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Orbifold principal bundles on an elliptic fibration and parabolic principal bundles on a Riemann surface

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Abstract

Let X be a compact Riemann surface and associated to each point p_i of a finite subset S of X is a positive integer m_i. Fix an elliptic curve C. To this data we associate a smooth elliptic surface Z fibered over X. The group C acts on Z with X as the quotient. It is shown that the space of all vector bundles over Z equipped with a lift of the action of C is in bijective correspondence with the space of all parabolic bundles over X with parabolic structure over S and the parabolic weights at any p_i being integral multiples of $1/m_i$. A vector bundle V over Z equipped with an action of C is semistable (respectively, polystable) if and only if the parabolic bundle on X corresponding to V is semistable (respectively, polystable). This bijective correspondence is extended to the context of principal bundles.

1. Introduction

Fix an elliptic curve C over $\mathbb{C}$. Given a h-pointed Riemann surface $(X, \{p_1, p_2, \cdots, p_h\})$ with a positive integer m_i for each marked point p_i, we construct an elliptic surface Z fibered over X. For any marked point p_i, the inverse image $f^{-1}(p_i)$, where f denotes the projection of Z to X, is $C/(\mathbb{Z}/m_i\mathbb{Z})$. For any nonmarked point, the fiber is a copy of C. The surface Z is constructed using logarithmic transformation on $X \times C$.

The group C acts on Z with X as the quotient. We consider vector bundles over Z equipped with a lift of the action of C, and call them orbifold bundles. We note that the terminology orbifold or orbifold bundle is usually used in the context of finite group actions. Here the role of the finite group is played by the group C.

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We construct a bijective correspondence between the set of orbifold bundles over Z and the set of parabolic bundles over X with parabolic structure at the marked pointed and the weights at each p_i being multiples of $1/m_i$ (Theorem 4.4). Given an orbifold vector bundle E over Z, the corresponding parabolic bundle over X is constructed using the subsheaves of E that are left invariant by the action of C.

Furthermore, this bijective correspondence takes semistable (respectively, polystable) orbifold bundles exactly to the parabolic semistable (respectively, polystable) bundles.

Let G be a connected semisimple algebraic group over $\mathbb{C}$. By an orbifold G–bundle over Z we mean a principal G–bundle over Z equipped with a lift of the action of C. In [1] parabolic G–bundles were defined.

In Theorem 5.1 we extend this bijective correspondence to principal G–bundle. Just as for vector bundles, this bijective correspondence preserves semistability and polystability.

2. Orbifold principal bundles

Fix a lattice $\Lambda := \mathbb{Z}+\mathbb{Z}\tau$ of $\mathbb{C}$, with the imaginary part of τ being nonzero. Therefore, $\mathbb{C}/\Lambda$ is an elliptic curve, which we will denote by C.

Let X be a compact Riemann surface and

$$S := \{p_1, p_2, \cdots, p_h\} \subset X$$

be a finite subset of h distinct points. To each point $p_i \in S$, we associate a positive integer m_i. Given this data, we have an elliptic fibration

$$f : Z \longrightarrow X \tag{1}$$

such that for any point $x \in X \setminus S$, we have $f^{-1}(x) = C$, and for any $p_i \in S$, the fiber over p_i is the quotient C by its subgroup $\mathbb{Z}/m_i\mathbb{Z}$, or in other words, the reduced inverse image

$$f^{-1}(p_i)_{\mathrm{red}} = \mathbb{C}/\left(\frac{\mathbb{Z}}{m_i} + \mathbb{Z}\tau\right), \tag{2}$$

and $f^{-1}(p_i) = m_i f^{-1}(p_i)_{\mathrm{red}}$. The details of this construction of elliptic fibration can be found in [2, p. 164]. However, we will briefly recall the construction.

To extend the trivial fibration $(X \setminus S) \times C$ over $X \setminus S$ to X satisfying the above conditions, it suffices to take a punctured disk around p_i and extend the family across the puncture.

Let $D = \{z \in \mathbb{C} \big| \, |z|^2 \leq 1\}$ be the unit disk and $D^0 := D \setminus \{0\}$ be the punctured disk. Let D' be another copy of the disk D and

$$g : D' \longrightarrow D \tag{3}$$

be the map defined by $z \longmapsto z^{m_i}$. Consider the following action of the cyclic group $\mathbb{Z}/m_i\mathbb{Z}$ on $D' \times C$. For $\beta \in \mathbb{Z}$, the action of β sends any $(s\,,c)$ to

$$\big(\exp(2\pi\sqrt{-1}\beta/m_i)s\,,c+\beta/m_i\big) \in D' \times C\,. \tag{4}$$

Since this is a diagonal action and the map g in (3) is invariant for the action on the factor D', the quotient

$$Z_{p_i} := \frac{D' \times C}{\mathbb{Z}/m_i\mathbb{Z}} \tag{5}$$

for the action in (4) has a map

$$f_i : Z_{p_i} \longrightarrow D\,. \tag{6}$$

It is easy to see that this extends the trivial fibration $D^0 \times C$ over D^0. This is an example of a logarithmic transformation. We recall that logarithmic transformations were introduced by K. Kodaira [7].

From the above construction it is immediate that the condition (2) is satisfied over 0 and we have $f_i^{-1}(0) = m_i f_i^{-1}(0)_{\mathrm{red}}$. For each $p_i \in S$, the fibration f_i is the local model for the elliptic fibration f in (1) around the point p_i.

