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Properties of extensions of algebraically maximal fields

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Abstract

We prove some properties similar to the theorem of Ax-Kochen-Ershov, in some cases of pairs of algebraically maximal fields of residue characteristic $p > 0$. These properties hold in particular for pairs of Kaplansky fields of equal characteristic, formally $\wp$-adic fields and finitely ramified fields. From that we derive results about decidability of such extensions.

1. Introduction

First we recall the theorem of Ax-Kochen-Ershov ([1], [2], [11], [12]). Let (K, v) and (K', v') be two henselian valued fields of residue characteristic zero. Then (K, v) and (K', v') are elementarily equivalent if and only if the value groups vK and $v'K'$ are elementarily equivalent, and the residue fields Kv and $K'v'$ are elementarily equivalent.

This theorem has also been proved for other classes of fields. We are interested in generalizing it to pairs of valued fields.

By a *pair* of valued fields we mean a structure $(K \subset L, v)$, with L a valued field, and K a subfield of L. Let $(K \subsetneq L, v)$ and $(K' \subsetneq L', v')$ be two pairs of henselian valued fields with residue characteristic zero. Is it true that $(K \subsetneq L, v) \equiv (K' \subsetneq L', v')$ if and only if $(vK \subset vL) \equiv (v'K' \subset v'L')$ and $(Kv \subset Lv) \equiv (K'v' \subset L'v')$?

In general, this is false. First, we recall that a pair $(K \subset L, v)$ of valued fields is *immediate* if $vK = vL$ and $Kv = Lv$. Françoise Delon proved the following (cf. [8]).

Let T_R (resp. T_G) be a theory of fields of characteristic zero (resp. totally ordered abelian groups), and T be the theory of all immediate pairs $(K \subsetneq L, v)$ of henselian

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fields such that $Kv \models T_R$ and $vK \models T_G$. Then T is not complete, and admits $2^{\aleph_0}$ distinct completions.

The failure comes from initial segments $v(l-K) = \{v(l-x) \mid x \in K\}$ of vL, for a fixed $l \in L$. With the set of all such initial segments, Françoise Delon interprets the theory of all symmetric and antireflexive graphs in T.

We need to consider those initial segments in the theory.

Note that for every $l \neq 0$ in L, $v(l-K) = v(l) + v(1-Kl^{-1})$. Hence we can generalize in a natural way the initial segments $v(l-K)$ by defining the sets $\mathcal{I}_n(K \subset L, v) := \{v(1-A) \mid A \subseteq L,\ A$ is a K-module of dimension n, disjoint from $K\}$, with n a positive integer. We also set $\mathcal{I}(K \subset L, v) = \{\mathcal{I}_n(K \subset L, v) \mid n \geq 1\}$.

If $vK = vL$, then $v(1-A)$ is an initial segment of vK. In general, we can have $\mathcal{I}_n(K \subset L, v) \neq \mathcal{I}_{n+1}(K \subset L, v)$: Baur proved that there exist pairs $(K \subset L, v)$ with $\mathcal{I}_2(K \subset L, v) \neq \mathcal{I}_3(K \subset L, v)$ (cf. [4], p. 33).

The family $\mathcal{I}(K \subset L, v)$ is interpretable in the theory of $(K \subset L, v)$. Hence, if $(K \subset L, v) \equiv (K' \subset L', v')$, then $(Kv \subset Lv) \equiv (K'v' \subset L'v')$ and $(vK \subset vL) \cup \mathcal{I}(K \subset L, v) \equiv (v'K' \subset v'L') \cup \mathcal{I}(K' \subset L', v')$. Is the converse true? In this paper, we focus on two special cases.

The first one is: $\mathcal{I}_1(K \subset L, v) = \{vK\}$. This is equivalent to K being dense in L, with the topology associated to v. In this case we say that the pair is *dense*.

The second one is: for every finitely generated K-submodule A of L, $v(1-A)$ has a greatest element. In this case, we say that the pair is *separated* or *vs-defectless*.

In both examples, the theory of $\mathcal{I}(K \subset L, v)$ is contained in the theory of $(vK \subset vL)$. We obtain generalizations of the theorem of Ax-Kochen-Ershov in the cases of dense and separated pairs of valued fields, and in other cases (see subsection 3.3).

In [17], those pairs have been studied only with residue characteristic 0. Here we generalize the results to some classes of fields with residue characteristic $p > 0$. In Section 4, we derive properties about decidability of such pairs.

Observe that we use the notation $(K \subsetneq L, v)$ only in the non-separated case. Indeed, from general properties of separated pairs (see, for example the main theorem of [9]), we deduce the following. If $(K \subset L, v)$ is a separated pair of valued fields, such that $vL = vK$ and $Kv = Lv$, then $K = L$.

2. Definitions - Classical results

The reader can find basic definitions about valuations and pseudo-Cauchy sequences in [10], [13], [15], [17], [19] (for example).

Let (K, v) be a valued field. We will use the following notations:

- v will be the valuation.
- vK will be the value group of (K, v).
- R_v will be its valuation ring, $R_v := \{x \in K \mid v(x) \geq 0\}$.
- M_v will be its maximal ideal, $M_v := \{x \in K \mid v(x) > 0\}$.
- Kv will be the residue field, $Kv := R_v/M_v$.
- For $x \in R_v$, x/v will be the class of x modulo M_v.

Let $U_v := R_v \cap (R_v\backslash\{0\})^{-1}$ be the group of all units of R_v. Then we know that $vK \simeq (K\backslash\{0\})/U_v$, and we can assume that v is the quotient map $K\backslash\{0\} \to (K\backslash\{0\})/U_v$. Therefore, we can assume that the language of valued fields is $\mathcal{L}_{VF} := (+, \cdot, 0, 1, R)$, where R is a unary predicate interpreted by: $\forall x \in K,\ R(x) \Leftrightarrow x \in R_v$. However, in our proofs we will use the symbol v of the valuation mapping because it is usual.
$\mathcal{L}_F$ will be the language of fields, and $\mathcal{L}_G$ will be the language of ordered groups.

2.1 Some valued fields

The results of Section 3 will apply to some classes of fields which we will define now.

Let (K, v) be a valued field of residue characteristic $p > 0$. (K, v) is called a *Kaplansky field* if it satisfies:

(i) $p \cdot vK = vK$

(ii) Any polynomial $X^{p^n} + a_{n-1}X^{p^{n-1}} + \cdots + a_1 X^p + a_0 X + b$ with coefficients in the residue field Kv has a root in Kv.

Note that, under assumption (i), (ii) is equivalent to "Kv does not admit a finite extension of degree divisible by p" ([15]). We see that the residue field of any Kaplansky field is perfect, and that any algebraically maximal Kaplansky field is perfect.

In [13], Theorem 5, Kaplansky proved that, for every Kaplansky field (K, v), the maximal immediate algebraic extension M of (K, v) is uniquely determined up to valuation preserving isomorphism. Furthermore, for every pseudo-Cauchy sequence of algebraic type (x_λ) having P for a minimal polynomial, and without a pseudo-limit in (K, v), M contains a pseudo-limit l of (x_λ) which is also a root of $P(x) = 0$ ([13], Theorem 3. For the definition of pseudo-Cauchy sequences, see [13], Section 2).

A valued field (K, v) is said to be *finitely ramified* if $\mathrm{char}K = 0 < p = \mathrm{char}(Kv)$, and there exists an integer $e \geq 1$ such that vp/e is the least positive element of vK. The integer e is called the *ramification index* of (K, v).

Note that the class of all finitely ramified fields is not elementary. However for any prime number p and any positive integer e, the class of all finitely ramified fields of residue characteristic p and with ramification index e is elementary.

A valued field (K, v) is called *unramified* if it is finitely ramified and its ramification index e is equal to 1.

A finitely ramified field (K, v) is called a *formally $\wp$-adic field* if Kv is a finite field.

