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Ample vector bundles with zero loci having a bielliptic curve section

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Abstract

Let X be a smooth complex projective variety and let $Z \subset X$ be a smooth submanifold of dimension ≥ 2, which is the zero locus of a section of an ample vector bundle $\mathcal{E}$ of rank $\dim X - \dim Z \geq 2$ on X. Let H be an ample line bundle on X whose restriction H_Z to Z is very ample. Triplets $(X, \mathcal{E}, H)$ as above are studied and classified under the assumption that Z is a projective manifold of high degree with respect to H_Z, admitting a curve section which is a double cover of an elliptic curve.

Introduction and statement of the result

In this paper we consider the following set-up. X is a smooth complex projective manifold of dimension n and $\mathcal{E}$ is an ample vector bundle of rank r, $2 \leq r \leq n-2$, on X, admitting a regular section s whose zero locus is a smooth submanifold Z. Moreover H is an ample line bundle on X and we assume that

(0.1) H_Z is very ample and (Z, H_Z) admits a bielliptic curve section C (of genus $g \geq 3$).

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Of course this includes the case in which the linear system $|H|$ contains $n-r-1$ smooth elements meeting transversally with Z along a smooth bielliptic curve C.

Our aim is to classify triplets $(X, \mathcal{E}, H)$ as in (0.1). By a bielliptic curve we mean, as in [2, p. 254], a smooth curve, not hyperelliptic, which is a double cover of a smooth curve of genus 1. For some properties of such curves concerning genus and gonality we refer to [2, Section 1]. The case when $n = 3$ and $\mathcal{E}$ and H are both very ample line bundles has been recently considered in [4]. Our point of view can be regarded as a natural generalization.

Our approach is inspired by [16], where a similar situation, with C being a smooth hyperelliptic curve, is considered. In particular, due to [16], the assumption that C is not hyperelliptic in the definition above of bielliptic curve is not a serious restriction. As in [16] a key role is played by previous results on ample vector bundles with a regular section vanishing on a special variety, especially those in [13], which we combine successfully with the classification of projective manifolds of high degree admitting a bielliptic curve among their curve sections ([2] and [3]); to do that we assume that $c_r(\mathcal{E})H^{n-r} \geq 18$. In fact it turns out that the most interesting situation occurs exactly when this is an equality. Our method relying on [13] can work also for lower values of $c_r(\mathcal{E})H^{n-r}$, since especially for $c_r(\mathcal{E})H^{n-r} \leq 8$ a partial classification of projective manifolds of degree ≤ 8 with a bielliptic curve section is available (see [2, Theorem 4.1] and [3, Theorem B]). However, in the range $6 \leq c_r(\mathcal{E})H^{n-r} \leq 17$, the situation is much more intricate, due to the possible appearance of reductions.

Our result is as follows.

Theorem

Let X be a smooth complex projective variety of dimension n, let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on X such that there exists a global section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a smooth submanifold of dimension $n-r \geq 2$ of X, and let H be an ample line bundle on X. Then the triplets $(X, \mathcal{E}, H)$ as in (0.1), *satisfying the condition $c_r(\mathcal{E})H^{n-r} \geq 18$, are the following*:

i) X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve isomorphic to C, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus r}$, and $H_F \cong \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of the bundle projection;

ii) X is a $\mathbb{P}^{n-1}$-bundle over an elliptic curve E, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(r-1)}$, and $H_F \cong \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of the bundle projection;

iii) there exists a surjective morphism from X to an elliptic curve E whose general fiber F is isomorphic to a smooth quadric $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}}(1)^{\oplus r}$, and $H_F \cong \mathcal{O}_{\mathbb{Q}}(1)$ for every such fiber F;

iv) X is a $\mathbb{P}^{n-1}$-bundle over an elliptic curve E, $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$, and $H_F \cong \mathcal{O}_{\mathbb{P}}(2)$ for every fiber F of the bundle projection;

v) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}}(3))$;

vi) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}}(3))$;

vii) X is a Fano manifold of index $n-1$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by an ample line bundle $\mathcal{L}$ such that $\mathcal{L}^n = 2$, $\mathcal{E} = \mathcal{L}^{\oplus(n-2)}$, and $H = 3\mathcal{L}$;

viii) $n-r = 2$, X is a Fano manifold, $K_X + \det \mathcal{E} = 0$, $H \in 3\mathrm{Pic}(X)$ and $c_{n-2}(\mathcal{E})H^2 = 18$.

The result is effective, in the sense that all cases in the list above do really occur. For cases i) – iv) see Proposition 2.1. For more information on case viii) see Section 3. Note that $r = n - 2$ in all cases iv)–viii), while cases v)– viii) come from the analysis of the lowest value of $c_r(\mathcal{E})H^{n-r} = 18$.

This paper is organized as follows: Section 1 is concerned with projective manifolds of degree ≥ 18 with a bielliptic curve section. In Section 2 we prove our Theorem. Section 3 is devoted to case viii). In the course of the paper we also have the opportunity to improve Theorem C in [13] (see Theorem 0.4) as well as Theorem A in [3] (see Proposition 1.1).

