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Mappings with dilatation in Orlicz spaces

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Abstract

We prove openness and discreteness for nonconstant mappings belonging to $W^{1,n}_{\text{loc}}(\Omega,\mathbb{R}^n)$, $n \geq 3$, with dilatation in certain Orlicz spaces which are strictly larger then all $L^p_{\text{loc}}(\Omega)$, $p > n-1$. This result contributes to decreasing the gap between known results and a conjecture of Iwaniec and Šverák.

1. Introduction

Let Ω be a connected open set in $\mathbb{R}^n$, $n \geq 2$, and $F : \Omega \to \mathbb{R}^n$ a mapping in the Sobolev space $W^{1,1}_{\text{loc}}(\Omega,\mathbb{R}^n)$. The mapping F is said to have *finite dilatation* if there exists a function K such that for almost all $x \in \Omega$, $1 \leq K(x) < \infty$ and

$$|DF(x)|^n \leq K(x) J_F(x),$$

where $|DF(x)| = \sup\{|DF(x)\xi| : |\xi| \leq 1\}$ is the operator norm of the differential $DF(x)$ of F at x and $J_F(x) = \det DF(x)$.

An important theorem due to Reshetnyak [9] states that if $F \in W^{1,n}_{\text{loc}}(\Omega,\mathbb{R}^n)$ is nonconstant and its dilatation K is bounded (i.e. F is quasiregular), then F is *continuous, open and discrete*, i.e. preimages and images of open sets are open and preimages of single points consist of isolated points. For more about quasiregular mappings, see e.g. Reshetnyak [10] and Rickman [11].

In 1993, Iwaniec and Šverák [4] conjectured that the conclusion of Reshetnyak's theorem is true whenever $F \in W^{1,n}_{\text{loc}}(\Omega,\mathbb{R}^n)$ and $K \in L^{n-1}_{\text{loc}}(\Omega)$, and proved it for $n = 2$. Their method does not directly generalize to higher dimensions. An example

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by Ball [1] shows that Reshetnyak's theorem fails if we only assume that $K \in L^p_{\mathrm{loc}}(\Omega)$ for all $p < n-1$.

By improving Reshetnyak's method, Heinonen and Koskela [2] verified the conjecture of Iwaniec and Šverák for $K \in L^p_{\mathrm{loc}}(\Omega)$, $p > n-1$, under the additional assumption that F is quasilight, i.e. that the preimage of each point is compact. The assumption of quasilightness was removed by Villamor and Manfredi [13].

Iwaniec-Koskela-Onninen [5] and Kauhanen-Koskela-Malý [6] weakened the assumption $F \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ while keeping $K \in L^p_{\mathrm{loc}}(\Omega)$, $p > n-1$. The borderline case $p = n-1$ was recently treated by Hencl and Malý [3] by a different method under the assumption of quasilightness.

In this paper, we combine the approach of Kauhanen, Koskela and Malý [6] with a refinement of Villamor and Manfredi's proof and verify the conjecture of Iwaniec and Šverák for a class of mappings with dilatation in Orlicz spaces, namely we prove the following theorem. (See the next section for the definitions of Young functions and Orlicz spaces.)

Theorem 1

Let Ψ be a doubling Young function such that $\Psi(t)/t^{n-1}$ is nondecreasing, $n \geq 3$, and

$$\int_1^\infty \left(\frac{t^{n-1}}{\Psi(t)} \right)^{1/n(n-2)} \frac{dt}{t} < \infty. \tag{1}$$

If $F \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a nonconstant mapping with dilatation belonging to the Orlicz space $L^\Psi_{\mathrm{loc}}(\Omega)$, such that

$$\lim_{\varepsilon \to 0+} \varepsilon \int_\Omega |DF(x)|^{n-\varepsilon}\, dx = 0, \tag{2}$$

then F is continuous, open and discrete.

