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Integral transforms of the Kontorovich-Lebedev convolution type

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Abstract

We deal with a class of integral transformations of the form

$$f(x)\to\frac{1}{2x}\prod_{n=1}^{\infty}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)\int_{\mathbb{R}_+^2}e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}f(u)h(y)dudy, x\in\mathbb{R}_+$$

in $L_2(\mathbb{R}_+;xdx)$, which is associated with the Kontorovich-Lebedev operator

$$K_{i\tau}[f]=\int_0^\infty K_{i\tau}(x)f(x)dx,\ \tau\in\mathbb{R}_+.$$

Necessary and sufficient conditions on h to establish that the transformation is unitary in $L_2(\mathbb{R}_+;xdx)$ are obtained. A reciprocal inversion formula and an example of the unitary convolution transformation are given.

1. Introduction

Let $f,\ h$ be functions defined on $\mathbb{R}_+ = [0,\infty)$. The following double integral, which we call the convolution of f and h, is denoted by $(f*h)(x),\ x\in\mathbb{R}_+$, and we define it as in [10], [14], [15] by

$$(f*h)(x)=\frac{1}{2x}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}f(u)h(y)dudy,\ x>0. \tag{1.1}$$

This convolution has a relationship with the Kontorovich-Lebedev transform [6], [15]

$$K_{i\tau}[f]=\int_0^\infty K_{i\tau}(x)f(x)dx,\ \tau\in\mathbb{R}_+, \tag{1.2}$$

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which contains as the kernel the modified Bessel function of the second kind $K_\nu(z)$ or the Macdonald function [1] of the pure imaginary index $\nu = i\tau$. The function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0, \tag{1.3}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. The Macdonald function has the asymptotic behaviour [1]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.4}$$

and near the origin

$$z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu) + o(1),\ z\to 0, \tag{1.5}$$

$$K_0(z) = -\log z + O(1),\ z\to 0. \tag{1.6}$$

The kernel of the Kontorovich-Lebedev operator (1.2) can be given by the following Fourier integral [1], [6]

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u}\cos \tau u du, x>0. \tag{1.7}$$

The product of the functions (1.7) of different arguments can be represented in turn by the Macdonald formula [cf. [1], [14]]

$$K_{i\tau}(x)K_{i\tau}(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} K_{i\tau}(u)\frac{du}{u}. \tag{1.8}$$

This is a key formula, which is used to prove the factorization property for the convolution (1.1), i.e.

$$K_{i\tau}[f*h] = K_{i\tau}[f]K_{i\tau}[h], \tau\in\mathbb{R}_+, \tag{1.9}$$

in terms of the Kontorovich-Lebedev operator (1.2) in appropriate Lebesgue spaces. In particular, this operator is well defined in the Banach ring $L^0(\mathbb{R}_+) \equiv L_1(\mathbb{R}_+; K_0(x)dx)$ (see [14, Section 15.4], [15, Section 4.3], [10], [14]) normed by

$$||f||_{L^0(\mathbb{R}_+)} = \int_0^\infty K_0(x)|f(x)|dx. \tag{1.10}$$

It is proved (see [14], Chapter 6) that the Kontorovich-Lebedev transform is a bounded operator from $L^0(\mathbb{R}_+)$ into the space of bounded continuous functions on $\mathbb{R}_+$ vanishing at infinity. Furthermore, the convolution (1.1) of two functions $f, h \in L^0(\mathbb{R}_+)$ belongs to $L^0(\mathbb{R}_+)$ and satisfies the norm inequality

$$||f*h||_{L^0(\mathbb{R}_+)} \le ||f||_{L^0(\mathbb{R}_+)}||h||_{L^0(\mathbb{R}_+)}. \tag{1.11}$$

However, if we define the operator (1.2) in $L_2(\mathbb{R}_+; xdx)$ as

$$K_{i\tau}[f] = \lim_{N\to\infty} \int_{1/N}^{N} K_{i\tau}(x)f(x)dx, \tag{1.12}$$

where the limit is taken in the mean square sense with respect to the norm of the space $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$, then (see [15], Theorem 2.4)

$$K_{i\tau} : L_2(\mathbb{R}_+; xdx) \leftrightarrow L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$$

is a bounded operator and forms an isometric isomorphism between these spaces with the Parseval identity of the form

$$\int_0^\infty x|f(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau. \tag{1.13}$$