Finally we would like to note that our method depending on [13] can be applied to other situation, e. g., the case when C is a trigonal curve. This will be done in a separate paper.

We use the standard notation from algebraic geometry. The tensor products of line bundles are denoted additively. The pull-back $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. In this paper we will use over and over the following fact coming from the Lefschetz–Sommese theorem [12, Theorem 1.1]. Let $\mathcal{E}$ be an ample vector bundle on a projective manifold X, having a section whose zero locus Z is a smooth submanifold of the expected dimension ≥ 2. Then the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism if $\dim Z \geq 3$ and an injection with torsion free cokernel if $\dim Z = 2$; moreover $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Z)$.

In Section 2 we will use also the following facts.

Lemma 0.2

Let $\mathcal{V}$ be a vector bundle over a smooth projective variety M, let $P := \mathbb{P}(\mathcal{V})$ be the associated projective bundle, with projection $\pi : P \to M$ and tautological line bundle ξ on P. Let L be an ample line bundle on M; then the line bundle $\xi + m\pi^*L$ is very ample for $m >> 0$.

Proof. Since L is ample, there exists an integer $n_0 > 0$ such that the sheaf $\mathcal{V}(nL) = \mathcal{V} \otimes (nL)$ is spanned for every $n \geq n_0$. Hence we can find a finite number of global sections that generate $\mathcal{V}(nL)$, that is, there exists a surjective morphism of sheaves $\mathcal{O}_M^{\oplus N} \to \mathcal{V}(nL)$ for some N. On the other hand, sL is very ample for some $s > 0$. Thus, from the exact sequence

$$(sL)^{\oplus N} \to \mathcal{V}\big((n+s)L\big) \to 0,$$

we infer that $\mathcal{V}((n+s)L)$ is very ample. Therefore $\mathcal{V}(mL)$ is very ample for any $m \geq n_0 + s$. Now, since $P = \mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{V}(mL))$ and ξ is the tautological line bundle on P associated to $\mathcal{V}$, we conclude that the tautological line bundle on P associated to $\mathcal{V}(mL)$ is $\xi + m\pi^*L$. Therefore $\xi + m\pi^*L$ is very ample for every $m \geq n_0 + s$. □

Lemma 0.3

Let X be a smooth projective variety of dimension $n \geq 4$, and let H be an ample line bundle on X. Assume that there exists a surjective morphism $f : X \to B$ onto a smooth irrational curve B such that the general fiber F of f is a smooth quadric $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$ with $H_F \cong \mathcal{O}_{\mathbb{Q}}(1)$. Then every fiber D of f is an irreducible quadric hypersurface in $\mathbb{P}^n$ having only isolated singularities, with $H_D \cong \mathcal{O}_D(1)$.

Proof. For the general fiber F, we have $(K_X)_F \cong K_F$; hence $(K_X + (n-2)H)_F \cong \mathcal{O}_{\mathbb{Q}}(-1)$. This implies that $K_X+(n-2)H$ is not nef. First we assume that $K_X+(n-1)H$ is not nef. Applying [7, Theorems 11.2 and 11.7] and noting that $h^1(\mathcal{O}_X) \geq 1$, B being irrational, we see that (X, H) is a scroll over a smooth curve. Take an arbitrary fiber $P(\cong \mathbb{P}^{n-1})$ of the scroll projection. Then $f(P)$ is a point of B, which contradicts the assumption that the general fiber of f is a smooth quadric $\mathbb{Q}^{n-1}$. From this we see that $K_X + (n-1)H$ is nef. Since $K_X + (n-2)H$ is not nef and $h^1(\mathcal{O}_X) \geq 1$, it follows from [7, Theorem 11.8] that one of the following holds:

(a) there exists an effective divisor E on X such that
$$\big(E, H_E, \mathcal{O}_E(E)\big) \cong \big(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(1), \mathcal{O}_{\mathbb{P}}(-1)\big);$$
(b) there exists a surjective morphism $\varphi : X \to C$ onto a smooth curve C such that every fiber G of φ is an irreducible quadric hypersurface in $\mathbb{P}^n$ having only isolated singularities, with $H_G \cong \mathcal{O}_G(1)$;
(c) (X, H) is a scroll over a smooth surface.

If (X, H) is as in (b), we can easily obtain $f = \varphi$, as desired. Therefore, from now on, we prove that cases (a) and (c) do not occur. If (X, H) is as in (c), then the same argument as above shows that the general fiber of f contains some fiber $\cong \mathbb{P}^{n-2}$ of the scroll projection. This is impossible, since $n \geq 4$. Now we consider case (a). Let $\sigma : X \to X'$ be the blowing-down of E to another smooth projective variety X'. Then $H = \sigma^* H' - \mathcal{O}_X(E)$ for some ample line bundle H' on X', so that
$$K_X + (n-1)H = \sigma^*(K_{X'} + (n-1)H').$$
This implies that $K_{X'} + (n-1)H'$ is nef. Moreover,
$$\sigma^*\big(K_{X'} + (n-2)H'\big) = K_X + (n-2)H - \mathcal{O}_X(E).$$
We note that $f(E)$ is a point of B. Since $(K_X + (n-2)H)_F \cong \mathcal{O}_{\mathbb{Q}}(-1)$ for the general fiber F of f, we conclude that $K_{X'} + (n-2)H'$ is not nef. Thus [7, Theorem 11.8] applies to (X', H') again. We claim that (X', H') is as in case (a). Indeed, in cases (b) and (c) there exists a curve $\Gamma \subset X'$ with $H'\Gamma = 1$ and $\Gamma \ni \sigma(E)$. But then $H'\Gamma > H\widetilde{\Gamma} > 0$ for the proper transform $\widetilde{\Gamma}$ of Γ on X. This is absurd. Repeating this procedure, we get a polarized manifold (X'', H'') which is not as in (a). We know that $K_{X''} + (n-1)H''$ is nef and that $K_{X''} + (n-2)H''$ is not nef. However, the same argument as above shows that neither (b) nor (c) occurs. This is a contradiction. □