Remark 2. For L^p spaces (i.e. $\Psi(t) = t^p$), the condition (1) is satisfied if and only if $p > n-1$. Other Young functions for which the theorem holds are e.g.

$$\Psi(t) = \begin{cases} t^{n-1}(\log t)^{n(n-2)+\varepsilon}, & t \geq e, \\ t^{n-1}, & 0 \leq t < e, \end{cases}$$

$$\Psi(t) = \begin{cases} t^{n-1}(\log t)^{n(n-2)}(\log\log t)^{n(n-2)+\varepsilon}, & t \geq e^e, \\ e^{n(n-2)}t^{n-1}, & 0 \leq t < e^e, \end{cases}$$

and other repeated logarithms with $\varepsilon > 0$. If $\varepsilon = 0$ in the above expressions, then the condition (1) just fails.

2. Young functions and auxiliary results

DEFINITION 3. A positive continuous convex function Ψ on $(0,\infty)$ satisfying

$$\lim_{t\to 0+} \frac{\Psi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Psi(t)}{t} = \infty \tag{3}$$

is called a *Young function*. If $\Psi(2t) \le C\Psi(t)$ for some constant C and all $t \in (0,\infty)$, then Ψ is said to be *doubling* (or satisfying the Δ_2-condition).

DEFINITION 4. The *Orlicz space* $L^{\Psi}(\Omega)$ is the set of all measurable functions with the Luxemburg norm

$$\|f\|_{L^{\Psi}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} \Psi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty,$$

where we interpret $\Psi(0) = 0$.

The following generalized Hölder inequality for Orlicz spaces is proved in the paper by O'Neil [7, Theorem 2.3].

Theorem 5

Let Ψ_1, Ψ_2 and Ψ_3 be Young functions such that

$$\Psi_1^{-1}(s)\Psi_2^{-1}(s) \le \Psi_3^{-1}(s).$$

If $f \in L^{\Psi_1}(\Omega)$ and $g \in L^{\Psi_2}(\Omega)$, then $fg \in L^{\Psi_3}(\Omega)$ and

$$\|fg\|_{L^{\Psi_3}(\Omega)} \le 2\|f\|_{L^{\Psi_1}(\Omega)}\|g\|_{L^{\Psi_2}(\Omega)}.$$

The following lemma shows that doubling Young functions can be assumed to be as smooth as needed. Note that for a Young function Ψ, the monotonicity of the function $t \mapsto \Psi(t)/t$ is a direct consequence of the convexity of Ψ.

Lemma 6

Let $\Psi \in C^k(0,\infty)$, $k \ge 0$, be a positive doubling function satisfying (3)*, such that the function $t \mapsto \Psi(t)/t$ is nondecreasing. Let*

$$\Psi_1(t) = \int_0^t \frac{\Psi(s)}{s}\, ds.$$

Then Ψ_1 is a doubling Young function, $\Psi_1 \in C^{k+1}(0,\infty)$ and $\Psi(t)/C \le \Psi_1(t) \le \Psi(t)$ for all $t \in (0,\infty)$. Moreover, $L^{\Psi}(\Omega) = L^{\Psi_1}(\Omega)$ and for all $f \in L^{\Psi}(\Omega)$, $\|f\|_{L^{\Psi_1}(\Omega)} \le \|f\|_{L^{\Psi}(\Omega)} \le 2\|f\|_{L^{\Psi_1}(\Omega)}$.

Proof. The fact that $\Psi_1 \in C^{k+1}(0,\infty)$ follows directly from the definition of Ψ_1. The convexity of Ψ_1 is a direct consequence of the fact that $\Psi(t)/t$ is nondecreasing. As for the doubling condition, we have

$$\Psi_1(2t) = \int_0^t \frac{\Psi(2s)}{s}\,ds \le C\int_0^t \frac{\Psi(s)}{s}\,ds = C\Psi_1(t).$$

To prove (3), let $\varepsilon, k > 0$ be arbitrary and find $t_1, t_2 > 0$ so that $\Psi(s)/s < \varepsilon$ for all $0 < s \le t_1$ and $\Psi(s)/s > k$ for all $s \ge t_2$. Then

$$\frac{\Psi_1(t)}{t} = \frac{1}{t}\int_0^t \frac{\Psi(s)}{s}\,ds \le \varepsilon \qquad \text{for } 0 < t \le t_1,$$
$$\frac{\Psi_1(t)}{t} \ge \frac{1}{t}\int_{t/2}^t \frac{\Psi(s)}{s}\,ds \ge \frac{k}{2} \qquad \text{for } t \ge 2t_2.$$

