Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. **54**, 3 (2003), 283–291

A note about the isotropy groups of 2-plane bundles over closed surfaces

A. Minatta

Mathematisches Institut der Universität Heidelberg, Heidelberg (Germany)

R. Piccinini

Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Milano (Italy) and Department of Mathematics and Statistics, Dalhousie University, Halifax (Canada)

M. Spreafico

Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Milano (Italy)

Received October 9, 2002. Revised January 24, 2003

Abstract

Let ξ be a 2-plane bundle over a closed surface S. The line bundles λ over S such that $\xi \otimes \lambda \cong \xi$ form a group $\mathfrak{I}(\xi)$ (the isotropy group of ξ); the scope of this paper is to describe $\mathfrak{I}(\xi)$.

1. Introduction

This note is devoted to answering the following question: given a vector bundle ξ and a line bundle λ over a Riemannian manifold M, when are ξ and $\xi \otimes \lambda$ equivalent? This problem arises from a different question, originally introduced in [8] with the intent of finding out when the gauge groups of two bundles over a convenient space X are conjugate as subgroups of the "local gauge group" associated to a covering of X; indeed, it turns out that for the case of vector bundles over a Riemannian manifold M, ξ and ξ' have conjugate gauge groups if, and only if, there is a line bundle λ over

Keywords: Plane-bundles, Gauge groups, Isotropy group.
MSC2000: 55R10; Secondary 55R25, 55R50, 14J60.
The first author is supported by INdAM.

M such that $\xi \otimes \lambda \cong \xi'$. Thus, it is important to find out what is the "size" of the *isotropy group* $\mathfrak{I}(\xi)$ of ξ, that is to say, the group of all line bundles λ over M such that $\xi \otimes \lambda \cong \xi$ (this situation was discussed thoroughly in [6]). A complete answer to the question raised at the beginning of the paper was given in [2] for the case of n-plane bundles over the n-real projective space; in that paper, the authors also introduce some techniques useful in dealing with the general problem.

In this note we discuss the case of 2-plane bundles over closed surfaces; we give a solution to the problem in the following result:

Theorem 1

Let ξ be a real 2-plane bundle over a closed surface S. Then, for any line bundle λ over S, $\xi \otimes \lambda$ and ξ are equivalent if, and only if,

$$w_1^2(\lambda) + w_1(\lambda)w_1(\xi) = 0$$

where $w_1(\lambda), w_1(\xi) \in H^1(S;\mathbb{Z}_2)$ are the first Stiefel-Whitney classes.

A remark is in order here. For the orientable case, the formula above is translated into $w_1(\lambda)w_1(\xi) = 0$; this indicates that the orientation bundle λ of each 2-plane bundle ξ belongs to the isotropy group $\mathfrak{I}(\xi)$. Actually, as noticed by M. Crabb [1], in such a case we can say much more, namely that ξ has a natural C_λ structure, as defined in [2]. Anyway, such bundles λ do not necessarily cover entirely $\mathfrak{I}(\xi)$ due to the fact that we are working with $\mathbb{Z}_2$ coefficients; this will be clear in the examples of Section 4.

Corollary 1

If ξ is an orientable 2-bundle over an orientable surface S, then

$$\mathfrak{I}(\xi) \;\cong\; H^1(S;\mathbb{Z}_2)$$

that is to say, the isotropy group of ξ is maximal.

The paper is organized as follows. In Section 2 we recall the general results necessary for the proof of Theorem 1 which will be dealt with in Section 3; some examples will be given in the last Section.

2. Auxiliary results

We begin by recalling that a closed surface S (i.e., a compact, connected 2-dimensional manifold without boundary) is homeomorphic to a 2-dimensional sphere, or to a connected sum of g tori, or to a connected sum of g real projective planes.