Finally we need the following result improving Theorem C of [13].

Theorem 0.4

Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on X such that there exists a global section whose zero locus Z is a smooth subvariety of X of dimension $n - r \geq 3$. Assume that (Z, H_Z) is a quadric fibration over a smooth curve B (in the sense of [12, p. 250]*) for some ample line bundle H on X. Then $(X, \mathcal{E}, H)$ is one of the following:*

(I) X is a $\mathbb{P}^{n-1}$-bundle over B, $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(r-1)}$, and $H_F = \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of the projection $X \to B$;

(II) there exists a surjective morphism $X \to B$ whose general fiber F is isomorphic to a smooth quadric $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$, $\mathcal{E}_F = \mathcal{O}_{\mathbb{Q}}(1)^{\oplus r}$, and $H_F = \mathcal{O}_{\mathbb{Q}}(1)$ for every such fiber F.

Proof. By [13, Theorem C] $(X, \mathcal{E}, H)$ is either as in (I), (II), or

(∗) $n - r = 3$, X is a $\mathbb{P}^{n-2}$-bundle over a geometrically ruled surface S over B and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}$, $H_F = \mathcal{O}_{\mathbb{P}}(1)$ for every fiber F of $X \to S$; moreover, the quadric fibration morphism $p : Z \to B$ is obtained by restricting to Z the composite $X \to S \to B$ of the two bundle projections.

So it is enough to show that case (∗) cannot occur. By contradiction, consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\subset} & X \\ {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow \\ B & \xleftarrow[q]{} & S \end{array}$$

where π and q are the bundle projections. Let $\mathcal{V} = \pi_* H$. Then $\mathcal{V}$ is an ample vector bundle on S of rank $n - 1$, $X = \mathbb{P}(\mathcal{V})$, H being the tautological line bundle. By the canonical bundle formula for projective bundles we have

$$K_X = -(n-1)H + \pi^*(K_S + \det \mathcal{V}).$$

Note that by assumption the restriction of $\mathcal{E} \otimes H^{-1}$ to every fiber F of π is trivial. Hence there exists a vector bundle $\mathcal{W}$ of rank $n - 3$ on S such that

$$\mathcal{E} = \pi^* \mathcal{W} \otimes H. \tag{0.4.1}$$

Therefore

$$K_X + \det \mathcal{E} + 2H = \pi^*(K_S + \det \mathcal{V} + \det \mathcal{W}).$$

By adjunction we have $(K_X + \det \mathcal{E})_Z = K_Z$. So, by restricting the expression above to Z we get

$$K_Z + 2H_Z = \big(\pi^*(K_S + \det \mathcal{V} + \det \mathcal{W})\big)_Z.$$

On the other hand, since (Z, H_Z) is a quadric fibration over B, we know that $K_Z + 2H_Z = p^* M$ for some line bundle M on B. By comparing these two expressions and taking into account the commutativity of the diagram above we get

$$\big(\pi^*(q^* M)\big)_Z = \big(\pi^*(K_S + \det \mathcal{V} + \det \mathcal{W})\big)_Z.$$

Now recall that the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is bijective by the Lefschetz–Sommese theorem, while $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ is injective since π makes (X, H) a scroll over S. We thus get

$$\det \mathcal{V} + \det \mathcal{W} = -K_S + q^* M.$$

Restricting this formula to any fiber f of q gives

$$\deg \mathcal{V}_f + \deg \mathcal{W}_f = 2. \tag{0.4.2}$$

Now let $Q_b = p^{-1}(b)$ ($b \in B$) be a general fiber of p, and set $f_b = q^{-1}(b)$. Due to the commutativity of the diagram above, we have a surjective morphism $\pi_{|Q_b} : Q_b \to f_b$ fibering Q_b over $f_b \cong \mathbb{P}^1$. Identify $\pi_{|Q_b}$ with the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor, denote by σ a fiber of it and let γ be a section corresponding to a fiber of the other projection. Then $(H_Z)_{Q_b} = [\sigma + \gamma]$. Hence $H\gamma = H_Z\gamma = (\sigma + \gamma)\gamma = 1$. Now, since $\mathcal{E}$ is ample we have $\deg \mathcal{E}_\gamma \geq \mathrm{rk}\mathcal{E}_\gamma = n - 3$. On the other hand, recalling (0.4.1) and taking into account that $(\pi_{|Q_b})_{|\gamma} : \gamma \to f_b$ is an isomorphism, we get

$$n - 3 = \mathrm{rk}\mathcal{E}_\gamma \leq \deg \mathcal{E}_\gamma = (n-3)H\gamma + \deg \mathcal{W}_{f_b} = (n-3) + \deg \mathcal{W}_{f_b}.$$