The two definitions (1.2) and (1.12) are equivalent, if $f \in L_2(\mathbb{R}_+; xdx) \cap L^0(\mathbb{R}_+)$. The inverse operator in the latter case is given by the formula $f(x) = \lim_{N\to\infty} f_N(x)$, where

$$f_N(x) = \frac{2}{\pi^2} \int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f]d\tau, \tag{1.14}$$

and the convergence is in the mean square sense with respect to the norm of $L_2(\mathbb{R}_+; xdx)$. It can be written for almost all $x \in \mathbb{R}_+$ in the equivalent form (see [15], formula (2.70))

$$f(x) = \frac{2}{x\pi^2} \frac{d}{dx} \int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f] dy d\tau. \tag{1.15}$$

The boundedness and inversion problems for convolution mappings of different classes of integral transformations are well known. We note that similar results were obtained for convolution transforms in [2], [3], [7], [9] related to the Mellin type and general Fourier type operators. The theory of the convolution (1.1) was developed by the author in [10], [11], [12], [13], [14], [16] and was used in some applications to the corresponding class of convolution integral equations.

Here we draw a parallel with results for the Fourier cosine and sine convolution transforms (cf. [4], [8]). Namely, we continue to study the Kontorovich-Lebedev convolution (1.1) in the weighted Lebesgue spaces. Furthermore, we investigate, for a fixed h, an integral transform of the convolution type $f \to g$, which contains a differential operator of the infinite order and can be written in the operational form as

$$g(x) = \frac{1}{2x} \prod_{n=1}^\infty \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)$$
$$\times \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} f(u)h(y)dudy, \; x > 0. \tag{1.16}$$

We obtain necessary and sufficient conditions on the kernel function $h \in L^0(\mathbb{R}_+)$ for the transformation (1.16) to be unitary on $L_2(\mathbb{R}_+; xdx)$, and compute its inverse. The Watson type theorem is proved. Finally we give a particular example of such convolution kernels.

2. Convolution properties

In this section we prove some important preliminary results, which give the norm estimates in $L_2(\mathbb{R}_+; xdx)$ for the convolution (1.1). For instance, appealing to Theorem 4.7 in [15] we obtain for $f, h \in L_2(\mathbb{R}_+)$ the following inequality

$$||f * h||_{L_2(\mathbb{R}_+;xdx)} \leq \frac{1}{2}\sqrt{\frac{\pi}{2}}||f||_{L_2(\mathbb{R}_+)}||h||_{L_2(\mathbb{R}_+)}. \tag{2.1}$$

Let us extend the norm inequality for convolution (1.1) if one of the functions, say h, belongs to $L^0(\mathbb{R}_+) \supset L_2(\mathbb{R}_+)$ and $f \in L_2(\mathbb{R}_+; xdx)$. We have

Lemma 1

Let $f \in L_2(\mathbb{R}_+; xdx)$ and $h \in L^0(\mathbb{R}_+)$. Then, convolution (1.1) exists for each $x > 0$ as the Lebesgue double integral and belongs to $L_2(\mathbb{R}_+; xdx)$. Moreover,

$$||f * h||_{L_2(\mathbb{R}_+;xdx)} \leq ||f||_{L_2(\mathbb{R}_+;xdx)}||h||_{L^0(\mathbb{R}_+)}. \tag{2.2}$$

Proof. Indeed, with Schwarz's inequality we deduce

$$|(f * h)(x)|^2 \leq \frac{1}{4x^2}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|\frac{dudy}{u}$$
$$\times\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy.$$

Since (see [1])

$$\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}\frac{du}{u} = 2K_0(\sqrt{x^2+y^2}), \tag{2.3}$$

it follows that

$$|(f * h)(x)|^2 \leq \frac{1}{2x^2}\int_0^\infty K_0(\sqrt{x^2+y^2})|h(y)|dy$$
$$\times\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy$$
$$\leq \frac{1}{2x^2}\int_0^\infty K_0(y)|h(y)|dy$$
$$\times\int_0^\infty\int_0^\infty u|f(u)|^2 e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}|h(y)|dudy. \tag{2.4}$$