The henselization of every valued field is uniquely determined up to valuation preserving isomorphism. Now, a finitely ramified field is algebraically maximal if and only if it is henselian (see, for example, [15]). Hence the immediate maximal algebraic extension of any finitely ramified field is uniquely determined up to valuation preserving isomorphism.

2.2 Enriched theories

In order to prove the results of Section 3, we introduce the following definitions.

DEFINITION 2.1 ([17], Définition 1.2.2). Let T be a theory of valued fields in a language $\mathcal{L}(T)$ containing $\mathcal{L}_{VF}$, and α, β be two n-ary symbols of $\mathcal{L}(T)$.
1) If:

$$\forall(x_1,\ldots,x_n)\ ,\ \alpha(x_1,\ldots,x_n) \Rightarrow \Big[\bigwedge_{1\le i\le n} x_i \in R_v \wedge \quad (\forall(y_1,\ldots,y_n) \qquad (*)$$
$$\bigwedge_{1\le i\le n} y_i \in R_v \wedge \bigwedge_{1\le i\le n} (y_i/v = x_i/v) \to \alpha(y_1,\ldots,y_n))\Big]$$

holds in T, then there exists an n-ary relation α/v on Kv with:

$$(\alpha/v)(x_1/v,\ldots,x_n/v) \Leftrightarrow \alpha(x_1,\ldots,x_n);$$

α will be called a *lifting* of the residue relation α/v.
2) If:

$$\forall(x_1,\ldots,x_n),\ \beta(x_1,\ldots,x_n) \Rightarrow \Big[\forall(y_1,\ldots,y_n) \qquad (**)$$
$$\bigwedge_{1\le i\le n} (v(y_i) = v(x_i)) \to \beta(y_1,\ldots,y_n)\Big]$$

holds in T, then there exists an n-ary relation $v\beta$ on vK with:

$$v\beta(v(x_1),\ldots,v(x_n)) \Leftrightarrow \beta(x_1,\ldots,x_n);$$

β will be called a *lifting* of the value relation $v\beta$.

DEFINITION 2.2 ([17], Définition 1.2.3). Let T be a theory of algebraically maximal valued fields. T will be called an *enriched theory* of valued fields if:

The language of T is $\mathcal{L}(T) = \mathcal{L}_{VF} \cup \{\alpha_i, \beta_i \mid i \in \mathbb{N}\} \cup \{F_i \mid 1 \le i \le n\}$, with $n < \omega$, where, in the theory T:

- every α_i is the lifting of a residue relation,
- every β_i is the lifting of a value relation,
- the interpretation of the F_i's in the models of T is an ascending chain of algebraically maximal valued subfields.

If T is an enriched theory of valued fields, let:

$\mathcal{L}_{RF}(T) := \mathcal{L}_F \cup \{\alpha_i/v \mid i \in \mathbb{N}\} \cup \{F_i/v \mid 1 \le i \le n\}$,
$T_{RF}(T)$ be the theory of all residue fields of models of T, in the language $\mathcal{L}_{RF}(T)$,
$\mathcal{L}_{VG}(T) := \mathcal{L}_G \cup \{v\beta_i \mid i \in \mathbb{N}\} \cup \{vF_i \mid 1 \le i \le n\}$,
$T_{VG}(T)$ be the theory of all value groups of models of T, in the language $\mathcal{L}_{VG}(T)$.

For example, the theory of all pairs of algebraically maximal fields $(K \subsetneq L, v)$, together with a valuation u finer than v is an enriched theory of valued fields. Indeed, let u/v be the quotient valuation, then $R_u := \{x \in R_v \mid x/v \in R_{u/v}\}$. Hence R_u can be interpreted by a lifting of a residue relation.

The enriched theories will enable us to generalize theorems on pairs in two ways. The first one is the generalization of 3.1 and 3.2 to separated n-tuples of valued fields by means of predicates of subfields (3.10 to 3.13). The second one is the generalization to some cases of neither dense nor separated pairs, by means of coarsenings of valuations, which can be encoded by residue relations (3.16, 3.17). Recall that a valuation v is said to be *coarser* than u if u is finer than v, i.e. $R_u \subset R_v$. We will denote $u > v$.

If f is an isomorphism of valued fields from (K, v) onto (K', v') then there exist two isomorphisms vf from vK onto $v'K'$, and f/v from Kv onto $K'v'$ such that:

- $\forall g \in vK, vf(g) = v(f(x))$ with $v(x) = g$
- $\forall z \in Kv, (f/v)(z) = f(x)/v$ with $x/v = z$.

DEFINITION 2.3. Let T be an enriched theory of valued fields and $F_1, \ldots, F_n$ be all the predicates of subfields of $\mathcal{L}(T)$. Let (K, v) be a model of T and (K_0, v) be a $\mathcal{L}(T)$-submodel of (K, v). We will say that (K_0, v) *has the linear disjointness property* if $\forall i, 1 \leq i \leq n, F_i$ and $F_{i+1} \cap K_0$ are linearly disjoint over $F_i \cap K_0$ (with $F_{n+1} = K$).

The following definition generalizes Définitions 1.4.1 in [17].

DEFINITION 2.4. Let T be an enriched theory of valued fields. T will be said to have the *residue-value extension property* if the following holds.
Let (K, v) and (K', v') be ω_1-saturated models of T, with $vK \equiv v'K'$ and $Kv \equiv K'v'$. Assume that there exists an $\mathcal{L}(T)$-isomorphism f_0 from K_0 onto K'_0 where (K_0, v) (resp. (K'_0, v')) is a countable $\mathcal{L}(T)$-submodel of (K, v) (resp. (K', v')), having the linear disjointness property, with $vK_0 \prec vK$ and $K_0v \prec Kv$ (resp. $v'K'_0 \prec v'K'$ and $K'_0v' \prec K'v'$). Let (K_1, v) be a countable $\mathcal{L}(T)$-submodel of (K, v) having the linear disjointness property, and such that $(K_0, v) \subset (K_1, v)$, $vK_1 \prec vK$ and $K_1v \prec Kv$. Let ϕ be an $\mathcal{L}_{VG}(T)$-isomorphism from vK_1 onto an $\mathcal{L}_{VG}(T)$-elementary substructure of $v'K'$ with $\phi\lceil vK_0 = vf_0$, and let ψ be an $\mathcal{L}_{RF}(T)$-isomorphism from K_1v onto an $\mathcal{L}_{RF}(T)$-elementary substructure of $K'v'$ with $\psi\lceil K_0v = f_0/v$. Then there exists an $\mathcal{L}(T)$-isomorphism f_1 from (K_1, v) onto an $\mathcal{L}(T)$-substructure of (K', v') such that $f_1\lceil K_0 = f_0$, $vf_1 = \phi$ and $f_1/v = \psi$.

For example, let p be a prime number, n be a positive integer and T be the theory of all finitely ramified algebraically maximal fields of residue characteristic p, with ramification index lower than n. Then T has the residue-value extension property (see [15]). The theory of all algebraically maximal Kaplansky fields also has the residue-value extension property ([15]).

Let (K, v) be a model of an enriched theory of valued fields. Then any elementary substructure of (K, v) has the linear disjointness property. Hence, by the theorem of Löwenheim-Skolem, there exist countable substructures (K_i, v) having the linear disjointness property and such that $vK_i \prec vK$, $K_i v \prec Kv$.

The proofs of the main results of this paper are based on the following theorem (cf. [6], Chapter 5).

Theorem 2.5

Let $\mathcal{L}$ be a language and M, M' be two ω_1-saturated $\mathcal{L}$-structures. Then the following (1), (2), (3) are equivalent.

(1) There exists a non-empty family of partial isomorphisms, between M and M', having the back and forth property.

(2) $M \equiv M'$.

(3) The family of all isomorphisms, from a countable elementary substructure of M onto a countable elementary substructure of M', is non-empty and has the back and forth property.

