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Curves on a smooth quadric

S. Giuffrida and R. Maggioni

Dipartimento di Matematica e Informatica, Viale A. Doria, 6

95125 Catania, Italia

E-mail: giuffrida@dmi.unict.it maggioni@dmi.unict.it

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Abstract

We associate to every curve on a smooth quadric a polynomial equation that defines it as a divisor; this polynomial is defined through a matrix. In this way we can study several properties of these curves; in particular we can give a geometrical meaning to the rank of the matrix which defines the curve.

Introduction

Plane curves have been studied very much since long time and are well known. The situation drastically changes when we consider non-degenerate space curves, let us say, of $\mathbb{P}^3$. The reason of this greater complexity is obvious: to any plane curve we can associate a principal ideal, while this is not the case for space curves.

In this paper we study properties of curves lying on a smooth quadric surface. In some sense this is just a step forward: after plane curves we increase by one the minimal degree of a surface containing the curves we consider. This step is meaningful because we associate to the quadric $\mathbf{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ a bi-graded ring $G(\mathbf{Q}) = k[u, u'; v, v']$ (the global ring of $\mathbf{Q}$), so that to any effective divisor of $\mathbf{Q}$ it corresponds a bi-graded bi-homogeneous polynomial which gives the equation of the divisor. Note that this bi-graduation of $G(\mathbf{Q})$ is induced by the Picard group of $\mathbf{Q}$, $\operatorname{Pic}\mathbf{Q} \cong \mathbb{Z}^2$.

Hence any curve lying on $\mathbf{Q}$ can be studied by means of a single equation, i.e. by a polynomial. The analogy with plane curves is not superficial; although it is explicitly considered in the last section, this analogy is understood in all the paper.

In Section 1 we fix the notation and give preliminary results. Section 2 deals with an important tool: the change of reference frame on $\mathbf{Q}$. Unfortunately, this change is very heavy and is not friendly to be used.

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To any curve of $\mathbf{Q}$ we associate a matrix H whose entries are in the fixed field k. Has the rank of H a geometrical meaning? The surprising reply is contained in Section 3. The last section contains the definition of affine quadric and precises the analogy between curves of the affine quadric and curves of the affine plane by means of a natural isomorphism between them.

As usual, for basic notions we refer to the Hartshorne's book [9].

1. Notation and preliminaries

Let $\mathbb{P}^1 = \mathbb{P}^1_k$ (k an algebraically closed field of characteristic zero) and let $\mathbf{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric and $\mathcal{O}_{\mathbf{Q}}$ be its structure sheaf. Since $\operatorname{Pic} \mathbf{Q} \cong \mathbb{Z} \times \mathbb{Z}$, as usual we can assume the classes of the two rulings as basis of $\operatorname{Pic} \mathbf{Q}$. If $D \subset \mathbf{Q}$ is a divisor, the class of D is singled out by a couple of integers (m, n), the *type* of D. If $D \subset \mathbf{Q}$ is a divisor of type (m, n), we denote by $\mathcal{O}_{\mathbf{Q}}(m, n)$ the corresponding sheaf, and, for any sheaf $\mathcal{F}$ on $\mathbf{Q}$ we set

$$\mathcal{F}(m, n) = \mathcal{F} \otimes \mathcal{O}_{\mathbf{Q}}(m, n)$$

We also use the notation

$$H^i(m, n) = H^i(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(m, n)), \quad h^i(m, n) = \dim_k H^i(m, n)$$

$$H^i(\mathcal{F}(m, n)) = H^i(\mathbf{Q}, \mathcal{F}(m, n)), \quad h^i(\mathcal{F}(m, n) = \dim_k H^i(\mathcal{F}(m, n))$$

for $i = 0, 1, 2$. The dimensions $h^i(m, n)$ $(i = 0, 1, 2)$ are easily computed (see [7], §1); note that for any divisor $D \subset \mathbf{Q}$ of type (m, n), effective or not, the Euler characteristic of $\mathcal{O}_{\mathbf{Q}}(m, n)$ is

$$\chi(\mathcal{O}_{\mathbf{Q}}(m, n)) = |(m + 1)(n + 1)|$$

since only one of $H^i(m, n)$ $(i = 0, 1, 2)$ can be different from zero.

Let us consider

$$S = H^0_*(m, n) = \bigoplus_{\substack{m \geq 0 \\ n \geq 0}} H^0(m, n)$$

S is in a natural way a k-algebra using product of sections; we call it the *global ring* of $\mathbf{Q}$ ([4]). It is easy to check that S is generated, as a k-algebra, by $H^0(1, 0)$ and $H^0(0, 1)$ (both vector spaces of dimension 2) since for every $m, n \geq 0$ the maps

$$H^0(m, n) \otimes H^0(1, 0) \to H^0(m + 1, n)$$

$$H^0(m, n) \otimes H^0(0, 1) \to H^0(m, n + 1)$$

given by the products are surjective (see Lemma 2.3 of [7] for a generalization).

S is a bi-graded k-algebra taking $H^0(m, n) = S_{(m,n)}$ as the bi-homogeneous component of degree (m, n). When $s \in H^0(m, n)$, with $(m, n) > (0, 0)$, its zero locus $(s)_0$ will be called a curve of type (m, n) or an (m, n)-curve; in particular $L = (l)_0$ and $L' = (l')_0$, with $l \in H^0(1, 0)$ and $l' \in H^0(0, 1)$ will be mentioned as $(1, 0)$-lines and

$(0,1)$-lines respectively. When no confusion can arise we will not distinguish between curves and their defining forms.

Let u, u' and v, v' be bases for $H^0(1,0)$ and $H^0(0,1)$; there is a bi-graded ring isomorphism

$$S \cong k[u,u'] \otimes k[v,v'] = k[u,u';v,v']$$

by which one can identify elements of $k[u,u';v,v']$ with elements of S. We deal only with bi-homogeneous ideals of S, i.e. ideals generated by elements which are homogeneous both with respect to u, u' and v, v'. From now on we will call them homogeneous ideals for short.

An ideal $\boldsymbol{a} \subset S$ is *irrelevant* when it contains either a power of $\boldsymbol{u} = (u,u')$ or a power of $\boldsymbol{v} = (v,v')$ (see [7], Section 1).

Let P be any point on $\mathbf{Q}$, i.e. the zero locus of an ideal $\boldsymbol{p} = (l(u,u'), l'(v,v'))$ where $l = au+bu'$ and $l' = cv+dv'$ are linear forms; the element $(b,-a;d,-c) \in k^2 \times k^2$ gives the coordinates of P as subvariety of $\mathbf{Q}$, with respect to the chosen basis. Of course for any $\rho, \rho' \in k \setminus \{0\}$ one has $(b,-a;d,-c) = (\rho b, -\rho a; \rho' d, -\rho' c)$.

