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Curves on a ruled cubic surface

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Abstract

For the general ruled cubic surface S (with a double line) in $\mathbb{P}^3 = \mathbb{P}^3_k$, k any algebraically closed field, we find necessary conditions for which curves on S can be the specialization of a flat family of curves on smooth cubics. In particular, no smooth curve of degree > 10 on S is such a specialization.

1. Introduction

Let k be an algebraically closed field, and let X_0 be the general ruled cubic surface in $\mathbb{P}^3 = \mathbb{P}^3_k$ (see Section 2 below). For any irreducible nonsingular (not necessarily complete) curve T over k with a special closed point $0 \in T$, consider families X of cubic surfaces, where X_t is nonsingular for $t \neq 0$ and the fibre over 0 is X_0. The aim of this paper is to determine which curves (locally Cohen-Macaulay subschemes of pure dimension 1) on X_0 can belong to a flat family of curves D in X for some family X. The language and techniques of *generalized divisors* as developed by Hartshorne ([6] and [5]) lend themselves well to this problem. In this language, the group of almost Cartier divisors for the ruled cubic was found in [5, 6], and the group of Cartier divisors on the smooth cubic surface is classically known ([4, V 4.8]).

This paper is organized as follows. In Section 2, we review the construction of the general ruled cubic X_0, briefly review Hartshorne's theory of generalized and almost Cartier divisors, and identify the almost Cartier divisor class group $\mathrm{APic}\, X_0$ on this

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surface. In Section 3 we base-extend the above family X so that there are 27 families of lines, which restrict on the general member of the family to the classical 27 lines on a smooth cubic surface, and we study the limits of these families on the special surface X_0. In Section 4 we compute the group of relatively almost Cartier divisors for the family X as in [6]. Now, if C is a curve on X_0 with no components supported on the double line, C comes from a flat family of divisors on X if it is the restriction of an effective Cohen-Macaulay relatively almost Cartier divisor on X. We show in Section 5 that this condition implies conditions on the divisor type of C. In Section 6, we specialize to the case where C is smooth, and we obtain an upper bound of 10 on the degree of C (see Theorem 6.1). This provides an example where the conjecture [2, Conjecture B′] of Ellia and Hartshorne is true, and it strengthens a result of Gruson and Peskine [3, B.4.], who proved that no *sufficiently general* smooth curve of degree > 10 on X_0 is such a limit.

2. Singular ruled cubic surface

In this section we review the construction of the general ruled cubic surface with a double line in $\mathbb{P}^3_k$ and calculate divisors on this surface.

We start with the projective plane $\mathbb{P}^2_k$ and a point Q in it. By blowing-up Q we obtain a surface S of degree 3 in $\mathbb{P}^4_k$ [4, II Exercise 7.7]. Let l be a line in $\mathbb{P}^2_k$ not containing Q. Its inverse image is a conic Γ in S which does not meet the exceptional curve E. Let N be the plane containing Γ and let $O \in N \setminus S$ be a point. Then we may project S with respect to O to $\mathbb{P}^3_k$. In this way we get a surface X_0 of degree 3 in $\mathbb{P}^3_k$ and a morphism $f : S \to X_0$, which sends Γ to a double line L and two points on Γ to two pinch points. The morphism f induces an isomorphism $f : S \setminus \Gamma \xrightarrow{\sim} X_0 \setminus L$. Note that the projection from O yields an involution σ on Γ (i.e. a one to one map σ on Γ such that $\sigma^2 = 1$). Another way to describe f is to consider the linear system of conics in P^2_k through Q, which meet l in a pair of the involution, i.e. a point on l and its image under σ. This gives a linear system $\mathcal{D}$ on S which induces f. Now the inverse images of lines in P^2_k through Q are all the lines in S, they meet E in one point, Γ in two, and cover S. These lines project to lines covering X_0 and meeting the lines $f(E)$ and L once. Hence X_0 is a ruled cubic surface in $\mathbb{P}^3_k$ with a double line L and two pinch points P and P' on L.

Alternatively, we may describe X_0 as follows: Let x, y, z, w be homogeneous coordinates of $\mathbb{P}^3_k$. After a suitable linear change of coordinates the surface X_0 can be defined by the equation $x^2z - y^2w = 0$. The rulings M_λ of X_0 are given by the equations $y = \lambda x$ and $z = \lambda^2 w$, together with $M_\infty (x = w = 0)$. Note that the only lines on X_0 are the rulings, L (given by $x = y = 0$) and the image $f(E)$ (given by $z = w = 0$) of E. Each point of L is contained in the rulings M_λ and its conjugate $M_{-\lambda}$ (the pinch points correspond to M_0 and M_∞, which are their own conjugates), and every ruling meets $f(E)$ (see Figure 1 after Proposition 10).