The translation action of the group C on itself induces an action of C on any quotient of it. Since C is abelian, left and right translations coincide. Therefore, C acts on the fibration Z in (1). So we have a homomorphism

$$\phi : C \longrightarrow \mathrm{Aut}(Z) \tag{7}$$

such that the projection f in (1) is invariant for this action of C on Z. For any point $x \in X \setminus S$, this action is free and transitive on $f^{-1}(x)$. But for any $p_i \in S$, although the action is transitive on $f^{-1}(p_i)_{\mathrm{red}}$, it is not free. More precisely, the subgroup $\mathbb{Z}/m_i\mathbb{Z}$ of C is the kernel. Therefore, we have $X = Z/C$.

It is easy to see that the surface Z is smooth. Indeed, it is immediate from its local description given in (6). We will now show that Z is a projective algebraic surface.

Let m be a common multiple of the numbers $m_1, m_2, \cdots, m_h$. Consider the subgroup

$$\Gamma := \mathbb{Z}/m\mathbb{Z} \subset C$$

defined using the inclusion of $\mathbb{R}$ in $\mathbb{C}$. Taking the quotient of Z by the action of Γ defined using ϕ in (7), we obtain a (ramified) Galois covering

$$g : Z \longrightarrow Z/\Gamma \tag{8}$$

with Galois group Γ. It is easy to see that $Z/\Gamma = X \times (C/\Gamma)$. Since Z/Γ is a projective manifold and the quotient map g is finite, we conclude that Z is algebraic. Indeed, using a criterion for ampleness [5, p. 65, Proposition 1.2(iv)] and the projection formula, it follows that for any coherent analytic sheaf F on Z and any ample line bundle L on

Z/Γ, we have $H^i(Z,\, F \bigotimes g^*L^{\otimes k}) \,=\, 0$ for all $i \,>\, 0$ and k sufficiently large. Therefore, a sufficiently large tensor power of g^*L is very ample, establishing projectivity of Z.

Let G be an algebraic group over $\mathbb{C}$. A *principal G–bundle* over Z is a complex manifold P equipped with a free right holomorphic action of G together with a smooth holomorphic surjective map

$$\gamma \,:\, P \longrightarrow Z \tag{9}$$

which is invariant for the action of G and $P/G \,=\, Z$ [12].

Since Z is algebraic, a G–bundle over Z is an algebraic bundle. An *isomorphism* between two such G–bundles P and P' is a biholomorphism between P and P' that commutes with the actions of G on P and P'.

DEFINITION 2.1. An orbifold G–bundle over Z is a principal G–bundle P with a holomorphic action of C on P satisfying the following two conditions:

1. the action of C on P commutes with the action of G;
2. the map γ in (9) is equivariant for the actions of C on P and Z.

From the above definition it follows immediately that for any $c \,\in\, C$, the automorphism of P defined by the action of c is a G–bundle isomorphism between P and the pullback bundle $\phi(c)^*P$, where ϕ is defined in (7).

If we set $G \,=\, \mathrm{GL}(n,\mathbb{C})$, then using the standard representation of $\mathrm{GL}(n,\mathbb{C})$, an orbifold $\mathrm{GL}(n,\mathbb{C})$–bundle over Z gets identified with a holomorphic vector bundle over Z of rank n equipped with a lift of the action of C. An orbifold $\mathrm{GL}(n,\mathbb{C})$–bundle will also be called an *orbifold vector bundle*.

In the next two sections we will identify orbifold vector bundles over Z with parabolic vector bundles over X.

3. Parabolic bundle associated to an orbifold bundle

Let V be an orbifold vector bundle of rank n over Z. Take any point $x \,\in\, X\setminus S$. Since the action of C on $f^{-1}(x)$ is free and transitive, the restriction of V to $f^{-1}(x)$ gets trivialized by the action of C on $V\big|_{f^{-1}(x)}$. More precisely, if we fix a point $y_0 \,\in\, f^{-1}(x)$, then for any point $y \,\in\, f^{-1}(x)$, the action of C on V identifies the fiber V_y with V_{y_0}. So $V\big|_{f^{-1}(x)}$ gets identified with the trivial vector bundle over $f^{-1}(x)$ with fiber V_{y_0}.

To understand the restriction of V to $f^{-1}(p_i)_{\mathrm{red}}$, where $p_i \,\in\, S$, we recall the local description of the fibration f given in (6).

Take any $p_i \,\in\, S$. Let

$$g_i \,:\, D' \times C \,\longrightarrow\, (D' \times C)/(\mathbb{Z}/m_i\mathbb{Z}) \,=\, Z_{p_i} \tag{10}$$

be the quotient map in (5). Consider the vector bundle g_i^*V over $D' \times C$ obtained by pulling back the restriction of V to $Z_{p_i} \,\subset\, Z$. The pullback of the action of C on V is an action of C on g_i^*V. Since this action descends to the translation action of C on $D' \times C$, the restriction of g_i^*V to $\{0\} \times C \,\subset\, D' \times C$ is trivial.

Since the group of automorphisms of a trivial bundle of rank n over C is $\mathrm{GL}(n,\mathbb{C})$, the action of $\mathbb{Z}/m_i\mathbb{Z}$ on $(g_i^*V)\big|_{\{0\}\times C}$ is given by a homomorphism of $\mathbb{Z}/m_i\mathbb{Z}$ into $\mathrm{GL}(n,\mathbb{C})$. We consider $\mathbb{C}^n$ as a $\mathbb{Z}/m_i\mathbb{Z}$–module using this homomorphism. Now, $\mathbb{C}^n$ decomposes as a direct sum of $\mathbb{Z}/m_i\mathbb{Z}$–modules of dimension one.

Let χ denote the character of $\mathbb{Z}/m_i\mathbb{Z}$ defined by

$$\alpha \longmapsto \exp(2\pi\sqrt{-1}\alpha/m_i)\,. \tag{11}$$

Any character of $\mathbb{Z}/m_i\mathbb{Z}$ is of the form χ^l, where $l \in [0\,, m_i - 1]$.