The cohomology ring $H^*(gT^2,\mathbb{Z}_2)$ has a basis $\{a^i, b^i | i = 1,\dots,g\}$ with the multiplication rules

$$(a^i)^2 = (b^i)^2 = a^i a^j = b^i b^j = a^i b^j = 0 \ (i \neq j) \text{ and } a^i b^i = b^i a^i = \{gT^2\}^*$$

where $\{gT^2\}^*$ is the dual of the fundamental class of gT^2, while the cohomology ring $H^*(g\mathbb{R}P^2, \mathbb{Z}_2)$ has a basis $\{a^1, \ldots, a^g\}$ with the multiplication rules

$$a^i a^j = 0 \ (i \neq j \ , \ i,j = 1, \ldots, g) \text{ and } (a^i)^2 = \{g\mathbb{R}P^2\}^*$$

where $\{g\mathbb{R}P^2\}^*$ is the dual of the $\mathbb{Z}_2$-fundamental class of $g\mathbb{R}P^2$ (see [11, 15.4.5]).

In order to study the isotropy group of a 2-plane bundle over a surface, we shall interpret $BO(2)$ as the total space of a $\mathbb{Z}_2$-bundle over $\mathbb{R}P^\infty$ with fibre $\mathbb{C}P^\infty$; this result is a consequence of a more general result which seems to be of interest on its own right and which we describe and prove in the sequel.

Let Γ and G be topological groups and let $\alpha : \Gamma \to \mathrm{Aut}G$ be a homomorphism into the automorphism group of G such that the left Γ-action

$$\Gamma \times G \to G, (a,g) \mapsto ag = \alpha_a(g)$$

is continuous. Let $M = \Gamma \times_\alpha G$ be the semi-direct product of Γ and G via α: its elements are pairs $(a,g) \in \Gamma \times G$ and its multiplication is given by

$$(a,g)(b,h) = (ab, \alpha_b(g)h)\,.$$

Notice that the functions

$$\iota : \Gamma \to M, a \mapsto (a, 1_G)\,,$$

$$\ell : G \to M \ , g \mapsto (1_\Gamma, g); p : M \to \Gamma, (a,g) \mapsto a$$

are (continuous) group homomorphisms.

Proposition 1

The classifying space BM is the total space of a Γ-bundle over $B\Gamma$ with fibre BG, projection $Bp : BM \to B\Gamma$ and a cross-section $B\iota : B\Gamma \to BM$.

Proof. We recall that for an arbitrary topological group G there exists a universal principal G-bundle $\xi_G = (EG, pG, BG)$. We refer the reader to [9] or [10, Appendix A] for the notation employed and the general facts about the construction of ξ_G; we also wish to note that we implicitly assume to be working in a convenient category of spaces (as in [10]) in order to guarantee, for instance, the continuity of the functions considered.

The homomorphisms ι, ℓ and p induce morphisms

$$E\iota : E\Gamma \hookrightarrow EM \ , \ E\ell : EG \hookrightarrow EM \ , \ Ep : EM \hookrightarrow E\Gamma \ ,$$

$$B\iota : B\Gamma \to BM \ , \ B\ell : BG \to BM \ , \ Bp : BM \to B\Gamma \ .$$

At this point we observe that there is an action

$$A : EG \times \Gamma \to EG \ , \ (x,a) \mapsto a^{-1}xa$$

and that this action passes to BG: the following diagram is commutative.

$$\begin{array}{ccc} EG \times \Gamma & \xrightarrow{A} & EG \\ {\scriptstyle p_G \times 1}\downarrow & & \downarrow{\scriptstyle p_G} \\ BG \times \Gamma & \xrightarrow{\bar{A}} & BG\,. \end{array}$$

These actions give rise to the quotient spaces $E\Gamma \times_\Gamma EG$ and $E\Gamma \times_\Gamma BG$ with the corresponding quotient maps

$$\pi : E\Gamma \times EG \to E\Gamma \times_\Gamma EG \ , \ \pi' : E\Gamma \times BG \to E\Gamma \times_\Gamma BG\,.$$

The group homomorphism

$$\psi : EM \to E\Gamma \times EG \ , \ y \mapsto \big(Ep(y), (Ep(y))^{-1}y\big)\,,$$

is indeed an isomorphism, with inverse

$$\phi : E\Gamma \times EG \to EM \ , \ (z,x) \mapsto E\iota(z)E\ell(x) \ :$$

$$\phi\psi(y) = \phi\big(Ep(y), (Ep(y))^{-1}y\big) = Ep(y)\big(Ep(y)\big)^{-1}y = y\,,$$

while

$$\begin{aligned} \psi\phi(z,x) &= \psi(E\iota(z)E\ell(x)) \\ &= \big(Ep(E\iota(z)E\ell(x)), ((Ep(E\iota(z)E\ell(x))\big)^{-1}(E\iota(z)E\ell(x)) \\ &= (E(p\iota)(z)E(p\ell)(x), \ldots) = (z, z^{-1}zx) = (z,x)\,. \end{aligned}$$