We note that we have used in (2.4) the elementary inequality

$$K_0(\sqrt{x^2+y^2}) \leq K_0(y),\ x, y > 0.$$

Hence multiplying both sides of (2.4) by x we integrate with respect to $x \in \mathbb{R}_+$. Inverting the order of integration by the Fubini theorem we invoke (2.3) to obtain

$$\int_0^\infty x|(f*h)(x)|^2 dx \leq \int_0^\infty K_0(y)|h(y)|dy \int_0^\infty \int_0^\infty uK_0(\sqrt{u^2+y^2})|f(u)|^2|h(y)|dudy$$
$$\leq \int_0^\infty u|f(u)|^2 du \left(\int_0^\infty K_0(y)|h(y)|dy\right)^2. \tag{2.5}$$

Now we recall norm (1.10) and write (2.5) in equivalent form (2.2). Lemma 1 is proved. □

The relationship between the convolution (1.1) and the Kontorovich-Lebedev transform (1.2) under the conditions of Lemma 1 is given by

Lemma 2

Let f, h be under conditions of Lemma 1. Then, the Kontorovich-Lebedev convolution $(f*h)(x)$ satisfies the factorization property (1.9). *Furthermore, for almost all $x > 0$ the generalized Parseval equality holds*

$$(f*h)(x) = \frac{2}{\pi^2} \lim_{N\to\infty} \int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f]K_{i\tau}[h]d\tau, \tag{2.6}$$

where the limit is taken with respect to the norm of $L_2(\mathbb{R}_+; xdx)$. In particular, it can be written in the form

$$(f*h)(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f]K_{i\tau}[h]d\tau, x > 0, \tag{2.7}$$

if the latter integral converges absolutely and uniformly on $x \geq x_0 > 0$.

Proof. By virtue of the formula (1.14) we have for almost all $x > 0$

$$x(f*h)(x) = \frac{2}{\pi^2} \frac{d}{dx} \int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f*h]dyd\tau. \tag{2.8}$$

Also, since $h \in L^0(\mathbb{R}_+)$ and $|K_{i\tau}(x)| \leq K_0(x)$ (see (1.7)), it follows from (1.10) that $|K_{i\tau}[h]| \leq ||h||_{L^0(\mathbb{R}_+)}$. Further, as is proved in [15], Lemma 2.5, the kernel

$$\int_0^x K_{i\tau}(y)dy \in L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$$

for each $x > 0$. Therefore we have that the product

$$K_{i\tau}[h] \int_0^x K_{i\tau}(y)dy \in L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau).$$

At the same time if we denote by

$$\theta_x(y) = \begin{cases} 1, & \text{if } y \in [0, x], \\ 0, & \text{if } y \in (x, \infty), \end{cases}$$

we see that $\theta_x(y) \in L^0(\mathbb{R}_+)$ and the factorization property (1.9) takes place in the Banach ring $L^0(\mathbb{R}_+)$ for the functions $h, \theta_x(y)$, namely

$$K_{i\tau}[h]K_{i\tau}[\theta_x] = K_{i\tau}[h * \theta_x].$$

Thus Lemma 1 and the Parseval equality (1.13) yield

$$\begin{aligned} &\frac{2}{\pi^2}\int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[h] K_{i\tau}[f] dy d\tau \\ &\quad = \frac{2}{\pi^2}\int_0^\infty \tau \sinh \pi\tau K_{i\tau}[f] K_{i\tau}[h * \theta_x] d\tau \\ &\quad = \int_0^\infty u f(u)(h * \theta_x)(u) du. \end{aligned} \tag{2.9}$$

Substituting the double integral (1.1) for $(h * \theta_x)(u)$ in (2.9) and inverting the order of integration by using Fubini's theorem, we get

$$\int_0^\infty u f(u)(h * \theta_x)(u) du = \int_0^x v(f * h)(v) dv.$$

Consequently, for almost all $x > 0$, we obtain

$$x(f * h)(x) = \frac{2}{\pi^2}\frac{d}{dx}\int_0^\infty \int_0^x \tau \sinh \pi\tau K_{i\tau}(y) K_{i\tau}[f] K_{i\tau}[h] dy d\tau \tag{2.10}$$

and comparing with (2.8) we verify the factorization equality (1.9) for $(f * h)(x)$ under conditions of the Lemma.

