, P\}$ gives rise to the two exact rows of diagram (3). Condition (1) gives

$$\mathrm{Ext}^1(\mathcal{O}_R(-P), \mathcal{O}_R(Z-F)) \cong \mathrm{H}^1(\mathcal{O}_R(Z+P-F)) = 0,$$

hence there exists $\phi_0 \in \mathrm{Hom}(\mathcal{I}_P \otimes \mathcal{O}_R, \mathcal{O}_R(Z-F))$ such that $\phi_0 \circ \tau = \sigma$. Moreover, any such morphism ϕ can be written $\phi = \phi_0 + \alpha \circ \pi$ for

$$\alpha \in \mathrm{Hom}(\mathcal{O}_R(-P), \mathcal{O}_R(Z-F)) \subset \mathrm{Hom}(\mathcal{I}_P \otimes \mathcal{O}_R, \mathcal{O}_R(Z-F)).$$

Let $\overline{\phi_0} : \mathcal{O}_R(-P) \to \mathcal{O}_Z(-P)$ be the morphism induced by ϕ_0. The snake lemma shows that the morphism $\phi_0 + \alpha \circ \pi$ is surjective if and only if $\overline{\phi_0} + \gamma \circ \alpha$ is. Tensoring with $\mathcal{O}_R(P)$, we view α as a global section of $\mathcal{O}_R(Z+P-F)$. The images of these global section under γ generate $\mathcal{O}_Z$ of by condition (2). Since Z is finitely supported, it follows that for a general such section $s \in H^0\mathcal{O}_R(Z+P-F)$, the global section $\gamma(s) + \overline{\phi_0}(1)$ is a unit in $\mathcal{O}_{Z,z}$ at each point $z \in Z$. Thus α with $\alpha(1) = s$ corresponds to a surjective morphism ϕ. $\square$

Remark 2.6. Neither hypothesis of Proposition 2.5 is necessary for the existence of a curve C with a given triple. For example, let $F \subset \mathbb{P}^3$ be a smooth surface of degree $d = \deg F$ containing a line L. The effective divisor $X = 2F$ on $\mathbb{P}^3$ is a ribbon over F which contains all double lines C supported on L. If $p_a(C) \neq 1 - d$, then $C \not\subset F$ so that $T(C) = \{Z, L, L\}$, where Z is an effective divisor of degree $d - 1 - p_a(C)$ on L. Since $\mathcal{O}_L(Z+L-F) \cong \omega_L(\deg Z + 4 - 2\deg F)$ it is clear that $H^1(\mathcal{O}_L(Z+L-F)) \neq 0$ and $H^0(\mathcal{O}_L(Z+L-F)) = 0$ for $d >> 0$.

Remark 2.7. Note that any of the following conditions imply those of Proposition 2.5:

(a) $\mathrm{H}^1(R, \mathcal{O}_R(Z + P - F)) = 0$ and $\mathcal{O}_R(Z + P - F)$ is generated by global sections.
(b) $\mathrm{H}^1(R, \mathcal{O}_R(Z + P - F - H)) = 0$ for some very ample divisor H on R.
(c) $\mathrm{H}^1(R, \mathcal{O}_R(P - F)) = 0$.

Indeed, the first condition is clearly stronger than the hypotheses of 2.5 and the second implies the first by Mumford regularity [17]. Finally, the exact sequence

$$0 \to \mathcal{O}_R(P - F) \to \mathcal{O}_R(Z + P - F) \xrightarrow{\gamma} \mathcal{O}_Z \to 0$$

combined with $\mathrm{H}^1(R, \mathcal{O}_R(P - F)) = 0$ implies that $\mathrm{H}^1(R, \mathcal{O}_R(Z + P - F)) = 0$ for any effective generalized divisor $Z \subset R$ and further that γ is surjective, which implies the second hypothesis of 2.5 because Z has finite length. In particular, if $X = 2H$ is a double plane in $\mathbb{P}^3$, then every triple $\{Z, R, P\}$ with Z Gorenstein arises from a curve $C \subset X$ since

$$\mathrm{H}^1(R, \mathcal{O}_R(P - F)) \perp H^0(R, \omega_R(1 - \deg P)) = H^0(R, \mathcal{O}_R(\deg R - \deg P - 2)) = 0$$

[12, Proposition 3.1].