Routine computations prove that the maps ψ and ϕ pass to the quotient spaces and are over the identity map $1_{B\Gamma}$. □

Corollary 2

The classifying space $BO(2)$ is the total space of a $\mathbb{Z}_2$-bundle over $B\mathbb{Z}_2 \cong \mathbb{R}P^\infty$ and fibre $BS^1 \cong \mathbb{C}P^\infty$. Furthermore, the inclusion $\iota : \mathbb{Z}_2 \to O(2)$ induces a section $s = B\iota : \mathbb{R}P^\infty \to BO(2)$.

Proof. With the notation of the previous Proposition we take $G = SO(2)$, $\Gamma = \mathbb{Z}_2$ and the action $\alpha : SO(2) \times \mathbb{Z}_2 \to SO(2)$ to be $\alpha_1 = 1_{SO(2)}$, $\alpha_{-1}(g) = g^{-1}$. Finally, we observe that $B(\mathbb{Z}_2 \times_\alpha SO(2)) \cong BO(2)$. □

Remark 1. If we take $G = SO(n)$ and $\Gamma = \mathbb{Z}_2$ we obtain $BO(n)$ as the total space of a $\mathbb{Z}_2$-bundle over $\mathbb{R}P^\infty$ and fibre $BSO(n)$.

Finally, we observe that the tail end of the exact sequence of homotopy groups for the $\mathbb{Z}_2$-bundle $\mathbb{C}P^\infty \xrightarrow{j} BO(2) \xrightarrow{q} \mathbb{R}P^\infty$, namely

$$0 \longrightarrow \pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z} \xrightarrow{j_2^*} \pi_2(BO(2)) \longrightarrow \pi_2(\mathbb{R}P^\infty) \cong 0 \longrightarrow$$
$$\longrightarrow \pi_1(\mathbb{C}P^\infty) \cong 0 \longrightarrow \pi_1(BO(2)) \xrightarrow{q_1^*} \pi_1(\mathbb{R}P^\infty) \cong \mathbb{Z}_2 \longrightarrow 0$$

shows that j_2^* and q_1^* are isomorphisms. Furthermore, we can describe the generators of $\pi_1(BO(2))$ and $\pi_2(BO(2))$ as follows: (i) The inclusion $c : \mathbb{R}P^1 \to BO(2)$ of the 1-cell can be decomposed as $c = su$, where s is the section defined before and u is the inclusion $\mathbb{R}P^1 \subset \mathbb{R}P^\infty$; then $[c] = s_*[u] = p_*^{-1}([u])$ generates $\pi_1(BO(2))$. (ii) Let $b : S^2 \to BO(2)$ be the inclusion of the 2-cell; then b factors through $\mathbb{C}P^\infty$ as $b = jv$ and $[b] = j_*([v])$ generates $\pi_2(BO(2))$.

Since 2-plane bundles over a space X are classified by $[X, BO(2)]$, we are going to study the set $[X, BO(2)]$ whenever X is a closed surface. Take the Puppe sequence of the map $S^1 \xrightarrow{f} \vee S^1$ used to construct the closed surface S (recall that S is a 2-sphere, gT^2 or $g\mathbb{R}P^2$) and apply the functor $[-, Y]_*$ to it to obtain the long exact sequence

$$\ldots \longrightarrow \oplus\, \pi_2(Y) \xrightarrow{f^*} \pi_2(Y) \xrightarrow{k^*} [S, Y]_* \xrightarrow{\bar{i}^*} \oplus\, \pi_1(Y) \xrightarrow{f^*} \pi_1(Y)\ .$$