Consider the quotient $(C\times\mathbb{C})/(\mathbb{Z}/m_i\mathbb{Z})$ for the diagonal action, where the action on C is the translation action and the action on $\mathbb{C}$ is the one defined by χ (any $t \in \mathbb{Z}/m_i\mathbb{Z}$ sends $c \in \mathbb{C}$ to $\chi(t)c$). This is the line bundle over the base

$$C/(\mathbb{Z}/m_i\mathbb{Z}) \;=\; f_i^{-1}(0)_{\mathrm{red}}$$

where f_i is defined in (6), associated to the principal $\mathbb{Z}/m_i\mathbb{Z}$–bundle

$$C \longrightarrow C/(\mathbb{Z}/m_i\mathbb{Z}) \tag{12}$$

for the character χ^{-1}. This line bundle is identified with the normal bundle of the divisor $f_i^{-1}(0)_{\mathrm{red}} \subset D\times C$.

To see this identification with the normal bundle, first observe that for the action of C on $D'\times C$, the induced action on the normal bundle of the (complex) submanifold $\{0\}\times C \;\subset\; D'\times C$ is given by the character χ. Now sending the tangent vector $\frac{\partial}{\partial z} \in T_0D'$, where z is the coordinate as in (3), to $\frac{\partial}{\partial z} \in T_0D$ we conclude that normal bundle of $\{0\}\times C$ in $D'\times C$ descends, by the quotient map in (12), to the normal bundle of $f_i^{-1}(0)_{\mathrm{red}} = C/(\mathbb{Z}/m_i\mathbb{Z})$ in $D\times C$.

Therefore, the restriction of the orbifold vector bundle V to the reduced fiber $f_i^{-1}(0)_{\mathrm{red}}$ is identified with a direct sum of the form

$$V\big|_{f_i^{-1}(0)_{\mathrm{red}}} \;=\; \bigoplus_{j=1}^{n} N^{i_j} \tag{13}$$

where N is the normal bundle of $f_i^{-1}(0)_{\mathrm{red}} \subset D\times C$ and $i_j \in [0\,, m_i - 1]$. From the construction of this decomposition it is evident that the decomposition is preserved by the action of C on $V\big|_{f_i^{-1}(0)_{\mathrm{red}}}$.

Note that N^{m_i} is the trivial bundle as the character χ is of order m_i. Using the adjunction formula [4, p. 146], the normal bundle N is identified with the restriction of the line bundle $\mathcal{O}_Z(f_i^{-1}(0)_{\mathrm{red}})$ to $f_i^{-1}(0)_{\mathrm{red}} \subset Z$.

Given an orbifold vector bundle V of rank n over Z, consider the direct image f_*V on X for the projection f in (1). Since f_*V is a torsionfree coherent sheaf, it defines a vector bundle over X. For any $x \in X$, there is a natural homomorphism

$$(f_*V)_x \longrightarrow H^0\big(f^{-1}(x)_{\mathrm{red}},\, V\big|_{f^{-1}(x)_{\mathrm{red}}}\big) \tag{14}$$

Since $V\big|_{f^{-1}(x)}$ is the trivial vector bundle of rank n for any $x \in X \setminus S$, we conclude that f_*V is a vector bundle of rank n over X. For any $x \in X \setminus S$, we have $(f_*V)_x \cong H^0(f^{-1}(x),\, V\big|_{f^{-1}(x)})$.

For any $p_i \in S$, using the decomposition (13) it follows that the image of $(f_*V)_{p_i}$ in $H^0(f^{-1}(p_i)_{\mathrm{red}},\, V\big|_{f^{-1}(p_i)_{\mathrm{red}}})$ for the homomorphism in (14) coincides with the subspace

$$\bigoplus_{\{i\in[1,n] \mid i_j=0\}} H^0(f^{-1}(p_i)_{\mathrm{red}},\, N^{i_j}) \subset \bigoplus_{j=1}^{n} H^0(f^{-1}(p_i)_{\mathrm{red}},\, N^{i_j})\,. \tag{15}$$

We will show that f_*V has a natural parabolic structure over the divisor $S \subset X$. For that we first recall from [9], [8] the definition of a parabolic bundle.

DEFINITION 3.1. A parabolic vector bundle on X with parabolic structure over the divisor S is a vector bundle E over X together with the following data:

1. for each $p_i \in S$, there is a filtration
$$E_{p_i} = F_i^1 \supset F_i^2 \supset F_i^3 \supset \cdots \supset F_i^{l_i} \supset F_i^{l_i+1} = 0$$
known as the quasiparabolic filtration,
2. and a sequence of rational numbers
$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \alpha_{i,3} < \cdots < \alpha_{i,l_i} < 1$$
corresponding to the filtration.

The numbers $\alpha_{i,j}$ are called the parabolic weights.

Let E_* denote the parabolic bundle defined above. The *parabolic degree* of E_* is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^{h}\sum_{j=1}^{l_i} \alpha_{i,j} n_{i,j}$$

where $n_{i,j} = \dim(F_i^j/F_i^{j+1})$ [9, Definition 1.11], [8, Definition 1.8].

Let $I \subset [1\,, n]$ be a subset. The subbundle

$$\bigoplus_{j\in I} N^{i_j} \subset \bigoplus_{j=1}^{n} N^{i_j} = V\big|_{f_i^{-1}(0)_{\mathrm{red}}}$$

over $f_i^{-1}(0)_{\mathrm{red}}$ constructed using the decomposition (13) will be denoted by $F_{i,I}$. Let $W_{I,i}$ denote the vector bundle over Z defined by the exact sequence

$$0 \longrightarrow W_{i,I} \longrightarrow V \longrightarrow (V\big|_{f_i^{-1}(0)_{\mathrm{red}}})/F_{i,I} \longrightarrow 0 \tag{16}$$

It was noted that the action of C on V preserves the decomposition (13). In particular, $F_{i,I}$ is preserved by the action. This immediately implies that the orbifold structure (i.e., the action of C) on V induces an orbifold structure on $W_{i,I}$.