Let us now turn to the description of almost Cartier divisors on X_0 and their properties. This was done by Hartshorne in [5, Section 6].

DEFINITION 2.1 (See [5, Section 2]). Let X be any Noetherian scheme satisfying G_1 (i.e. every local ring of dimension ≤ 1 is Gorenstein) and the Serre property S_2. Let $\mathcal{K}_X$ be the sheaf of total quotient rings. A subsheaf $\mathcal{I} \subseteq \mathcal{K}_X$ is an *almost Cartier divisor* if

(a) $\mathcal{I}$ is a coherent reflexive nondegenerate (i.e. for every generic point η of X, $\mathcal{I}_\eta = \mathcal{K}_{X,\eta}$) $\mathcal{O}_X$-module.
(b) There is a closed subset $Z \subset X$ of codimension ≥ 2 such that $\mathcal{I}$ restricted to $X \setminus Z$ is Cartier (that is locally principal).

Remark 2.2. There is an exact sequence [5, 6.3]

$$0 \longrightarrow \operatorname{APic} X_0 \longrightarrow \operatorname{Pic} S \oplus \operatorname{Div} \Gamma / f^* \operatorname{Div} L \longrightarrow \operatorname{Pic} \Gamma / f^* \operatorname{Pic} L \longrightarrow 0,$$

which allows us to represent an element of $\operatorname{APic} X_0$ as a triple (a, b, α), where $(a, b) \in \mathrm{Z} \oplus \mathrm{Z} \cong \operatorname{Pic} S$ and $\alpha \in \operatorname{Div} \Gamma / f^* \operatorname{Div} L$, such that $a = \deg\, \alpha \pmod 2$ (in fact $\operatorname{Pic} \Gamma / f^* \operatorname{Pic} L \cong \mathrm{Z}/2\mathrm{Z}$).

We also recall the following proposition from [5, Section 6].

Proposition 2.3

Let $D \in \operatorname{APic} X_0$ be an effective divisor represented by (a, b, α); then

(a) $\deg D = 2a - b$.
(b) $p_a(D) = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}b(b-1) + \frac{1}{2}(a - h(\alpha))$, *where $h(\alpha)$ is the least degree of an effective divisor representing α in* $\operatorname{Div} \Gamma / f^* \operatorname{Div} L$.
(c) *D is represented by an effective divisor if and only if*
 (i) $a > b, a > 0$, *or*
 (ii) $a = \alpha = 0, b < 0$, *or*
 (iii) $a = b > 0$ *and* $h(\alpha) \leq a$.
(d) *D is represented by a smooth curve if and only if*
 (i) $D = f(E)$, *or*
 (ii) *D is a union of rulings meeting L at distinct points, or*
 (iii) $a > b \geq 0$ *and* $h(\alpha) = a$.

3. Degeneration of the 27 lines

We will now turn our attention to a family X of cubic surfaces over a smooth irreducible (not necessarily complete) curve T with point 0, where X_t is smooth for $t \neq 0$ and X_0 is the ruled cubic from Section 2. In this section, we examine the possible ways in which the 27 lines on the general surface can degenerate on X_0. We start by introducing some notation.

DEFINITION 3.1 (*Cf. [6], paragraphs before Proposition 1.3)* . Let $X \subset \mathbb{P}^3_T$ be a flat family of surfaces over a nonsingular irreducible curve T with special point 0, and let D be an almost Cartier divisor on X with associated sheaf $\mathcal{L} = \mathcal{L}(D)$, such that $\mathcal{L}$ is invertible at every generic point of X_0. Define the *restriction* of $\mathcal{L}$ to X_0 as the sheaf $\mathcal{L}_0 = (\mathcal{L} \otimes \mathcal{O}_{X_0})^{\vee\vee}$. $\mathcal{L}$ is *relatively almost Cartier* if its restriction to X_0 is invertible at all points of X_0 of codimension 1. Define the group of relatively almost Cartier classes by $\mathrm{RAPic}\,(X, X_0)$, or, if no confusion will result, $\mathrm{RAPic}\,X$.

Remarks 3.2.

1. If D is effective, then the divisor D_0 defined by the restriction $\mathcal{L}_0$ is the scheme obtained by throwing away any embedded components from the scheme-theoretic intersection $D \cap X_0$.
2. By [6], Proposition 1.3, the operation of restriction defines a group mapping ρ' : $\mathrm{RAPic}\,X \to \mathrm{APic}\,X_0$.