We can embed $\mathbf{Q} \hookrightarrow \mathbb{P}^3$ by the Segre map, so that every subscheme $X \subset \mathbf{Q}$ can be seen as a subscheme of $\mathbb{P}^3$. Hence we can consider both the ideal sheaves of X,

$$\overline{\mathcal{I}}_X \subset \mathcal{O}_{\mathbf{Q}}, \quad \mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^3}$$

If $X \subset \mathbf{Q}$ is a curve — i.e. an effective divisor — then the graded Betti numbers of a minimal free resolution of $\mathcal{I}_X$ are well known (see [5]).

Considered as subscheme of $\mathbf{Q}$, a curve Γ of type (m,n) is the zero locus of a homogeneous polynomial of degree (m,n)

$$F(u,u';v,v') = \sum_{\substack{i=0,\dots,m\\ j=0,\dots,n}} h_{ij}\, u^{m-i} u'^{i}\, v^{n-j} v'^{j}$$

Setting

$$\boldsymbol{u}^{\,m} = (u^m, u^{m-1}u', \dots, u'^m), \ \boldsymbol{v}^{\,n} = (v^n, v^{n-1}v', \dots, v'^n)$$

and $H = (h_{ij}) \in k^{m+1,n+1}$, the equation of Γ can be written

$$F(u,u';v,v') = \boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$$

Notice that, given the equation $F = 0$ of Γ, we can explicitly write a minimal set of generators for the ideal of Γ as a subscheme of $\mathbb{P}^3$: consider a basis of the image of the map (suppose $m \le n$)

$$H^0(n-m,0) \otimes F \to H^0(n,n)$$

these elements give, by the Segre embedding, $n-m+1$ forms of degree n in $k[x_0,x_1,x_2,x_3]$ which, together with the equation of $\mathbf{Q}$, generate the ideal of $\Gamma \subset \mathbb{P}^3$.

2. Changes of frame in $\mathbf{Q}$

A change of coordinates, from $(u, u'; v, v')$ to $(r, r'; s, s')$, is given by a couple of automorphisms of $H^0(1,0)$ and $H^0(0,1)$, hence by two invertible 2×2 matrices P, Q such that, setting $\boldsymbol{r} = (r, r')$, $\boldsymbol{s} = (s, s')$

$$^t\boldsymbol{u} = P\,^t\boldsymbol{r}, \qquad ^t\boldsymbol{v} = Q\,^t\boldsymbol{s}$$

We want to determine how the matrix H of an (m,n)-curve Γ changes under this transformation. Setting $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can compute

$$^t\boldsymbol{u}^{\,m} = (P\,^t\boldsymbol{r}\,)^m = \begin{pmatrix} (ar+br')^m \\ (ar+br')^{m-1}(cr+dr') \\ \cdots \\ (cr+dr')^m \end{pmatrix} = P^{(m)}\,^t\boldsymbol{r}^{\,m}$$

where $P^{(m)}$ is the $(m+1)\times(m+1)$ matrix whose columns are the $(m+1)$-tuples of the coefficients of $r^m, r^{m-1}r', \ldots, r'^m$. The $(i+1)$-th row of $P^{(m)}$ consists of the $m+1$ monomials of the product $(a+b)^{m-i}(c+d)^i$ for $i = 0, \ldots, m$. For instance, for $m = 3$, we get

$$P^{(3)} = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d+2abc & 2abd+b^2c & b^2d \\ ac^2 & 2acd+bc^2 & ad^2+2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}$$

Repeating the same argument for $^t\boldsymbol{v}^{\,n}$ we get the new equation for Γ:

$$\Gamma: \quad \boldsymbol{r}^{\,m} H'\,^t\boldsymbol{s}^{\,n} = 0$$

with $H' = {}^tP^{(m)}HQ^{(n)}$.

Now we want to describe the matrices like $P^{(m)}$ in a more general setting.

Let us consider a k-vector space $V = <\boldsymbol{x}, \boldsymbol{y}>$ with basis $\boldsymbol{x}, \boldsymbol{y}$ and the automorphism $p : V \to V$ determined by the matrix P with respect to this basis. For any $m > 0$ we get an automorphism

$$S_m p : S_m V \to S_m V$$

where $S_m V$ is the m-th symmetric power and $S_m p$ is induced by p. Let $S_m P$ be the matrix associated to $S_m p$ with respect to the basis $(\boldsymbol{x}, \boldsymbol{y})^m$. A direct computation shows that

$$P^{(m)} = {}^t(S_m\,{}^tP) \quad \text{so that} \quad \boldsymbol{u}^{\,m} = \boldsymbol{r}^{\,m}(S_m\,{}^tP)$$

Note that the relation

$$H' \cong H \iff \text{there exist } P, Q \in GL_2(k) \text{ such that } H' = {}^tP^{(m)}HQ^{(n)}$$

is an equivalence relation.

$$\begin{tikzcd} V \arrow[r, "P"] \arrow[dr, "RP"'] & V \arrow[d, "R"] \\ & V \end{tikzcd} \qquad \begin{tikzcd} S_m V \arrow[r, "S_m P"] \arrow[dr, "S_m R \cdot S_m P"'] & S_m V \arrow[d, "S_m R"] \\ & S_m V \end{tikzcd}$$

Figure 1

This can be deduced by the commutative diagrams in Figure 1, where $V = < \boldsymbol{x}, \boldsymbol{y} >$, $P, R \in GL_2(k)$; note that, by the functoriality of S_m, one has

$$S_m R \cdot S_m P = S_m(R \cdot P)$$

In particular $S_m P$ is an invertible matrix.

Proposition 2.1

Let $V = < \boldsymbol{x}, \boldsymbol{y} >$ be a 2-dimensional vector space; then for any $r > 0$ there is an equivariant $GL(V)$-isomorphism

$$\bigwedge^{r+1}(S_r V) \cong S_{\binom{r+1}{2}}(\bigwedge^2 V)$$

Proof. (Boffi [1]) We had a couple of proofs of this proposition by G. Boffi. The first one uses properties of plethysms to show that both 1-dimensional vector spaces are isomorphic to the same Shur module $L_{\boxed{\,}\boxed{\,}\}\binom{r+1}{2}} V$.