Notice that if $\pi_1(Y)$ is Abelian and S is orientable, then f^* is trivial; moreover, if $\pi_1(Y) \cong \mathbb{Z}_2$, then f^* is trivial also in the non-orientable case. Hence, we obtain the following two short exact sequences corresponding to $Y = BO(2)$ and $Y = BO$, respectively:

$$0 \longrightarrow \pi_2(BO(2)) \cong \mathbb{Z} \xrightarrow{k^*} [S, BO(2)]_* \xrightarrow{\bar{i}^*}$$

$$\xrightarrow{\bar{i}^*} \oplus\, \pi_1(BO(2)) \cong \oplus \mathbb{Z}_2 \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \pi_2(BO) \cong \mathbb{Z}_2 \xrightarrow{k^*} [S, BO]_* \xrightarrow{\bar{i}^*} \oplus\, \pi_1(BO) \cong \oplus \mathbb{Z}_2 \longrightarrow 0\ . \tag{2}$$

The second exact sequence shows that the cardinality of $\widetilde{KO}(S) = [S, BO]_*$ is

$$\begin{cases} 2^{2g+1}\,, & \text{if } S = gT^2 \\ 2^{g+1}\,, & \text{if } S = g\mathbb{R}P^2. \end{cases}$$

3. Proof of the theorem

The argument is divided in two parts. In the first, we show that the stable class of ξ is completely described by its characteristic classes. This implies that the stable isotropy group of ξ (namely, the group $\mathfrak{I}_s(\xi)$ of all line bundles λ over S such that $\xi \otimes \lambda$ is stably equivalent to ξ) is completely determined by the given relation on the characteristic classes. In the second part we show that in the present case the stable and unstable isotropy groups coincide.

Part 1 - *The Stiefel-Whitney class*

$$w : \widetilde{KO}(S) \to \widetilde{H}^*(S; \mathbb{Z}_2)$$

is a bijection.

Because $\widetilde{KO}(S)$ and $\widetilde{H}^*(S;\mathbb{Z}_2)$ have the same cardinality it is enough to show that w is surjective.

The cohomology ring of $BO(2)$ is given by

$$H^*(BO(2);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2]$$

and the classes w_1, w_2 are algebraically independent (see [4, 20, 5.2]); moreover, if γ_2 is the canonical universal 2-plane bundle over $BO(2)$, $w(\gamma_2) = 1 + w_1 + w_2$; hence, if $f : S \to BO(2)$ classifies a 2-bundle ξ over S, $w(\xi) = 1 + f^*(w_1) + f^*(w_2)$.

Take the surjection

$$\bar{i}^* : [S, BO(2)]_* \longrightarrow \oplus\pi_1(BO(2))$$

(see the short exact sequence (1) of Section 2) and, in view of Hopf's Representation Theorem [3, II, 7.4.4], we have that

$$H^1(S;\mathbb{Z}_2) \cong H^1(\vee S^1;\mathbb{Z}_2) \cong \oplus[S^1, K(\mathbb{Z}_2, 1)] \cong \oplus\pi_1(BO(2)) \ ;$$

hence, for every $x \in H^1(S;\mathbb{Z}_2)$, there exists a map $f : S \to BO(2)$ such that $f^*(w_1) = x$.

Because S is obtained by the attachment of a 2-disk D^2 to a wedge of circles $\vee S^1$, there is a natural co-operation of S^2 on S

$$\theta : S \longrightarrow S \vee S^2$$

obtained by pinching the cone $CS^1 \cong D^2$ half way through its height, or in other words, by collapsing to a point the circle $S^1_{1/2}$ inserted half way on CS^1 (see [10, 4.2.1]); this cooperation gives rise to an action

$$\theta_* : [S, BO(2)]_* \times \pi_2(BO(2)) \to [S, BO(2)]_* \ , \ ([f], [h]) \mapsto [\sigma(f \vee h)\theta]$$

where σ is the folding map. The exact (reduced) cohomology sequence of the pair $(S, S^1_{1/2})$ shows that

$$(\sigma(f \vee b)\theta)^*(w_1) = f^*(w_1)$$

$$(\sigma(f \vee b)\theta)^*(w_2) = f^*(w_2) + b^*(w_2)$$

where $b : S^2 \to BO(2)$ is the inclusion of the 2-cell.