Proposition and Definition 3.3 (*Cf. [6], Proposition 1.4 and 1.5) Let $X \subset \mathbb{P}^3_T$ be a flat family of surfaces over a nonsingular irreducible curve T with special point 0, and let D be an almost Cartier divisor on X with associated sheaf $\mathcal{L} = \mathcal{L}(D)$. Then the following conditions are equivalent:*

1. *$\mathcal{L} \otimes \mathcal{O}_{X_0}$ is a reflexive sheaf.*
2. *$\mathcal{L}$ is a Cohen-Macaulay sheaf at all points of X_0.*
3. *The fractional ideal $\mathcal{I}_D \cong \mathcal{L}(-D)$ is a Cohen-Macaulay sheaf at all points of X_0.*
   In this case we say that D is a **Cohen-Macaulay divisor of X along X_0**.
   If in addition D is effective and contains no irreducible component of X_0, then the above are also equivalent to
4. *D is a Cohen-Macaulay scheme at all points of X_0.*
5. *The scheme D has no embedded points.*
   If D is effective without vertical components, and if either (a) X_t is smooth for $t \neq 0$ or (b) the curve $D_t \subset X_t$ is smooth for $t \neq 0$, then the above are equivalent to
6. *The divisors D_t form a flat family of curves in $\mathbb{P}^3$.*
7. *The arithmetic genus is constant over the family, i.e. $p_a(D_0) = p_a(D_t)$ for all $t \neq 0$ in T.*

Remark 3.4. Hartshorne states the above in a slightly more general context, specifically for X a three-dimensional Gorenstein scheme and X_0 a Cartier divisor on X.

Now suppose that we are in the situation of smooth cubics degenerating to the ruled cubic X_0, as described at the beginning of this section. By making a suitable base extension if necessary [6, 1.8], we may assume that there are effective CM almost Cartier Divisors E_i, G_i for $i = 1, \ldots, 6$ and F_{ij} for $1 \leq i < j \leq 6$ on the whole family X with no vertical components, which restricted to the nonsingular cubic surfaces are the 27 lines with the notation of [4, V 4.9]. This is the kind of family that we are going to study in the rest of the paper.

Let G be a CM almost Cartier divisor on X such that G_t is a line for every $t \in T \setminus \{0\}$ (27 possibilities, no vertical components). Then the lines form a flat family [6, 1.4 and 1.5] and G_0, the special member, is a line on the ruled cubic X_0. Now we know from Section 2 all the lines on X_0. Which of these will G_0 be? To answer this question we apply two principles.

Remark 3.5.

(1) A *triangle* Δ_t on the general fibre X_t is a scheme consisting of three distinct lines each meeting the other two. A *family of triangles* is an effective almost Cartier divisor Δ on X such that Δ_t is a triangle for every $t \in T \setminus \{0\}$. Since Δ_t is contained in a plane, its degeneration Δ_0 is also contained in a plane by semicontinuity. Therefore Δ is a CM divisor; in fact, it is Cartier on each fibre. (Δ is CM on X_0 because the ruled cubic is irreducible, so that the scheme-theoretic intersection $\Delta \cap X_0$ is already contained in the complete intersection of X_0 with a plane, which has no embedded components.)

(2) If two lines meet on the general fibre, then they also meet on the special fibre X_0, by [6, 3.1].

Lemma 3.6

No family of triangles on the family X can degenerate to a multiplicity-3 structure on the line L in the special fibre X_0.

Proof. A family of triangles Δ on the family X has degree three and is cut out by a plane on every fibre. Now assume that Δ degenerates to a triple structure on L. Then there must be a plane meeting X_0 at L only. This is impossible, because the equations of the rulings of X_0 in Section 2 show that every plane containing L also contains a ruling. □

Lemma 3.7

Two incident lines on the general fibre of the family X cannot degenerate to the same ruling on the special fibre X_0.

Proof. The two lines are part of a family of triangles Δ. The double structure on the ruling has to be contained in a plane (3.5 (1)), therefore its arithmetic genus is 0 [1, p. 116, and Exercise III-28]. But by 2.3 the 2-structure on the ruling has arithmetic genus -1 (in fact a ruling is represented by the triple $(1, 1, Q)$ where Q is the intersection point of the ruling with L). This is impossible. □

Lemma 3.8

At least one of the 27 lines on the general fibre of the family X degenerates to the line $f(E)$ on the special fibre X_0.

Proof. Assume that none of the 27 lines degenerate to $f(E)$. Then the only a priori possible degenerations of triangles on X_0 are two: the double line L and a ruling M_λ or the reduced line L and a multiplicity 2 structure on a ruling M_λ. Only the first of the two possibilities can occur (see 3.7). Now by 3.6 there is a line, E_1 say (after relabeling the 27 lines if necessary [4, V 4.10]), that degenerates to a ruling. Thus the triangles $E_1F_{1i}G_i$ show that the lines F_{1i}, G_i degenerate to L for $2 \le i \le 6$.

Consider the triangles $G_1F_{1i}E_i$: Where does G_1 and E_i degenerate for $2 \le i \le 6$? We have precisely two cases both of which will lead to impossibilities.

