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Interpolation properties of a scale of spaces

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Abstract

A scale of function spaces is considered which proved to be of considerable importance in analysis. Interpolation properties of these spaces are studied by means of the real interpolation method. The main result consists in demonstrating that this scale is interpolated in a way different from that for L^p spaces, namely, the interpolation space is not from this scale.

1. Introduction

We consider the spaces $A_{p,r}$ endowed with the norms

$$\|f\|_{A_{p,r}} = \left(\int_0^\infty \left(\frac{1}{u}\int_{u\le|\xi|\le 2u} |f(\xi)|^p d\xi\right)^{r/p} du\right)^{1/r} \qquad (1 \le p, r < \infty),$$

$$\|f\|_{A_{\infty,r}} = \left(\int_0^\infty \left(\sup_{u\le|\xi|\le 2u} |f(\xi)|\right)^{r} du\right)^{1/r} \qquad (1 \le r < \infty),$$

$$\|f\|_{A_{p,\infty}} = \sup_{u>0}\left(\frac{1}{u}\int_{u\le|\xi|\le 2u} |f(\xi)|^p d\xi\right)^{1/p} \qquad (1 \le p < \infty).$$

The case $p = r = \infty$ is naturally reduced to $A_{\infty,\infty} = L^\infty$; moreover for any $1 \le p < \infty$ we have $A_{p,p} = L^p$. In the case $r = 1$ we denote $A_{p,1} = A_p$.

Mainly the spaces A_p were applied to different problems in analysis. Multiplier properties of A_p were considered in [7], though the spaces A_p have first appeared in a paper by D. Borwein [5] devoted to summability problems. The A_∞ – with the Fourier transform $\widehat{f}$ instead of f – has an even longer history. It was introduced in a

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paper by A. Beurling [4] dealing with spectral synthesis problems. A detailed survey of applications of A_∞ is given in [1]. In [8] the A_p spaces gave a good example of applying a general theorem on representation of continuous linear functionals. These spaces in the sequence version were mainly used in problems of integrability of trigonometric series. In [9] the asymptotic behaviour of the Fourier transform was studied for functions with derivative in A_p or more general space. The latter results were modified and used in approximation problems in [6]. The role of the second parameter r in $A_{p,r}$ is not quite clear in these or different problems.

In [10] the results from [7] were generalized to more general spaces, but in addition interpolation properties of A_p as well as $A_{p,r}$ were studied by means of L^p interpolation properties of vector-valued functions (see, e.g., [3, Theorem 5.1.2.]). Here we study the same object by means of K-functionals, that is, the real interpolation method.

In [10] a misleading statement is given; what is really obtained is Theorem 2.2 of the present paper. However the main result here is Theorem 2.1 in which not only the interpolation space for any pair $A_{p_0,r}$ and $A_{p_1,r}$ is described but examples are given to show that their interpolation properties differ from those for L^p spaces. For the sake of completeness we also prove, in the same manner, Theorem 2.2.

The paper is organized as follows. In the second Section we give necessary notation and formulate the results. In the next Section the proofs are given.

2. Results

Recall that the interpolation space $(X_1, X_2)_{\theta,p}$ of the couple X_1 and X_2 consists of all $f \in X_1 + X_2$ such that

$$\|f\|_{\theta,p} = \left(\int_0^\infty \left[t^{-\theta} K(f,t;X_1,X_2)\right]^p \frac{dt}{t} \right)^{1/p} < \infty \quad (0 < \theta < 1,\ 1 \le p < \infty),$$

where

$$K(f,t;X_1,X_2) = \inf_{f=g+h} (\|g\|_{X_1} + t\|h\|_{X_2})$$

is the K-functional of the two spaces.

It is well-known that for $0 < \theta < 1, 1 \le p_0 < p_1 \le \infty$, and $1/p = (1-\theta)/p_0 + \theta/p_1$ we have

$$(L^{p_0}, L^{p_1})_{\theta,p} = L^p.$$

One may expect that the most important spaces $A_p = A_{p,1}$ are interpolated in the same, so to say natural way. It turns out that this is not the case for them, moreover for most of $A_{p,r}$ which are of the vector-valued type. The "L^p type" interpolation occurs when the first parameter is fixed (see Theorem 2.2); unfortunately this case seems to be of less importance to date. In the case when the second parameter is fixed while the first one varies, the interpolation space is of very special structure and contains an "irregular" part as regarded to $A_{p,r}, p \neq r$.