The well-known formula

$$w_k(\xi \otimes \lambda) = \sum_{j=0}^{k} \begin{pmatrix} r-j \\ k-j \end{pmatrix} w_j(\xi) w_1^{k-j}(\lambda)$$

(see [6]), where r is the rank of ξ ($r = 2$ in the present case) gives rise to the relation between the Stiefel-Whitney classes of ξ and λ stated in Theorem 1.

Part 2 - *For each 2-plane bundle ξ over S, $\mathfrak{I}(\xi) = \mathfrak{I}_s(\xi)$.*

We begin by observing that the statement above is a consequence of [2, Proposition 3.6] whenever $w_1(\xi) = w_1(TS)$.

Now suppose that $w_1(\xi) \neq w_1(TS)$; we are going to prove that each stable class contains only one equivalence class of bundles.

The fibration of Corollary 2 gives rise to the exact sequence of sets

$$[S, \mathbb{C}P^\infty] \xrightarrow{j_*} [S, BO(2)] \xrightarrow{p_*} [S, \mathbb{R}P^\infty] \longrightarrow 0$$

in which the function p_* is surjective because the projection $p : BO(2) \to \mathbb{R}P^\infty$ has a cross-section. Then, as a set $[S, BO(2)]$ is the union of the counterimages of the elements of $[S, \mathbb{R}P^\infty]$ via p_*; however, we must establish with precision what is the counterimage of an element $[f] \in [S, \mathbb{R}P^\infty]$. To this end, we follow the ideas described by L.L. Larmore in [5], from which we extract the following. For any space Y, we choose a Postnikov tower for Y, namely: (i) for each $n \geq 0$, a space $(Y)_n$ and a map $P_n : Y \to (Y)_n$ which induces an isomorphism in homotopy through dimension n, and with $\pi_k((Y)_n) = 0$ for every $k > n$; (ii) for each $n \geq 1$, a fibration $p_n : (Y)_n \to (Y)_{n-1}$ with fibre $K(\pi_n((Y)_n), n)$ and $p_n P_n = P_{n-1}$. For every $n \geq 1$ let $G_n(Y)$ be a sheaf over $(Y)_1$ with stalk $\pi_n(Y)$; next, given a pair of spaces (X, A) and a map $f : X \to (Y)_n$, we define $\pi_n(Y, f)$ to be the sheaf over X obtained as the pull back via $p_2 \ldots p_n f$ of the sheaf $G_n(Y)$. Larmore's main result (see [5, (1.3)]) is as follows: for every $f : X \to (Y)_n$, there exists a spectral sequence

$$E_2^{p,q} = \begin{cases} H^p(X, A; \pi_q(Y, f)) & 2 \leq q \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$d_r : E_r^{p,q} \longrightarrow E_r^{+r, q+r-1} \ , \ r \geq 2$$

and $E_\infty^{n,n}$ is a group in a bijective correspondence with the set of all rel.A homotopy classes of maps $g : X \to (Y)_n$ such that $p_n f \sim_{rel.A} p_n g$. More precisely, let $[X; (Y)_n : f|A]$ be the set of all homotopy classes rel.A of maps $g : X \to (Y)_n$ such that $f|A = g|A$; the map $p_n : (Y)_n \to (Y)_{n-1}$ induces a function

$$(p_n)_* : [X; (Y)_n : f|A] \longrightarrow [X; (Y)_{n-1} : p_n f|A] \ ;$$

then, $E_\infty^{n,n} = (p_n)_*^{-1}([p_n f])$.

At this point we specialize the space Y to be $BO(2)$, X to be the surface S and A, the base point of S. In order to build the Postnikov tower for $BO(2)$ we consider the fibration $\mathbb{C}P^\infty \longrightarrow BO(2) \longrightarrow \mathbb{R}P^\infty$ and set $(BO(2))_0 = *$, $(BO(2))_1 = \mathbb{R}P^\infty$, $(BO(2))_2 = BO(2)$, and $p_2 = p$. For $Y = BO(2)$, the spectral sequence collapses (see [5, (4.4.1)]) and hence, for a given $f : S \to (BO(2))_2 = BO(2)$, the set $p_*^{-1}([pf]_*)$ is in a bijective correspondence with $H^2(S; \pi_2(BO(2), f))$. As $\pi_1(BO(2))$ acts non-trivially on $\pi_2(BO(2))$, we reformulate the sheaf $\pi_2(BO(2), f)$ in order to take into consideration this action; we proceed as follows: first of all take the $\mathbb{Z}_2$-bundle $\mathbb{Z}^T$ over $\mathbb{R}P^\infty$ obtained by changing the fibre of the universal bundle over $\mathbb{R}P^\infty$ into $\mathbb{Z}$, with the action