First case. G_1 degenerates to a ruling and E_i to L for $2 \leq i \leq 6$. The triangles $E_2F_{23}G_3$ and $E_4F_{45}G_5$ show, using 3.6 and 3.7, that the lines F_{23} and F_{45} degenerate to distinct rulings. But this means that the triangle $F_{16}F_{23}F_{45}$ degenerates to the reduced line L and two rulings, which is not contained in a plane, an impossibility.

Second case. G_1 degenerates to L and E_i to rulings for $2 \leq i \leq 6$. The triangles $E_2F_{23}G_3$ and $E_4F_{45}G_5$ show in this case that F_{23} and F_{45} degenerate to L. Then the triangle $F_{16}F_{23}F_{45}$ degenerates to a multiplicity 3 structure on L, which is impossible by 3.6. □

Lemma 3.9

Two lines on the general fibre of the family X cannot degenerate to $f(E)$ on the special fibre X_0.

Proof. Let D be an almost Cartier divisor on the whole family X such that D_t is the union of two lines, for $t \neq 0$ and D_0 a 2-structure on $f(E)$. D is a CM divisor because it is locally Cartier everywhere (here we use the fact that X_0 is nonsingular along $f(E)$). The arithmetic genus of D_t is 0 (if the two lines meet) or -1 (else) for $t \neq 0$ [4, V 4.8]. But the genus of D_0 is -2 by 2.3 ($f(E)$ is represented by $(0, -1, 0)$). This contradicts the fact that a CM divisor on the family gives a flat family of curves [6, 1.5]. □

Proposition 3.10

After relabeling, if necessary, the 27 *lines degenerate as follows (see* Figure 1 below).

(i) E_6, G_i for $1 \leq i \leq 5$ and F_{ij} for $1 \leq i < j \leq 5$ degenerate to L.
(ii) G_6 degenerates to $f(E)$.
(iii) E_i for $1 \leq i \leq 5$ degenerate to (possibly coincident) rulings.
(iv) F_{i6} for $1 \leq i \leq 5$ degenerate to the conjugate rulings of (iii).

Proof. By 3.8 and relabeling the 27 lines if necessary we may assume that G_6 degenerates to $f(E)$ and by 3.9 this is the only line degenerating to $f(E)$. Consider the triangle $G_6F_{16}E_1$; then F_{16} and E_1 must degenerate to conjugate rulings. In fact none of these lines can degenerate to L, by 3.5 (2). Lemma 3.7 tells us that F_{16} and E_1 have to degenerate to distinct rulings. These must meet by 3.5 (2), hence the rulings have to be conjugated. The same holds for F_{i6} and E_i for $1 \leq i \leq 5$. Now by 3.9 E_6 degenerates to either a ruling or to L. In the first case we come to a contradiction as follows. The triangle $E_6F_{i6}G_i$ shows that G_i degenerates to $f(E)$ for $1 \leq i \leq 5$, a contradiction to 3.9. Hence E_6 must degenerate to L as well as G_i for $1 \leq i \leq 5$ (using the same triangles). Finally the triangles $E_iF_{ij}G_j$ show that also F_{ij} degenerates to L for $1 \leq i < j \leq 5$. This concludes our proof. □

4. Almost Cartier divisors on X

Assume that X and X_0 are as usual, with the base curve T base-extended as above so that there are 27 irreducible families of lines. In this section we compute the group of almost Cartier divisors APic (X) and the subgroup of *relatively almost Cartier divisors* RAPic (X), which are those almost Cartier divisors on X whose restriction to X_0 is almost Cartier (see [6, 1.3]). Now, since X_0 is irreducible, the group of almost Cartier divisors with no vertical components APic $(X)/f^*\text{Pic}\,(T)$ is isomorphic to Z^7 by [6, 1.2 and 1.1]. We will denote this group APic (X) in the sequel omitting $f^*\text{Pic}\,(T)$. Similarly we will write simply RAPic (X) for the subgroup of relative almost Cartier divisors with no vertical components. What are the generators of RAPic (X)?