Note that $f_*W_{i,I}$ is a subsheaf of f_*V, with the inclusion map obtained from the inclusion of $W_{i,I}$ in V in (16). Clearly $f_*W_{i,I}$ and f_*V coincide over $X \setminus \{p_i\}$. The parabolic structure on f_*V will be constructed using these subsheaves.

The parabolic weights on f_*V at the point p_i are $\{i_j/m_i\}_{j=1}^n$, where i_j are as in (13). Take any $t \in \{i_j/m_i\}$. Set

$$I := \{j \in [1\,,n] \,\big|\, i_j \geq tm_i\}$$

The subspace in the quasiparabolic filtration of $(f_*V)_{p_i}$ corresponding to the parabolic weight t is the image in $(f_*V)_{p_i}$ of $(f_*W_{i,I})_{p_i}$. In other words, the image of the fiber $(f_*W_{i,I})_{p_i}$ in $(f_*V)_{p_i}$ (for the inclusion map of the sheaf $f_*W_{i,I}$ in f_*V) is one of the terms in the quasiparabolic filtration over p_i (see Definition 3.1). Furthermore, all terms in the quasiparabolic filtration over p_i arise this way.

Thus we have constructed a vector bundle with parabolic structure from an orbifold bundle. We will denote by V_* the parabolic bundle constructed from V.

For a projective manifold Y, let $\mathrm{CH}^1(Y)$ denote the group of divisors on Y modulo rational equivalence. Let $\mathrm{CH}^1(Y)_{\mathbb{Q}} := \mathrm{CH}^1(Y) \bigotimes_{\mathbb{Z}} \mathbb{Q}$ denote the rational Chow group. For a divisor D, its cycle class in $H^2(Y, \mathbb{Q})$ will be denoted by $[D]$. The first Chern class is a homomorphism from coherent sheaves on Y to $\mathrm{CH}^1(Y)_{\mathbb{Q}}$. If we identify $\mathrm{CH}^1(Y)$ with the Picard group of line bundles on Y, then the homomorphism defined by the first Chern class is simply taking the top exterior power of a coherent sheaf.

Let Z be the elliptic surface constructed in Section 2 using the above m_i, $1 \leq i \leq h$.

Lemma 3.2

For first Chern class $c_1(V) \in CH^1(Z)_{\mathbb{Q}}$ coincides with

$$f^*c_1(f_*V) + \sum_{i=1}^{h}\sum_{j=1}^{n} \frac{i_j}{m_i} f^{-1}(p_i) \in CH^1(Z)_{\mathbb{Q}}$$

where the integers i_j are as in (13).

Proof. Consider the canonical homomorphism

$$\psi : f^*f_*V \longrightarrow V$$

It was noted earlier that for any $x \in X \setminus S$, the homomorphism in (14) is an isomorphism and the restriction $V\big|_{f^{-1}(x)}$ is trivial. Therefore, ψ is an isomorphism on $f^{-1}(X \setminus S)$. Since $m_i f^{-1}(p_i)_{\mathrm{red}} = f^{-1}(p_i)$ and $f^*c_1(W) = c_1(f^*W)$ for any vector bundle W on X, it suffices to show that $c_1(V/f^*f_*V) \in \mathrm{CH}^1(Z)_{\mathbb{Q}}$ coincides with $\sum_{i=1}^h \sum_{j=1}^n i_j f^{-1}(p_i)_{\mathrm{red}}$.

To calculate the quotient of the homomorphism ψ, we recall the construction in (5) and (6) of the local model of the fibration f around $p_i \in S$.

Since a complex $\mathbb{Z}/m_i\mathbb{Z}$–module of dimension n decomposes as a direct sum of n one dimensional modules, the question of determining the quotient reduces to the case of line bundles.

Consider the character χ^l of $\mathbb{Z}/m_i\mathbb{Z}$, where χ is defined in (11) and $l \in [0, m_i - 1]$. The group $\mathbb{Z}/m_i\mathbb{Z}$ acts on $\mathbb{C}$ using the character χ^l, and it acts on $D' \times C$ as the Galois group for the covering g_i defined in (10). The quotient of $(D' \times C) \times \mathbb{C}$ by the diagonal action of $\mathbb{Z}/m_i\mathbb{Z}$ is a line bundle, which we will denote by L^l, over Z_{p_i}, the quotient of $D' \times C$ by $\mathbb{Z}/m_i\mathbb{Z}$ as in (5). Consider $f_i^*(f_i)_*L^l$, where f_i is the projection as in (6). It is easy to see that $f_i^*(f_i)_*L^l$ is identified with the line bundle $\mathcal{O}_{Z_i}(-lf_i^{-1}(0)_{\mathrm{red}})$. In particular, the support of the quotient $L^l/f_i^*(f_i)_*L^l$ is $lf_i^{-1}(0)_{\mathrm{red}}$.

The restriction of L^l to $f_i^{-1}(0)_{\mathrm{red}}$ is of finite order (the order divides m_i). In particular, the rational Chern class of $L^l\big|_{f_i^{-1}(0)_{\mathrm{red}}}$ vanishes. Combining this with the observations in the previous paragraph we conclude that

$$c_1(V/f^*f_*V) = \sum_{i=1}^{h}\sum_{j=1}^{n} i_j f^{-1}(p_i)_{\mathrm{red}}$$

This completes the proof of the lemma. $\square$

The Lemma 3.2 compares the parabolic degree V_* with the degree of V.