Next, the fact that $\Psi(t)/t$ is nondecreasing and the doubling property of Ψ_1 give

$$\Psi_1(t) \ge \int_{t/2}^t \frac{\Psi(s)}{s}\,ds \ge \frac{t}{2}\,\frac{\Psi\left(\frac{1}{2}t\right)}{\frac{1}{2}t} = \Psi\Big(\frac{1}{2}t\Big) \ge \frac{\Psi(t)}{C}, \tag{4}$$

$$\Psi_1(t) = \int_0^t \frac{\Psi(s)}{s}\,ds \le t\frac{\Psi(t)}{t} = \Psi(t). \tag{5}$$

Finally, if $\lambda > \|f\|_{L^{\Psi_1}(\Omega)}$, then (4) implies

$$1 \ge \int_\Omega \Psi_1\left(\frac{f(x)}{\lambda}\right)dx \ge \int_\Omega \Psi\left(\frac{f(x)}{2\lambda}\right)dx$$

and hence $2\lambda \ge \|f\|_{L^\Psi(\Omega)}$. Taking infimum over all possible λ shows that $\|f\|_{L^\Psi(\Omega)} \le 2\|f\|_{L^{\Psi_1}(\Omega)}$. Similarly, the inequality $\|f\|_{L^{\Psi_1}(\Omega)} \le \|f\|_{L^\Psi(\Omega)}$ follows from (5). $\square$

Lemma 7

Let Ψ be a continuous doubling function on $(0,\infty)$, such that $\Psi(t)/t^p$ is nondecreasing. Let $0 < \alpha < 1 - 1/p$ and define Ψ_1 by $\Psi_1^{-1}(s) = s^\alpha \Psi^{-1}(s)$. Then Ψ_1 is a continuous doubling function on $(0,\infty)$ satisfying (3). Moreover, the function $t \mapsto \Psi_1(t)/t$ is increasing.

Proof. The continuity is clear. A simple calculation shows that the doubling condition for Ψ is equivalent to $2\Psi^{-1}(s) \le \Psi^{-1}(Cs)$ for some $C > 1$ and all $s \in (0,\infty)$. We then have

$$2\Psi_1^{-1}(s) = 2s^\alpha\Psi^{-1}(s) \le s^\alpha\Psi^{-1}(Cs) \le \Psi_1^{-1}(Cs),$$

i.e. Ψ_1 is doubling. To prove (3), note that

$$\frac{\Psi_1(t)}{t} = \frac{s}{\Psi_1^{-1}(s)} = \frac{s^{1-\alpha}}{\Psi^{-1}(s)} = \left(\frac{\Psi(u)}{u^p}\right)^{1-\alpha} u^{(1-\alpha)p-1}, \tag{6}$$

where $s = \Psi_1(t)$ and $u = \Psi^{-1}(\Psi_1(t)) \to 0+$, as $t \to 0+$. Hence,

$$\lim_{t\to 0+}\frac{\Psi_1(t)}{t} = \lim_{u\to 0+}\left(\frac{\Psi(u)}{u^p}\right)^{1-\alpha} u^{(1-\alpha)p-1} = 0,$$

and similarly for $t \to \infty$. Finally, (6) also shows that $\Psi_1(t)/t$ is increasing. $\square$

We shall also need the following Sobolev type inequality. For the readers convenience, we repeat the short proof.

Lemma 8

Let B and $B_0 \subset B \subset \mathbb{R}^n$ be balls, $n \geq 3$, $p > 1$ and $1 < q < np/(n-p)$. Then there exists $C > 0$ such that for all $u \in W^{1,p}(B)$,