We close this section with a discussion of which triples correspond to curves on the doubling X of a smooth quadric surface $Q \subset \mathbb{P}^3$, using the standard isomorphism $\mathrm{Pic}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}$ [6, II, 6.6.1].

Example 2.8: Let $\{Z, R, P\}$ be a triple where $R \subset P$ are curves on a smooth quadric surface $Q \subset \mathbb{P}^3$ and P has type (a, b). We consider various cases.

(1) Suppose that $a, b > 0$ and $R \neq P$. Then every triple $\{Z, R, P\}$ with Z Gorenstein arises from a curve $C \subset X$. Indeed, the exact sequence

$$0 \to \mathcal{O}_Q(P - R - Q) \to \mathcal{O}_Q(P - Q) \to \mathcal{O}_R(P - Q) \to 0$$

combined with $H^1(\mathcal{O}_Q(P - Q)) \perp H^1(\mathcal{O}_Q(-P)) = 0$ [6, III, Exercise 5.6] and $H^2(\mathcal{O}_Q(P - R - Q)) \perp H^0(\mathcal{O}_Q(R - P)) = 0$ show that $H^1(\mathcal{O}_R(P - Q)) = 0$ as in Remark 2.7(c). In particular, taking Z to be empty we see that the normal bundle $\mathcal{N}_{P,\mathbb{P}^3}$ always splits when restricted to a *proper* subcurve R, while this is quite rare when $R = P$ [13].

(2) If $a = b$ and $R = P$, then P is a complete intersection and $\{\emptyset, P, P\}$ arises from a curve by Remark 2.4, however not every triple $\{Z, P, P\}$ arises from a curve: if P has type $(1, 1)$ and $\deg(Z) = 1$, then the triple $\{Z, P, P\}$ could only be associated

to a curve of degree 4 and genus 2 by (2.1), but there is no such curve in $\mathbb{P}^3$ [8, 3.1 and 3.3]. On the other hand, if $\deg(Z) > 1$ and P is a smooth conic, there exists a curve $C \subset X$ with $T(C) = \{Z, P, P\}$ by Proposition 2.5.

(3) If $1 = a < b$ and $R = P$ is a *smooth* rational curve, then the triple $\{Z, P, P\}$ arises from a curve if and only if Z is nonempty. (condition (2) of 2.5 fails when $\deg(Z) = 1$, but any nonzero map ϕ in diagram (3) is surjective in this case). The normal bundle splits when $k = \mathbb{C}$ [13, Theorem 1], but the top row of diagram (3) does not.

(4) If $a = 0$ and $R \subset P$ is a disjoint union of *reduced* lines, then it is easy to check that the triple $\{Z, R, P\}$ arises from a curve $C \subset X$ if and only if $Z \cap L \neq \emptyset$ for each line $L \subset R$ if and only if $\mathrm{H}^1(\mathcal{O}_R(Z + P - Q)) \cong \mathrm{H}^1(\mathcal{O}_R(Z - Q)) = 0$.

3. Hilbert Schemes

In this section we use triples to study flat families of curves on a projective ribbon X over a smooth surface F. We first stratify the Hilbert schemes $H_{d,g}(X)$ via the invariants of the associated triples to obtain locally closed subsets $H_{z,r,p}(X) \subset H(X)$ along with natural transformations $H_{z,r,p}(X) \xrightarrow{t} D_{z,r,p}(F)$ to the corresponding Hilbert flag schemes of triples. We then show that t has the structure of an open immersion followed by an affine bundle projection over the locus $V \subset D_{z,r,p}$ corresponding to triples satisfying the conditions of Proposition 2.5. This enables us to prove that one can construct specializations of curves on X lifting specializations of triples - see Corollary 3.4 and the examples following it.