Recall that the non-increasing rearrangement of a measurable function f is defined by

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)| \quad (0 < t < \infty).$$

Given $p_0 < p_1$, denote $\overline{p} = p_0 p_1/(p_1 - p_0)$. By $a \asymp b$ we denote a two-sided estimate $c_1 b \le a \le c_2 b$ with c_1, c_2 being two constant independent of essential parameters. Denote by I_j the (doubled) dyadic interval $I_j = \{\xi : 2^j \le |\xi| \le 2^{j+1}\}$. Let $f_j = f \cdot \chi_{I_j}$, where χ_E is the indicator function of a set E.

Theorem 2.1

Let $0 < \theta < 1, 1 \le p_0 < p_1 \le \infty$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then

(i) The following equivalence holds for all $r \ge 1$:

$$(1)\quad K(f,t;A_{p_0,r},A_{p_1,r}) \asymp \Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_0)} \Big(\int_0^{2^j t^{\overline{p}}} [f_j^*(\tau)]^{p_0} d\tau\Big)^{r/p_0}\Big)^{1/r} + t\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_1)} \Big(\int_{2^j t^{\overline{p}}}^{\infty} [f_j^*(\tau)]^{p_1} d\tau\Big)^{r/p_1}\Big)^{1/r} \quad (0 < t < \infty).$$

(ii) If $1 \le r < p$, then

$$A_{p,r} \subset (A_{p_0,r}, A_{p_1,r})_{\theta,p},$$

and the converse embedding is invalid: there exists a function

$$f \in (A_{p_0,r}, A_{p_1,r})_{\theta,p} \setminus A_{p,r}.$$

(iii) If $r = p$, then

$$(A_{p_0,p}, A_{p_1,p})_{\theta,p} = A_{p,p} = L^p.$$

(iv) If $p < r < \infty$, then

$$(A_{p_0,r}, A_{p_1,r})_{\theta,p} \subset A_{p,r},$$

and the converse embedding is invalid: there exists a function

$$f \in A_{p,r} \setminus (A_{p_0,r}, A_{p_1,r})_{\theta,p}.$$

Theorem 2.2

Let $1 \le r_0 < r_1 \le \infty, 0 < \theta < 1$, and $1/r = (1-\theta)/r_0 + \theta/r_1$. Then for all $1 \le p \le \infty$

$$(2)\quad (A_{p,r_0}, A_{p,r_1})_{\theta,r} = A_{p,r}.$$

3. Proofs

Proof of Theorem 2.1 First, note that the norm in $A_{p,r}$ can be represented in a usual way via doubled dyadic intervals $I_j = \{\xi : 2^j \le |\xi| \le 2^{j+1}\}$:

$$\|f\|_{A_{p,r}} \asymp \Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p)}\Big(\int_{I_j} |f(\xi)|^p d\xi\Big)^{r/p}\Big)^{1/r} \qquad (1 \le p \le \infty,\ 1 \le r < \infty). \tag{3}$$

Now (i) is a simple consequence of (3) and the known formula for the K-functional for (L^{p_0}, L^{p_1}) couple (see, e.g., [3, p. 124]):

$$K(g,t;L^{p_0},L^{p_1}) \asymp \Big(\int_0^{t^{\overline{p}}} [g^*(\tau)]^{p_0} d\tau\Big)^{1/p_0} + t\Big(\int_{t^{\overline{p}}}^\infty [g^*(\tau)]^{p_1} d\tau\Big)^{1/p_1};$$

unfortunately, a misprint occurs in [3], namely, t is omitted before the last term on the right; in [2, p. 308] the result needed is given in a more general form.

Observe that for $t > 2^{1/\overline{p}}$ the right side of (1) is equivalent to $\|f\|_{A_{p_0,r}}$.