$$\mathbb{Z}_2 \times \mathbb{Z} \longrightarrow \mathbb{Z} \ , \ (\pm 1, n) \mapsto \pm n \ ;$$

next, take $f : S \longrightarrow BO(2)$ corresponding to $w_1(\xi) \in H^1(S;\mathbb{Z}_2)$ (see Part 1) and pull back $\mathbb{Z}^T$ over S via the map pf to obtain a $\mathbb{Z}_2$-bundle which we call $\mathbb{Z}^T[x]$; finally, observe that $\pi_2(BO(2), f)$ corresponds to the $\mathbb{Z}_2$-bundle $\mathbb{Z}^T[x]$. Therefore, $p_*^{-1}([pf]_*)$ is in a bijective correspondence with $H^2(S;\mathbb{Z}^T[x])$.

Observe that $H^2(S;\mathbb{Z}^T[x])$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, according to the twisting of ξ being compatible or not with that of TS; since $w_1(\xi) \neq w_1(TS)$, we have the second alternative and so, $p_*^{-1}([pf]_*)$ consists of two elements η_1 and η_2 which are distinguished by the obstruction class $\mathfrak{o}_2$. Reduction modulo 2 induces an isomorphism $H^2(S;\mathbb{Z}^T[x]) \to H^2(S;\mathbb{Z}_2)$ which maps $\mathfrak{o}_2(\eta)$ onto $w_2(\eta)$ (see [7, Theorem 12.1]). Since at least one of the elements η_1 and η_2 has a non-vanishing second Stiefel-Whitney class, it follows that $\mathfrak{o}_2(\eta) \neq 0$ implies that $w_2(\eta) \neq 0$. This means that η_1, η_2 have different second Stiefel-Whitney classes and hence, are not stably equivalent.

Notice that $p_*^{-1}([pf]_*) \in [S, BO(2)]_*$; however, we must study its corresponding class in $[S, BO(2)] \cong [S, BO(2)]_*/\pi_1(BO(2))$. This action must be taken into consideration; it effectively corresponds to a change of sign, and hence, it has no effect in the interesting case, namely $H^2(S;\mathbb{Z}^T[x])) \cong \mathbb{Z}_2$.

We notice the following alternative argument for orientable bundles over orientable surfaces: because the complexified bundle of any real line bundle is trivial,

$$\xi \otimes \lambda = \xi \otimes_{\mathbb{C}} \lambda_{\mathbb{C}} = \xi \ .$$

4. Low genus surfaces

In this last Section we give explicit results for low genus surfaces, namely the 2-sphere, the torus and the connected sum of 2 torus, the projective plane and the connected sum of two real projective planes (i.e. the Klein bottle). These are collected in the two tables at the end of the paper, constructed as follows.

First of all, we enumerate the line bundles by their first Stiefel-Whitney classes (thus, 0 represents the trivial bundle); β and α are the degree 2 cohomology generators of S for the orientable and non orientable surfaces, respectively; the 2-plane bundles are represented by their first Stiefel-Whitney class $w_1 \in H^1(S;\mathbb{Z}_2)$ and by the obstruction class $\mathfrak{o}_2 \in H^*(S;\mathbb{Z}^T(\xi))$, which corresponds to the Euler class for orientable bundles and to the second Stiefel-Whitney class for non-orientable bundles. Moreover, $x, y \in H^1(S;\mathbb{Z}_2)$, and finally, $\mathbb{N}[a]$ denotes the quotient of the group $\mathbb{Z}[a]$ under the action of $\mathbb{Z}_2$ which consists in changing sign (this comes from the action of $\pi_1(BO(2))$ over $[S, BO(2)]_*$).

We complete the paper with a few comments on particular bundles. For $S = S^2$, β corresponds to the first Chern class of the canonical complex Hopf line bundle over $\mathbb{C}P^1 = S^2$ since in this case the group of local coefficients is trivial; on the other hand, the tangent bundle is represented by 2β, and is trivial in the K-ring.