We begin with the relative version of the constructions from Section 2. Define the contravariant functor $H : Sch_k \to Sets$ by

$$H(S) = \left\{ \begin{array}{ccc} C & \subset & X \times S \\ & \searrow & \downarrow \\ & \text{flat} & S \end{array} \;\middle|\; \begin{array}{c} \text{The sheaves } \mathcal{O}_{C \cap (F \times S)} \text{ and} \\ \mathcal{E}_C = \mathcal{E}xt^1_{\mathcal{O}_{F \times S}}(\mathcal{I}_{C \cap (F \times S)}, \mathcal{O}_{F \times S}) \\ \text{are flat over } S \end{array} \right\}.$$

For a morphism $T \xrightarrow{\phi} S$ in Sch_k, we naturally define $H(\phi)(C) = C_T = C \times_S T$ via the pull-back. The flatness of $\mathcal{O}_{C_T}$ and $\mathcal{O}_{C_T \cap (F \times T)}$ are immediate. To see that $\mathcal{E}_{C_T}$ is also flat over T, observe that $\mathcal{E}xt^2(\mathcal{I}_{C_s \cap F}, \mathcal{O}_F) = 0$ on fibres, so the theorem on base change for the $\mathcal{E}xt$ functors [4, 5] tells us that $\mathcal{E}_C$ commutes with base change - the natural map $(\mathrm{Id}_F \times \phi)^* \mathcal{E}_C \to \mathcal{E}xt^1_{\mathcal{O}_{F \times T}}(\mathcal{I}_{C_T \cap (T \times F)}, \mathcal{O}_{F \times T}) = \mathcal{E}_{C_T}$ is an isomorphism.

Letting D denote the Hilbert flag functor of triples $Z \subset R \subset P$ on F with Z zero-dimensional and $R \subseteq P$ effective Cartier divisors, we have the following result.

Theorem 3.1

There is a natural transformation $H \xrightarrow{t} D$ such that for any closed point $s \in Sch_k$ and $C \in H(s)$, $t(s)(C) = T(C)$.

Proof. Given a flat family $C \subset X \times S$ in $H(S)$, we produce a flag $t(C) = \{Z, R, P\}$ in $D(S)$ as follows. To construct P, we first show that the sheaf

$$\mathcal{H}_C = \mathcal{H}om_{\mathcal{O}_{F\times S}}(\mathcal{I}_{C\cap(F\times S)}, \mathcal{O}_{F\times S})$$

is invertible on $F \times S$. We have already seen that $\mathcal{E}_C = \mathcal{E}xt^1_{\mathcal{O}_{F\times S}}(\mathcal{I}_{C\cap(F\times S)}, \mathcal{O}_{F\times S})$ is flat over S (by definition of $H(S)$) and its formation commutes with base change. The theorem of base change for the functors $\mathcal{E}xt^i$ [4, 5] implies that $\mathcal{H}_C$ itself is flat over S and commutes with base change. In particular, the natural map

$$\mathcal{H}_C \otimes k(s) \to \mathcal{H}om(\mathcal{I}_{C_s\cap F}, \mathcal{O}_F)$$

is an isomorphism for every closed point $s \in S$. Thus the restriction of $\mathcal{H}_C$ to each fibre is an invertible sheaf, hence so is $\mathcal{H}_C$. By a standard argument [14, 7.4.1], the inclusion $\mathcal{I}_{C\cap(F\times S)} \hookrightarrow \mathcal{O}_{F\times S}$ defines a global section of $\mathcal{H}_C$ whose zero scheme is an effective Cartier divisor $P \subset F \times S$, flat over S.

Now define $Z \subset F \times S$ to be the residual scheme to P in $C \cap (F \times S)$, so that $\mathcal{I}_{C\cap(F\times S)} = \mathcal{I}_P\mathcal{I}_Z$. That Z is flat over S follows from the isomorphism $\mathcal{O}_Z(-P) \cong \mathcal{I}_{P,C\cap(F\times S)}$ and [14, 7.4.1].

Finally, define $R \subset X \times S$ to be the residual scheme to the intersection of C with $F \times S$. The exact sequence

$$0 \to \mathcal{O}_R(-F \times S) \to \mathcal{O}_C \to \mathcal{O}_{C\cap(F\times S)} \to 0$$

shows that R is flat over S, and that for each $s \in S$ the fibre R_s is the residual scheme to the intersection of C_s with F. Since $Z_s \subseteq R_s \subseteq P_s$ for each $s \in S$, we have $Z \subseteq R \subseteq P$.