Then $\deg \mathcal{W}_f \geq 0$ for every fiber f of q. So, recalling (0.4.2), we conclude that $\deg \mathcal{V}_f \leq 2$. On the other hand $\deg \mathcal{V}_f \geq \mathrm{rk}\mathcal{V}_f = n - 1$, since $\mathcal{V}$ is ample. This clearly gives a contradiction, since $n \geq 5$. □

1. Projective manifolds of high degree with a bielliptic curve section

In this Section we recall the classification of projective manifolds of degree ≥ 18 admitting a bielliptic curve among their curve sections, improving the known results in dimension ≥ 3. For convenience let us denote by $(Z, \mathcal{H})$ such a projective manifold, i. e., $Z \subset \mathbb{P}^N$ is a smooth projective variety of dimension $k \geq 2$, $\mathcal{H} = (\mathcal{O}_{\mathbb{P}^N}(1))_Z$, and C is a bielliptic curve section of $(Z, \mathcal{H})$; moreover $\mathcal{H}^k \geq 18$. Thus, by [2, Theorem 3.5] and [3, Theorem A] we know that $(Z, \mathcal{H})$ is one of the following pairs:

(1) a scroll over a smooth curve isomorphic to C;
(2) a quadric fibration over an elliptic curve E;
(3) $k \geq 3$, $\mathcal{H}^k = 18$, and $-K_Z = (k-2)\mathcal{H}$;
(4) $k = 2$, $\mathcal{H}^2 = 18$, and $(Z, \mathcal{H}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(3,3))$;
(5) $k = 2$, $\mathcal{H}^2 = 18$, and Z is a double plane.

Now let us discuss case (5) and provide more information with respect to the rough description given in [2]. According to this description ([2, pp. 274–275]), there is a morphism $\pi : Z \to \mathbb{P}^2$ of degree 2 branched along a smooth curve of degree $2b$, for some $b \geq 1$, such that $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}}(3)$. So the general element of the linear subsystem $\pi^*|\mathcal{O}_{\mathbb{P}}(3)|$ of $|\mathcal{H}|$ is a bielliptic curve. Since $\pi_*\mathcal{O}_Z = \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-b)$, by the projection formula we get

$$h^0(\mathcal{H}) = h^0(\pi_*\mathcal{H}) = h^0(\pi_*\pi^*\mathcal{O}_{\mathbb{P}}(3)) = h^0(\mathcal{O}_{\mathbb{P}}(3)) + h^0(\mathcal{O}_{\mathbb{P}}(3-b)).$$

So, if $b \geq 4$ we have $h^0(\mathcal{H}) = h^0(\mathcal{O}_{\mathbb{P}}(3))$, hence the morphism $\varphi_{\mathcal{H}}$ factors through π, which is of degree 2. This contradicts the very ampleness of $\mathcal{H}$ (the generic very ampleness in [2], since there the authors deal with the reduction of $(Z, \mathcal{H})$). Therefore $b \leq 3$. Note that if $b = 1$, then $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{H} = \pi^*\mathcal{O}_{\mathbb{P}}(3) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)$, so $(Z, \mathcal{H})$ is as in case (4). We thus conclude that (5) gives rise to the following two possibilities:

(5′) $b = 2$, Z is a Del Pezzo surface with $K_Z^2 = 2$ and $\mathcal{H} = -3K_Z$, since π is given by $|-K_Z|$;

(5″) $b = 3$, Z is a K3 surface and $\mathcal{H} = 3\pi^*\mathcal{O}_{\mathbb{P}}(1)$.

In both cases we have seen that $\mathcal{H} = 3h$, where $h = \pi^*\mathcal{O}_{\mathbb{P}}(1)$.

In case (3) Z is a Fano manifold. By using the numerical condition $\mathcal{H}^k = 18$, it is easy to see that $k-2$ is in fact the index of Z. Therefore $(Z, \mathcal{H})$ is a Mukai manifold of dimension ≥ 3 and degree 18. According to [3, pp. 102–103] for $k = 3$, the pair $(Z, \mathcal{H})$ is an extension of a surface as in case (5). Conversely, we have

Remark. Let $(Z, \mathcal{H})$ be as in case (3). Then there are $k-2$ elements of $|\mathcal{H}|$ meeting transversally along a smooth surface S such that the pair $(S, \mathcal{H}_S)$ is as in case (5″).