It is possible to justify the differentiation under the integral sign in (2.10) by the absolute and uniform convergence of the differentiated integral. Therefore in this case we write (2.10) in the form (2.7). However, comparing with (1.15) we find, that for the sequence f_N of $L_2(\mathbb{R}_+; xdx)$- functions, which is defined by (1.14) the differentiation in (2.10) is performed and

$$\begin{aligned} (f_N * h)(x) &= \frac{2}{\pi^2}\int_0^\infty \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f_N] K_{i\tau}[h] d\tau \\ &= \frac{2}{\pi^2}\int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f] K_{i\tau}[h] d\tau, \end{aligned} \tag{2.11}$$

where

$$K_{i\tau}[f_N] = \begin{cases} K_{i\tau}[f], & \text{if } \tau \in [0, N], \\ 0, & \text{if } \tau \in (N, \infty). \end{cases}$$

Indeed, the integral (2.11) converges uniformly with respect to x over any finite interval $(0, N)$. Further, invoking (2.2) we derive

$$||f * h - f_N * h||_{L_2(\mathbb{R}_+;xdx)} = ||(f - f_N) * h||_{L_2(\mathbb{R}_+;xdx)} \leq ||f - f_N||_{L_2(\mathbb{R}_+;xdx)} ||h||_{L^0(\mathbb{R}_+)} \to 0, N \to \infty.$$

Thus the limit of $(f_N * h)(x)$ with respect to the norm in $L_2(\mathbb{R}_+; xdx)$ coincides with $(f * h)(x)$ and we prove (2.6) and complete the proof of Lemma 2. □

Corollary 1

Under conditions of Lemma 1 the Parseval equality (1.13) *takes the form*

$$\int_0^\infty x|(f * h)(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f] K_{i\tau}[h]|^2 d\tau.$$

In particular, for $f \in L^0(\mathbb{R}_+) \cap L_2(\mathbb{R}_+; xdx)$ it gives

$$\int_0^\infty x|(f * \overline{f})(x)|^2 dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^4 d\tau.$$

3. A Watson type theorem

In this section we study the problem of inverting the convolution transformation (1.16) in $L_2(\mathbb{R}_+; xdx)$, where $h(x)$ belongs to a class of the so-called Kontorovich-Lebedev kernels. We will show that this class is significant in the theory of the transformation (1.16) as, for instance, a class of general Fourier kernels related to the Fourier and Mellin type convolution transforms (cf. [7, Chapter VIII], [3], [4], [8]).

Definition 1. A function $h \in L^0(\mathbb{R}_+)$ is said to be a Kontorovich-Lebedev kernel if it satisfies the following convolution equation

$$(h * \overline{h})(x) = \frac{4}{\pi^2} K_0(x), \ x > 0. \tag{3.1}$$

Combining this with the factorization property (1.9) and formula (2.16.48.14) in [5] we obtain

$$K_{i\tau}[h * \overline{h}] = |K_{i\tau}[h]|^2 = K_{i\tau} \left[\frac{4}{\pi^2} K_0(x) \right] = \frac{1}{\cosh^2(\pi\tau/2)}, \tag{3.2}$$

or

$$|K_{i\tau}[h]| = \frac{1}{\cosh(\pi\tau/2)}. \tag{3.3}$$

Now we are ready to prove an analogue of Watson's theorem [9], [14] about necessary and sufficient conditions for convolution integral transformations of the Kontorovich-Lebedev type to be unitary in $L_2(\mathbb{R}_+; xdx)$.