Summing up, to any $C \in H(S)$ we can associate a triple $t(C) = \{Z, R, P\}$ in $D(S)$, where $Z \subseteq R \subseteq P$ are closed subschemes of $F \times S$ which are flat over S, and this construction is compatible with base change. Moreover, when S is a closed point, this construction agrees with that of Proposition 2.1. Thus we have the natural transformation $t : H \to D$ as desired. □

Both H and D are represented by quasiprojective schemes. This is well known for D. Using Mumford's flattening stratification, we see H is representable by a subscheme of the Hilbert scheme of curves in X. Since D is represented by a disjoint union of locally closed subschemes $D_{z,r,p}$ where $\{z, r, p\}$ vary in the set of possible Hilbert polynomials for Z, R and P, the same is true of H in taking $H_{z,r,p} = t^{-1}(D_{z,r,p})$. When the triples satisfy the vanishing of Proposition 2.5, the map t has an especially nice structure.

Theorem 3.2

Let $V \subset D_{z,r,p}$ be the open subscheme corresponding to triples $\{Z, R, P\}$ satisfying $H^1(\mathcal{O}_R(Z + P - F)) = 0$. Then the map $t^{-1}(V) \to V$ is the composition of an open immersion and an affine bundle projection.

Proof. Given a triple $\{Z, R, P\} \in D(S)$, we define

$$\mathcal{O}_R(Z - F \times S) = \mathcal{H}om_{\mathcal{O}_R}(\mathcal{I}_{Z,R}, \mathcal{O}_R(-F \times S)).$$

If $s \in S$ is a closed point, then $\mathcal{E}xt^1_{\mathcal{O}_{R_s}}(\mathcal{I}_{Z_s,R_s}, \mathcal{O}_{R_s}(-F)) = 0$ because R_s is Gorenstein. It follows [4, 5] that $\mathcal{O}_R(Z - F \times S)$ is flat over S and commutes with base change: for every morphism $g : T \to S$ in Sch_k the pull back of $\mathcal{O}_R(Z - F \times S)$ is $\mathcal{O}_{R_T}(Z_T - F \times T)$. Hence there is a functor A that assigns to the scheme S the set of flat families of flags $Z \subset R \subset P \subset F \times S$ with Hilbert polynomials z,r,p along with a morphism $\phi : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F \times S)$. Furthermore, the exact sequences

$$0 \to \mathcal{O}_R(-F \times S) \xrightarrow{\tau} \mathcal{I}_P \otimes \mathcal{O}_R \xrightarrow{\pi} \mathcal{O}_R(-P) \to 0. \tag{4}$$

and

$$0 \to \mathcal{O}_R(-F \times S) \xrightarrow{\sigma} \mathcal{O}_R(Z - F \times S) \to \mathcal{E}xt^1(\mathcal{O}_{Z_S}, \mathcal{O}_{R_S}(-F \times S)) \to 0 \tag{5}$$

(obtained by dualizing $0 \to \mathcal{I}_{Z,R} \to \mathcal{O}_R \to \mathcal{O}_Z \to 0$) are both compatible with base change, so A has a subfunctor M corresponding to morphisms ϕ satisfying $\phi \circ \tau = \sigma$.

Now we claim that $H_{z,r,p}$ is an open subfunctor of M. Indeed, given $C \in H(S)$, we may write a commutative diagram with exact rows and columns analogous to diagram (2):

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & \mathcal{I}_C \otimes \mathcal{O}_R \arrow[r] \arrow[d] & \mathcal{I}_{Z,R}(-P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_R(-F \times S) \arrow[r, "\tau"] \arrow[d] & \mathcal{I}_P \otimes \mathcal{O}_R \arrow[r] \arrow[d, "\phi"] & \mathcal{O}_R(-P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_R(-F \times S) \arrow[r, "\sigma"] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}_Z(-P) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{6}$$

As in the proof of 2.1, we obtain a morphism $\psi : \mathcal{L} \to \mathcal{O}_R(Z - F \times S)$. These sheaves are flat over S and compatible with pull back. Since ψ induces isomorphisms ψ_s on the fibres by the proof of 2.1, ψ is an isomorphism. Thus the diagram gives us a morphism $\phi : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F \times S)$ with $\phi \circ \tau = \sigma$, and we obtain a natural transformation from H to M that makes H into a subfunctor of M. It is open because it corresponds to the open condition that the map ϕ be surjective.