Proof. Let C be a bielliptic curve section of $(Z, \mathcal{H})$. By the Bertini theorem the general element of $|\mathcal{H} - C|$ is a smooth hypersurface of Z. Then by induction we see that there are $k-2$ elements of $|\mathcal{H} - C|$ meeting transversally along a smooth surface S such that $C \in |\mathcal{H}_S|$. Now, since

$$K_S = (K_Z + (k-2)\mathcal{H})_S = 0 \quad \text{and} \quad h^1(\mathcal{O}_S) = h^1(\mathcal{O}_Z) = 0,$$

we conclude that S is a K3 surface and so $(S, \mathcal{H}_S)$ is as in case (5″). □

This allows us to improve [3, Theorem A] by ruling out case A.3. In fact we have

Proposition 1.1

Case (3) *does not occur.*

Proof. Suppose that $(Z, \mathcal{H})$ is as in case (3), and consider $(S, \mathcal{H}_S)$ as in the Remark above. Then S is a double cover of $\mathbb{P}^2$. Let $\pi : S \to \mathbb{P}^2$ be the corresponding morphism. Then $\mathcal{H}_S = 3h$, where $h = \pi^*\mathcal{O}_{\mathbb{P}}(1)$. Consider the restriction homomorphism $\gamma : \mathrm{Pic}(Z) \to \mathrm{Pic}(S)$. By the Lefschetz theorem we know that: (a) γ is injective, and (b) Cokerγ is torsion free. Now, since $3h$ extends to $\mathcal{H} \in \mathrm{Pic}(Z)$, (b) says that h itself extends to an element $\widetilde{h} \in \mathrm{Pic}(Z)$. Then $(\widetilde{h})_S = h$. Furthermore, since $(3\widetilde{h})_S = 3h = \mathcal{H}_S$, (a) implies that $\mathcal{H} = 3\widetilde{h}$. In particular $\widetilde{h}$ is ample. But then, since $k \geq 3$, we would get

$$18 = \mathcal{H}^k = (3\widetilde{h})^k = 3^k\widetilde{h}^k \geq 27,$$

which is a contradiction. □

2. Proof of the Theorem

As we said, our approach is inspired by [16]. Under the assumption in (0.1), and the further restriction given by $c_r(\mathcal{E})H^{n-r} \geq 18$, we know that (Z, H_Z) is one of the pairs $(Z, \mathcal{H})$ in (1), (2), (4), (5′), (5″) of Section 1, with $k = n - r$.

In case (1), we can use [12, Theorem B] for $n - r \geq 3$ and [12, Remark 3.2] for $n - r = 2$. Note that Z cannot be $\mathbb{F}_0$, since $g(C) \geq 3$. More generally, we could use [11] for $n - r = 2$; in this case, for the same reason Z cannot be also $\mathbb{F}_1$. Hence we see that $(X, \mathcal{E})$ is as in case i) of the Theorem. Note that if H_Z is very ample, then the general section of $H_Z^{\oplus(n-r-1)}$ vanishes along a bielliptic curve.

In case (2), by using Theorem 0.4 for $n-r \geq 3$ and [5] for $n-r=2$, we get the following possibilities:

(2a) X is a $\mathbb{P}^{n-1}$-bundle over E, $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(r-1)}$, and $H_F = \mathcal{O}_{\mathbb{P}}(1)$, for every fiber F;

(2b) there exists a surjective morphism $X \to E$ whose general fiber F is isomorphic to a smooth quadric $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$, $\mathcal{E}_F = \mathcal{O}_{\mathbb{Q}}(1)^{\oplus r}$, and $H_F = \mathcal{O}_{\mathbb{Q}}(1)$, for every such fiber F;

(2c) $n-r=2$, X is a $\mathbb{P}^{n-1}$-bundle over E, $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$, and $H_F = \mathcal{O}_{\mathbb{P}}(2)$ for every fiber F.

For $n-r \geq 3$, cases (2a), (2b) come from Theorem 0.4 (recall that the analog of case (2c) cannot happen for $n-r \geq 3$, as explained in [12, (4.4)]). For $n-r=2$ cases (2a), (2b), (2c) correspond to (a), (c) and (b) in [5, Theorem], respectively. Note that the special subcase described there in (b) cannot occur in our setting, since E is elliptic. Cases (2a), (2b),(2c) give cases ii), iii), iv) in the Theorem, respectively. We note that if H_Z is very ample, then the general section of $H_Z^{\oplus(n-r-1)}$ vanishes along a bielliptic curve in all these cases.

To complete the analysis of cases (1) and (2) we add the following

Proposition 2.1

Let $(X, \mathcal{E}, H)$ be a triplet as in cases i) – iv) *of the Theorem. Then there exists a very ample line bundle $H^\sharp$ on X such that $(X, \mathcal{E}, H^\sharp)$ has the same type as $(X, \mathcal{E}, H)$. In particular $(X, \mathcal{E}, H^\sharp)$ satisfies condition* (0.1).