Let (K, v) and (K', v') be ω_1-saturated models of an enriched theory of valued fields. Let f_0, K_0, K'_0, K_1 be like in Definition 2.4. Assume that: $vK \equiv v'K'$. Hence the family of all isomorphisms between countable elementary substructures of vK and $v'K'$ respectively has the back and forth property. Hence there exists an isomorphism ϕ extending vf_0 to vK_1. In the same way, if $Kv \equiv K'v'$, then there exists an isomorphism ψ extending f_0/v to K_1v.

Theorem 2.6

Let T be an enriched theory having the residue-value extension property. Let (K', v') be a model of T and let (K, v) be an $\mathcal{L}(T)$-substructure of (K', v') having the linear disjointness property, such that $vK \prec v'K'$ and $Kv \prec K'v'$. Then $(K, v) \prec (K', v')$.

Proof. The proof is similar to Démonstration 1.6.2 in [17]. □

Theorem 2.7

Let T be an enriched theory having the residue-value extension property, and (K, v), (K', v') be two models of T such that $vK \equiv v'K'$ and $Kv \equiv K'v'$. Assume that there exists an isomorphism f_0 from K_0 onto K'_0 where (K_0, v) (resp. (K'_0, v')) is a countable $\mathcal{L}(T)$-substructure of (K, v) (resp. (K', v')), having the linear disjointness property, with $vK_0 \prec vK$ and $K_0v \prec Kv$ (resp. $v'K'_0 \prec v'K'$ and $K'_0v' \prec K'v'$). Then $(K, v) \equiv (K', v')$.

Proof. Let $\mathcal{I}$ be the family of all isomorphisms f_1 from K_1 onto K_1' such that (K_1, v) (resp. (K_1', v')) is a countable $\mathcal{L}(T)$-substructure of (K, v) (resp. (K', v')), having the linear disjointness property, with $vK_1 \prec vK$, $v'K_1' \prec v'K'$, $K_0v \prec Kv$ and $K_1'v' \prec K'v'$. Then $\mathcal{I}$ is not empty, because it contains f_0. By Definition 2.4, $\mathcal{I}$ has the back and forth property. Now, by Theorem 2.5, we have $(K, v) \equiv (K', v')$. □

We know that f_0 exists in the case of Kaplansky fields of positive characteristic (see [7], Proposition 5.15). In the case of unramified fields, we know that the algebraic closure of $(\mathbb{Q}, v_p)$ in (K, v) is isomorphic to the algebraic closure of $(\mathbb{Q}, v_p)$ in (K', v'), with v_p the p-adic valuation (see [11] and [12]). But, in the case of finitely ramified fields, f_0 need not exist. For example, in [3], p. 192, Basarab defines two algebraically maximal finitely ramified fields (K, v) and (K', v') such that:

- $Kv \simeq K'v'$, $vK \simeq v'K' \simeq \mathbb{Z}$,
- the ramification index of K and K' is 2,
- $(K, v) \not\equiv (K', v')$.

Proposition 2.8

Let $T \subset T'$, be two enriched theories of valued fields such that:

i) $\mathcal{L}(T') = \mathcal{L}(T) \cup \{\alpha_i, \beta_i \mid i \in \mathbb{N}\}$,
ii) for all $i \in \mathbb{N}$, the α_i's satisfy $(*)$ and the β_i's satisfy $(**)$ of Definition 2.1,
iii) $T' \lceil \mathcal{L}(T) = T$,
iv) T' is generated by T and the interpretations of $T_{RF}(T')$ and $T_{VG}(T')$,
v) T has the residue-value extension property.

Then T' has the residue-value extension property.

Proof. The proof is similar to the proof of Propriété 1.4.4. in [17]. □

Definition 2.9 ([17], Définition 1.4.5). Let T be an enriched theory of valued fields and T^* be a theory of pairs of valued fields in the language $\mathcal{L}_{VF} \cup \{E\}$ (where E is interpreted in all models $(K \subset L, v)$ of T^* by: $E(x)$ iff $x \in K$). A theory T' will be called the *expansion* of T to T^* if it satisfies:

(a) $\mathcal{L}(T') = \mathcal{L}(T) \cup \{E\}$,
(b) $(K \subset L, v)$ is a model of T' if and only if:

- the $\mathcal{L}_{VF} \cup \{E\}$-reduct of $(K \subset L, v)$ is a model of T^*,
- $T' \models \forall x,\ F_n(x) \to E(x)$ (hence for every i, $1 \leq i \leq n$, $T' \models \forall x,\ F_i(x) \to E(x)$)
- (K, v) is a model of T,
- (L, v) is a model of T.

Definition 2.10. Let $\mathcal{C}$ be a class of valued fields. We will say that $\mathcal{C}$ has *uniqueness of the maximal immediate algebraic extensions* if the maximal immediate algebraic extension of any $(K, v) \in \mathcal{C}$ is unique up to valuation preserving isomorphism.

We recall that the henselization of any valued field is uniquely determined up to valuation preserving isomorphism. But in general, the maximal immediate algebraic extensions are not (see [13] Section 5, [16] and [20]).

For example, the class of all valued fields of residue characteristic 0, the class of all Kaplansky fields and the class of all finitely ramified fields have uniqueness of the maximal immediate algebraic extensions.

DEFINITION 2.11. Let (K, v) be a valued field and T be a theory of valued fields. We will say that (K, v) is *between two models of* T if there exist two models (K_0, v_0) and (K_1, v_1) of T such that $(K_0, v_0) \subset (K, v) \subset (K_1, v_1)$ (with $v_0 = v\lceil K_0$, and $v = v_1\lceil K$).

Assume that all the models of the theory T are finitely ramified fields. Then the class of all valued fields which are between two models of T has uniqueness of the maximal immediate algebraic extensions, because every subfield of a finitely ramified field is finitely ramified. The same holds if T is a theory of fields of characteristic 0.

2.3 Liftings

In practice the construction of the residue-value extension requires torsion-free quotients of value groups. In the case of elementary substructures, this condition is satisfied. Otherwise, we can drop this hypothesis if the pair $(K \subset L, v)$ contains a lifting of the pair of value groups $(vK \subset vL)$. We know that such a lifting exists when the pair is ω_1-saturated and vL/vK is torsion-free, but a value lifting may exist in other cases.

In the same way, we will require that the pair of residue fields be a separable pair. If the residue field is perfect, this condition holds. However, it is possible to omit this hypothesis by introducing liftings of residue fields (in the case $\mathrm{char}K = \mathrm{char}Kv$), in definitions.

Recall that a multiplicative subgroup G of $K\backslash\{0\}$ is called a *group of representatives for the value group* or a *lifting* of vK if there exists an isomorphism f from vK onto G such that $v \circ f = \mathrm{Id}$. The isomorphism f is called a *cross-section.*

Let $(K \subset L, v)$ be an ω_1-saturated pair of valued fields such that vL/vK is torsion-free. Then L contains a pair of multiplicative groups $(G \subset H)$ such that v is an isomorphism from H onto vL, and the inverse-image of vK is $G \subset K$ ([14]).

DEFINITION 2.12. Let T be an enriched theory of valued fields, and $F_1, \ldots, F_n$ be all the predicates of subfields of $\mathcal{L}(T)$. We let VG and RF be new unary predicates. An $\mathcal{L}(T) \cup \{VG\}$ (resp. $\mathcal{L}(T) \cup \{RF\}$, resp. $\mathcal{L}(T) \cup \{VG, RF\}$)-model of T will be a model of T containing a lifting VG of vK (resp. a lifting RF of Kv, resp. liftings VG of vK and RF of Kv), such that for every i, $1 \leq i \leq n$, $F_i \cap VG$ is a lifting of vF_i (resp. $F_i \cap RF$ is a lifting of $F_i v$).