The second proof is based on the classical characters' theory (see for instance [11], Chapter 1). We omit these proofs since they would take us far from the object of this paper. □

Corollary 2.2

Let $V = < \boldsymbol{x}, \boldsymbol{y} >$ be a k-vector space, $p : V \to V$ an isomorphism and P its matrix with respect to the basis $\boldsymbol{x}, \boldsymbol{y}$; then $\det(P^{(m)}) = \det(S_m P) = (\det P)^{\binom{m+1}{2}}$

Proof. The linear function p induces the commutative square

$$\begin{tikzcd} \bigwedge^{m+1}(S_m V) \arrow[r, "\bigwedge^{m+1} S_m p"] \arrow[d] & \bigwedge^{m+1}(S_m V) \arrow[d] \\ S_{\binom{m+1}{2}}(\bigwedge^2 V) \arrow[r, "S_{\binom{m+1}{2}}(\bigwedge^2 p)"'] & S_{\binom{m+1}{2}}(\bigwedge^2 V) \end{tikzcd}$$

of isomorphisms. By Proposition 2.1 the scalar which determines the isomorphism in the upper row of the diagram, $\bigwedge^{m+1} S_m p$, is just that of the lower row, $S_{\binom{m+1}{2}}(\bigwedge^2 p)$, which is clearly $(\bigwedge^2 p)^{\binom{m+1}{2}}$, so

$$\bigwedge^{m+1} S_m p = (\bigwedge^2 p)^{\binom{m+1}{2}} . \ \square$$

Remark 2.3. To give a base change on $\mathbf{Q}$ is equivalent to give two invertible matrices $P, Q \in k^{2,2}$, i. e. projectivities of $\mathbb{P}(H^0(1,0))$ and $\mathbb{P}(H^0(0,1))$. Each of these projectivities is given by three couples of corresponding lines, so to the lines L_1, L_2, L_3 of type $(1,0)$ there correspond the lines R_1, R_2, R_3 of the same type and similarly to the $(0,1)$-lines L_1', L_2', L_3' there correspond R_1', R_2', R_3'. Hence the change of base is fixed by three couples of corresponding points, $L_i \cap L_i'$ and $R_i \cap R_i'$, no two collinear $(i = 1,2,3)$. Note that in this way nine couples of corresponding points are fixed: to $P_{ij} = L_i \cap L_j'$ it corresponds $P_{ij}' = R_i \cap R_j'$ $(i,j = 1,2,3)$. So, given a curve $\Gamma \subset \mathbf{Q}$, one can choose the new basis in such a way that Γ passes through the points of coordinates, say, $(0,1;0,1), (1,0;1,0), (1,1;1,1)$. With this choice one gets a matrix $H' = (h_{ij}')$ having $h_{00}' = h_{mn}' = 0, \sum h_{ij}' = 0$.

Remark 2.4. Given a curve $\Gamma \subset \mathbf{Q}$ having equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$, by a base change we get a new matrix, $H' = {}^tP^{(m)} H Q^{(n)}$ having the same rank of H since $P^{(m)}, Q^{(n)}$ are invertible.

3. The rank of H

Let $\Gamma \subset \mathbf{Q}$ be an (m,n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$ with $H \in k^{m+1,n+1}$. We want to study the geometric meaning of the rank of H. We deal with the rows of H, but of course similar arguments hold for columns.

Let us begin with the case $\operatorname{rank} H = 1$. The reader can find in [6], Section 1, a detailed discussion on the case when a given curve $\Gamma \subset \mathbf{Q}$ contains a $(1,0)$-line or a $(0,1)$-line, with multiplicity $s \geq 1$.

Theorem 3.1

Let $\Gamma \subset \mathbf{Q}$ be an (m,n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$. Then Γ is union of $m \cdot n$ lines if and only if $\operatorname{rank} H = 1$.

Proof. If Γ contains the $(1,0)$-line $L : au + bu' = 0$, then the equation of Γ can be written

$$(au + bu')\boldsymbol{u}^{\,m-1} K\, {}^t\boldsymbol{v}^{\,n} = 0$$

where $K \in k^{m,n+1}$. Setting

$$H = \begin{pmatrix} H^0 \\ \vdots \\ H^m \end{pmatrix} \qquad K = \begin{pmatrix} K^0 \\ \vdots \\ K^{m-1} \end{pmatrix}$$

a simple computation gives

$$H = \begin{pmatrix} aK^0 \\ aK^1 + bK^0 \\ \vdots \\ aK^{m-1} + bK^{m-2} \\ bK^{m-1} \end{pmatrix}$$

hence the rows of H are linear combinations of the rows of K. We conclude by repeating this argument m times.

Conversely, in order that a line L, of equation $au + bu' = 0$, be contained in Γ it must happen (see [6], Proposition 1.1 and Remark 1.2) that

$$(b,-a)^m \begin{pmatrix} \rho_0 \boldsymbol{v} \\ \rho_1 \boldsymbol{v} \\ \vdots \\ \rho_m \boldsymbol{v} \end{pmatrix} = \mathbf{0}, \qquad \text{with} \qquad \mathbf{0} \neq \boldsymbol{v} \in k^{n+1},\ (\rho_0, \rho_1, \ldots, \rho_m) \neq \mathbf{0}$$

hence $(b, -a)$ must be a solution of the homogeneous equation

$$(y,-x)^m \begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = 0$$

which has m non-zero solutions. The same for $(0,1)$-lines. □

One can easily find examples of curves Γ not containing lines, whose matrix has not maximal rank (see [6], Remark 1.3). In fact the geometric meaning of dropping rank is not given by the property of containing lines, which is a very special case; so we give the following definition which introduces the property which is the true responsible for the matrix of the curve to drop rank.

DEFINITION 3.2. We call *horizontal i-grid* of type $h \times k$ the complete intersection of a $(h,0)$-curve with an (i,k)-curve. Similarly a *vertical j-grid* of type $h \times k$ is the complete intersection of a $(0,k)$-curve with a (h,j)-curve. Note that horizontal 0-grids are also vertical 0-grids; hence they will be called 0-grids. A grid is contained in a curve Γ when it is a subscheme of Γ.

The following theorem is useful to understand the relevance of the grids for the rank of the matrix of a curve.

Theorem 3.3

Let $\Gamma \subset \mathbf{Q}$ be a reduced curve of type (m,n) not containing lines, of matrix H. Then the following are equivalent:

1) *every pair (L_1, L_1') consisting of a $(1,0)$-line and a $(0,1)$-line, such that $L_1 \cap L_1' \in \Gamma$ and meeting Γ in distinct points, determine a 0-grid of type $m \times n$ contained in Γ;*
2) *Γ contains a 0-grid of type $m \times n$;*
3) $\operatorname{rank} H = 2$.

Proof. Let $P_{11} \in \Gamma$ be a generic point, belonging to the $(1,0)$-line $L_1 : a_1u + b_1u' = 0$ and to the $(0,1)$-line $L'_1 : a'_1v + b'_1v' = 0$, such that

$$L_1 \cap \Gamma = \{P_{11}, P_{12}, \dots, P_{1n}\} \quad \text{and} \quad L'_1 \cap \Gamma = \{P_{11}, P_{21}, \dots, P_{m1}\}$$

consist of distinct points. Let $P_{1j} \equiv (b_1, -a_1; b'_j, -a'_j)$ $j = 1, 2, \dots, n$.