Now we prove (ii). Using (1) and Minkowski's inequality, we obtain

$$\begin{aligned}
&\|f\|_{(A_{p_0,r},A_{p_1,r})_{\theta,p}} \\
&\le c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_0)}\Big(\int_0^\infty t^{-\theta p}\Big(\int_0^{2^j t^{\overline{p}}} [f_j^*(\tau)]^{p_0} d\tau\Big)^{p/p_0}\frac{dt}{t}\Big)^{r/p}\Big)^{1/r} \\
&\quad + c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_1)}\Big(\int_0^\infty t^{(1-\theta)p}\Big(\int_{2^j t^{\overline{p}}}^\infty [f_j^*(\tau)]^{p_1} d\tau\Big)^{p/p_1}\frac{dt}{t}\Big)^{r/p}\Big)^{1/r}.
\end{aligned} \tag{4}$$

Now the substitution $t^{\overline{p}} \to t$ and Hardy's inequality (see, e.g., [2, p. 124]) show that the first sum in (4) is at most

$$\begin{aligned}
&c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_0)}\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^{2^j t} [f_j^*(\tau)]^{p_0} d\tau\Big)^{p/p_0} dt\Big)^{r/p}\Big)^{1/r} \\
&\le c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p)}\Big(\int_{I_j} |f(\xi)|^p d\xi\Big)^{r/p}\Big)^{1/r} \le c\|f\|_{A_{p,r}}.
\end{aligned}$$

Since the rearrangement is non-increasing, the same substitution yields that the second sum in (4) is bounded by

$$\begin{aligned}
&c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_1)}\Big(\int_0^\infty \Big(\frac{1}{t}\int_{2^j t}^\infty [f_j^*(\tau)]^{p_1} d\tau\Big)^{p/p_1} dt\Big)^{r/p}\Big)^{1/r} \\
&\le c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p)}\Big(\int_0^\infty \Big(\sum_{k=0}^\infty 2^k [f_j^*(2^k t)]^{p_1}\Big)^{p/p_1} dt\Big)^{r/p}\Big)^{1/r} \\
&\le c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p)}\Big(\sum_{k=0}^\infty 2^{kp/p_1}\int_0^\infty [f_j^*(2^k t)]^p dt\Big)^{r/p}\Big)^{1/r} \\
&= c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p)}\Big(\sum_{k=0}^\infty \frac{1}{2^{k(1-p/p_1)}}\int_{I_j} |f(\xi)|^p d\xi\Big)^{r/p}\Big)^{1/r} \le c\|f\|_{A_{p,r}}.
\end{aligned}$$

Therefore, we get

$$\|f\|_{(A_{p_0,r},A_{p_1,r})_{\theta,p}} \le c\|f\|_{A_{p,r}}, \tag{5}$$

which proves the embedding $A_{p,r} \subset (A_{p_0,r}, A_{p_1,r})_{\theta,p}$.

To complete the second item, let us show that there exists a function $f \in (A_{p_0,r}, A_{p_1,r})_{\theta,p}$ such that $f \notin A_{p,r}$. Let f be of the form

$$f = \sum_{j=1}^{\infty} c_j \chi_{(2^j,2^j(1+a_j))},$$

where sequences $\{c_j\}, \{a_j\}$ will be specified later on. We now only point out that $a_j < 1$, $\{a_j\}$ is decreasing and tends to zero. It is easy to see that

$$\|f\|_{A_{s,r}} \asymp \left(\sum_{j=1}^{\infty} 2^j c_j^r a_j^{r/s}\right)^{1/r}. \tag{6}$$

Next, a simple calculation shows that

$$\left(\int_0^{2^j t^{\overline{p}}} [f_j^*(\tau)]^{p_0} d\tau\right)^{r/p_0} = \begin{cases} (2^j)^{r/p_0} c_j^r t^{\overline{p}r/p_0}, & 0 < t \le a_j^{1/\overline{p}}, \\ c_j^r (2^j a_j)^{r/p_0}, & t > a_j^{1/\overline{p}}, \end{cases}$$

and

$$\left(\int_{2^j t^{\overline{p}}}^{\infty} [f_j^*(\tau)]_1^p d\tau\right)^{r/p_1} \le c_j^r (2^j a_j)^{r/p_1} \chi_{(0,a_j^{1/\overline{p}})}(t).$$