If $S = T$, the bundles a_1 and a_2 are obtained respectively as pull backs of the Möbius strip over the circle via the projection of the two fundamental classes.

If $S = \mathbb{R}P^2$, the line bundle e_1 is the real Hopf bundle H, $H - 1$ generates the K-ring, and the bundle $1 + e_1 + n\alpha$ is stably equivalent to H or $3H$ respectively, depending on the parity of n; this can be shown by computing the Stiefel-Whitney classes classes (see [2]). In this case the tangent bundle of $\mathbb{R}P^2$ is $1+e_1+2\alpha$ (local coefficients).

Orientable bundles

S	$Vect_2^+(S)$	$\mathfrak{I}$
S^2	$\mathbb{N}[\beta]$	$\mathfrak{I}(1+n\beta)=\{0\}$
T	$\mathbb{N}[\beta]$	$\mathfrak{I}(1+n\beta)=\mathbb{Z}_2[a_1,b_1]$
$T\#T$	$\mathbb{N}[\beta]$	$\mathfrak{I}(1+n\beta)=\mathbb{Z}_2[a_1,b_1,a_2,b_2]$
$\mathbb{R}P^2$	$\mathbb{Z}_2[\alpha]$	$\mathfrak{I}(1+n(\text{mod}2)\alpha)=\{0\}$
K	$\mathbb{Z}_2[\alpha]$	$\mathfrak{I}(1+n(\text{mod}2)\alpha)=\{0,e_1+e_2\}$

Non orientable bundles

S	$Vect_2^-(S)$	$\mathfrak{I}$
S^2	$\emptyset$	
T	$(\mathbb{Z}_2[a_1,b_1]-\{0\})\times\mathbb{Z}_2[\beta]$	$\mathfrak{I}(1+x+n(\text{mod}2)\beta)=\{0,x\}$
$T\#T$	$(\mathbb{Z}_2[a_1,a_2,b_1,b_2]-\{0\})\times\mathbb{Z}_2[\beta]$	$\mathfrak{I}(1+x+n(\text{mod}2)\beta)=\{y\|xy=0\}$
$\mathbb{R}P^2$	$\{e_1\}\times\mathbb{N}[\alpha]$	$\mathfrak{I}(1+e_1+n\alpha)=\{0,e_1\}$
K	$(\mathbb{Z}_2[e_1,e_2]-\{0\})\times\mathbb{N}[\alpha]$	$\mathfrak{I}(1+e_1+n\alpha)=\{0,e_2\}$ $\mathfrak{I}(1+e_2+n\alpha)=\{0,e_1\}$ $\mathfrak{I}(1+e_1+e_2+n\alpha)=\{0,e_1+e_2\}$

References

1. M.C. Crabb, private communication.
2. M.C. Crabb, M. Spreafico, and W.A. Sutherland, Enumerating projectively equivalent bundles, *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 223–242.
3. P.J. Hilton and S. Wylie, *Homology Theory: An Introduction to Algebraic Topology,* Cambridge University Press, Cambridge, 1960.
4. D. Husemoller, *Fibre Bundles, Third Edition*, Springer Verlag, New York, Heidelberg, 1994.
5. L.L. Larmore, Twisted cohomology and enumeration of vector bundles, *Pacific J. Math.* **30** (1969), 437–457.
6. M. Marcolli and M. Spreafico, Gauge groups and characteristic classes, *Exposition. Math.* **15** (1997), 229–249.
7. J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, 1974.
8. C. Morgan and R.A. Piccinini, Conjugacy classes of groups of bundle automorphisms, *Manuscripta Math.* **63** (1989), 233–244.
9. R.A. Piccinini and M. Spreafico, The Milgram-Steenrod construction of classifying spaces for topological groups, *Exposition. Math.* **16** (1998), 97–130.
10. R.A. Piccinini and M. Spreafico, *Conjugacy Classes in Gauge Groups*, Queen's Papers in Pure and Applied Mathematics 111, 1998.
11. R. Stöker and H. Zieschang, *Algebraische Topologie*, B.G. Teubner, Stuttgart, 1988.