Cutting Γ with the $(0,1)$-line L'_j, containing P_{1j}, of equation $a'_jv + b'_jv' = 0$, we have the non-zero homogeneous polynomials

$$\boldsymbol{u}^{\,m} H \begin{pmatrix} b'_j \\ -a'_j \end{pmatrix}^n \qquad j = 1, \dots, n \tag{$*$}$$

whose roots give the points of $L'_j \cap \Gamma$. Γ contains the $m \times n$ 0-grid generated by the sides $L_1 \cap \Gamma$ and $L'_1 \cap \Gamma$ if and only if the polynomials $(*)$ have the same roots, and this happens if and only if there exist n non-zero constants $\rho_1, \rho_2, \dots, \rho_n$ such that

$$\rho_i H \begin{pmatrix} b'_i \\ -a'_i \end{pmatrix}^n = \rho_j H \begin{pmatrix} b'_j \\ -a'_j \end{pmatrix}^n$$

Defining the matrix

$$B = \left(\begin{pmatrix} b'_1 \\ -a'_1 \end{pmatrix}^n \begin{pmatrix} b'_2 \\ -a'_2 \end{pmatrix}^n \dots \begin{pmatrix} b'_n \\ -a'_n \end{pmatrix}^n \right)$$

which has rank n, one sees that Γ contains the above $m \times n$ 0-grid if and only if $\operatorname{rank}(HB) = 1$.

Considering the linear applications associated to H and B we have the commutative diagram

$$\begin{tikzcd} k^n \arrow[r, "\varphi_B"] \arrow[dr, "\varphi_{HB}"'] & k^{n+1} \arrow[d, "\varphi_H"] \\ & k^{m+1} \end{tikzcd}$$

Figure 2

Since B has maximal rank, if $\operatorname{rank}(HB) = 1$ then $\operatorname{rank} H \leq 2$ hence $\operatorname{rank} H = 2$ because Γ has no lines (Proposition 3.1). Hence 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

Finally, assuming $\operatorname{rank} H = 2$, we choose a generic point $P_{11} \in \Gamma$. Using the above notation we must show that $\operatorname{rank}(HB) = 1$, i.e. $\operatorname{Im} \varphi_{HB} \neq \operatorname{Im} \varphi_H$. If not, supposing $\operatorname{Im} \varphi_{HB} = \operatorname{Im} \varphi_H$, there are i and j, with $1 \leq i < j \leq n$, such that

$$\operatorname{Im} \varphi_{HB} = < \varphi_{HB}(\boldsymbol{e}_{\,i}), \varphi_{HB}(\boldsymbol{e}_{\,j}) > = < H \begin{pmatrix} b'_i \\ -a'_i \end{pmatrix}^n, H \begin{pmatrix} b'_j \\ -a'_j \end{pmatrix}^n >$$

By a base change we can set

$$\begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b'_i \\ -a'_i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b'_j \\ -a'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that Γ passes through the points $(1,0;1,0)$, $(1,0;0,1)$ and the matrix H has $h_{00} = h_{0n} = 0$.

Now,

$$\varphi_{HB}(\boldsymbol{e}_{\,i}) = \varphi_H \begin{pmatrix} b'_i \\ -a'_i \end{pmatrix}^n = \varphi_H \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} h_{00} \\ h_{10} \\ \dots \\ h_{m0} \end{pmatrix} = \begin{pmatrix} 0 \\ h_{10} \\ \dots \\ h_{m0} \end{pmatrix}$$

and similarly

$$\varphi_{HB}(\boldsymbol{e}_{\,j}) = \begin{pmatrix} 0 \\ h_{1n} \\ \dots \\ h_{mn} \end{pmatrix}$$

hence the first and the last column of H are independent, so $\operatorname{rank} H = 2$ implies that the first row of H is zero. But this means that Γ must contain the line $u' = 0$, a contradiction. $\square$

So we have proven a characterization for 0-grids which are maximal for the curve Γ. But we can say more about 0-grids of type $h \times n$ or of type $m \times k$; for this we need more tools.

Let $R \subset k^{m+1}$ be the vector space of relations among the rows of H, and let $\dim R = r + 1$, so that $\operatorname{rank} H = m - r$. R determines in $\mathbb{P}^m$ a linear subspace $\mathcal{R}$ of dimension r. Consider the rational normal curve $X_m \subset \mathbb{P}^m$ and, following the notation of [8], let $S_k(X_m)$ be the variety of secant k-planes of X_m (every secant k-plane meets X_m in $k+1$ points, so that $S_0(X_m) = X_m$). Recall that the varieties $S_k(X_m)$ can be obtained as determinantal varieties from the catalecticant matrix (also called Hankel matrix)

$$T = \begin{pmatrix} x_0 & x_1 & \dots & x_d \\ x_1 & x_2 & \dots & x_{d+1} \\ \dots & & \dots & \dots \\ x_{m-d} & x_{m-d+1} & \dots & x_m \end{pmatrix}$$

built up as square as possible, whose entries are the indeterminates $x_0, x_1, \dots, x_m$. In particular $S_0(X_m) = X_m$ is the vanishing locus of the 2-minors, and of course has dimension 1, $S_1(X_m)$ is the vanishing locus of the 3-minors, and has dimension 3, ..., $S_{s-1}(X_m)$ (with s such that either $m = 2s$ or $m = 2s+1$) is the zero locus of the $(s+1)$-minors. Notice that $S_s(X_m) = \mathbb{P}^m$, $S_{s-1}(X_m)$ is a hypersurface when m is even, and for $i \le s-1$ $\dim S_i(X_m) = 2i+1$ (see [8] Lec. 9, Lec. 11 Proposition 11.32).

We want to decompose R (and hence $\mathcal{R}$) into the direct sum of subspaces, in order to obtain a kind of filtration.

DEFINITION 3.4. Let $\mathit{\Gamma} \subset \mathbf{Q}$ be an (m,n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$; let $R \subset k^{m+1}$ be the vector subspace generated by the relations among the rows of H and let $\mathcal{R} \subset \mathbb{P}^m$ be the corresponding linear subspace. We consider the points of the variety $S_i(X_m) \subset \mathbb{P}^m$ as elements of k^{m+1}. We set

- $R_0 =< R \cap S_0(X_m) >$, $\mathcal{R}_0 \subset \mathbb{P}^m$ the corresponding linear subspace.
- R_1 is generated by elements of $R \cap S_1(X_m)$ and is such that $R_0 \oplus R_1 =< R \cap S_1(X_m) >$, $\mathcal{R}_1 \subset \mathbb{P}^m$ the corresponding linear subspace.
- ...
- for any $i :\ 0 < i \leq s = [m/2]$: R_i is generated by elements of $R \cap S_i(X_m)$ and is such that $R_0 \oplus R_1 \oplus \ldots \oplus R_i =< R \cap S_i(X_m) >$, $\mathcal{R}_i \subset \mathbb{P}^m$ the corresponding linear subspace.