We will construct a polarization on Z. Let γ_1 (respectively, γ_2) denote the positive generator of $H^2(X, \mathbb{Z})$ (respectively, $H^2(C/\Gamma, \mathbb{Z})$), where Γ as in (8). Let q_1 (respectively, q_2) be the projection of $Z/\Gamma = X \times (C/\Gamma)$ to X (respectively, C/Γ). Let

$$\xi := (q_1 \circ g)^*\gamma_1 + \frac{(q_2 \circ g)^*\gamma_2}{m} \in H^2(Z, \mathbb{Q}) \tag{17}$$

be the polarization on Z, where g as in (8) and $m = \#\Gamma$. It is easy to check that the polarization ξ does not depend on the choice of the common multiple m.

The degree of a coherent sheaf W on Z is defined to be $(c_1(W) \cup \xi) \cap [Z] \in \mathbb{Q}$.

Corollary 3.3

The degree of an orbifold vector bundle V coincides with the parabolic degree of the corresponding parabolic bundle V_*.

Proof. From the definition of the polarization ξ it is immediate that for any vector bundle W on X, the degree of f^*W coincides with the degree of W. Also recall that $m_i[f^{-1}(p_i)_{\mathrm{red}}] = [f^{-1}(x)]$. In view of these, comparing Lemma 3.2 with the definition of parabolic degree of V_* we immediately conclude that the parabolic degree of V_* coincides with the degree of V. This completes the proof of the corollary. $\square$

In the next section we will describe the inverse construction, that is, starting with a parabolic bundle we will construct an orbifold bundle. We will also compare their semistability conditions.

4. From parabolic bundle to orbifold bundle

In order to construct an orbifold bundle from a given parabolic on X, we first analyze further the relationship between V and f_*V, where V is an orbifold vector bundle over Z.

Note that V is a subsheaf of $f^*(\mathcal{O}_X(S) \bigotimes f_*V)$. Indeed, this is an immediate consequence of the observation in the previous section that the support of the quotient $L^l/f_i^*(f_i)_*L^l$ is $lf_i^{-1}(0)_{\mathrm{red}}$. The tensor product of the fiber $\mathcal{O}_X(S)_{p_i}$ with the quasi-parabolic filtration of the fiber $(f_*V)_{p_i}$ defines a filtration of $(\mathcal{O}_X(S) \bigotimes f_*V)_{p_i}$. The pullback of which induces a filtration of $f^*(\mathcal{O}_X(S) \bigotimes f_*V)\big|_{f^{-1}(p_i)}$. In terms of this filtration it is easy to trace the elementary transformations on $f^*(\mathcal{O}_X(S) \bigotimes f_*V)$ that give back V. Our next step is to do that.

Let E_* be a parabolic vector bundle, as in Definition 3.1, of rank n over X with parabolic structure over S and underlying vector bundle E. We assume that for each $p_i \in S$, the parabolic weights are of the form

$$\alpha_{i,j} = \frac{\lambda_{i,j}}{m_i} \tag{18}$$

where $\lambda_{i,j}$ is an integer. Note that the collection $\{\alpha_{i,j}\}$ do not determine $\{\lambda_{i,j}\,, m_i\}$. After fixing positive integers c_i for p_i, we can replace $\{\lambda_{i,j}\,, m_i\}$ by $\{c_i\lambda_{i,j}\,, c_i m_i\}$. Given a collection of parabolic weights $\alpha_{i,j}$, fix $\{\lambda_{i,j}\,, m_i\}$ satisfying (18).

For notational simplicity, we will denote $E \bigotimes \mathcal{O}_X(S)$ by W. For any pair $(i\,, j)$, consider the subspace of the fiber W_{p_i} defined by $F_i^j \bigotimes \mathcal{O}_X(S)_{p_i}$, where F_i^j as in Definition 3.1. Denoting this subspace by W_i^j we observe that $f^*W_i^j$ is a subbundle of $(f^*W)\big|_{f^{-1}(p_i)} \cong f^*W_{p_i}$ over $f^{-1}(p_i)$.

For notational simplicity, we will denote the divisor $(m_i - \lambda_{i,j})f^{-1}(p_i)_{\mathrm{red}}$ of Z by $D_{i,j}$ (recall that $\alpha_{i,j} = \lambda_{i,j}/m_i$). Let $\overline{W}_i^j$ denote the restriction of $f^*W_i^{j+1}$ to $D_{i,j}$ (note that $W_i^{m_i+1} = 0$). Recall that $f^*W_i^j$ is a vector bundle over $f^{-1}(p_i) = m_i f^{-1}(p_i)_{\mathrm{red}}$. So $\overline{W}_i^j$ is the restriction of this vector bundle to the subscheme $D_{i,j} \subset f^{-1}(p_i)$.

Now we construct a vector bundle V_i^j on X using the exact sequence

$$0 \longrightarrow V_i^j \longrightarrow f^*W \longrightarrow (f^*W)\big|_{D_{i,j}}/\overline{W}_i^j \longrightarrow 0 \tag{19}$$

where $i \in [1\,, h]$ and $j \in [1\,, m_i]$.

Note that since f^*W is a pullback of a vector bundle on X, it has an obvious orbifold structure (that is, an action of the elliptic curve C). The subsheaf V_i^j is clearly preserved by this action. Indeed, this is an immediate consequence of the observation that the action of C on $f^*W\big|_{D_{i,j}}$ preserves the subbundle $\overline{W}_i^j$.

Therefore, the intersection

$$\overline{V} := \bigcap_{i,j} V_i^j \tag{20}$$

is an orbifold bundle. Note that $\overline{V}$ and f^*W coincide on $Z \setminus f^{-1}(S)$.