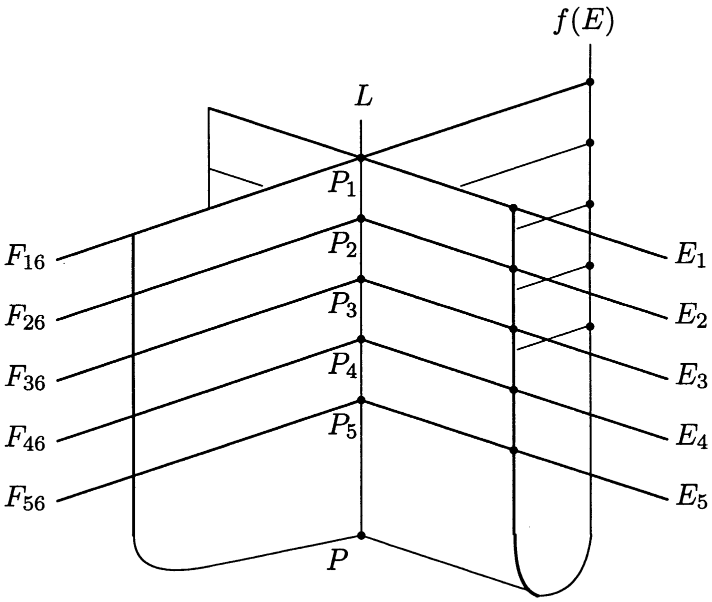

Figure 1: Lines on X_0

Lemma 4.1

The group RAPic (X) *is a subgroup of index* 2 *of* APic (X)*, and it is free of rank* 7 *generated by* $H, G_6, \{E_i : 1 \leq i \leq 5\}$*, where* H *is the class of a hyperplane section of* X.

Proof. Let P_i be the intersection of the restriction of E_i to X_0 with the line L for $1 \leq i \leq 5$, see Figure 1 after Proposition 3.10. Clearly E_i is locally Cartier everywhere except at P_i, $1 \leq i \leq 5$. Hence $E_i \in \mathrm{RAPic}\,(X)$. Since H is a Cartier divisor, $H \in \mathrm{RAPic}\,(X)$. The divisor G_6 also belongs to $\mathrm{RAPic}\,(X)$ because it is locally Cartier everywhere (since X_0 is regular along $f(E)$).

Set $S := E_i : 1 \leq i \leq 5$. Now, for $t \neq 0$, consider the restriction map $\cdot_t$: $\mathrm{RAPic}\,(X) \subset \mathrm{APic}\,(X) \longrightarrow \mathrm{APic}\,(X_t)$ in the remarks following Definition 3.1. By looking at the images of the classes H, G_6, and the elements of S on a general fibre, we see that $< H, G_6, S >$ is free of rank 7 and that it is a subgroup of index 2 in $\mathrm{APic}\,(X)$. Thus the index formula for the chain of inclusions of subgroups

$$< H, G_6, S > \subseteq \mathrm{RAPic}\,(X) \subset \mathrm{APic}\,(X)$$

yields the equality $< H, G_6, S >= \mathrm{RAPic}\,(X)$, because there are elements in $\mathrm{APic}\,(X)$ not in $\mathrm{RAPic}\,(X)$ (for example E_6). □

Let $D := hH + gG_6 + \sum_{i=1}^{5} e_i E_i \in \mathrm{RAPic}\,(X)$, where $h, g, e_1, ..., e_5 \in \mathbb{Z}$. The restriction of D to a general fibre ($t \neq 0$) has divisor type

$$D_t = (3h + 2g; h + g - e_1, h + g - e_2, \ldots, h + g - e_5, h), \tag{1}$$

hence we know the degree and arithmetic genus of D_t (if effective). Let us now turn to the restriction of D to the special fibre (see Section 2), we have

$$\begin{aligned} H_0 &: (2, \quad 1, \quad 0) \\ (E_i)_0 &: (1, \quad 1, \quad P_i) \\ (F_{i6})_0 &: (1, \quad 1, -P_i) \quad \text{if } P_i \text{ is not a pinch point, } (1,1,P_i) \text{ else} \\ (G_6)_0 &: (0, -1, \quad 0) \end{aligned}$$

for $1 \leq i \leq 5$, where two or more of the five points P_i may coincide and may coincide also with the pinch points P and P'. Hence

$$D_0 = \left(2h + \sum_{i=1}^{5} e_i\,, h - g + \sum_{i=1}^{5} e_i\,, \sum_{i=1}^{5} e_i P_i\right). \tag{2}$$

By 2.3 we can find the degree and arithmetic genus of D_0 if it is effective.

5. Cohen-Macaulay divisors in RAPic (X)

We determine the Cohen-Macaulay relatively almost Cartier divisors with no vertical components on X.

EXAMPLE 5.1: An effective divisor of class $nH, n > 0$, gives a CM family of complete intersections, since the genus of a complete intersection depends only on the degrees of the intersected surfaces. For a more interesting example, let C_0 be an effective divisor on X_0 which is residual to a ruling of type $(1,1,P)$ in a complete intersection. Then the divisor type of C is $(2n-1,n-1,Q)$, where n is the degree of the other surface in the complete intersection and Q is the point conjugate to P. Then C_0 is the limit of a family of divisors on a family of smooth cubics containing L, and one can verify that the genus formula given in Proposition 2.3 gives the same answer as the usual genus formula on a smooth cubic.