Therefore,

$$\begin{aligned} K(f,t;&A_{p_0,r}, A_{p_1,r}) \\ &\asymp \left(\sum_{j=1}^{\infty}\Big((2^j c_j^r t^{\overline{p}r/p_0} + 2^j c_j^r a_j^{r/p_1} t^r)\chi_{(0,a_j^{1/\overline{p}})}(t) + 2^j c_j^r a_j^{r/p_0} \chi_{(a_j^{1/\overline{p}},\infty)}(t)\Big)\right)^{1/r} \\ &\asymp \left(\sum_{k=1}^{\infty}\Big(t^{\overline{p}r/p_0}\sum_{j=1}^{k} 2^j c_j^r + t^r \sum_{j=1}^{k} 2^j c_j^r a_j^{r/p_1} + \sum_{j=k+1}^{\infty} 2^j c_j^r a_j^{r/p_0}\Big)\chi_{(a_{k+1}^{1/\overline{p}},a_k^{1/\overline{p}})}(t) \right. \\ &\quad \left. + \|f\|_{A_{p_0,r}} \chi_{(a_1^{1/\overline{p}},\infty)}(t)\right)^{1/r}. \end{aligned}$$

From this we get

$$\begin{aligned} &\|f\|^p_{(A_{p_0,r},A_{p_1,r})_{\theta,p}} \\ &\le c\sum_{k=1}^{\infty}\left\{\Big(\sum_{j=1}^{k} 2^j c_j^r\Big)^{p/r} \int_{a_{k+1}^{1/\overline{p}}}^{a_k^{1/\overline{p}}} t^{\overline{p}-1} dt + \Big(\sum_{j=1}^{k} 2^j c_j^r a_j^{r/p_1}\Big)^{p/r} \int_{a_{k+1}^{1/\overline{p}}}^{a_k^{1/\overline{p}}} t^{\frac{\overline{p}(p_1-p)}{p_1}-1} dt \right. \\ &\quad \left. + \Big(\sum_{j=k+1}^{\infty} 2^j c_j^r a_j^{r/p_0}\Big)^{p/r} \int_{a_{k+1}^{1/\overline{p}}}^{a_k^{1/\overline{p}}} \frac{dt}{t^{\frac{\overline{p}(p-p_0)}{p_0}+1}}\right\} + c\|f\|^{p/r}_{A_{p_0,r}} \\ &\le c\sum_{k=1}^{\infty}\left\{\Big(\sum_{j=1}^{k} 2^j c_j^r\Big)^{p/r} (a_k - a_{k+1}) + \Big(\sum_{j=1}^{k} 2^j c_j^r a_j^{r/p_1}\Big)^{p/r} \Big(a_k^{\frac{p_1-p}{p_1}} - a_{k+1}^{\frac{p_1-p}{p_1}}\Big) \right. \\ &\quad \left. + \Big(\sum_{j=k+1}^{\infty} 2^j c_j^r a_j^{r/p_0}\Big)^{p/r} \Big(\frac{1}{a_{k+1}}\Big)^{\frac{p-p_0}{p_0}+\frac{1}{\overline{p}}} \Big(a_k^{1/\overline{p}} - a_{k+1}^{1/\overline{p}}\Big)\right\} + c\|f\|^{p/r}_{A_{p_0,r}}. \end{aligned} \tag{7}$$

We are now in a position to specify the sequences $\{c_j\}, \{a_j\}$ so that the right side of (7) is finite, while the series in (6) diverges for $s = p$. Set

$$c_j = 2^{-j/r}, \quad a_j = (j \log(j+1))^{-p/r}.$$

Then, in view of (6),

$$\|f\|^r_{A_{p_0,r}} \asymp \sum_{j=1}^{\infty} \frac{1}{(j \log(j+1))^{p/p_0}} < \infty,$$

while

$$\|f\|^r_{A_{p,r}} \asymp \sum_{j=1}^{\infty} \frac{1}{j \log(j+1)} = \infty.$$

It remains to show that the series on the right side of (7) converges. Simple estimates yield

$$\begin{aligned} a_k - a_{k+1} &\leq \frac{c}{k^{p/r+1}(\log(k+1))^{p/r}}, \\ a_k^{\frac{p_1-p}{p_1}} - a_{k+1}^{\frac{p_1-p}{p_1}} &\leq \frac{c}{k^{\frac{p(p_1-p)}{rp_1}+1}(\log(k+1))^{\frac{p(p_1-p)}{rp_1}}}, \\ \left(\frac{1}{a_{k+1}}\right)^{\frac{p-p_0}{p_0}+\frac{1}{\overline{p}}} \left(a_k^{1/\overline{p}} - a_{k+1}^{1/\overline{p}}\right) &\leq ck^{\frac{p(p-p_0)}{rp_0}-1}(\log(k+1))^{\frac{p(p-p_0)}{rp_0}}, \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{j=1}^{k} 2^j c_j^r\right)^{p/r} &= k^{p/r}, \\ \left(\sum_{j=1}^{k} 2^j c_j^r a_j^{r/p_1}\right)^{p/r} &\leq c \frac{k^{\frac{p(p_1-p)}{rp_1}}}{(\log(k+1))^{p^2/rp_1}}, \\ \left(\sum_{j=k+1}^{\infty} 2^j c_j^r a_j^{r/p_0}\right)^{p/r} &\leq \frac{c}{k^{\frac{p(p-p_0)}{rp_0}}(\log(k+1))^{p^2/rp_0}}. \end{aligned}$$