T will be said to have the *residue-value extension property with value lifting* if the following holds:

Let (K, v) and (K', v') be ω_1-saturated $\mathcal{L}(T) \cup \{VG\}$-models of T, with $vK \equiv v'K'$ and $Kv \equiv K'v'$. Assume that there exists an $\mathcal{L}(T) \cup \{VG\}$-isomorphism f_0 from K_0 onto K'_0 where (K_0, v) (resp. (K'_0, v')) is a countable $\mathcal{L}(T)$-submodel of (K, v) (resp. (K', v')), having the linear disjointness property, with $vK_0 \prec vK$ and $K_0 v \prec Kv$

(resp. $v'K'_0 \prec v'K'$ and $K'_0v' \prec K'v'$). Let (K_1, v) be a countable $\mathcal{L}(T) \cup \{VG\}$-submodel of (K, v) having the linear disjointness property, and such that $(K_0, v) \subset (K_1, v)$, $vK_1 \prec vK$ and $K_1v \prec Kv$. Let ϕ be an $\mathcal{L}_{VG}(T)$-isomorphism from vK_1 onto an $\mathcal{L}_{VG}(T)$-elementary substructure of $v'K'$ with $\phi\lceil vK_0 = vf_0$, and let ψ be an $\mathcal{L}_{RF}(T)$-isomorphism from K_1v onto an $\mathcal{L}_{RF}(T)$-elementary substructure of $K'v'$ with $\psi\lceil K_0v = f_0/v$. Then there exists an $\mathcal{L}(T) \cup \{VG\}$-isomorphism f_1 from (K_1, v) onto an $\mathcal{L}(T) \cup \{VG\}$-substructure of (K', v') such that $f_1\lceil K_0 = f_0$, $vf_1 = \phi$ and $f_1/v = \psi$. In the same way, we define the *residue-value extension property with residue lifting* and the *residue-value extension property with residue and value lifting.*

Note that in any ω_1-saturated model such that the vF_{i+1}/vF_i's are torsion-free, the value lifting always exists, hence we have the following.

Theorem 2.13

Let T be a theory having the residue-value extension property with value lifting such that, for $1 \leq i \leq n$, vF_{i+1}/vF_i's are torsion-free. Let (K', v') be a model of T (in a language without lifting of the value group) and (K, v) be a $\mathcal{L}(T)$-substructure of (K', v') having the linear disjointness property such that $vK \prec v'K'$ and $Kv \prec K'v'$. Then $(K, v) \prec (K', v')$.

Proof. The proof is the same as Démonstration 1.6.1 in [17]. □

Theorems 2.6, 2.7 and Property 2.8 remain valid with residue-value extension property with value lifting (resp. with residue lifting).

3. Elementarily equivalent pairs

3.1 Separated pairs

Theorem 3.1

Let T be an enriched theory of algebraically maximal valued fields with perfect residue fields and having the residue-value extension property.

Let T_1 be the restriction of T to $\mathcal{L}_{VF}$-formulas and T' be the expansion of T to separated pairs $(K \subset L, v)$ such that vL/vK is torsion-free.

Assume that the class of all the valued fields which are between two models of T_1 has uniqueness of the maximal immediate algebraic extensions.

Then T' has the residue-value extension property.

Proof. Let $(K_0 \subset L_0, v)$ be a countable T'-submodel of $(K \subset L, v)$, having the linear disjointness property, and let f_0 be an $\mathcal{L}(T')$-isomorphism from $(K_0 \subset L_0, v)$ into an ω_1-saturated model $(K' \subset L', v')$. We set $(K'_0 \subset L'_0, v') := f(K_0 \subset L_0, v)$. Assume that $(v'K'_0 \subset v'L'_0) \prec (v'K' \subset v'L')$ and $(K'_0v' \subset L'_0v') \prec (K'v' \subset L'v')$.

Let $(K_1 \subset L_1, v)$ be a countable model of T', having the linear disjointness property, with $L_0 \subset L_1$, $(vK_0 \subset vL_0) \prec (vK_1 \subset vL_1)$, $(K_0/v \subset L_0/v) \prec (K_1/v \subset L_1/v)$. Let ϕ be an isomorphism from $(vK_1 \subset vL_1)$ onto an $\mathcal{L}_{VG}(T')$-elementary substructure

$(v'K' \subset v'L')$ with $\phi\lceil vK_0 = vf_0$, and let ψ be an isomorphism from $(K_1v \subset L_1v)$ onto an $\mathcal{L}_{RF}(T')$-elementary substructure of $(K'v' \subset L'v')$ with $\psi\lceil K_0v = f_0/v$. We have to construct an extension f_1 of f_0 to $(K_1 \subset L_1, v)$ such that $vf_1 = \phi$, $f_1/v = \psi$.

In the same way as in [17], Démonstration 1.5.2, we can prove that $vK_1L_0 = vL_0 + vK_1$, $(K_1L_0)v = (K_1v)(L_0v)$ and that there exists $K_1' \subset K'$ and an $\mathcal{L}(T')$-isomorphism f' from K_1L_0 onto $K_1'L_0'$ such that $f'\lceil L_0 = f_0$.

We set $L_{01} := K_1L_0$. Then by hypothesis, the maximal immediate algebraic extension $\overline{L_{01}}^m$ of L_{01} is unique, up to valuation preserving isomorphism. Hence f' extends to $\overline{L_{01}}^m$. Now, $(\overline{L_{01}}^m \subset L_1)$ is separable, $L_{01}v = (K_1v)(L_0v)$ is perfect, and $vL_{01} = vL_0 + vK_1$, therefore vL_1/vL_{01} is torsion-free. Hence this isomorphism extends to L_1, by using the construction of [15] between (L_1, v) and (L', v'). We recall that this construction consists in extending the isomorphism to any finitely generated extension of L_{01}. Then by using ω_1-saturation it extends to L_1. By hypothesis, the maximal immediate algebraic extension of any finitely generated extension of L_{01} is uniquely determined, up to valuation preserving isomorphism. Let f_1 be this extension. Then f_1 is an isomorphism of valued fields. The same arguments as in [17], Démonstration 1.5.2, prove that f_1 is an $\mathcal{L}(T')$-isomorphism. □

This is true in the case of finitely ramified fields and in the case of fields of residue characteristic 0, because any subfield is algebraically maximal if and only if it is henselian.

Theorem 3.2

Let T be an enriched theory of algebraically maximal valued fields with perfect residue fields and having the residue-value extension property with value lifting.

Let T_1 be the restriction of the theory T to $\mathcal{L}_{VF}$-formulas, and T' be the expansion of T to separated pairs $(K \subset L, v)$ such that $(K \subset L, v)$ contains a lifting of the value group, where the lifting of every subfield is contained in this subfield.

Assume that either T_1 is a theory of algebraically maximal Kaplansky fields of positive characteristic or that the class of all valued fields which are between two models of T_1 has uniqueness of the maximal immediate algebraic extensions.

Then T' satisfies the residue-value extension property with value lifting.

Proof. The proof is similar to the proof of Theorem 3.1, but in the case of the Kaplansky fields of positive characteristic, we extend f' in another way. In this case, K_0 contains a lifting k_0 of K_0v that extends to a lifting k_1 of K_1v and to a lifting l_0 of L_0v. By linear disjointness, k_1l_0 is a lifting of $(K_1v)(L_0v) = (K_1L_0)v$. L_1 is Kaplansky, hence k_1l_0 extends to a lifting l_1 of L_1v, that contains a subfield l_{01} such that l_{01} is algebraic over l_1k_0 and l_{01} is closed under extensions of degree p. Let $\overline{K_1L_0}^h$ be the henselization of K_1L_0, and let $L_{01} = \overline{K_1L_0}^h(l_{01})$. L_{01} is algebraic over $\overline{K_1L_0}^h$, hence f' extends to L_{01}. Now, L_{01} is Kaplansky therefore the maximal immediate algebraic extension $\overline{L_{01}}^m$ of L_{01} is unique, up to valuation preserving isomorphism, and f' extends to $\overline{L_{01}}^m$. Moreover, $\overline{L_{01}}^m$ is perfect, because L_{01} is Kaplansky. Now, we can extend f' to L_1, by using liftings of the residue fields and of the value groups.