It is easy to see that if we set E_* to be the parabolic bundle constructed in the previous section from an orbifold bundle V, then $\overline{V}$ in (20) is canonically isomorphic, as on orbifold bundle, with V. To prove this it is enough to consider the local model in (6) around $p_i \in S$. Using the decomposition of the $\mathbb{Z}/m_i\mathbb{Z}$–module $\mathbb{C}^n$ into a direct sum of one dimensional modules the problem reduces to the case of a line bundle. Now it is straight-forward to construct the isomorphism in question using the local model of V described in the proof of Lemma 3.2 in terms of the character χ.

Also, the parabolic bundle corresponding to $\overline{V}$ by the construction in the previous section coincides with E_*. Therefore, these two constructions are inverses of each other.

From the constructions it is evident that subsheaves of the parabolic bundle E_* are in one-to-one correspondence with the coherent subsheaves of $\overline{V}$ left invariant by the action of C.

We recall that a torsionfree coherent sheaf V on Z is called *semistable* if for any coherent subsheaf F of V, the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}$$

is valid. If the strict inequality is valid for all F with V/F torsionfree, then V is called *stable*. If V is a direct sum of stable sheaves with same degree/rank quotient, then V is called *polystable*. We will call an orbifold bundle V on Z to be *orbifold semistable* if for any coherent subsheaf F of V left invariant by the action of C, the above inequality is valid. Similarly, define *orbifold stability* and *orbifold polystability* by restricting to only invariant subsheaves for the action of C.

We recall that a parabolic bundle E_* is called *parabolic semistable* if for any subbundle F of the vector bundle underlying E_*, the inequality $\text{par-deg}(F_*)/\text{rank}(F_*) \leq \text{par-deg}(E_*)/\text{rank}(E_*)$ is valid, where F_* is the parabolic structure induced on F [9, Definition 1.13(ii)], [8, Definition 1.10(2)]. If the strict inequality is valid for any proper subbundle F, then E_* is called *parabolic stable*. If E_* is a direct sum of parabolic stable bundles with same par-deg/rank quotient, then E_* is called *parabolic polystable*.

In view of Corollary 3.3 we have the following proposition.

Proposition 4.1

A parabolic bundle E_* is parabolic semistable (respectively, parabolic stable) if and only if the corresponding orbifold vector bundle $\overline{V}$ is orbifold semistable (respectively, orbifold stable). Similarly, E_* is parabolic polystable if and only if $\overline{V}$ orbifold polystable.

The above proposition can be strengthened as follows.

Proposition 4.2

A parabolic bundle is parabolic semistable if and only if the corresponding orbifold vector bundle is semistable in the usual sense.

Proof. In view of Proposition 4.1 it suffices to show that an orbifold semistable bundle is semistable in the usual sense.

If V is a vector bundle which is not semistable, then it has a unique maximal unstable subsheaf F (the first term in the Harder–Narasimhan filtration) [6, p. 174, Theorem 7.15]. If V is an orbifold bundle, from the uniqueness of F it follows that the action of C on V leaves F invariant. Therefore, V cannot be orbifold semistable. This completes the proof of the proposition. $\square$

Although the corresponding assertion in the context of stable bundles is not valid, it remains valid for polystable bundles, as shown in the following proposition.

Proposition 4.3

A parabolic bundle is parabolic polystable if and only if the corresponding orbifold vector bundle is polystable in the usual sense.

Proof. All we need to show is that any orbifold bundle which is orbifold polystable is actually polystable in the usual sense.

If V is an orbifold bundle which is orbifold polystable, then it follows that V admits a Hermitian–Yang–Mills connection [13, p. 878, Theorem 1]. Set the group A in [13] to be C and the Higgs field to be zero. Since C is compact, given any Kähler form on Z, we can integrate (average) it over all automorphisms of Z given by the action of C. This average is a Kähler form on Z invariant under the action of C. Now [13, Theorem 1] says that an orbifold polystable bundle V admits a Hermitian–Yang–Mills connection which is invariant under the action of C on V.

On the other hand, a vector bundle over Z admits a Hermitian–Yang–Mills connection if and only if it is polystable [3, p. 1, Theorem 1], [13, Theorem 1]. This completes the proof of the proposition. $\square$

Combining the constructions in Section 3 and in this section, and Propositions 4.2 and 4.3, we have

Theorem 4.4

There is a bijective correspondence between the space of isomorphism classes of orbifold vector bundles over Z and parabolic bundles over X with parabolic structure over S satisfying the condition that at each $p_i \in S$, the parabolic weights are of the form λ/m_i, where $\lambda \in \mathbb{Z}$. An orbifold vector bundle is semistable (respectively, polystable) in the usual sense if and only if the corresponding parabolic bundle is parabolic semistable (respectively, parabolic polystable).

In view of the above theorem it is natural to ask whether a given vector bundle over Z can admit more than one orbifold structure. The following proposition answers this negatively.

Proposition 4.5

Let V be a holomorphic vector bundle over Z admitting an orbifold structure. Then V admits exactly one orbifold structure.

Proof. Take any $x \in X \setminus S$. The group C acts freely and transitively on $f^{-1}(x)$ and the restriction of V to $f^{-1}(x)$ is the trivial bundle.

To prove the proposition it suffices to show that for the trivial vector bundle $W := C \times \mathbb{C}^n$ of rank n over C, there is exactly one lift of the translation action of C on itself to W. The trivial action of C on $\mathbb{C}^n$ gives an action of C on W. Using this action we see that any action of C on W gives a map from C to $\mathrm{GL}(n,\mathbb{C})$. More precisely, fix a point $y \in C$. If we have an action of C on W, then send any $z \in C$ to the isomorphism of the fiber W_y with W_z defined by the action of the (unique) element in C that sends y to z. But $W_z = \mathbb{C}^n = W_y$. So this isomorphism is an element of $\mathrm{GL}(n,\mathbb{C})$. This defines a holomorphic map μ from C to $\mathrm{GL}(n,\mathbb{C})$.