Hence, the series on the right side of (7) is at most

$$c \sum_{k=1}^{\infty} \frac{1}{k(\log(k+1))^{p/r}} < \infty,$$

as required.

To prove (iii), it suffices to demonstrate that the estimate converse to (5) holds. Indeed, in this case (1) and change of variables $2^j t^{\overline{p}} \to t$ yield

$$
\begin{aligned}
(8)\qquad & \|f\|_{(A_{p_0,p},A_{p_1,p})_{\theta,p}} \\
&\asymp c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-p/p_0)}\int_0^\infty t^{-\theta p}\Big(\int_0^{2^j t^{\overline{p}}}[f_j^*(\tau)]^{p_0}d\tau\Big)^{p/p_0}\frac{dt}{t}\Big)^{1/p} \\
&\quad + c\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-p/p_1)}\int_0^\infty t^{(1-\theta)p}\Big(\int_{2^j t^{\overline{p}}}^\infty[f_j^*(\tau)]^{p_1}d\tau\Big)^{p/p_1}\frac{dt}{t}\Big)^{1/p} \\
&\asymp c\Big(\sum_{j\in\mathbb{Z}}\int_0^\infty\Big(\frac{1}{t}\int_0^t[f_j^*(\tau)]^{p_0}d\tau\Big)^{p/p_0}dt\Big)^{1/p} \\
&\quad + c\Big(\sum_{j\in\mathbb{Z}}\int_0^\infty\Big(\frac{1}{t}\int_t^\infty[f_j^*(\tau)]^{p_1}d\tau\Big)^{p/p_1}dt\Big)^{1/p} \\
&\ge c\Big(\sum_{j\in\mathbb{Z}}\int_0^\infty[f_j^*(t)]^p dt\Big)^{1/p} + c\Big(\sum_{j\in\mathbb{Z}}\int_0^\infty[f_j^*(2t)]^p dt\Big)^{1/p} \ge c\|f\|_{L^p},
\end{aligned}
$$

and we are done.

We now consider the last case $p<r<\infty$. Applying (1) and Hölder's inequality yields

$$
\begin{aligned}
(9)\qquad \|f\|_{A_{p,r}} &\le c\Big(\sum_{j\in\mathbb{Z}} 2^j\Big(\int_0^2[f_j^*(2^j t)]^p dt\Big)^{r/p}\Big)^{1/r} \\
&\le c\Big(\sum_{j\in\mathbb{Z}} 2^{jp/r}\int_0^2[f_j^*(2^j t)]^p dt\Big)^{1/p} \\
&\le c\Big(\int_0^2\Big(\sum_{j\in\mathbb{Z}} 2^j[f_j^*(2^j t)]^r\Big)^{p/r}dt\Big)^{1/p} \\
&\le c\Big(\int_0^2\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_0)}\Big(\frac{1}{t}\int_0^{2^j t}[f_j^*(\tau)]^{p_0}d\tau\Big)^{r/p_0}\Big)^{p/r}dt\Big)^{1/p} \\
&\le c\Big(\int_0^\infty t^{-\theta p}\Big(\sum_{j\in\mathbb{Z}} 2^{j(1-r/p_0)}\Big(\int_0^{2^j t^{\overline{p}}}[f_j^*(\tau)]^{p_0}d\tau\Big)^{r/p_0}\Big)^{p/r}\frac{dt}{t}\Big)^{1/p} \\
&\le c\|f\|_{(A_{p_0,r},A_{p_1,r})_{\theta,p}},
\end{aligned}
$$

which gives the imbedding $(A_{p_0,r},A_{p_1,r})_{\theta,p}\subset A_{p,r}$. To complete the proof, we show that there exists a function $f\in A_{p,r}\setminus(A_{p_0,r},A_{p_1,r})_{\theta,p}$. As above, let f be of the form