This construction is the same as the construction in the proof of the theorem of Ax-Kochen-Ershov in the case of Kaplansky fields of positive characteristic or in the case of fields of residue characteristic 0 (cf. [7], Proposition 5.15). □

In particular, these theorems apply to theories of finitely ramified fields with fixed (or bounded) residue characteristic and ramification index.

In the same way as we proved Theorem 3.2 in the case of Kaplansky fields of positive characteristic, we may prove the following.

Theorem 3.3

Let:

- *T be an enriched theory of valued fields having the residue-value extension property with residue lifting and value lifting,*
- *T_1 be the restriction of the theory T to $\mathcal{L}_{VF}$-formulas,*
- *T' be the expansion of T to separated pairs $(K \subset L, v)$ such that vL/vK is torsion-free, and $(K \subset L, v)$ contains a lifting of the pair of residue fields, where the lifting of every subfield is contained in this subfield.*

Assume that T_1 is a theory of algebraically maximal Kaplansky fields of positive characteristic.

Then T' satisfies the residue-value extension property with residue lifting and value lifting.

Theorem 3.4

Let T be an enriched theory of valued fields with perfect residue fields and having the residue-value extension property.

Let T_1 be the restriction of T to $\mathcal{L}_{VF}$-formulas, and T' be the expansion of T to the separated pairs $(K \subset L, v)$ such that vL/vK is torsion-free. Assume that either T_1 is a theory of algebraically maximal Kaplansky fields of positive characteristic or that the class of all the valued fields which are between two models of T_1 has uniqueness of the maximal immediate algebraic extensions.

Let $(K \subset L, v)$ and $(K' \subset L', v')$ be two separated pairs with (K, v), (K', v') models of T and (L, v), (L', v') models of T_1. Assume that:

1) $(K \subset L, v) \subset (K' \subset L', v')$
2) $(Kv \subset Lv) \prec (K'v' \subset L'v')$ *in* $\mathcal{L}_{RF}(T')$
3) $(vK \subset vL) \prec (v'K' \subset v'L')$ *in* $\mathcal{L}_{VG}(T')$
4) (K, v) *is a substructure of* (K', v') *having the linear disjointness property*
5) $v'L'/v'K'$ *is torsion-free.*

Then $(K \subset L, v) \prec (K' \subset L', v')$ *in* $\mathcal{L}(T')$.

Proof. L and K' are linearly disjoint over K because $(Kv \subset Lv) \prec (K'v' \subset L'v')$ and $(K \subset L, v)$ is separated. So $(K \subset L, v)$ is a substructure of $(K' \subset L', v')$ having the linear disjointness property. Now, the expansion of T to separated pairs verifying 5) has the residue-value extension property (cf. Theorem 3.1). The result follows from Theorem 2.6 and Theorem 2.13. □

Corollary 3.5

Let $(K \subset L, v)$ and $(K' \subset L', v')$ be two separated pairs of algebraically maximal Kaplansky fields of positive characteristic, such that:

- $(K \subset L, v) \subset (K' \subset L', v')$,
- $(vK \subset vL) \prec (v'K' \subset v'L')$ *in* $\mathcal{L}_G \cup \{E\}$*, where* E *is interpreted by* $E(x) \Leftrightarrow x \in vK$ *(resp.* $v'K'$*),*
- $(Kv \subset Lv) \prec (K'v' \subset L'v')$ *in* $\mathcal{L}_F \cup \{E\}$*, where* E *is interpreted by* $E(x) \Leftrightarrow x \in Kv$ *(resp.* $K'v'$*),*
- $v'L'/v'K'$ *is torsion-free.*

Then $(K \subset L, v) \prec (K' \subset L', v')$ *in* $\mathcal{L}_{VF} \cup \{E\}$*, where* E *is interpreted by* $E(x) \Leftrightarrow x \in K$ *(resp.* K'*).*

Theorem 3.6

Let T be an enriched theory of algebraically maximal valued fields with perfect residue fields and having the residue-value extension property (resp. with value lifting). Assume that for any two models (K, v), (K', v') of T if $vK \equiv v'K'$ and $Kv \equiv K'v'$, then $(K, v) \equiv (K', v')$.

Let T_1 be the restriction of T to $\mathcal{L}_{VF}$-formulas, and T' be the expansion of T to the separated pairs $(K \subset L, v)$ such that vL/vK is torsion-free. Assume that the class of all valued fields which are between two models of T_1 has uniqueness of maximal immediate algebraic extensions (resp. T_1 is a theory of algebraically maximal Kaplansky fields of positive characteristic).

Then for any two models $(K \subset L, v)$, $(K' \subset L', v')$ of T', such that $(vK \subset vL) \equiv (v'K' \subset v'L')$ and $(Kv \subset Lv) \equiv (K'v' \subset L'v')$, we have $(K \subset L, v) \equiv (K' \subset L', v')$.

Proof. We can assume that (K, v) and (K', v') are ω_1-saturated. Hence the family of all isomorphisms between countable elementary substructures of (K, v) and (K', v') has the back and forth property. By Theorem 2.5, there exists an isomorphism f_0 from K_0 onto K'_0 such that (K_0, v) (resp. (K'_0, v')) is a countable $\mathcal{L}(T)$-substructure of (K, v) (resp. (K', v')), having the linear disjointness property, whith $vK_0 \prec vK$, $v'K'_0 \prec v'K'$, $K_0v \prec Kv$ and $K'_0v' \prec K'v'$. The isomorphism f_0 can be extended to some $(K_1 \subset L_1, v)$ like in the proof of Theorems 3.1 and 3.2. Now, we conclude by Theorem 2.7. □

This holds if T is the theory of all unramified fields with perfect residue field, or a theory of unramified formally $\wp$-adic fields with fixed residue field.

3.2 Dense pairs

Theorem 3.7

Let T be an $\mathcal{L}_{VF}$-theory of algebraically maximal valued fields having the residue-value extension property (resp. the residue-value extension property with value lifting). Assume that for every immediate pair $(M \subset N, v)$ of valued fields, if $N \models T$, then the maximal immediate algebraic extension of (M, v) is uniquely determined up to valuation preserving isomorphism. Then the theory of dense pairs $(K \subsetneq L, v)$ of models of T has the residue-value extension property (resp. the residue-value extension property with value lifting).

Proof. This proof is the same as F. Delon's proof in the case of valued fields of residue characteristic 0 (see [17], proof of Théorème 2.1). □

Theorem 3.8

Let T be a theory of valued fields having the residue-value extension property (resp. the residue-value extension property with value lifting). Assume that for every immediate pair $(M \subset N, v)$ of valued fields, if $N \models T$, then the maximal immediate algebraic extension of (M, v) is uniquely determined up to valuation preserving isomorphism. Let $(K \subsetneq L, v)$ and $(K' \subsetneq L', v')$ be two dense pairs of models of T such that:

- $(K \subsetneq L, v) \subset (K' \subsetneq L', v')$
- *L and K' are linearly disjoint over K*
- $Kv \prec K'v'$
- $vK \prec v'K'$.

Then $(K \subset L, v) \prec (K' \subset L', v')$.

Proof. Follows from Theorems 3.7, 2.6 and Theorem 2.13. □

These results apply to any theory of algebraically maximal Kaplansky fields, to any theory of algebraically maximal finitely ramified fields with perfect residue field with fixed residue characteristic, and bounded ramification index.

Theorem 3.9

Let T be a theory of valued fields with residue-value extension property. Assume that for any two models (K, v), (K', v') of T, if $vK \equiv v'K'$ and $Kv \equiv K'v'$, then $(K, v) \equiv (K', v')$.