It remains to show that when we take inverse images over $V \subset D$, the induced map $M_V \xrightarrow{t} V$ has the structure of an affine bundle. Let $U \subset V$ be an affine open set equipped with universal flat flag

$$\begin{array}{ccccccc} Z & \subset & R & \subset & P & \subset & F \times U \\ & & & & & & \downarrow f \\ & & & & & & U. \end{array}$$

Since $\mathrm{H}^1(\mathcal{O}_{R_u}(Z_u + P_u - F)) = 0$ for each $u \in U$, we deduce [6, III, 8.5 and 12.9] that $R^1 f_* \mathcal{O}_R(Z + P - F) = 0$ and hence

$$\mathrm{Ext}^1(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \cong H^1(\mathcal{O}_R(Z + P - F)) = 0.$$

In particular, there exists $\phi_0 : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F)$ such that $\phi_0 \circ \tau = \sigma$.

By Lemma 3.3 below, the functor $G : Sch_U \to Sets$ given by

$$G(T) = \mathrm{Hom}_{R_T}(\mathcal{O}_{R_T}(-P_T), \mathcal{O}_{R_T}(Z_T - F \times T))$$

is represented by the geometric vector bundle $B \xrightarrow{p} U$ with sheaf of sections $\mathcal{E} = f_* \mathcal{H}om_{\mathcal{O}_R}(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F))$ and hence comes equipped with a universal map $\alpha : \mathcal{O}_{R_B}(-P_B) \to \mathcal{O}_{R_B}(Z_B - F)$ on the pullback of the universal flag to B. To finish the proof, we observe that the pair $(B, \phi = p^*(\phi_0) + \alpha \circ \pi)$ represents M_U. To see this, let S be a scheme, $Z_S \subset R_S \subset P_S \subset F \times S$ a flag corresponding to a map $h : S \to D$ that factors through U, and let $\psi : \mathcal{I}_{P_S} \otimes \mathcal{O}_{R_S} \to \mathcal{O}_{R_S}(Z_S - F_S)$ be a map satisfying $\psi \circ \tau_S = \sigma_S$. By construction the map $\psi - h^*(\phi_0)$ is the image of a map in $\mathrm{Hom}(\mathcal{O}_{R_S}(-P_S), \mathcal{O}_{R_S}(Z_S - F_S))$, hence the universal property of $B \to S$ yields a unique lifting $\tilde{h} : S \to B$ of h. It is clear from construction that $\psi = \tilde{h}^*(\phi)$, thus (B, ϕ) represents M_U. □

The following lemma, which we used in the above proof, is immediate from the theorems of base change for cohomology and for the $\mathcal{E}xt$ functors [4, 5].

Lemma 3.3

Let $f : R \to U$ be a morphism of locally Noetherian schemes over k, and let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on R. Let $G = G_{\mathcal{F},\mathcal{G}} : Sch_U \to Sets$ be the contravariant functor that to a locally Noetherian U-scheme T associates the set

$$G(T) = \mathrm{Hom}_{R_T}(\mathcal{F}_T, \mathcal{G}_T)$$

where R_T, $\mathcal{F}_T$, $\mathcal{G}_T$ are the base extensions to T. Suppose that f is projective and flat, and $\mathcal{F}$,$\mathcal{G}$ are flat over U. Furthermore, suppose that for every point $u \in U$:

(1) $\mathcal{E}xt^1_{\mathcal{O}_{R_u}}(\mathcal{F}_u, \mathcal{G}_u) = 0$;

(2) $H^1(R_u, \mathcal{H}om_{\mathcal{O}_{R_u}}(\mathcal{F}_u, \mathcal{G}_u)) = 0$.

Then the sheaf $\mathcal{E} = f_* \mathcal{H}om_{\mathcal{O}_R}(\mathcal{F}, \mathcal{G})$ is locally free on U, and G is represented by the geometric vector bundle over U whose sheaf of sections is $\mathcal{E}$.

We note the following important consequence of Theorem 3.2, and give examples of how it can be used to prove the existence of interesting specializations of curves.