Theorem

Let $h(x) \in L^0(\mathbb{R}_+)$. The transformation $f \to g$ given by formula (1.16) *is unitary on $L_2(\mathbb{R}_+; xdx)$ whose inverse can be written in the symmetric form*

$$f(x) = \lim_{N\to\infty} \frac{1}{2x} \prod_{n=1}^{N} \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) \times \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} g(u)\overline{h(y)}dudy, \tag{3.4}$$

if, and only if, h is a Kontorovich-Lebedev kernel. The convergence in (3.4) *is with respect to the norm in $L_2(\mathbb{R}_+; xdx)$.*

Proof. Sufficiency: Let h be a Kontorovich-Lebedev kernel. Applying Lemma 2 we find that convolution transform (1.16) can be written in the form

$$g(x) = \lim_{N\to\infty} \frac{2}{\pi^2 x} \prod_{n=1}^{N} \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x) K_{i\tau}[f] K_{i\tau}[h] d\tau. \tag{3.5}$$

Indeed, the corresponding integral (2.7) is absolutely and uniformly convergent on $x > 0$. We show this fact by using Schwarz's inequality and invoking (3.3). Precisely, we have

$$\int_0^\infty \tau \sinh \pi\tau \left|K_{i\tau}(x) K_{i\tau}[f] K_{i\tau}[h]\right| d\tau \le \left(\int_0^\infty \tau \sinh \pi\tau \left|K_{i\tau}[f]\right|^2 d\tau\right)^{1/2} \times \left(2\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) \left|K_{i\tau}(x)\right|^2 d\tau\right)^{1/2} < \infty, \tag{3.6}$$

since the first integral in the right-hand side of inequality (3.6) represents the norm of $K_{i\tau}[f]$ in $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$ and the second one is convergent for each $x > 0$ by virtue of the asymptotic expansion (1.148) in [15] for the Macdonald function $K_{i\tau}(x)$ when $\tau \to +\infty$.

Furthermore, if we denote by

$$g_N(x) = \frac{2}{\pi^2 x} \prod_{n=1}^{N} \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x) K_{i\tau}[f] K_{i\tau}[h] d\tau, \tag{3.7}$$

then we can invert the order of operators in (3.7) and it formally becomes

$$g_N(x) = \frac{2}{\pi^2 x} \int_0^\infty \tau \sinh \pi\tau \prod_{n=1}^{N} \left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right) [K_{i\tau}(x)] K_{i\tau}[f] K_{i\tau}[h] d\tau. \tag{3.8}$$

Hence applying (1.3) we see that

$$\prod_{n=1}^{N}\left(1+\frac{x\left(x-\frac{d}{dx}-x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)[K_{i\tau}(x)] = K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right).$$

Consequently, the interchange of the order of integration and the N-th product in (3.8) is performed if the following integral

$$\int_0^\infty \tau\sinh\pi\tau K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]K_{i\tau}[h]d\tau$$

converges uniformly on $x>0$. Splitting the latter integral into two $\int_0^E$ and $\int_E^\infty$ it is not difficult to verify the interchange in the first integral over any finite interval $[0,E]$, since the integrand is analytic with respect to $x>0$. To change the order in the second integral we show that

$$\left|\int_E^\infty \tau\sinh\pi\tau K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]K_{i\tau}[h]d\tau\right| \to 0, E\to\infty,$$

uniformly for all $x>0$. In a similar manner we use the uniform estimate (1.100) in [15] for the Macdonald function and via Schwarz's inequality we obtain

$$\left|\int_E^\infty \tau\sinh\pi\tau K_{i\tau}(x)\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]K_{i\tau}[h]d\tau\right|$$
$$\le K_0(x_0\cos\delta)\left(2\int_E^\infty \tau\tanh\left(\frac{\pi\tau}{2}\right)(1+\tau^2)^{2N}e^{-2\delta\tau}d\tau\right)^{1/2}$$
$$\times\, ||K_{i\tau}[f]||_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)} \to 0,$$

when $E\to\infty$ and $x\ge x_0>0,\ \delta\in\left(0,\frac{\pi}{2}\right)$. We employ now the elementary infinite product

$$\cosh\left(\frac{\pi\tau}{2}\right) = \prod_{n=1}^{\infty}\left(1+\frac{\tau^2}{(2n-1)^2}\right), \tag{3.9}$$

which converges uniformly on $0\le\tau\le A,\ A>0$. Hence due to equalities (1.13), (3.2) and Levi's theorem we deduce

$$\lim_{N\to\infty}\frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\left|\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]K_{i\tau}[h]\right|^2 d\tau$$
$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\,|K_{i\tau}[f]|^2\,d\tau = \int_0^\infty x|f(x)|^2dx.$$