Remark 3.5. While R_0 is univocally determined, this is not the case for R_i when $i > 0$. For instance it can happen that we find $\gamma_1, \ldots, \gamma_r$, with $\gamma_i \in (R \cap S_1(X_m)) \setminus R_0$, which are not linearly independent. In fact we can construct a curve $\mathit{\Gamma}$ in which this situation occurs. Suppose $m = 8$ and take 10 points in X_8, say $P_i \equiv (b_i, -a_i)^8$ for $i = 1, \ldots, 10$. These points are linearly dependent, so we have a relation $\sum_{i=1}^{10} q_i P_i = \mathbf{0}$ with $q_i \neq 0$. Define the five 9-tuples

$$\gamma_1 = q_1 P_1 + q_2 P_2, \quad \ldots, \quad \gamma_5 = q_9 P_9 + q_{10} P_{10}$$

and observe that $\gamma_1, \ldots, \gamma_5$ are linearly dependent. Now construct a matrix $H \in k^{9,r}$ (say with $r \geq 9$) such that $\gamma_i H = \mathbf{0}$ for $i = 1, \ldots, 5$ and $R_0 = \{0\}$. Now $\dim R_1 < 5$.

Remark 3.6. It is easy to check that for any $P_1, \ldots, P_{i+1} \in X_m$ there are just two possibilities for the linear space they generate, denote it by $\mathcal{L}(P_1, \ldots, P_{i+1})$:

$$\mathcal{L}(P_1, \ldots, P_{i+1}) \cap \mathcal{R}_i = \begin{cases} \emptyset & \\ 1 & \text{point} \end{cases}$$

In fact, if $\gamma, \gamma' \in \mathcal{L}(P_1, \ldots, P_{i+1}) \cap \mathcal{R}_i$, then the line $\mathcal{L}(\gamma, \gamma')$ meets the linear subspaces $\mathcal{L}(P_1, \ldots, \widehat{P_j}, \ldots, P_{i+1})$ hence is contained in $\mathcal{R}_{i-1}$, a contradiction because by definition $\mathcal{R}_{i-1} \cap \mathcal{R}_i = \emptyset$.

Remark 3.7. Note that if $\delta \in R_i \cap S_i(X_m)$ then there exists a unique secant i-plane containing δ, unless $m = 2s$ and $i = s$. In fact in this case $S_s(X_m) = \mathbb{P}^m$ and for any point $D \in \mathbb{P}^m$ there exists a pencil of secant s-planes.

The geometrical meaning of a generator that we choose in R_0, say $(b, -a)^m \in X_m \cap R$ is that the line of equation $au + bu' = 0$ is a component of the curve $\mathit{\Gamma}$.

As a consequence, in the sequel we shall consider curves of $\mathbf{Q}$ not containing any line. With this choice we get $R_1 =< R \cap S_1(X_m) >$. Now we see how the generators that we choose in $R_1, R_2, \ldots$, determine grids contained in $\mathit{\Gamma}$.

Proposition 3.8

Let $\mathit{\Gamma} \subset \mathbf{Q}$ be an (m,n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$ not containing lines. If $\mathit{\Gamma}$ contains a 0-grid of type $h \times n$ ($h < m$) then there exist $h-1$ independent relations among the rows of H. The same result holds for 0-grids of type $m \times k$ ($k < n$) and columns of H.

Proof. Suppose first that the h $(1,0)$-lines of the grid are distinct, and let $a_iu+b_iu'=0$ $(i=1,\dots,h)$ be their equations. By assumption the polynomials

$$(b_i,-a_i)^m H\,{}^t\boldsymbol{v}^{\,n} \quad (i=1,\dots,h)$$

have the same roots, i.e. they have proportional coefficients. Hence

$$(b_i,-a_i)^m H=\rho_i(b_1,-a_1)^m H \quad (i=2,\dots,h)$$

and we have the $h-1$ relations

$$\gamma_i=(b_i,-a_i)^m-\rho_i(b_1,-a_1)^m$$

If $\gamma_2,\dots,\gamma_h$ were linearly dependent one had a relation among h points of X_m, with $h<m$, a contradiction.

Suppose now that the h lines of the grid are not distinct. For the sake of simplicity we take $h=2$: we have a $2\times n$ 0-grid having the line L of equation $au+bu'=0$ with multiplicity 2. Call $(d_i,-c_i)$, $i=1,\dots,n$, the roots of the homogeneous polynomial $(b,-a)^m H\,{}^t\boldsymbol{v}^{\,n}$; each $(0,1)$-line of equation $c_iv+d_iv'=0$ meets Γ at the point $(b,-a;d_i,-c_i)$ with multiplicity 2. Hence the polynomials $\boldsymbol{u}^{\,m}H\,{}^t(d_i,-c_i)^n$, $i=1,\dots,n$, have the double root $(b,-a)$; so, using derivatives (see [6] Section 1):

$$\Big(\frac{\partial}{\partial u}\boldsymbol{u}^{\,m}\Big)_{(b,-a)}H\begin{pmatrix}d_i\\-c_i\end{pmatrix}^n=0;\quad \Big(\frac{\partial}{\partial u'}\boldsymbol{u}^{\,m}\Big)_{(b,-a)}H\begin{pmatrix}d_i\\-c_i\end{pmatrix}^n=0;\quad i=1,\dots,n$$

This means that the polynomials

$$\Big(\frac{\partial}{\partial u}\boldsymbol{u}^{\,m}\Big)_{(b,-a)}H\,{}^t\boldsymbol{v}^{\,n},\ \Big(\frac{\partial}{\partial u'}\boldsymbol{u}^{\,m}\Big)_{(b,-a)}H\,{}^t\boldsymbol{v}^{\,n}$$

have the same roots; now the conclusion follows as above. The same holds for vertical 0-grids and columns of H. □

Notice that the $h-1$ relations among the rows of H in the above proposition are elements of $R\cap S_1(X_m)$: γ_i belongs to the secant line P_1P_i of X_m. A kind of converse of the above proposition holds.

Proposition 3.9

Let $\Gamma\subset\mathbf{Q}$ be an (m,n)-curve not containing lines; with the above notation let $\gamma\in R_1\cap S_1(X_m)$. Then Γ contains a horizontal 0*-grid of type* $2\times n$ *determined by* γ.