Let T' be the expansion of T to dense pairs $(K \subsetneq L, v)$. Assume that for every immediate pair $(M \subset N, v)$ of valued fields, if $N \models T$, then the maximal immediate algebraic extension of (M, v) is uniquely determined up to valuation preserving isomorphism. Then for any two models $(K \subset L, v)$, $(K' \subset L', v')$ of T', such that $vK \equiv v'K'$ and $Kv \equiv K'v'$, we have $(K \subset L, v) \equiv (K' \subset L', v')$.

Proof. The proof is the same as the proof of Theorem 3.6, by taking Theorem 3.7 instead of Theorems 3.1 and 3.2. □

3.3 Application to n-tuples and to some neither dense nor separated pairs

Proposition 3.10

Assume that either T is a theory of algebraically maximal Kaplansky fields of positive characteristic or the class of all valued fields which are between two models of T_1 has uniqueness of the maximal immediate algebraic extensions. Let T_{trip} be the theory of all 3-tuples $(K \subsetneq M \subset L, v)$ of valued fields with K, L and M models of T and such that:

- *$(K \subsetneq M, v)$ is dense*
- *$(M \subset L, v)$ is separated*
- *vL/vK is torsion-free.*

Then T_{trip} has the residue-value extension property.

Proof. Follows from Theorems 3.7, 3.1 and 3.2. □

Proposition 3.11

Assume that either T is a theory of algebraically maximal Kaplansky fields of positive characteristic or the class of all the valued fields which are between two models of T_1 has uniqueness of the maximal immediate algebraic extensions. Let $(K \subsetneq M \subset L, v) \subset (K' \subsetneq M' \subset L', v')$ be 3-tuples of models of T, let $\alpha_i, i \in \mathbb{N}$, be relations in M, and $\alpha'_i, i \in \mathbb{N}$, be relations in M'.

Assume that:

1) *$\alpha_i, \alpha'_i, i \in \mathbb{N}$ are liftings of residue relations*
2) *$(K \subset M, v)$ and $(K' \subset M', v')$ are dense*
3) *$(M \subset L, v)$ and $(M' \subset L', v')$ are separated*
4) *$(vK \subset vL) \prec (v'K' \subset v'L')$*
5) *$(Kv \subset Lv, \alpha_i/v, i \in \mathbb{N}) \prec (K'v' \subset L'v', \alpha'_i/v', i \in \mathbb{N})$*
6) *vL/vK is torsion-free*
7) *M and K' are linearly disjoint over K.*

Then $(K \subset M \subset L, v, \alpha_i, i \in \mathbb{N}) \prec (K' \subset M' \subset L', v', \alpha'_i, i \in \mathbb{N})$.

Proof. By Proposition 3.10, T_{trip} has the residue-value extension property. By Proposition 2.8, the theory T_1 obtained by adding the relations $\alpha_i, i \in \mathbb{N}$, also has the residue-value extension property. The elementary inclusion follows from Theorem 2.6. □

Proposition 3.12

Let T be either a theory of algebraically maximal Kaplansky fields of positive characteristic or a theory of unramified fields with perfect residue fields. Let $(K \subsetneq M \subset L, v)$ and $(K' \subsetneq M' \subset L', v')$ be 3-tuples of models of T, let $\alpha_i, i \in \mathbb{N}$, be relations in M, and $\alpha'_i, i \in \mathbb{N}$, be relations in M'.

Assume that:

1) $\alpha_i, \alpha'_i, i \in \mathbb{N}$ *are liftings of residue relations*
2) $(K \subset M, v)$ *and* $(K' \subset M', v')$ *are dense*
3) $(M \subset L, v)$ *and* $(M' \subset L', v')$ *are separated*
4) $(vK \subset vL) \equiv (v'K' \subset v'L')$
5) $(Kv \subset Lv, \alpha_i/v, i \in \mathbb{N}) \equiv (K'v' \subset L'v', \alpha'_i/v', i \in \mathbb{N})$
6) vL/vK *is torsion-free.*

Then $(K \subset M \subset L, v, \alpha_i, i \in \mathbb{N}) \equiv (K' \subset M' \subset L', v', \alpha'_i, i \in \mathbb{N})$.

Proof. Follows from Theorems 3.9, 3.6 and 2.7. □

Remark 3.13. Propositions 3.10, 3.11 and 3.12 remain valid in the case of separated n-tuples.

In [17], the dense closure was defined in the case of henselian fields of residue characteristic 0. But the proofs only need general properties of valuations. So we may generalize it, by means of the following proposition.

Proposition 3.14

Let $(K \subset L, v)$ *be a pair of algebraically maximal Kaplansky fields of positive characteristic such that* vK *is cofinal in* vL*. For* $l \in L$*, let* $I(K,l) = \{v(l-x) \mid x \in K, x \neq l\}$*, and let* K^* *be the definable set* $K^* = \{l \in L \mid I(K,l) = vK\}$*. Then:*

a) K^* *is a subfield of* L*,*
b) $(K \subset K^*, v)$ *is dense,*
c) *every subfield* L_1 *of* L*, such that* $(K \subset L_1, v)$ *is dense, is contained in* K^**,*
d) (K^*, v) *is an algebraically maximal Kaplansky field.*

Proof.

a), b), c): cf. [17], Theorem 3.3.4.

d) K^* is Kaplansky because $K^*v = Kv$ and $vK^* = vK$.

Let (x_α) be a pseudo-Cauchy sequence in K^* of algebraic type and without a pseudo-limit in K^*, with $P(X) = a_nX^n + \cdots + a_1X + a_0$ for a minimal polynomial. L is an algebraically maximal Kaplansky field, hence (x_α) admits a pseudo-limit l in L such that $P(l) = 0$. Since $(K \subset K^*, v)$ is dense, we have $I(K^*, l) = I(K, l)$. So we can assume that (x_α) is a pseudo-Cauchy sequence of K. Recall that the hypothesis "(x_α) without pseudo-limit in K" implies that (v_α) is cofinal in $I(K,l)$, (with $v_\alpha = v(x_\alpha - x_{\alpha+1})$). We can also assume that $v(l) \geq 0$ (this is possible by multiplying l by an element of K), and that $v(a_i) \geq 0$, $0 \leq i \leq n$.

If $I(K,l) = vK$, then by the definition of K^*, $l \in K^*$. So we may assume that $I(K,l)$ is not cofinal in vK.

We form a Taylor expansion for the polynomial P :

$$P(x) - P(x_\alpha) = (x - x_\alpha)P_{(1)}(x_\alpha) + \cdots + (x - x_\alpha)^n P_{(n)}(x_\alpha),$$

where $P_{(i)}$ is the i^{th} formal derivative of P. The sequence $(v(P_{(i)}(x_\alpha)))$ is eventually constant, let $vP_{(i)}$ be the asymptotic value. Then by [13], Lemma 8, there exists some p^h such that for all sufficiently large α

$$vP(x_\alpha) = vP_{(p^h)} + p^h v_\alpha < \min_{i \neq p^h} vP_{(i)} + iv_\alpha, \quad \text{and} \quad vP_{(i)}(l) = vP_{(i)}.$$

Let g' and g be elements of vK such that $g' > I(K,l)$ and $g > \max(vP_{(i)} + ig')$. Let $P_g(X) = a_{n,g}X^n + \cdots + a_{1,g}X + a_{0,g}$ be a polynomial in $K[X]$ such that, for $1 \leq i \leq n$, $v(a_i - a_{i,g}) \geq g$. The sequences $a_{i,g}x_\alpha^i$ are increasing, but, a priori, this doesn't imply that the sequence $vP_g(x_\alpha)$ is increasing. Now, by hypothesis, for $1 \leq i \leq n$, $v(P_{(i)} - P_{g,(i)}) \geq g$. Therefore, for all sufficiently large α and for all i, $1 \leq i \leq n$, $vP_{g,(i)} + iv_\alpha = vP_{(i)} + iv_\alpha$. So $vP_g(x_\alpha) = vP_{g,(p^h)} + p^h v_\alpha$. This shows that the sequence $vP_g(x_\alpha)$ is increasing, and (x_α) is of algebraic type in K. Since K is algebraically maximal, there exists a pseudo-limit $x_g \in K$ of (x_α). Then for all α, $v(l - x_g) > v(l - x_\alpha)$. Consequently $v(l - x_g) > I(K,l)$: a contradiction. Therefore, $I(K,l) = vK$, and $l \in K^*$. □

DEFINITION 3.15 ([17], Définition 3.3.4). Let $(K \subset L, v)$ be a pair of algebraically maximal fields with vK cofinal in vL. Then the set $K^* = \{l \in L \mid I(K,l) = vK\}$ will be called the *dense closure* of (K,v) in (L,v).