Since $\mathrm{GL}(n,\mathbb{C})$ is an affine variety, there is no nonconstant map from C to $\mathrm{GL}(n,\mathbb{C})$. Now, since $\mu(y) = \mathrm{Id}$, we conclude that the action of C on W coincides with the trivial action. This completes the proof of the proposition. □

In the next section we will relate the parabolic analog of principal bundles over X with the orbifold principal bundles over Z introduced in Section 2.

5. Parabolic principal bundles

Let G be a connected semisimple algebraic group over $\mathbb{C}$. We will recall from [1] the definition of a parabolic G–bundle.

Let $\mathrm{Rep}(G)$ denote the category of all finite dimensional complex left G–modules. Note that $\mathrm{Rep}(G)$ is equipped with the operations of taking dual, direct sum and tensor product.

Let $\mathrm{PVect}(X,S)$ denote the space of parabolic vector bundles over X, of arbitrary rank, with parabolic structure over the divisor S. This has direct sum operation, tensor product operation, and the operation of taking dual (see [14]). For any integer $N \geq 2$, let $\mathrm{PVect}(X,S,N) \subset \mathrm{PVect}(X,S)$ be the subset consisting of all parabolic bundles that have the property that all the parabolic weights are integral multiples of $1/N$. The subset $\mathrm{PVect}(X,S,N)$ is closed under all the three operations on $\mathrm{PVect}(X,S)$ mentioned above.

A *parabolic G–bundle* over X gives a covariant functor $\mathcal{F}$ from the category $\mathrm{Rep}(G)$ of left G–modules to the category $\mathrm{PVect}(X,S)$ of parabolic bundles compatible with the operations of taking dual, direct sum and tensor product and satisfying the condition that there is an integer N such that the image of $\mathcal{F}$ is contained in $\mathrm{PVect}(X,S,N)$ (see [1, p. 343, Definition 2.5] for the details).

It should be pointed out that the above definition of a parabolic G–bundle given in [1] was very much inspired by [10]. The motivation for such a definition can be found in [10], where usual G–bundles were defined as functor with the above compatibility properties. Below we very briefly recall this definition of [10].

Nori proves that the collection of principal G–bundles over an irreducible projective variety M are in bijective correspondence with the collection of $\mathbb{C}$–additive functors

$$\mathcal{F} : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(M)\,,$$

where $\mathrm{Vect}(M)$ denotes the category of vector bundles over M, satisfying the following properties:

1. The rank of the vector bundle $\mathcal{F}(V)$ coincides with the dimension of the G–module V.
2. A morphism of vector bundles is said to be *strict* if the cokernel is also locally free. Let
$$f : V \longrightarrow W$$
be a homomorphism of G–modules. Then the corresponding homomorphism of vector bundles
$$\mathcal{F}(f) : \mathcal{F}(V) \longrightarrow \mathcal{F}(W)$$
is strict. In other words, the cokernel of $\mathcal{F}(f)$ is locally free. Note that this implies that both the image and the kernel of $\mathcal{F}(f)$ are both locally free.
3. The kernel of the homomorphism $\mathcal{F}(f)$ (which is a vector bundle by the previous condition) coincides with $\mathcal{F}(\mathrm{kernel}(f))$ and the cokernel of $\mathcal{F}(f)$ coincides with $\mathcal{F}(\mathrm{cokernel}(f))$. The rank of the vector bundle $\mathcal{F}(V)$ coincides with the dimension of the G–module V.
4. For any two G–modules V and W,
$$\mathcal{F}(V \otimes W) = \mathcal{F}(V) \otimes \mathcal{F}(W)$$
and $\mathcal{F}(V^*) = \mathcal{F}(V)^*$. Furthermore, $\mathcal{F}(\mathbb{C})$, where $\mathbb{C}$ is the trivial G–module, is the trivial line bundle $\mathcal{O}_M$.
5. For any two G–modules V and W, the map
$$\mathcal{F}(\mathrm{Hom}(V\,, W)) = \mathcal{F}(V^* \otimes W) \longrightarrow \mathcal{F}(V^*) \otimes \mathcal{F}(W) = \mathrm{Hom}(\mathcal{F}(V)\,, \mathcal{F}(W))$$
is injective.

Given such a functor $\mathcal{F}$, there is a G–bundle E, unique up to a unique isomorphism, such that $\mathcal{F} \cong \mathcal{F}(E)$ [10, p. 34, Proposition 2.9]. For any principal G–bundle E, consider the functor

$$\mathcal{F}(E) : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}(M)$$

that sends a G–module V to the vector bundle

$$\mathcal{F}(E)(V) := E \times^G V := \frac{E \times V}{G}\,,$$

where the quotient is for the twisted diagonal action of G on $E \times V$. We recall that the action of any $\alpha \in G$ sends a point $(y\,, v) \in E \times V$ to $(y\alpha\,, \alpha^{-1}(v))$, where $\alpha^{-1}(v) \in V$ is defined using the left G–module structure of V. This functor $\mathcal{F}(E)$ has all the above properties. The bijective correspondence between functors and G–bundles sends the G–bundle E to this functor $\mathcal{F}(E)$.

Let Y be a polarized smooth projective variety. Given a G–bundle E_G over Y and a maximal parabolic subgroup Q of G, let $E_G(Q)$ denote a reduction of the structure group of E_G to Q over an open set $U \subseteq X$ with the property that $\mathrm{codim}(X-U) \geq 2$. The principal G–bundle E_G is called *semistable* (respectively, *stable*) if for every Q and every such reduction, the line bundle over U associated to $E_G(Q)$ for any character of Q, dominant with respect to a Borel subgroup contained in Q, is of nonpositive degree (respectively, strictly negative degree) [11, p. 282, Definition 3.7]. Let $H \subset G$ be a maximal reductive subgroup of a parabolic subgroup of G. The principal bundle E_G is called *polystable* if there is such a H and a reduction of the structure group $E_G(H) \subset E_G$ (over Y) of E_G to H such that $E_G(H)$ is a stable principal H–bundle, and furthermore, for any character of M the corresponding line bundle associated to $E_G(M)$ is of degree zero [11, p. 285, Definition 3.16], [1, p. 346, Definition 3.1].