Proof. Suppose first $\gamma\in P_1P_2$ with $P_1,P_2\in X_m$, $P_1\neq P_2$; hence

$$\gamma=\rho(b_1,-a_1)^m-(b_2,-a_2)^m$$

this means that

$$\rho(b_1,-a_1)^m H=(b_2,-a_2)^m H$$

that is the lines L_1,L_2 of equations $a_iu+b_iu'=0$ cut on Γ a 0-grid.

Suppose now that $\gamma \in T_P(X_m)$, where $T_P(X_m)$ is the tangent line to X_m at the point $P \equiv (b,-a)^m$. The three points P, $(\frac{\partial}{\partial u}\boldsymbol{u}^{\,m})_{(b,-a)}$, $(\frac{\partial}{\partial u'}\boldsymbol{u}^{\,m})_{(b,-a)}$ belong to the line $T_P \subset \mathbb{P}^m$ (see [2], page 380), so the relation $\gamma \in T_P$ says that the polynomials

$$(b,-a)^m H\,{}^t\boldsymbol{v}^{\,n}, \quad \left(\frac{\partial}{\partial u}\boldsymbol{u}^{\,m}\right)_{(b,-a)} H\,{}^t\boldsymbol{v}^{\,n}, \quad \left(\frac{\partial}{\partial u'}\boldsymbol{u}^{\,m}\right)_{(b,-a)} H\,{}^t\boldsymbol{v}^{\,n}$$

have the same roots $(d_i,-c_i)$, $i=1,\ldots,n$.

Each line $c_iv+d_iv'=0$ is tangent to Γ at the point $(b,-a;d_i,-c_i)$, hence the line $au+bu'=0$, with multiplicity 2, gives a 0-grid on Γ. $\square$

If $\gamma \in R_1 \cap S_1(X_m)$ belongs to the secant line P_1P_2, one can consider the cone C_1 projecting X_m from P_1. If $\gamma' \in C_1 \cap R$, $\gamma \neq \gamma'$, then γ and γ' determine a 0-grid of type $3 \times n$. In fact, if $\gamma' \in P_1P_3$, then $(b_3,-a_3)^mH, (b_2,-a_2)^mH, (b_1,-a_1)^mH$ are proportional. In this way we can embed each 0-grid in a maximal one involving, say, the points $P_1,P_2,\ldots,P_r \in X_m$. Note that changing the vertex of the projection, among the points $P_1,\ldots,P_r$, the 0-grid does not change.

Summing all we have seen that the elements of $R_1 \cap S_1(X_m)$ determine horizontal 0-grids on the curve Γ. Of course the same holds for vertical 0-grids and the relations among the columns of H.

Now we consider the general case, that is we study the horizontal h-grids of Γ.

Theorem 3.10

Let $\Gamma \subset \mathbf{Q}$ be an (m,n)-curve not containing lines; with the above notation, for any integer h such that $h+1 \leq (m-1)/2$, there exists a bijection between the horizontal h-grids of type $(h+2)\times n$ contained in Γ and the elements of $R_{h+1} \cap S_{h+1}(X_m)$.

Proof. Let $\delta \in R_{h+1} \cap S_{h+1}(X_m)$; δ belongs to a secant $(h+1)$-plane Λ of X_m. Suppose first that Λ cuts X_m in $h+2$ distinct points, say $\delta \in \mathcal{L}(P_1,\ldots,P_{h+2})$ with $P_i \equiv (b_i,-a_i)^m \in X_m$ $(i=1,\ldots,h+2)$:

$$\delta = \lambda_1(b_1,-a_1)^m + \ldots + \lambda_{h+1}(b_{h+1},-a_{h+1})^m - (b_{h+2},-a_{h+2})^m$$

the equality $\delta H = \mathbf{0}$ gives

$$(b_{h+2},-a_{h+2})^m H = \lambda_1(b_1,-a_1)^m H + \ldots + \lambda_{h+1}(b_{h+1},-a_{h+1})^m H$$

Let L_i be the $(1,0)$-lines of equation $a_iu+b_iu'=0$ $(i=1,\ldots,h+2)$ and set for $i=1,\ldots,h+1$

$$L_i \cap \Gamma = \{P_{i1},\ldots,P_{in}\}$$

The curves of type (h,n) passing through the points P_{ij} $(i=1,\ldots,h+1; j=1,\ldots,n)$ move in a linear system Φ of dimension h; otherwise one could find a curve of Φ passing through one more point for each of the lines $L_1,\ldots,L_{h+1}$, and this curve should contain $h+1$ $(1,0)$-lines.

The equation of Φ can be written:

$$\sum_{i=1}^{h+1} \mu_i (a_1 u + b_1 u') \cdot \ldots \cdot (\widehat{a_i u + b_i} u') \cdot \ldots \cdot (a_{h+1} u + b_{h+1} u') \cdot (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = 0$$

using the $h+1$ curves which are union of the $(1,0)$-lines $L_1, \ldots, \widehat{L}_i, \ldots, L_{h+1}$ and the $(0,1)$-lines passing through the points of $L_i \cap \Gamma$ (where $\hat{}$ means omitted).

The relation δ gives the equality

$$(b_{h+2}, -a_{h+2})^m H\, {}^t\boldsymbol{v}^{\,n} = \sum_{i=1}^{h+1} \lambda_i (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} \tag{†}$$

We look for the points cut on L_{h+2} by Φ

$$\begin{cases} a_{h+2} u + b_{h+2} u' = 0 \\ \sum_{i=1}^{h+1} \mu_i (a_1 u + b_1 u') \cdot \ldots \cdot (\widehat{a_i u + b_i} u') \cdot \ldots \cdot (a_{h+1} u + b_{h+1} u') \cdot (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = 0 \end{cases}$$

from which, setting $c_{ij} = a_i b_j - a_j b_i \neq 0$, we have

$$\sum_{i=1}^{h+1} \mu_i (c_{1h+2} \cdot \ldots \cdot \hat{c}_{ih+2} \cdot \ldots \cdot c_{h+1h+2}) \cdot (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = 0 \tag{‡}$$

Intersecting Γ with L_{h+2} we have

$$(b_{h+2}, -a_{h+2})^m H\, {}^t\boldsymbol{v}^{\,n} = 0$$

from which, by (†), we have

$$\sum_{i=1}^{h+1} \lambda_i (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = 0$$

Now, comparing with (‡), we see that the points of $L_{h+2} \cap \Gamma$ are cut by the curve of type (h, n) obtained by the equation of Φ giving to the parameters the values:

$$\mu_i = \frac{\lambda_i c_{ih+2}}{\prod_{j=1}^{h+1} (c_{jh+2})} \qquad i = 1, \ldots, h+1$$

Hence, starting from the relation $\delta \in R_{h+1} \cap S_{h+1}(X_m)$ among the rows of H, we have constructed a horizontal h-grid of type $(h+2) \times n$ contained in Γ.