Before to prove Theorem 3.16, observe the following. Let (K,v) be an algebraically maximal valued field and w be a valuation on K, with w coarser than v (i.e. $R_v \subset R_w$). Then, it is routine to prove that (K,w) and $(Kw, v/w)$ are algebraically maximal.

In the same way, by means of general properties of valued fields, one can prove that if (K,v) is a valued field and u, w are valuations coarser than v, then:

If $p \cdot vK = vK$, then $p \cdot wK = wK$ and $p \cdot (v/w)(Kw) = (v/w)(Kw)$.

If any equation $X^{p^n} + a_{n-1}X^{p^{n-1}} + \cdots + a_1X^p + a_0X + b$ with coefficients in Kw has a root in Kw, then any equation $X^{p^n} + a_{n-1}X^{p^{n-1}} + \cdots + a_1X^p + a_0X + b$ with coefficients in Kv has a root in Kv.

Hence if (K,v) and (K,w) are Kaplansky fields and $w < u < v$ (i.e. $R_w \supset R_u \supset R_v$), then (K,u), $(Kw, u/w)$, $(Ku, v/u)$, and $(Kw, v/w)$ also are Kaplansky fields.

Theorem 3.16

Let $(K \subset L, v)$ and $(K' \subset L', v')$ be two pairs of algebraically maximal Kaplansky fields of characteristic $p > 0$ such that:

- *the valued fields $(K,0)$, $(L,0)$, $(K',0)$ and $(L',0)$ are Kaplansky fields, where 0 is the trivial valuation defined by: $\forall x,\ 0(x) = 0$,*
- *vL/vK is torsion-free,*
- *$(Kv \subset Lv) \equiv (K'v' \subset L'v')$.*

Assume that there exist valuations $v = v_0 > v_1 > \cdots > v_n > v_{n+1} = 0$ in $(K \subset L, v)$ (resp. $v' = v'_0 > v'_1 > \cdots > v'_n > v'_{n+1} = 0$ in $(K' \subset L', v')$) such that for all i, $2 \leq i \leq n+1$:

a) $(w_iKv_i \subset w_iLv_i) \equiv (w'_iK'v'_i \subset w'_iL'v'_i)$, where $w_i = v_{i-1}/v_i$ and $w'_i = v'_{i-1}/v'_i$,
b) either w_iKv_i is cofinal in w_iLv_i, or $w_iKv_i = 0$,
c) $((Kv_i)^*, w_i) = (Kv_i, w_i)$ iff $((K'v'_i)^*, w_i) = (K'v'_i, w'_i)$,
d) $((Kv_i)^* \subset Lv_i, w_i)$ and $((K'v'_i)^* \subset L'v'_i, w'_i)$ are separated.

Then $(K \subset L, v) \equiv (K' \subset L', v')$.

Proof. Observe that from hypothesis it follows that for or all i, j, $0 \leq i < j \leq n+1$, $(Kv_j, v_i/v_j)$, $(Lv_j, v_i/v_j)$, $(K'v'_j, v'_i/v'_j)$ and $(L'v'_j, v'_i/v'_j)$ are algebraically maximal Kaplansky fields. By Proposition 3.14, $((Kv_{i+1})^*, w_{i+1})$ and $((K'v'_{i+1})^*, w'_{i+1})$ are algebraically maximal Kaplansky fields. By Proposition 3.12

$$(Kv_1 \subset (Kv_1)^* \subset Lv_1, w_1) \equiv (K'v'_1 \subset (K'v_{1'})^* \subset L'v'_1, w'_1).$$

Now, assume that

$$(Kv_i \subset (Kv_i)^* \subset Lv_i, v_0/v_i, \cdots, w_i) \equiv (K'v'_i \subset (K'v'_i)^* \subset L'v'_i, v_0/v'_i, \cdots, w'_i).$$

By Proposition 3.12,

$$(Kv_{i+1} \subset (Kv_{i+1})^* \subset Lv_{i+1}, v_0/v_{i+1}, \cdots, w_{i+1}) \equiv$$
$$(K'v'_{i+1} \subset (K'v'_{i+1})^* \subset L'v'_i, v'_0/v'_{i+1}, \cdots, w'_{i+1}).$$

If $i = n$, we have in particular $(K \subset L, v) \equiv (K' \subset L', v')$. □

Corollary 3.17

Let $(K \subsetneq L, v)$ and $(K' \subsetneq L', v')$ be two immediate pairs of algebraically maximal Kaplansky fields of characteristic $p > 0$ such that:

- *the valued fields $(K, 0)$, $(L, 0)$, $(K', 0)$ and $(L', 0)$ are Kaplansky fields, where 0 is the trivial valuation defined by $\forall x,\ 0(x) = 0$,*
- *vK and $v'K'$ are p-divisible,*
- *$Kv \equiv K'v'$.*

Assume that there exist valuations $v = v_0 > v_1 > \cdots > v_n > v_{n+1} = 0$ (resp. $v' = v'_0 > v'_1 > \cdots > v'_n > v'_{n+1} = 0$ in $(K' \subset L', v')$) such that for all $i, 2 \leq i \leq n+1$:

a) $(w_i)(Kv_i) \equiv (w'_i)(K'v'_i)$ where $w_i = v_{i-1}/v_i$ and $w'_i = v'_{i-1}/v'_i$
b) $((Kv_i)^*, w_i) = (Kv_i, w_i)$ iff $((K'v'_i)^*, w'_i) = (K'v'_i, w'_i)$
c) $((Kv_i)^* \subset Lv_i, w_i)$ and $((K'v'_i)^* \subset L'v'_i, w'_i)$ are separated.

Then $(K \subset L, v) \equiv (K' \subset L', v')$.

4. Decidability

Let T be an enriched theory of valued fields. We will denote by $\mathcal{F}_T$ the set of all formulas $\bigwedge_{i=1}^{n}(A_i \vee B_i)$ of T ($n \in \mathbb{N}$), where:

- the A_i's are interpretations of residue formulas such that neither A_i nor $\neg A_i$ belong to $T_{RF}(T)$,
- the B_i's are interpretations of value formulas such that neither B_i nor $\neg B_i$ belong to $T_{VG}(T)$.

Note that, the hypothesis being as in Theorem 3.6 or 3.9, T' is complete if and only if $T_{RF}(T')$ and $T_{VG}(T')$ are both complete. If this holds, then $\mathcal{F}_{T'} = \emptyset$.

Proposition 4.1

Let T be as in Theorem 3.9, and let T' be the theory of all dense pairs $(K \subsetneq L, v)$ of models of T, in the language $\mathcal{L}(T') = \mathcal{L}(T) \cup \{E\}$, such that for every predicate of subfield F_i of $\mathcal{L}(T)$, and every $l \in L$, $F_i(l) \Rightarrow l \in K$. Assume that "$(K, v)$ model of T" can be expressed by an enumerable recursive scheme $\mathcal{E}_0$ of axioms. If $T_{RF}(T')$, $T_{VG}(T')$ and $\mathcal{F}_{T'}$ are decidable, then so is T'.

Proof. The sets $T_{RF}(T')$ and $T_{VG}(T')$ are interpretable in T', hence we can assume that $T_{RF}(T') \subset T'$ and $T_{VG}(T') \subset T'$. "$(K \subset L, v)$ dense" can be axiomatized by the axiom Ξ: $\forall l_1 \in L$, $\forall l_2 \in L$, $\exists x \in K$, $v(l_1 - x) > v(l_2)$.