On the other hand,

$$\frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\left|\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)K_{i\tau}[f]K_{i\tau}[h]\right|^2 d\tau$$
$$= \int_0^\infty x|g_N(x)|^2dx \tag{3.10}$$

and

$$\lim_{N\to\infty}\int_0^\infty x|g_N(x)|^2dx = \int_0^\infty x|f(x)|^2dx. \tag{3.11}$$

It is clear that

$$K_{i\tau}[g_N] = K_{i\tau}[f]K_{i\tau}[h]\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)$$

and invoking (3.9) we verify the pointwise convergence to a function $K_{i\tau}[g]$, namely

$$\begin{aligned}\lim_{N\to\infty}K_{i\tau}[g_N] &= \lim_{N\to\infty}K_{i\tau}[f]K_{i\tau}[h]\prod_{n=1}^{N}\left(1+\frac{\tau^2}{(2n-1)^2}\right)\\ &= K_{i\tau}[f]K_{i\tau}[h]\cosh\left(\frac{\pi\tau}{2}\right) = K_{i\tau}[g].\end{aligned} \tag{3.12}$$

Using again the Parseval identity (1.13) we show that the sequence $g_N(x)$ converges in mean to a function $g(x)$ say, of the space $L_2(\mathbb{R}_+; xdx)$, which we call the convolution transform (1.16). From (3.10), (3.12) we have

$$\int_0^\infty x|g_N(x)-g(x)|^2dx = \frac{2}{\pi^2}\int_0^\infty \tau\sinh\pi\tau\,|K_{i\tau}[g_N]-K_{i\tau}[g]|^2\,d\tau. \tag{3.13}$$

The right-hand side of (3.13) tends to zero when $\tau\to\infty$ via the dominated convergence theorem since $|K_{i\tau}[g_N]| \le |K_{i\tau}[f]|$. Thus we establish that $g_N(x)\to g(x)$ in $L_2(\mathbb{R}_+; xdx)$ and passing to the limit in (3.11) we obtain the Parseval equality for the transform (1.16)

$$\int_0^\infty x|g(x)|^2dx = \int_0^\infty x|f(x)|^2dx.$$

However formulas (1.9), (3.2) and (3.12) yield

$$K_{i\tau}[g]K_{i\tau}[\overline{h}] = K_{i\tau}[f]\frac{1}{\cosh(\pi\tau/2)}. \tag{3.14}$$

Therefore, it is equivalent to the identity

$$K_{i\tau}[f] = K_{i\tau}[g]K_{i\tau}[\overline{h}]\cosh\left(\frac{\pi\tau}{2}\right), \tag{3.15}$$

where both sides are $L_2(\mathbb{R}_+; \tau\sinh\pi\tau d\tau)$-functions. Thus, in the same manner as above it corresponds to (3.4) and gives the inversion formula of the transform (1.16).

Necessity: We suppose that $h\in L^0(\mathbb{R}_+)$ is a function such that the convolution transformation (1.16) is unitary on $L_2(\mathbb{R}_+; xdx)$ and whose inverse is given by (3.4). Here we cannot apply directly relations (3.12) and (3.15) since there is no guarantee that, for instance, the right-hand side of (3.15) is a function from $L_2(\mathbb{R}_+; \tau\sinh\pi\tau d\tau)$. We only know that $h\in L^0(\mathbb{R}_+)$ and consequently $K_{i\tau}[h]$ is bounded. But since (3.4) takes place for any $g\in L_2(\mathbb{R}_+; xdx)$ let us consider

$$g(x) = \frac{4\sqrt{2}}{\pi^3\sqrt{\pi}}\left(\frac{e^{-x}}{\sqrt{x}} * K_0(x)\right). \tag{3.16}$$