Proof. Let L be an ample line bundle on the base curve C or E of X. In cases i) and ii) simply take $H^\sharp = H + m\pi^*L$ with $m >> 0$ and use Lemma 0.2 with $M = C$ or E respectively, according to the two cases. Similarly, in case iv) take $H^\sharp = 2(\xi + m\pi^*L)$ with $m >> 0$. Finally, let $(X, \mathcal{E}, H)$ be as in case iii). Then by Lemma 0.3 we know that every fiber G of $f : X \to E$ is an irreducible hyperquadric of $\mathbb{P}^n$ having only isolated singularities. Let $\mathcal{V} := f_*H$. Then $\mathcal{V}$ is a vector bundle of rank $n+1$ on the elliptic curve E, X is embedded fiberwise in $P = \mathbb{P}(\mathcal{V})$ (i. e., f is induced by the bundle projection $\pi : P \to E$), and ξ, the tautological line bundle on P, satisfies $\xi_X = H$. Then we can apply Lemma 0.2 again and put $H^\sharp = (\xi + m\pi^*L)_X$ with $m >> 0$. □

To consider cases $(5')$ and $(5'')$ recall that $H_Z = 3h$. Look at the restriction homomorphism $\theta : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$. Since Coker$\theta$ is torsion free and $3h$ extends to an element of $\mathrm{Pic}(X)$, we conclude that h itself extends to an element $\mathcal{L} \in \mathrm{Pic}(X)$. Then $(\mathcal{L})_Z = h$. Furthermore, since $(3\mathcal{L})_Z = 3h = H_Z$, the injectivity of θ implies that $H = 3\mathcal{L}$. Therefore $\mathcal{L}$ is ample, H being so. Now, in case $(5')$ $(Z, \mathcal{L}_Z)$ is a Del Pezzo manifold of dimension 2. Then, by applying [13, Theorem 4 and Remark in Section 2] to the triplet $(X, \mathcal{E}, \mathcal{L})$ we conclude that X is a Fano manifold of index $n-1$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by the ample line bundle $\mathcal{L}$, $\mathcal{L}^n = 2$, and $\mathcal{E} = \mathcal{L}^{\oplus(n-2)}$. As we already observed, $H = 3\mathcal{L}$. This gives case vii) in the Theorem. Note that this case is effective. In fact we have $c_{n-2}(\mathcal{E})H^2 = \mathcal{L}^{n-2}(3\mathcal{L})^2 = 9\mathcal{L}^n = 18$. Moreover $H = 3\mathcal{L} = K_X + (n+2)\mathcal{L}$ is even very ample, since $\mathcal{L}$ is ample and spanned (e. g., see [14, Theorem 1.2]).

To describe $(X, \mathcal{E})$ in case $(5'')$ note that $0 = K_Z = (K_X + \det \mathcal{E})_Z$. Hence, due to the injectivity of the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ we get $K_X + \det \mathcal{E} = 0$. It thus follows that X is a Fano manifold. By recalling that $H \in 3\mathrm{Pic}(X)$, this gives case viii) in the Theorem.

Finally consider case (4). For short set $h = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$. We know that $(Z, H_Z) = (\mathbb{P}^1 \times \mathbb{P}^1, 3h)$. Thus the same argument as before relying on the Lefschetz-Sommese theorem shows that $H = 3\mathcal{L}$, for some ample line bundle $\mathcal{L}$ on X. In the present case we have $-K_Z = 2h = (2\mathcal{L})_Z$, hence $(Z, 2\mathcal{L}_Z)$ is a Del Pezzo manifold of dimension 2, with $2\mathcal{L}$ being an ample line bundle on X. Then, by applying [13, Theorem 4 and Remark in Section 2] to the triplet $(X, \mathcal{E}, 2\mathcal{L})$ we conclude that $(X, \mathcal{E}, H)$ is one of the following:

(4a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}, \mathcal{O}_{\mathbb{P}}(3))$;
(4b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}, \mathcal{O}_{\mathbb{Q}}(3))$.

These give cases v) and vi) in the Theorem respectively. Note that both cases are effective. Actually if $\pi : Z \to \mathbb{P}^2$ is a morphism of degree 2 representing $Z = \mathbb{P}^1 \times \mathbb{P}^1$ as a double cover of $\mathbb{P}^2$, then the general element of the linear subsystem $\pi^*|\mathcal{O}_{\mathbb{P}^2}(3)|$ of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)|$ is a bielliptic curve.

Remark. If we simply look at the pair $(X, \mathcal{E})$ under the assumption that a regular section of $\mathcal{E}$ vanishes on $Z = \mathbb{P}^1 \times \mathbb{P}^1$, then according to [10, Theorem B] there is one more case to consider in addition to (4a), (4b): namely, X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ for every fiber F of the bundle projection. This case however cannot occur in our setting; the reason is that the line bundle extending $-K_Z$ is not ample on X, contrary to what we proved before. For instance, this can be easily checked on the concrete example of $(X, \mathcal{E})$ constructed in [10, Section 4], by contrasting the expression of $-(K_X + \det \mathcal{E})$ with the ampleness conditions.