Conversely let Δ be the (h, n)-curve cutting on the $(1,0)$-lines $L_1, \ldots, L_{h+2}$ (L_i of equation $a_i u + b_i u' = 0$) the given h-grid of Γ. Δ belongs to the linear system Φ of (h, n)-curves passing through the points of $L_i \cap \Gamma$ $(i = 1, \ldots, h+1)$; hence, writing Φ as in the direct part of the theorem, one gets the equation of Δ for $\mu_i = \overline{\mu}_i$:

$$\Delta : \sum_{i=1}^{h+1} \overline{\mu}_i \prod_{\substack{j=1 \\ j \neq i}}^{h+1} (a_j u + b_j u') \cdot (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = 0$$

Since $\Delta \cap L_{h+2} = \mathit{\Gamma} \cap L_{h+2}$, one gets

$$\sum_{i=1}^{h+1} \overline{\mu}_i \frac{\prod_{j=1}^{h+1} c_{jh+2}}{c_{ih+2}} \cdot (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = \rho(b_{h+2}, -a_{h+2})^m H\, {}^t\boldsymbol{v}^{\,n}$$

Renaming parameters we get

$$\sum_{i=1}^{h+1} \lambda_i (b_i, -a_i)^m H\, {}^t\boldsymbol{v}^{\,n} = \rho(b_{h+2}, -a_{h+2})^m H\, {}^t\boldsymbol{v}^{\,n}$$

So we have found the relation

$$\delta = \sum_{i=1}^{h+1} \lambda_i (b_i, -a_i)^m - \rho(b_{h+2}, -a_{h+2})^m \in R_{h+1} \cap S_{h+1}(X_m)$$

Now we go to the non-reduced case, when the $(h+1)$-plane $\mathit{\Lambda}$ meets X_m in less than $h+2$ distinct points. If $\mathit{\Lambda} \cap X_m$ contains the point $P \equiv (a,b)^m$ r times, then the points

$$\left(\frac{\partial^{r-1}\boldsymbol{u}^{\,m}}{\partial u^{r-1}}\right)_P, \left(\frac{\partial^{r-1}\boldsymbol{u}^{\,m}}{\partial u^{r-2}\,\partial u'}\right)_P, \dots, \left(\frac{\partial^{r-1}\boldsymbol{u}^{\,m}}{\partial u'^{\,r-1}}\right)_P \qquad \text{of } \ \mathbb{P}^m$$

are r independent generators of $\mathit{\Lambda}$ (see [2], page 380 for a non-homogeneous version of this fact).

In order to make the argument more understandable we consider the case $h = 1$ and suppose that $\mathit{\Lambda}$ is the osculating 2-plane to X_m at $P \equiv (a,b)^m$. Moreover we can assume, up to a coordinate change, that $P \equiv (0,1)^m$ and that the $(1,0)$-line $L:\ u = 0$ cuts $\mathit{\Gamma}$ in n points $P_j \equiv (0,1;\alpha_j,\beta_j)$ with $\beta_j \neq 0$ $(j = 1,\dots,n)$; we set $b_j = \alpha_j/\beta_j$. This means that the polynomial $H^m\, {}^t\boldsymbol{v}^{\,n}$ has degree n with respect to v. Hence the relation $\delta \in \mathit{\Lambda} \cap R_2$ can be uniquely written in the form

$$\delta = d_0(0,\dots,0,0,1) + d_1(0,\dots,0,1,0) - (0,\dots,1,0,0)$$

so that $\delta H = 0$ means that $H^{m-2} = d_1 H^{m-1} + d_0 H^m$. Now, as in the reduced case, we look for the pencil Φ of $(1,n)$-curves of non-homogeneous equation

$$G(u,v) = (u,1)\begin{pmatrix} K^0 \\ K^1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix}^n = 0$$

passing through $P_1, \dots P_n$ and having intersection multiplicity ≥ 2 with $\mathit{\Gamma}$ at each point P_j.

The equation of $\mathit{\Gamma}$ has the following non-homogeneous form

$$F(u,v) = (u,1)^m H \begin{pmatrix} v \\ 1 \end{pmatrix}^n = 0$$

so that a $(1,n)$-curve passes through $P_1,\dots,P_n$ if and only if the polynomials $H^m\,{}^t\boldsymbol{v}^{\,n}$, $K^1\,{}^t\boldsymbol{v}^{\,n}$ have the same roots, i.e. $K^1=\rho H^m$. Calling $\mathcal{O}_{P_j}=K[u,v]_{(u,v-b_j)}$ we must impose (see for instance [3]) for the intersection multiplicity that

$$I(P_j,F\cap G)=\dim_k\frac{\mathcal{O}_{P_j}}{(F,G)}\geq 2\quad j=1,\dots,n$$

Now

$$I(P_j,F\cap G)=I(P_j,(\rho F-G)\cap F)=I(P_j,u\cap F)+I(P_j,(\rho\overline{F}-K^0\,{}^t\boldsymbol{v}^{\,n})\cap F)$$

where we have set $F:=u\overline{F}+H^m\,{}^t\boldsymbol{v}^{\,n}$.

Suppose first that $I(P_j,u\cap F)=1$ for $j=1,\dots,n$. So we must require that $I(P_j,(\rho\overline{F}-K^0\,{}^t\boldsymbol{v}^{\,n})\cap F)\geq 1$. This happens if and only if $(\rho H^{m-1}-K^0)\,{}^t\boldsymbol{v}^{\,n}\in(u,v-b_j)$, $j=1,\dots,n$. This means that the polynomial $(\rho H^{m-1}-K^0)\,{}^t\boldsymbol{v}^{\,n}$ must have the roots $b_1,\dots,b_n$, so $\rho H^{m-1}-K^0=\lambda H^m$. Hence Φ has the equation

$$(u,1)\begin{pmatrix}\rho H^{m-1}-\lambda H^m\\ \rho H^m\end{pmatrix}\begin{pmatrix}v\\1\end{pmatrix}^n=0$$

One can check that the same result follows also in the case when the line L cuts Γ in less than n distinct points: if P_j has algebraic multiplicity σ in $L\cap\Gamma$ then one must require that $I(P_j,(\rho\overline{F}-K^0\,{}^t\boldsymbol{v}^{\,n})\cap F)\geq\sigma$ in order that F,G have the correct intersection multiplicity at P_j.