Corollary 3.4

Let $Y \subset D_{z,r,p}$ be an irreducible subset of triples satisfying the conditions of Proposition 2.5. If $C_0 \in H_{z,r,p}(X)$ is a curve with $T(C_0) \in Y$, then C_0 is a specialization of a curve $C \in H_{z,r,p}(X)$ for which $T(C)$ is general in Y.

Example 3.5: Let W be a quasi-primitive triple line of type $(0,b)$ in $\mathbb{P}^3$ for some $b \geq 0$ [3, 18]. Then the underlying double line D necessarily lies on a smooth quadric surface Q [18, 1.5] and hence W lies on the double quadric $X = 2Q$. The associated triple is $T(W) = \{Z, L, D\}$, where L is the support of W and $Z \subset L$ is a divisor of degree $b+2$ by the genus formula (1), since $g(W) = -2-b$ [18, 2.3a]. If H denotes the hyperplane divisor, then

$$\mathrm{H}^1(\mathcal{O}_L(Z+D-Q-H)) \cong \mathrm{H}^1(\mathcal{O}_L(b-1)) = 0$$

since $b \geq 0$ and Remark 2.7(b) applies. We deduce from Corollary 3.4 that W is the limit of a family of curves on $2Q$ whose general member is the disjoint union of a line and a double line. This generalizes the deformation used in the proof of [18, 3.3].

Example 3.6: This is the example that inspired the present paper. Let $R = P$ be a double line $2L$ on the smooth quadric surface $Q \subset \mathbb{P}^3$. Let $c \geq b \geq 0$ be integers and let $Z \subset R$ be a divisor consisting of $c-b$ simple points and $b+2$ double points which are not contained in L. One can show that the triple $\{Z, R, P\}$ arises from a general quasiprimitive 4-line C of type $(0,b,c)$ and further that $H^1(R, \mathcal{O}_R(Z+P-Q)) = 0$ [19, 3.2]. It is not difficult to deduce from Theorem 3.2 that, since the general member of $|\mathcal{O}_Q(0,2)|$ is a disjoint union of two lines, C must be a limit of disjoint unions of double lines on $X = 2Q$. However, we needed to know *which* unions of double lines could be used. For this we apply Corollary 3.4. Let $p_1, p_2, \ldots, p_{c-b}$ be the reduced points of Z, and for each double point z_j of Z choose a plane H_j which contains z_j and is transverse to $L = R_{red}$. Let $\{L_t : t \in \mathbb{P}^1\}$ be the family of rulings on Q with $L_0 = L$, and define

$$Z_t = \{p_1, \ldots, p_{c-b}\} \ \cup \ \bigcup_{j=1}^{b+2} [H_j \cap (L_t \cup L_0)].$$

Setting $R_t = L_t \cup L_0$, we obtain a family of triples $\{Z_t, R_t, R_t\}$ specializing to $\{Z, R, P\}$ in $D_{z,r,p}$. Moreover, we have the vanishing

$$H^1(\mathcal{O}_{R_t}(Z_t + D_t - Q) \cong H^1(\mathcal{O}_{L_0}(c+2-2)) \oplus H^1(\mathcal{O}_{L_t}(b+2-2)) = 0$$

for all t. Thus we may apply Corollary 3.4 to see that C is a limit of two double lines: one has triple $\{A_0, L_0, L_0\}$, where A_0 consists of $c+2$ points (hence has genus $-1-c$ by formula (1)) and the other has triple $\{A_t, L_t, L_t\}$ in which A_t consists of $b+2$ points (hence has genus $-1-b$).

EXAMPLE 3.7: It is not true that all 4-lines C on $X = 2Q$ are limits of disjoint unions of double lines. Recall that a *thick* 4-line is a curve of degree 4 supported on a line L and containing the first infinitesimal neighborhood $L^{(2)}$ [3]. We claim that such a curve is **not** a flat limit of disjoint unions of double lines on X. To see this, we first note that the family of double lines of genus g_1 with fixed support is irreducible of dimension $1 - 2g_1$ by [18, 1.6]. Since the lines on Q form a one-dimensional family, the disjoint unions of two double lines of genera g_1 and g_2 form a family of dimension $4 - 2g_1 - 2g_2 = 2 - 2g$.