Lemma 5.2

Let

$$D := \sum_{i=0}^{s} f_i(F_{i6}+G_6) + \sum_{j=s+1}^{6} e_jE_j \in \mathrm{RAPic}\,(X)$$

be an effective divisor with $0 \leq s \leq 5$, $f_0 = e_6 = 0$. *Then* D *is* CM *if and only if* $\sum_{i=0}^{s} f_i^2 + \sum_{j=s+1}^{6} e_j^2 - h(\alpha) = 0$, *where* $h(\alpha)$ *is the least degree of an effective divisor on* Γ *representing the third component of* D_0 *(see* 2.3*).*

Proof. Set $f := \sum_{i=0}^{s} f_i$ and $e := \sum_{j=s+1}^{6} e_j$; then, by (2) in section 4 we have $D_0 = (f+e, e, \alpha)$, where $\alpha := \sum_{i=0}^{s}(-1)^{p_i} f_iP_i + \sum_{j=s+1}^{6} e_jP_j$ and $p_i = 0$ if P_i is a pinch point, $p_i = 1$ else. Hence we may compute the arithmetic genus with 2.3

$$p_a(D_0) = \frac{1}{2}\big(f^2 + 2fe - 2f - e + 2 - h(\alpha)\big).$$

On the general fibre we have (1) of Section 4:

$$D_t = (3f, f+f_1, .., f+f_s, f-e_{s+1}, .., f-e_5, f).$$

By [4, V 4.8] the arithmetic genus is

$$p_a(D_t) = \frac{1}{2}\left(f^2 - \sum_{i=0}^{s} f_i^2 - \sum_{j=s+1}^{6} e_j^2 + 2fe - 2f - e + 2\right).$$

Now D is CM if and only if the two arithmetic genera are equal. □

Note that the lemma holds also for $D := \sum_{i\in A} f_i(F_{i6}+G_6) + \sum_{j\in B} e_jE_j$, where A and B are nonempty complementary subsets of $\{0,1,...,6\}$.

Lemma 5.3

A divisor $D \in \mathrm{APic}\,(X)$ *is* CM *if and only if its negative* $-D$ *is.*

Proof. Since X is a Gorenstein scheme, this follows from [5, 1.14], which says that a finitely generated module over a local Gorenstein ring is CM if and only if its dual is. □

Remark 5.4. Let $D := hH + gG_6 + \sum_{i=1}^{5} e_iE_i \in \mathrm{RAPic}\,(X)$. Since X is regular everywhere except along the double line L, the divisor D is locally Cartier and hence CM except possibly at the points where it meets L, namely $P_1, ..., P_5$. So we only have to check at these points which are the CM divisors in RAPic$\,(X)$.

Proposition 5.5

Let $D := hH + gG_6 + \sum_{i=1}^{5} e_iE_i \in \mathrm{RAPic}\,(X)$ *be a divisor. Keeping the above notation, assume that* $P_1, ..., P_5$ *are distinct points. Then* D *is* CM *if and only if* $|e_i| \leq 1$ *for* $1 \leq i \leq 5$.

Proof. By 5.4 we only have to compute the CM divisors at $P_1, ..., P_5$. Without loss of generality we may restrict ourselves to the computation at P_1. Now at P_1, D and e_1E_1 are the same, hence it suffices to compute the conditions for e_1E_1 to be CM.

Case $e_1 \geq 0$. Then e_1E_1 is CM if and only if $e_1^2 - h(\alpha) = 0$, by 5.2. Since $\alpha = e_1P_1$ we have $h(\alpha) = e_1$ (if P_1 is not a pinch point) or $h(\alpha) = 0$ (if P_1 is a pinch point and e_1 is even) or $h(\alpha) = 1$ (if P_1 is a pinch point and e_1 is odd). Hence D is CM at P_1 if and only if $e_1 = 0, 1$.

Case $e_1 \leq 0$. Set $e := -e_1$; then eE_1 is effective and dual to e_1E_1. The same argument as before applied to eE_1 and 5.3 yield that D is CM if and only if $e_1 = -1, 0$. □

Proposition 5.6

Let $D := hH + gG_6 + \sum_{i=1}^{5} e_iE_i \in \mathrm{RAPic}\,(X)$ *be a divisor, and assume that precisely* $2 \leq r \leq 5$ *of the points* $P_1, ..., P_5$ *are equal, say* $P_1 = \cdots = P_r$ *(after relabeling if necessary).*

(a) *Suppose the coincident point is not a pinch point. Then* D *is* CM *at* P_1 *if and only if* $\sum_{i=1}^{r} e_i^2 - |\sum_{i=1}^{r} e_i| = 0$.

(b) *Suppose that the coincident point* $P_1 = \cdots = P_r$ *is a pinch point. Then* D *is* CM *at* P_1 *if and only if* $\sum_{i=1}^{r} e_i^2 = 0, 1$.