It is easily seen from (1.11) and Lemma 1 that $g(x) \in L_2(\mathbb{R}_+; xdx) \cap L^0(\mathbb{R}_+)$. Consequently, there exists a function $f(x)$ which corresponds to (3.16) by the inversion formula (3.4). At the same time, by virtue of the factorization property (1.9) in the ring $L^0(\mathbb{R}_+)$ by using formulas (2.16.6.4) in [5] and (3.2) we find

$$K_{i\tau}[g] = \frac{4\sqrt{2}}{\pi^3\sqrt{\pi}} K_{i\tau}\left[\left(\frac{e^{-x}}{\sqrt{x}} * K_0(x)\right)\right] = \frac{1}{\cosh^2(\pi\tau/2)\cosh\pi\tau}. \tag{3.17}$$

Therefore, formula (3.14) yields

$$K_{i\tau}[f] = \frac{K_{i\tau}[\overline{h}]}{\cosh(\pi\tau/2)\cosh\pi\tau} \in L_2(\mathbb{R}_+; \tau\sinh\pi\tau d\tau) \tag{3.18}$$

since $K_{i\tau}[\overline{h}]$ is bounded. Combining with (3.17) and (1.9) we obtain

$$\frac{1}{\cosh^2(\pi\tau/2)\cosh\pi\tau} = \frac{K_{i\tau}[h * \overline{h}]}{\cosh\pi\tau},$$

and immediately arrive at (3.2), (3.3). The theorem is proved. □

Finally, we give some ideas to construct a Kontorovich-Lebedev kernel (3.1) and an example of this function with reciprocal formulas of the convolution transformation (1.16), (3.4). In view of the formula (2.16.2.1) in [5]

$$\frac{2}{\pi}\int_0^\infty K_{i\tau}(x)dx = \frac{1}{\cosh(\pi\tau/2)},$$

we see that the function $h(x) = 2/\pi \in L^0(\mathbb{R}_+)$ is a Kontorovich-Lebedev kernel. Moreover, after calculating the inner integral in (1.16), (3.4) with respect to y (cf. [15, Section 4.5]) we deduce the following pair of reciprocal convolution transformations of the Kontorovich-Lebedev type

$$g(x) = \lim_{N\to\infty}\frac{2}{\pi x}\prod_{n=1}^{N}\left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)\int_0^\infty K_1(\sqrt{x^2+u^2})\frac{xu}{\sqrt{x^2+u^2}}f(u)du,$$

$$f(x) = \lim_{N\to\infty}\frac{2}{\pi x}\prod_{n=1}^{N}\left(1 + \frac{x\left(x - \frac{d}{dx} - x\frac{d^2}{dx^2}\right)}{(2n-1)^2}\right)\int_0^\infty K_1(\sqrt{x^2+u^2})\frac{xu}{\sqrt{x^2+u^2}}g(u)du,$$

which are unitary in the space $L_2(\mathbb{R}_+; xdx)$.

Corollary 2

Let $h, l \in L^0(\mathbb{R}_+)$ be the Kontorovich-Lebedev kernels. If $m(x), x \in \mathbb{R}_+$ is a solution of the convolution integral equation in the Banach ring $L^0(\mathbb{R}_+)$

$$\frac{2}{\pi}\int_0^\infty \frac{u}{\sqrt{x^2+u^2}}K_1(\sqrt{x^2+u^2})m(u)du = \varphi(x), \tag{3.19}$$

where $\varphi(x) = (h * l)(x)$, then $m(x)$ is a Kontorovich-Lebedev kernel.

Indeed, in terms of the Kontorovich-Lebedev transform (1.2) (see (1.9)) equation (3.19) can be written as follows

$$K_{i\tau}[m]\frac{1}{\cosh(\pi\tau/2)} = K_{i\tau}[\varphi] = K_{i\tau}[h]K_{i\tau}[l].$$

Thus via (3.2) we finally derive

$$|K_{i\tau}[m]| = \cosh\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[h]K_{i\tau}[l]| = \frac{1}{\cosh(\pi\tau/2)}.$$

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