On the other hand, the thick 4-lines on fixed support L are determined by surjections in

$$\mathrm{Hom}(\mathcal{I}_{L^{(2)}}, \mathcal{O}_L(-g-1)) \cong \mathrm{Hom}(\mathcal{O}_L(-2)^3, \mathcal{O}_L(-g-1)) \cong H^0(\mathcal{O}_L(-g+1)^3)$$

by [3, §4], hence these form an irreducible family of dimension $5-3g$. We are interested in the subset of those which send the equation of X to zero. If $L = \{x = y = 0\}$ and $Q = \{xz - yw = 0\}$, then $X = \{x^2z^2 - 2xyzw + y^2w^2 = 0\}$ and hence the thick 4-lines with support L lying on X correspond to the triples $\{(a, b, c) \in H^0(\mathcal{O}_L(-g+1)^3) : az^2 - 2zwb + cw^2 = 0\}$. These form a vector subspace of codimension $-g+4$ (provided char $k \neq 2$), hence the family has dimension $1 - 2g$. Varying the support line L on Q, we obtain a family of dimension $2 - 2g$ and conclude that the general thick 4-line C cannot be the limit of a family whose general member is a disjoint union of two double lines.

References

1. D. Bayer and D. Eisenbud, Ribbons and their canonical embeddings, *Trans. Amer. Math. Soc.* **347** (1995), 719–756.
2. E. Ballico and G. Bolondi, The variety of module structures, *Arch. Math. (Basel)* **54** (1990), 397–408.
3. C. Bănică and O. Forster, Multiplicity structures on space curves, The Lefschetz centennial conference, Part I, *Contemp. Math.* **58** (1986), 47–64.
4. C. Bănică, M. Putinar, and G. Schumacher, Variation der globalen ext in deformationen kompakter komplexer Räume, *Math. Ann.* **250** (1980), 135–155.
5. T. de Jong and D. van Straten, Deformations of the normalization of hypersurfaces, *Math. Ann.* **288** (1990), 527–547.
6. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
7. R. Hartshorne, Generalized divisors on Gorenstein schemes, *K-Theory* **8** (1994) 287–339.
8. R. Hartshorne, The genus of space curves, *Ann. Univ. Ferrara Sez. VII (N.S.)* **40** (1994), 207–223.
9. R. Hartshorne, On the connectedness of the Hilbert scheme of curves in $\mathbb{P}^3$, *Comm. Algebra* **28** (2000), 6059–6077.
10. R. Hartshorne, Questions of connectedness of the Hilbert scheme of curves in $\mathbb{P}^3$, Preprint Math.AG/0104265. Proceedings of the Abhyankar Conference, Purdue Univ. 2000, Springer Verlag, 2001.
11. R. Hartshorne, M. Martin-Deschamps, and D. Perrin, Triades et familles de courbes gauches, *Math. Ann.* **315** (1999), 397–468.
12. R. Hartshorne and E. Schlesinger, Curves in the double plane, *Comm. Algebra* **28** (2000), 5655–5676.

13. K. Hulek, The normal bundle of a curve on a quadric, *Math. Ann.* **258** (1981-82), 201–206.
14. J. Kollár, *Rational Curves on Algebraic Varieties*, Springer-Verlag, Berlin Heidelberg New York, 1996.
15. M. Martin-Deschamps and D. Perrin, *Sur la Classification des Courbes Gauches,* Astérisque 184–185, 1990.
16. J. Migliore, *Introduction to Liaison Theory and Deficiency Modules,* Progress in Mathematics **165**, Birkhäuser, Boston, 1998.
17. D. Mumford, *Lectures on Curves on an Algebraic Surface,* Math. Studies 59, Princeton University Press, Princeton, NJ, 1966.
18. S. Nollet, The Hilbert schemes of degree three curves, *Ann. Sci. École Norm. Sup. 4* **30** (1997) 367–384.
19. S. Nollet and E. Schlesinger, Hilbert schemes of degree four curves, *Compositio Math.,* to appear.
20. D. Perrin, Un pas vers la connexité du schéma de Hilbert: les courbes de Koszul sont dans la composante des extrémales, *Collect. Math.* **52** (2001), 295–319.
21. E. Sernesi, Topics on Families of Projective Schemes, Queen's Papers in Pure and Applied Mathematics 73, Queen's University Press, Kingston, 1986.