A parabolic G–bundle $\mathcal{F}$ is called *semistable* (respectively, *polystable*) if the image of the functor is contained is semistable (respectively, polystable) parabolic vector bundles [1, p. 347, Definition 3.3].

We introduce a couple of notations for the benefit of the exposition. Let $\mathcal{M}$ denote the set of isomorphism classes of parabolic vector bundles over X with parabolic structure over S such that the parabolic weights at every $p_i \in S$ are all integral multiples of $1/m_i$ (the same set in Theorem 4.4). It is easy to see that $\mathcal{M}$ is closed under all the three operations, namely, direct sum, dual and tensor product. Let $\mathcal{M}_G$ denote the set of all isomorphism classes of parabolic G–bundles such that the image of the functor is contained in $\mathcal{M}$.

Theorem 5.1

There is a bijective correspondence between $\mathcal{M}_G$ and the space of all isomorphism classes of orbifold G–bundles over Z. An orbifold G–bundle is semistable (respectively, polystable) in the usual sense if and only if the corresponding parabolic G–bundle is parabolic semistable (respectively, parabolic polystable).

Proof. The set of all orbifold vector bundles over Z is equipped with the operations of taking dual, direct sum and tensor product. The bijective correspondence in Theorem 4.4 is actually compatible with these operations. Clearly direct sum of orbifold bundles over Z corresponds to the direct sum of the corresponding parabolic bundles over X. For the other two operations, we need to consider the local model given in (5) and in the proof of Lemma 3.2.

Since a complex $\mathbb{Z}/m_i\mathbb{Z}$–module of dimension n decomposes as a direct sum of n one dimensional modules, the question reduces (as in Lemma 3.2) to the case of line bundles ($n = 1$). But for line bundles, the compatibility of the bijective correspondence in Theorem 4.4 with the operations of taking dual and tensor product is obvious.

Now the bijective correspondence asserted in the theorem is quite clear. Take an orbifold G–bundle E_G over Z. For any G–module $V \in \mathrm{Rep}(G)$, consider the corresponding vector bundle

$$\overline{V} := E_G \times^G V$$

over Z associated to E_G for V. Since the action of C on E_G commutes with the action of G, the vector bundle $\overline{V}$ is equipped with an action of C. Let $V_* \in \mathcal{M}$ denote the parabolic vector bundle over X associated, by Theorem 4.4, to the orbifold bundle $\overline{V}$.

The earlier observation that the bijective correspondence in Theorem 4.4 is compatible with all the three operations imply that the map

$$F(E_G) : \mathrm{Rep}(G) \longrightarrow \mathcal{M}$$

that sends a G–module V to the parabolic bundle V_* constructed above defines a parabolic G–bundle. The other conditions in the definition of a parabolic G–bundle [1, p. 343, Definition 2.5] are satisfied.

So we have a map from the space of orbifold G–bundles over Z to $\mathcal{M}_G$ that sends any E_G to the parabolic G–bundle defined by the functor $F(E_G)$.

For the converse direction, we start with a parabolic G–bundle over X defined by a covariant functor $\mathcal{F}$ from $\mathrm{Rep}(G)$ to $\mathcal{M}$. So, using Theorem 4.4, for any $V \in \mathrm{Rep}(G)$ we have an orbifold vector bundle $\overline{V}$ over Z corresponding to the parabolic bundle $\mathcal{F}(V)$. Again from the earlier observations we know that this map that sends V to $\overline{V}$ is compatible with the three operations.

We know from [10] that such a map defines a principal G–bundle over Z [10, p. 32, Lemma 2.3]. We need to show that this G–bundle, which we will denote by E_G, has an orbifold structure. Recall that according to Proposition 4.5, the G–bundle E_G can have at most one orbifold structure.

Take a point $c \in C$ of the elliptic curve. Consider the automorphism $\phi(c) \in \mathrm{Aut}(Z)$, where ϕ as in (7). Since $\overline{V}$ is an orbifold bundle, we are given with an isomorphism of $\overline{V}$ with $\phi(c)^*\overline{V}$. The point to note about this isomorphism is that it is compatible with the operations of taking dual, direct sum and tensor product. Indeed, this is an immediate consequence of the earlier observation that the bijective correspondence in Theorem 4.4 is compatible with these operations. Therefore, we obtain an automorphism

$$\delta_c : E_G \longrightarrow \phi(c)^*E_G$$

of G–bundles [10, p. 34, Proposition 2.9(a)].

From the construction of δ_c it is immediate that δ_0 is the identity automorphism and $\delta_{b+c} = \delta_b \circ \delta_c$. Consequently, these isomorphisms δ_c together define an action of C on E_G lifting the action of C on Z. In other words, E_G is an orbifold G–bundle.

Therefore, we have a map from $\mathcal{M}_G$ to orbifold G–bundles over Z that sends any $\mathcal{F}$ to the orbifold G–bundle E_G constructed from it. It is easy to check that this map is the inverse of the earlier constructed map from orbifold G–bundles over Z to $\mathcal{M}_G$.

Recall the definition of parabolic semistability and parabolic polystability given earlier. From the second part of Theorem 4.4 it is immediate that an orbifold G–bundle over Z is semistable (respectively, polystable) in the usual sense if and only if the corresponding parabolic bundle over X is parabolic semistable (respectively, parabolic polystable). This completes the proof of the theorem. $\square$

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