Proof. (a). The divisor D at $P_1 = \cdots = P_r$ is the same as $D' := \sum_{i=1}^{r} e_iE_i$. Hence D is CM at P_1 if and only if D' is. The class of D' in $\mathrm{APic}\,(X_0)$ is given by (e, e, eP_1), where $e := \sum_{i=1}^{r} e_i$. Note that $h(eP_1) = |e|$ because P_1 is not a pinch point, by assumption. We want to represent D' by an effective divisor in order to be able to apply 5.2.

A) *Case where* $e_i \geq 0$ *for* $1 \leq i \leq r$. The divisor D' is effective, thus D' is CM at P_1 if and only if $\sum_{i=1}^{r} e_i^2 - e = 0$, by 5.2.

B) *Case where* $e_i \leq 0$ *for* $1 \leq i \leq r$. Here we apply 5.3 and reduce to case A). Then D' is CM at P_1 if and only if $\sum_{i=1}^{r} e_i^2 - |e| = 0$, by 5.2.

C) *Case where* $e_1 \leq 0$ *and* $e_i \geq 0$ *for* $2 \leq i \leq r$. Let $f_1 := -e_1$. If we add f_1 times the hyperplane section $E_1 + F_{16} + G_6$ (which is principal) to D' (or D), we do not affect its class and $D'' := f_1(F_{16} + G_6) + \sum_{i=2}^{r} e_iE_i$ is effective. Apply 5.2. Then D' is CM at P_1 if and only if $\sum_{i=1}^{r} e_i^2 - |e| = f_1^2 + \sum_{i=2}^{r} e_i^2 - |e| = 0$.

D) *Case where* $2 < r$, $e_1, e_2 \leq 0$ *and* $e_i \geq 0$ *for* $3 \leq i \leq r$. If we cannot reduce to one of the previous cases by dualizing and relabeling, we will set $f_1 := -e_1$ and $f_2 := -e_2$. By adding f_1 times the hyperplane section $E_1 + F_{16} + G_6$ and f_2 times $E_2 + F_{26} + G_6$ we may represent D' by the effective divisor $f_1(F_{16} + G_6) + f_2(F_{16} + G_6) + \sum_{i=3}^{r} e_iE_i$. Again 5.2 gives that D' is CM at P_1 if and only if $\sum_{i=1}^{r} e_i^2 - |e| = f_1^2 + f_2^2 + \sum_{i=3}^{r} e_i^2 - |e| = 0$.

Any divisor D' can be brought into the form of one of the above four cases just by dualizing and relabeling. This concludes the proof of (a).

For the proof of (b) we follow the proof of (a). The only difference is that here, since P_1 is a pinch point, $h(eP_1) = 0$ if e is even and 1 if e is odd. □

Corollary 5.7

Let $D := hH + gG_6 + \sum_{i=1}^{5} e_i E_i \in \mathrm{RAPic}\,(X)$*, let* $E_i \cap L = P_i$ *for* $1 \leq i \leq 5$ *and denote by* P_0, P_6 *the two pinch points of* X_0*. Let* $S := \{1, ..., 5\}$ *and* $S_c := \{i \in S : \exists j \in S \setminus \{i\}$ *such that* $P_i = P_j\}$*. Define the disjoint (possibly empty) subsets* S_0, S_1, S_2, S_3 *as follows:*

$S_0 := \{i \in S : P_i = P_0\}$, $S_1 := \{i \in S : P_i = P_6\}$,
$S_2 := \{i \in S : P_i = P_j,\ j \in S_c \setminus (S_0 \cup S_1)\}$,
$S_3 := \{i \in S : P_i = P_k,\ k \in S_c \setminus (S_0 \cup S_1 \cup S_2)\}$.

Then D *is* CM *if and only if*

(i) $|e_i| \leq 1$ *for* $i \in S \setminus (\bigcup_{l=0}^{3} S_l)$.
(ii) $\sum_{i \in S_l} e_i^2 - |\sum_{i \in S_l} e_i| = 0$ *for* $l = 2, 3$.
(iii) $\sum_{i \in S_l} e_i^2 = 0$ *or* 1 *for* $l = 0, 1$.

Proof. One only has to find necessary and sufficient conditions for D to be Cohen-Macaulay at the points P_1 to P_5. At the points P_i with $i \in S \setminus (\cup_{l=0}^{3} S_l)$ the necessary and sufficient condition (i) is due to the proof of 5.5. If $i \in S_2 \cup S_3$, then (ii) is necessary and sufficient because of 5.6 (a). At the points P_i with $i \in S_0 \cup S_1$ the divisor D is CM if and only if (iii) holds, by 5.6 (b). □

Remark 5.8. From the Corollary 5.7 follows that whenever $D := hH + gG_6 + \sum_{i=1}^{5} e_i E_i \in \mathrm{RAPic}\,(X)$ is CM, the absolute value of the e_i's cannot be larger than 1. This implies that $0 \leq h(\alpha) \leq 5$, where $D_0 = (a, b, \alpha)$ is the restriction of D to X_0 (see 2.3). Thus we get a strong necessary condition for an effective divisor on X_0, having no components supported on the double line, to be a limit of a flat family of effective divisors on smooth cubics. Except in special easy cases (such as those in 5.1), we do not know of any conditions on a specific effective divisor on X_0 which will guarantee that it is realizable as such a limit.