Now we show that the curve of Φ having intersection multiplicity ≥ 3 at one of the points P_j has the same multiplicity at any P_i. Denoting again by G the polynomial of Φ we get

$$\rho F-G=u[\rho(u^{m-1}H^0+\ldots+uH^{m-2})+\lambda H^m]\,{}^t\boldsymbol{v}^{\,n}:=uF_1$$

hence

$$I(P_j,F\cap G)=I(P_j,(\rho F-G)\cap F)=I(P_j,u\cap F)+I(P_j,F_1\cap F)$$

Now we compute

$$\rho F_1-\lambda G=[\rho^2u(u^{m-2}H^0+\ldots+H^{m-2})-\lambda u(\rho H^{m-1}-\lambda H^m)]\,{}^t\boldsymbol{v}^{\,n}:=uF_2$$

so that $I(P_j,F\cap G)=2I(P_j,u\cap F)+I(P_j,F_2\cap F)$. In order that $I(P_j,F_2\cap F)\geq 1$ we require that

$$(\rho^2H^{m-2}-\lambda\rho H^{m-1}+\lambda^2H^m)\,{}^t\boldsymbol{v}^{\,n}\in(u,v-b_j)$$

Using the relation δ, i.e. $H^{m-2}=d_1H^{m-1}+d_0H^m$, we have

$$\left[\rho(\rho d_1-\lambda)H^{m-1}+(\rho^2d_0+\lambda^2)H^m\right]\,{}^t\boldsymbol{v}^{\,n}\in(u,v-b_j)$$

Since $H^m\, {}^t\boldsymbol{v}^{\,n} \in (v - b_j)$ for every j and H^{m-1} is not proportional to H^m, for some j we must have $H^{m-1}\, {}^t\boldsymbol{v}^{\,n} \notin (v - b_j)$: this means $\rho d_1 - \lambda = 0$ so that at each point P_j $(j = 1, \ldots, n)$ one gets $I(P_j, F_2 \cap F) \geq 1$.

Conversely, if the $(1, n)$-curve Δ cutting on Γ the 1-grid is given, the last steps of the previous proof give

$$\rho H^{m-2} - \lambda H^{m-1} = H^m$$

and this is the relation in $R_2 \cap S_2(X_m)$ corresponding to the 1-grid. □

4. The affine quadric

The quadric $\mathbf{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ has the standard covering given by the four open subsets

$$U_{00} = \{(u, u'; v, v') \mid u \neq 0, v \neq 0\} \quad U_{01} = \{(u, u'; v, v') \mid u \neq 0, v' \neq 0\}$$

$$U_{10} = \{(u, u'; v, v') \mid u' \neq 0, v \neq 0\} \quad U_{11} = \{(u, u'; v, v') \mid u' \neq 0, v' \neq 0\}$$

Each of these open sets is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$, and can be considered an *affine quadric*. Referring to U_{11}, we see that its ring is $k[u, v]$, considered as a bi-graded ring. We use the isomorphism $\pi_{11} : k[u, v] \to k[x, y]$ defined by $u \mapsto x, v \mapsto y$, to identify affine curves in U_{11} with affine plane curves. In the usual embedding $\mathbf{Q} \subset \mathbb{P}^3$, given by $x = uv',\ y = u'v,\ z = uv,\ t = u'v'$, this corresponds to the stereographic projection of $\mathbf{Q}$ from the point $(1, 0; 1, 0)$ to the plane $z = 0$.

If $\Gamma \subset \mathbf{Q}$ is an (m, n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$, its restriction Γ_a to U_{11} has equation $f(u, v) = (u, 1)^m H\, {}^t(v, 1)^n = 0$; the corresponding plane curve $\pi_{11}(\Gamma_a)$ has equation $f(x, y) = (x, 1)^m H\, {}^t(y, 1)^n = 0$.

This plane curve is naturally bi-graded: if Γ does not contain as components neither the line $u' = 0$ nor the line $v' = 0$, then its bi-degree is (m, n). As to the total degree we must look at the matrix $H = (h_{ij})$. Setting $r = min\{i + j \mid h_{ij} \neq 0\}$ we have $\deg f(x, y) = m + n - r$. Note that, up to a change of frame in $\mathbf{Q}$, we always can suppose that $\pi_{11}(\Gamma_a)$ has degree $m + n$.

Of course every polynomial in x, y can be seen as a bi-graded one, and can be written in matricial form (this is a classical point of view; see also [10], where the bi-graded polynomial is called 'generating function' of the matrix).

It is possible to describe the geometrical meaning of the rank of the matrix of a plane curve $f(x, y)$. Observe that, in the isomorphism π_{11}, to lines of U_{11} of equation $u = \lambda, v = \mu$, there correspond lines of equation $x = \lambda, y = \mu$ respectively.

Conversely, to a generic plane line $L : ax + by + c = 0$ it corresponds the $(1, 1)$-curve $\pi_{11}^{-1}(L) : au + bv + c = 0$. In particular, to a $(1, 1)$-curve tangent to Γ at P it corresponds a line tangent to $\pi_{11}(\Gamma_a)$ at $\pi_{11}(P)$.

Let $\Gamma \subset \mathbf{Q}$ be an (m, n)-curve of equation $\boldsymbol{u}^{\,m} H\, {}^t\boldsymbol{v}^{\,n} = 0$. The relations among the lines of H depend on the existence of ℓ-grids.

The image by π_{ij} $(i, j = 0, 1)$ of a horizontal, say, ℓ-grid of type $h \times n$ in general is an ℓ-grid of the same type, cut on $\pi_{ij}(\Gamma_a)$ by h lines L_a of equation $x = \lambda_a$ $(a = 1, \ldots, h)$ by a plane curve of bi-degree (ℓ, n).

But in few particular cases one has to pay attention. For instance, if a horizontal grid contains the line $u' = 0$, then this line 'disappears' if we project Γ by π_{11}; it happens that the points of this line are 'pushed to infinity' in direction of $x = 0$. One can recover these points looking at the tangent lines to $\pi_{11}(\Gamma_a)$ at x_∞. Of course this problem can be bypassed by a suitable change of frame on $\mathbf{Q}$. The next example clarifies this situation.

EXAMPLE 4.1: Let $\Gamma \subset \mathbf{Q}$ be the $(3,3)$-curve associated to the rank 3 matrix

$$H = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \end{pmatrix}.$$

One sees that among the rows of H there is the relation $\gamma = (2,0,0,-1) = 2(1,0)^3 - (0,1)^3$; so there exists a 0-grid of type 2×3 cut on the lines $u = 0, u' = 0$ by the $(0,3)$-curve of equation $v(v^2 - v'^2) = 0$. The plane curve $\pi_{11}(\Gamma_a)$ has equation:

$$x^3y^3 + x^2y^3 - x^3y + x^2y^2 - xy^2 + 2y^3 + x^2 + xy - 2y = 0\,.$$

This curve has triple points at x_∞, y_∞; cutting with the line $x = 0$, image of $u = 0$, one gets the points $(0,0), (0,1), (0,-1)$. The remaining three points of the grid are recovered noting that the lines $y = 0,\ y = 1,\ y = -1$ through the previous points are tangent at x_∞.

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