$$\|u\|_{L^q(B)} \leq C\big(\|\nabla u\|_{L^p(B)} + \|u\|_{L^p(B_0)}\big).$$

Proof. Assume that this is not true. Then there exist $u_k \in W^{1,p}(B)$ so that $\|u_k\|_{L^q(B)} = 1$ and $\|\nabla u_k\|_{L^p(B)} + \|u_k\|_{L^p(B_0)} \leq 1/k$. By the local weak sequentional compactness of $W^{1,p}(B)$, we can find a subsequence, also denoted u_k, so that $u_k \to u_0$ weakly in $W^{1,p}(B)$. The weak lower semicontinuity of the L^p-norm implies $\|\nabla u_0\|_{L^p(B)} = \|u_0\|_{L^p(B_0)} = 0$ and hence $u_0 = 0$. On the other hand, since the embedding $W^{1,p}(B) \subset L^q(B)$ is compact (by e.g. Theorem 2.5.1 in Ziemer [14]), a subsequence of $\{u_k\}_{k=1}^{\infty}$ converges to u_0 in $L^q(B)$ and $\|u_0\|_{L^q(B)} = 1$. This contradicts $u_0 = 0$. □

3. Proof of Theorem 1

Definition 9. Let Ψ be a Young function and $K \subset B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ be compact. The Ψ-*capacity* of K with respect to the ball $B(x_0, 2R)$ is

$$\operatorname{cap}_\Psi(K, B(x_0, 2R)) = \inf \|\nabla\phi\|_{L^\Psi(B(x_0,2R))},$$

where the infimum is taken over all continuous functions $\phi \in W^{1,1}(B(x_0, 2R))$ with compact support in $B(x_0, 2R)$ and $\phi \geq 1$ on K.

The 1-dimensional Hausdorff measure is denoted H^1. The following theorem is due to Onninen [8].

Theorem 10

Let Ψ be a doubling Young function satisfying $\Psi(t) \geq Ct^{n-1}$ and

$$\int_1^\infty \left(\frac{t}{\Psi(t)}\right)^{1/(n-2)} dt < \infty.$$

If $K \subset B(x_0, R) \subset \mathbb{R}^n$, $n \geq 3$, is compact, then

$$H^1(K) \leq C \operatorname{cap}_\Psi(K, B(x_0, 2R)). \tag{7}$$

Remark 11. In the definition of Ψ-capacity in Onninen [8], only Lipschitz continuous functions are considered. However, the Lipschitz continuity is not used anywhere in the proof. In fact, one only needs that all $x \in K$ are Lebesgue points of ϕ in order to obtain the fundamental inequality

$$1 \leq C \sum_{j=0}^{\infty} \frac{R}{2^j} \fint_{B(x,10R/2^j)} |\nabla\phi(y)|\, dy,$$

whose integration with respect to the Frostman measure leads to (7).

Proof of Theorem 1. The proof is based on the ideas of Manfredi and Villamor [13]. Note first, that by Theorem 1.3 in Iwaniec–Koskela–Onninen [5] and Theorem 1.5 in Kauhanen-Koskela-Malý [6], F is continuous and sense preserving. The openness and discreteness of F then follows from the Titus-Young theorem (Theorem A in [12]) if we can show that $H^1(F^{-1}(b)) = 0$ for every $b \in \mathbb{R}^n$ (so that $F^{-1}(b)$ is totally disconnected).

Exhausting Ω by countably many compacts and covering each of them by finitely many small balls, we can write Ω as a countable union of balls $B_j = B(x_j, r_j)$ so that $\overline{B(x_j, 2r_j)} \subset \Omega$ and $F(B_j) \subset B\big(b_j, \frac{1}{2}e^{-e}\big)$, $b_j \in \mathbb{R}^n$. By the σ-subadditivity of the Hausdorff measure, it suffices to show that $H^1(B_j \cap F^{-1}(b)) = 0$.

Let $B = B(x_0, r)$ be one of the balls B_j. Replacing F by $F - b$, we can assume that $b = 0 \in F(B)$ and $F(B) \subset B(0, e^{-e})$. In [13], Villamor and Manfredi constructed a family of C^2-smooth radially symmetric n-superharmonic functions Φ_a on $B(0, e^{-e})$, $0 < a < e^{-e}$, with (among others) the following properties,

$$\log(1/a) \le \Phi_a(y) \le \log(1/a) + \frac{1}{2} + \log 2 \qquad \text{for } |y| \le a, \tag{i}$$

$$\Phi_a(y) = \log(1/|y|) \qquad \text{for } a \le |y| \le e^{-e}, \tag{ii}$$

$$\Phi_a(y) \ge e \qquad \text{for } |y| \le e^{-e}. \tag{iii}$$

Then, under the assumption $F \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, they prove that for every nonnegative function $\eta \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla(\log \Phi_a \circ F)(x)|^n \frac{\eta(x)^n}{K(x)}\, dx \le C \int_\Omega K(x)^{n-1} |\nabla \eta(x)|^n\, dx, \tag{8}$$

where C is independent of a, see (4.1) in [13]. Under the assumptions $F \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and (2) considered here, the estimate (8) is proved in Kauhanen-Koskela-Malý [6], formulas (3.1) and (3.2). In the rest of the proof, all constants C will be independent of a, they may, however, depend on F, B and other fixed data.