3. More on case viii)

In this Section we provide examples and more details about case viii) in our Theorem. First of all we prove the following

Proposition 3.1

If $(X, \mathcal{E}, H)$ *is as in case* viii) *and* $n \geq 6$, *then* $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Proof. Let $C \subset X$ be any rational curve. Since $K_X + \det \mathcal{E} = 0$ we have

$$-K_X C = \deg \mathcal{E}_C \geq \mathrm{rk}\mathcal{E}_C = n - 2.$$

In particular this says that the pseudoindex of X is $\geq n-2$. Thus, if $n \geq 7$ a result of Wiśniewski [17, Theorem A] shows that $\mathrm{Pic}(X) \cong \mathbb{Z}$. Now let $n = 6$. If $\mathrm{Pic}(X) \not\cong \mathbb{Z}$, then by applying [1, Lemma 5.3] we conclude that $X = \mathbb{P}^3 \times \mathbb{P}^3$ with $\mathcal{E} = \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1,1)^{\oplus 4}$. Of course the line bundle H is of the form $H = \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(a,b)$ for some positive integers a, b. Thus a straightforward computation gives

$$c_4(\mathcal{E})H^2 = (\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1,1))^4(\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(a,b))^2 = 2\binom{4}{2}ab + 4(a^2+b^2).$$

But then the right hand term cannot be 18, being a multiple of 4. This concludes the proof. □

To show that case viii) is effective for every $n \geq 4$, here we produce two examples.

EXAMPLE 3.2: (a) Let $\Pi : X \to \mathbb{P}^n$, $n \geq 4$, be a double cover branched along a smooth sextic hypersurface and let $\mathcal{L} = \Pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Set $\mathcal{E} = \mathcal{L}^{\oplus(n-2)}$ and $H = 3\mathcal{L}$. Then $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by $\mathcal{L}$. Moreover

$$K_X = \Pi^*\big(K_{\mathbb{P}^n} + \mathcal{O}_{\mathbb{P}^n}(3)\big) = -(n-2)\mathcal{L}.$$

Hence X is a Mukai n-fold. We have $K_X + \det\mathcal{E} = 0$. Note that $\mathcal{L}$ is an ample and spanned line bundle, and $\mathcal{L}^n = 2$. Thus $\mathcal{E}$ is ample and spanned, hence the general section of $\mathcal{E}$ vanishes along a smooth surface Z. By adjunction

$$K_Z = (K_X + \det\mathcal{E})_Z = 0, \quad \text{and} \quad h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_X) = 0,$$

hence Z is a K3 surface. Moreover $H = 3\mathcal{L} = K_X + (n+1)\mathcal{L}$ is very ample (e. g., see [14, Theorem 1.2]). We have

$$c_{n-2}(\mathcal{E})H^2 = \mathcal{L}^{n-2}(3\mathcal{L})^2 = 9\mathcal{L}^n = 18.$$

Finally note that the pull-back via Π of the general smooth cubic lying in a general plane of $\mathbb{P}^n$ is a smooth bielliptic curve. The discussion above shows that $(X, \mathcal{E}, H)$ is a triplet satisfying (0.1), as in case viii).

(b) Let $(X, \mathcal{L})$ be a Del Pezzo manifold of dimension $n \geq 4$ and degree $\mathcal{L}^n = 1$. Recall that $\mathrm{Pic}(X) \cong \mathbb{Z}$, generated by $\mathcal{L}$, which is an ample line bundle with the base locus of $|\mathcal{L}|$ consisting of a single point. In particular $(X, \mathcal{L})$ has a regular smooth ladder [6, (1.4)]. Moreover $2\mathcal{L}$ is spanned and $3\mathcal{L}$ is very ample [15, Theorem 1.2]. Set $\mathcal{E} = 2\mathcal{L} \oplus \mathcal{L}^{\oplus(n-3)}$ and $H = 3\mathcal{L}$. Then $\mathcal{E}$ is ample, H is very ample, and $\det\mathcal{E} = (n-1)\mathcal{L} = -K_X$; moreover, although not spanned, $\mathcal{E}$ has a regular section defining a smooth surface Z. Since

$$K_Z = (K_X + \det\mathcal{E})_Z = 0 \quad \text{and} \quad h^1(\mathcal{O}_Z) = h^1(\mathcal{O}_X) = 0,$$

Z is a K3 surface. We have

$$c_{n-2}(\mathcal{E})H^2 = 2\mathcal{L}^{n-2}(3\mathcal{L})^2 = 18\mathcal{L}^n = 18.$$

Taking the intersection Y of $n-3$ general elements of $|\mathcal{L}|$, we obtain a Del Pezzo 3-fold $(Y, \mathcal{L}_Y)$ with $\mathcal{L}_Y^3 = 1$, which is a double cover of the cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ via the morphism φ given by $|2\mathcal{L}_Y|$. Then $Z \in |2\mathcal{L}_Y|$, and $\varphi_{|Z} : Z \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ with $\mathcal{L}_Z = \varphi_{|Z}^*\mathcal{O}_{\mathbb{P}^2}(1)$. Thus $H_Z = 3\mathcal{L}_Z = \varphi_{|Z}^*\mathcal{O}_{\mathbb{P}^2}(3)$; hence the general element of the linear subsystem $\varphi_{|Z}^*|\mathcal{O}_{\mathbb{P}^2}(3)|$ of $|H_Z|$ is a bielliptic curve. We thus conclude that $(X, \mathcal{E}, H)$ is another triplet satisfying (0.1) as in case viii).