6. Specialization to smooth curves

Let X denote as usual a family of cubic surfaces parametrized by the irreducible nonsingular curve T. In this section we look at families of curves on X which specialize to a smooth curve on the ruled cubic surface X_0. Observe that if $D_0 \in \mathrm{APic}\,(X_0)$ is a smooth curve in X_0 which comes from a flat family of curves D_t on X_t, then we may assume that there is an effective CM divisor $D \in \mathrm{RAPic}\,(X)$ with no vertical components which restricts to D_t on X_t (see [6, 1.6 and 1.6.1]).

Theorem 6.1

Let $D_t \subseteq \mathbb{P}_k^3$ *be a flat family of curves included in* X *such that* D_0 *is a smooth curve in the ruled cubic* X_0 *represented by* (a, b, α) *(see 2.3). Then* $0 \leq a \leq 5$*, and in particular* $\deg D_0 \leq 10$.

Proof. In the observation preceding the theorem we saw that there is an effective CM divisor $D \in \text{RAPic}(X)$ with no vertical components which restricts to the given curves D_t on X_t. By Section 4, $D := hH + gG_6 + \sum_{i=1}^{5} e_i E_i \in \text{RAPic}(X)$. Let $E_i \cap L = P_i$ for $1 \le i \le 5$. Proposition 2.3 tells us that there are only three types of classes of smooth curves on X_0.

(i) $D_0 = f(E)$, represented by $(0,-1,0)$ has degree 1 and $a = 0$. The only family of curves degenerating to $f(E)$ is G_6 (see 3.10).

(ii) D_0 is a union of rulings meeting at distinct points of L. The families degenerating to such a curve are of the form $\sum_{i=1}^{s} \delta_i (H - G_6 - E_i) + \sum_{j=s+1}^{5} \delta_j E_j$ with $\delta_i = 0, 1$ and $0 \le s \le 5$ (after relabeling if necessary), hence $0 \le \deg D_0 = \sum_{i=1}^{5} \delta_i = a \le 5$ (see 2.3).

(iii) D_0 is represented by (a, b, α) with $a > b \ge 0$ and $h(\alpha) = a$. On the other hand the restriction of the family D to X_0 is given by

$$\left(2h + \sum_{i=1}^{5} e_i, h + \sum_{i=1}^{5} e_i - g, \sum_{i=1}^{5} e_i P_i\right).$$

By replacing these values for a, b, and α we obtain the following inequalities:

$$c \le -g < -\left(c + \sum_{i=1}^{5} e_i\right) \le c + 5, \tag{3}$$

where $c := -(\sum_{i=1}^{5} e_i + h)$. Now Remark 5.8 shows that $|\sum_{i=1}^{5} e_i| \le 5$ and that generally $0 \le h(\alpha) \le 5$. Therefore we get in our situation $1 \le a = h(\alpha) \le 5$ and, using (3), deg $D_0 = 3h + \sum_{i=1}^{5} e_i + g \le 10$ (see 2.3 and Table 1 below). □

The family of lines G_6 has degree 1 and arithmetic genus 0. What are the possible degree d and genus p_a of families of curves with limit a curve of type (ii) in 2.3 ? Write $D := \sum_{i=1}^{s} \delta_i (H - G_6 - E_i) + \sum_{j=s+1}^{5} \delta_j E_j$ with $\delta_i = 0, 1$ and $0 \le s \le 5$ as in the proof of 6.1. Using 5.7 we obtain $(d, p_a) = (1,0), (2,-1), (3,-2), (4,-3), (5,-4)$.

Now let $D := hH + gG_6 + \sum_{i=1}^{5} e_i E_i \in \text{RAPic}(X)$ be an effective CM divisor with no vertical components and special fibre D_0 a smooth curve of type (iii) in 2.3. Since $e := \sum_{i=1}^{5} e_i$ ranges between -5 and 5 (see 5.7), we are able to write Table 1 of all possible values of degree d, $h(\alpha)$ and arithmetic genus p_a of D_0. We use (3) to compute h and g. Then these values feed 2.3 with which we obtain d, $h(\alpha)$ and p_a.

d	p_a	$h(\alpha)$
10	6	5
9	6	5
8	3	4
	5	5
7	3	4
		5
6	0	5
	1	3
	2	4

d	p_a	$h(\alpha)$
5	0	4
	1	3
4	0	2
		3
3	0	2
2	0	1

Table 1: Specialization to smooth curves of type (iii) of 2.3 on X_0

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