1) Denote by $\mathcal{S}$ the closed under deduction theory generated by $\{\Xi\} \cup \mathcal{E}_0 \cup T_{RF}(T') \cup T_{VG}(T')$. Let A be a formula of the language $\mathcal{L}(T')$, such that $A \notin \mathcal{S}$ and $\neg A \notin \mathcal{S}$. Denote by $\mathcal{F}_A$ the set of all formulas $F = B_1 \wedge B_2$, with

$$B_1 \in \mathcal{L}_{RF}(T'), B_2 \in \mathcal{L}_{VG}(T'), B_1 \notin T_{RF}(T'),$$
$$\neg B_1 \notin T_{RF}(T'), B_2 \notin T_{VG}(T'), \neg B_2 \notin T_{VG}(T'),$$

and such that $\mathcal{S} \cup \{A\} \vdash F$.

Now, we prove that $\mathcal{S} \cup \mathcal{F}_A \vdash A$. Assume that this doesn't hold. Hence $\mathcal{S} \cup \mathcal{F}_A \cup \{\neg A\}$ is consistent. Let T'' be a complete theory containing $\mathcal{S} \cup \mathcal{F}_A \cup \{\neg A\}$. By Theorem 3.9, T'' is generated by

$$\{\Xi\} \cup \mathcal{E}_0 \cup T_{RF}(T'') \cup T_{VG}(T'') = \mathcal{S} \cup (T_{RF}(T'') \backslash T_{RF}(T')) \cup (T_{VG}(T'') \backslash T_{VG}(T')).$$

Therefore, there are formulas $B_1 \in (T_{RF}(T'') \backslash T_{RF}(T')) \cup \{\text{tautology}\}$, $B_2 \in (T_{VG}(T'') \backslash T_{VG}(T')) \cup \{\text{tautology}\}$ such that $\mathcal{S} \cup \{B_1 \wedge B_2\} \vdash \neg A$. Hence $\mathcal{S} \cup \{A\} \vdash (\neg B_1) \vee (\neg B_2)$. Now, $T' \subset T''$, hence $T_{RF}(T') \subset T_{RF}(T'')$ and $T_{VG}(T') \subset T_{VG}(T'')$. Consequently, $\neg B_1 \notin T_{RF}(T')$ and $\neg B_2 \notin T_{VG}(T')$. It follows: $(\neg B_1) \vee (\neg B_2) \in \mathcal{F}_A$. Hence $\mathcal{S} \cup \mathcal{F}_A \cup \{B_1 \wedge B_2\}$ is not consistent. Now, $\mathcal{S} \cup \mathcal{F}_A \cup \{B_1 \wedge B_2\} \subset T''$, hence T'' is not consistent: a contradiction.

2) From 1) it follows that T' is generated by $\{\Xi\} \cup \mathcal{E}_0 \cup T_{RF}(T') \cup T_{VG}(T') \cup \mathcal{F}_{T'}$.

In order to prove that T' is decidable, let A be a formula of $\mathcal{L}(T')$. We derive all the formulas deduced from $\{\Xi\} \cup \mathcal{E}_0 \cup T_{RF}(T') \cup T_{VG}(T') \cup \mathcal{F}_{T'}$, and all the formulas of $\mathcal{F}_A$ and of $\mathcal{F}_{\neg A}$. If A or $\neg A$ belongs to T', then we deduce it from $\{\Xi\} \cup \mathcal{E}_0 \cup T_{RF}(T') \cup T_{VG}(T') \cup \mathcal{F}_{T'}$. Otherwise, $\mathcal{F}_A$ and $\mathcal{F}_{\neg A}$ contain formulas that don't belong to $\mathcal{F}_{T'}$. This procedure is recursive.

Corollary 4.2

Let T_1 be either a theory of algebraically maximal Kaplansky fields or a theory of unramified fields with perfect residue fields. Let T be the expansion of T_1 to dense pairs $(K \subsetneq L, v)$. If $T_{RF}(T)$, $T_{VG}(T)$ and $\mathcal{F}_T$ are decidable, then so is T.

"$(K \subset L, v)$ separated" can be axiomatized by an enumerable recursive scheme of axioms which expresses that all the sets $v(1 - A)$ have a maximal element, where A is a finitely generated K-module. "vL/vK torsion-free" can also be axiomatized by an enumerable recursive scheme of axioms. It follows that, in the same way as above (by taking Theorem 3.6 and Proposition 3.12 respectively instead of Theorem 3.9), we can prove the following propositions.

Proposition 4.3

Let T be as in Theorem 3.6, and let T' be the theory of all separated pairs $(K \subsetneq L, v)$ of models of T, in the language $\mathcal{L}(T') = \mathcal{L}(T) \cup \{E\}$, such that for every predicate of subfield F_i of $\mathcal{L}(T)$, and every $l \in L$, $F_i(l) \Rightarrow l \in K$. Assume that "$(K, v)$ model of T" can be expressed by an enumerable recursive scheme of axioms. If $T_{RF}(T')$, $T_{VG}(T')$ and $\mathcal{F}_{T'}$ are decidable, then so is T'.

Corollary 4.4

Let T_1 be either an enriched theory of algebraically maximal Kaplansky fields of positive characteristic or an enriched theory of unramified fields with perfect residue fields. Let T be the expansion of T_1 to separated pairs $(K \subset L, v)$ such that either vL/vK is torsion-free, or $(K \subset L, v)$ contains a lifting $(G \subset H)$ of the pair $(vK \subset vL)$ with $G = K \cap H$. If $T_{RF}(T)$, $T_{VG}(T)$ and $\mathcal{F}_T$ are decidable, then so is T.

Proposition 4.5

Let T_1 be either an enriched theory of algebraically maximal Kaplansky fields of positive characteristic or an enriched theory of unramified fields with perfect residue fields. Let T be the theory of all 3-tuples $(K \subsetneq M \subset L, v)$ which are models of T_1 with $(K \subsetneq M, v)$ dense, $(M \subset L, v)$ separated and either vL/vK torsion-free, or any $(K \subset L, v)$ which is a model of T contains a lifting $(G \subset H)$ of the pair $(vK \subset vL)$ such that $G = K \cap H$. If $T_{RF}(T)$, $T_{VG}(T)$ and $\mathcal{F}_T$ are decidable, then so is T.

The same proof works in case of henselian fields of residue characteristic 0. The tools are Théorème 2.1, Théorème 1.5.1 and Proposition 3.4.1 of [17] respectively.

Proposition 4.6

Let T be a theory of dense pairs $(K \subsetneq L, v)$ of henselian fields of residue characteristic 0. Then the same conclusion as in Proposition 4.2 holds.

Proposition 4.7

Let T be an $\mathcal{L}_{VF}$-theory of separated pairs of henselian fields of residue characteristic 0. Assume that either for every pair $(K \subset L, v)$ which is a model of T, vL/vK is torsion-free, or every pair $(K \subset L, v)$ which is a model of T contains a lifting $(G \subset H)$ of the pair $(vK \subset vL)$ such that $G = K \cap H$. If $T_{RF}(T)$, $T_{VG}(T)$ and $\mathcal{F}_T$ are decidable, then so is T.

Proposition 4.8

Let T be a theory of 3-tuples $(K \subsetneq M \subset L, v)$ of henselian fields of residue characteristic 0 with $(K \subsetneq M, v)$ dense, $(M \subset L, v)$ separated. Assume either that for any 3-tuple $(K \subsetneq M \subset L, v)$ which is a model of T, vL/vK is torsion-free, or that any 3-tuple $(K \subsetneq M \subset L, v)$ which is a model of T contains a lifting $(G \subset H)$ of the pair $(vK \subset vL)$ such that $G = K \cap H$. If $T_{RF}(T)$, $T_{VG}(T)$ and $\mathcal{F}_T$ are decidable, then so is T.

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