On the other hand, we can also prove the following

Proposition 3.3

Assume that $(X, \mathcal{E}, H)$ *is as in case* viii). *If* $\mathrm{Pic}(X) \cong \mathbb{Z}$, *then* $\mathcal{E}$ *can never be very ample.*

Proof. By Section 2 we know that (Z, H_Z) is $(Z, \mathcal{H})$ as in case $(5'')$ of Section 1. First of all, it should be emphasized that the line bundle $\mathcal{L}$ extending h to X (see Section 2) is the ample generator of $\mathrm{Pic}(X)$, because $H = 3\mathcal{L}$ and $H_Z^2 = 18$. So we can write $-K_X = i\mathcal{L}$, where i is the index of X. Take a general element $B \in |h|$. Then, since $K_Z = 0$ and $h^2 = 2$, B is a smooth curve of genus two, hence hyperelliptic. Since $K_X + \det \mathcal{E} = 0$, we have

$$(\det \mathcal{E})B = (-K_X)B = i\mathcal{L}B = i\mathcal{L}_Z B = ih^2 = 2i.$$

Now suppose to the contrary that $\mathcal{E}$ is very ample. Then, by [8, Corollary 1] we get $(\det \mathcal{E})B \geq 3n-4$. Combining this with the above equality, we obtain $i \geq n + n/2 - 2$. Moreover, since $n \geq 4$, we conclude that $i \geq n$. This tells us that X is either $\mathbb{P}^n$ or $\mathbb{Q}^n$ with $\mathcal{L} = \mathcal{O}_X(1)$ very ample. However, in either event $\mathcal{O}_Z(1)^2 = \mathcal{L}_Z^2 = h^2 = 2$, which implies that $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. This is a contradiction. □

Finally we characterize the triplets $(X, \mathcal{E}, H)$ exhibited in Example 3.2 as follows.

Proposition 3.4

Let $(X, \mathcal{E}, H)$ be as in case viii) *of the Theorem. If $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $\mathcal{E}$ is a direct sum of line bundles, then $(X, \mathcal{E}, H)$ is as in* (a) *or* (b) *of Example* 3.2.

Proof. As in the proof of Proposition 3.3, we know that (Z, H_Z) is as in case $(5'')$ and that the ample line bundle $\mathcal{L}$ such that $\mathcal{L}_Z = h$ is the ample generator of $\mathrm{Pic}(X)$; moreover $H = 3\mathcal{L}$. Thus, since $\mathcal{E}$ is ample and a direct sum of line bundles we can write

$$\mathcal{E} = \bigoplus_{i=1}^{n-2} (t_i \mathcal{L}),$$

for some positive integers t_i $(i = 1, \dots, n-2)$. Therefore

$$t := \sum_{i=1}^{n-2} t_i \geq n-2.$$

From the relation $K_X + \det \mathcal{E} = 0$ we thus get $-K_X = t\mathcal{L}$. Hence t, which is the index of X, satisfies the condition $n-2 \leq t \leq n+1$. This leads to the following possibilities:

(a) $t = n+1$ and $X = \mathbb{P}^n$;
(b) $t = n$ and $X = \mathbb{Q}^n$;
(c) $t = n-1$ and $(X, \mathcal{L})$ is a Del Pezzo manifold;
(d) $t = n-2$ and $(X, \mathcal{L})$ is a Mukai manifold.

Cases (a) and (b) cannot occur. Otherwise every summand of $\mathcal{E}$ would be a very ample line bundle, which contradicts Proposition 3.3. In case (c) we have $\mathcal{E} = (2\mathcal{L}) \oplus \mathcal{L}^{\oplus(n-3)}$. From the equality

$$18 = c_{n-2}(\mathcal{E})H^2 = 2\mathcal{L}^{n-2}(3\mathcal{L})^2 = 18\mathcal{L}^n,$$

we deduce that $\mathcal{L}^n = 1$. This means that $(X, \mathcal{E}, H)$ is as in Example 3.2,(b). Finally, in case (d) we get $\mathcal{E} = \mathcal{L}^{\oplus(n-2)}$, and the equality

$$18 = c_{n-2}(\mathcal{E})H^2 = \mathcal{L}^{n-2}(3\mathcal{L})^2 = 9\mathcal{L}^n$$

shows that $\mathcal{L}^n = 2$. Thus $(X, \mathcal{L})$ is a Mukai manifold of degree 2. In other words its genus is $g = 1 + 1/2\,\mathcal{L}^n = 2$. We know that there are $n-2$ sections of $\mathcal{L}$ whose zero loci meet along a smooth surface Z. By applying [9, Corollary 2.4.7] we thus conclude that the linear system $|\mathcal{L}|$ defines a morphism $\varphi : X \to \mathbb{P}^n$, whose degree is two, since $\mathcal{L}^n = 2$ and $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Let Δ be the branch divisor of φ. Then Δ is a smooth hypersurface of some degree $2b$. Moreover, from the formula expressing the canonical bundle

$$\varphi^*\mathcal{O}_{\mathbb{P}^n}(2-n) = -(n-2)\mathcal{L} = K_X = \varphi^*(K_{\mathbb{P}^n} + \mathcal{O}_{\mathbb{P}^n}(b)) = \varphi^*\mathcal{O}_{\mathbb{P}^n}(b-n-1),$$

we get $b = 3$. This shows that $(X, \mathcal{E}, H)$ is as in Example 3.2,(a). □

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