$$
f=\sum_{j=1}^\infty c_j\chi_{(2^j,2^j(1+a_j))}.
$$

It is clear that $(f_j)^*(2^j t) = c_j \chi_{(0,a_j)}(t)$, and hence,

$$\|f\|_{(A_{p_0,r},A_{p_1,r})_{\theta,p}} \geq c\Big(\int_0^2 \Big(\sum_{j\in\mathbb{Z}} 2^j [f_j^*(2^j t)]^r\Big)^{p/r} dt\Big)^{1/p}$$

$$\geq c\sum_{k=1}^{\infty}\int_{a_{k+1}}^{a_k}\Big(\sum_{j=1}^{k} 2^j [f_j^*(2^j t)]^r\Big)^{p/r} = \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} 2^j c_j r\Big)^{p/r}(a_k - a_{k+1}).$$

Set now

$$c_j = 2^{-j/r}, a_j = (j^{p/r}\log(j+1))^{-1}.$$

Then, in view of (6),

$$\|f\|_{A_{p,r}}^r \asymp \sum_{j=1}^{\infty}\frac{1}{j(\log(j+1))^{r/p}} < \infty,$$

while

$$\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{k} 2^j c_j r\Big)^{p/r}(a_k - a_{k+1}) \asymp \sum_{j=1}^{\infty}\frac{1}{j\log(j+1)} = \infty.$$

The theorem is proved. □

Remark 3.1. In the proof of (ii) and (iv) estimates from above and from below, respectively, were used for the norm in the interpolation space (see (4) and what follows, and (9)). The counterexamples given above show that the corresponding bounds are in principle impossible to be two-sided, except the only case $r = p$ (see (8)).

Proof of Theorem 2.2 Keeping in mind reiteration, we first consider the case $r_0 = 1$ and $r_1 = \infty$. Denoting

$$T_p f(u) = \Big(\frac{1}{u}\int_{u\leq|\xi|\leq 2u}|f(\xi)|^p d\xi\Big)^{1/p},$$

let us show that

$$\int_0^t (T_p f)^*(\tau)d\tau \leq K(f,t;A_{p,1},A_{p,\infty}) \leq 25\int_0^t (T_p f)^*(\tau)d\tau. \tag{10}$$

The left side of (10) is trivial. To prove the right side of (10), set

$$g(x) = f\cdot\chi_{(-t,t)}(x), \quad h(x) = f(x) - g(x).$$

Then

$$\|g\|_{A_{p,1}} = \int_0^t T_p f(u)du \leq \int_0^t (T_p f)^*(\tau)d\tau.$$

We now observe that

$$T_pf(u) \le \inf_{s\in(u/2,u)} \left(\frac{1}{u}\int_{s\le|\xi|\le 4s} |f(\xi)|^p d\xi\right)^{1/p} \le \inf_{s\in(u/2,u)} \big(T_pf(s) + 2T_pf(2s)\big)$$
$$\le \big(T_pf(s) + 2T_pf(2s)\big)^*(u/2) \le (T_pf)^*(u/4) + 2(T_pf)^*(u/2) \le 3(T_pf)^*(u/4).$$

Since $h(x) = 0$ for $x \in (-t, t)$, we obtain

$$\|h\|_{A_{p,\infty}} = \sup_{u>0} \left(\frac{1}{u}\int_{u\le|\xi|\le 2u} |h(\xi)|^p d\xi\right)^{1/p} = \sup_{u>t/2} T_pf(u) \le 3(T_pf)^*(t/8),$$

and thus

$$K(f, t; A_{p,1}, A_{p,\infty}) \le \|g\|_{A_{p,1}} + t\|h\|_{A_{p,\infty}} \le \int_0^t (T_pf)^*(\tau)d\tau + 3t(T_pf)^*(t/8)$$
$$\le \int_0^t (T_pf)^*(\tau)d\tau + 24\int_0^{t/8} (T_pf)^*(\tau)d\tau \le 25\int_0^t (T_pf)^*(\tau)d\tau,$$

as required.

It is clear that (10) proves the theorem when $r_0 = 1$ and $r_1 = \infty$. One can then apply the Holmstedt reiteration theorem (see, e.g., [2, p. 307]) to describe the K-functional for any of the couples (A_{p,r_0}, A_{p,r_1}) and get (2) for all $1 < r_0 < r_1 < \infty$. □

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