Let K be a compact subset of $F^{-1}(0) \cap B$ and choose a nonnegative function $\eta \in C_0^\infty(B)$ so that $\eta \ge 1$ on K. Following [13], we consider the functions

$$V_a(x) = \frac{\eta(x)(\log \Phi_a \circ F)(x)}{\log\log(1/a)}$$

which are continuous by (iii) and satisfy $V_a \ge 1$ on K by (i). Moreover, as $F \in W^{1,n-1}_{\mathrm{loc}}(\Omega)$ (by (2)) and $\log \Phi_a$ are Lipschitz (using the C^2-smoothness of Φ_a and (iii)), we have $V_a \in W^{1,n-1}_0(B)$ and

$$\nabla V_a(x) = \frac{\nabla(\log \Phi_a \circ F)(x)\eta(x) + (\log \Phi_a \circ F)(x)\nabla \eta(x)}{\log\log(1/a)}.$$

Let $\Psi_1(t) = \Psi(t^n)$. It is easily verified that Ψ_1 is also a doubling Young function and that $K^{1/n} \in L^{\Psi_1}(B)$. Moreover, the function $\Psi_1(t)/t^{n(n-1)}$ is nondecreasing. Let Ψ_2 be defined by $\Psi_2^{-1}(s) = s^{1/n}\Psi_1^{-1}(s)$ and let

$$\Psi_3(t) = \int_0^t \frac{\Psi_2(s)}{s}\, ds.$$

As $n \geq 3$, Lemmas 6 and 7 imply that Ψ_3 is a doubling Young function. Let us compute the $L^{\Psi_3}(B)$-norm of ∇V_a. We have

$$\|\nabla V_a\|_{L^{\Psi_3}(B)} \leq \frac{\big\||\nabla(\log\Phi_a \circ F)|\eta\big\|_{L^{\Psi_3}(B)} + \big\|(\log\Phi_a \circ F)|\nabla\eta|\big\|_{L^{\Psi_3}(B)}}{\log\log(1/a)}. \tag{9}$$

As $\Psi_3(t) \leq \Psi_2(t)$ by Lemma 6, we have $s^{1/n}\Psi_1^{-1}(s) \leq \Psi_3^{-1}(s)$ and an application of the generalized Hölder inequality (Theorem 5) yields

$$\big\||\nabla(\log\Phi_a \circ F)|\eta\big\|_{L^{\Psi_3}(B)} \leq 2\big\||\nabla(\log\Phi_a \circ F)|K^{-1/n}\eta\big\|_{L^n(B)}\,\|K^{1/n}\|_{L^{\Psi_1}(B)}, \tag{10}$$

where $\|K^{1/n}\|_{L^{\Psi_1}(B)} < \infty$ by the assumption. By (8), we have

$$\big\||\nabla(\log\Phi_a \circ F)|K^{-1/n}\eta\big\|_{L^n(B)} \leq C\Big(\int_B K(x)^{n-1}|\nabla\eta(x)|^n\,dx\Big)^{1/n}, \tag{11}$$

and hence the first norm on the right-hand side in (9) is bounded from above by a constant independent of a. It remains to estimate the second term on the right-hand side in (9). Another application of the generalized Hölder inequality (Theorem 5) shows that

$$\begin{aligned}\big\|(\log\Phi_a \circ F)|\nabla\eta|\big\|_{L^{\Psi_3}(B)} &\leq C\|\log\Phi_a \circ F\|_{L^{\Psi_3}(B)}\\ &\leq 2C\|\log\Phi_a \circ F\|_{L^n(B)}\|1\|_{L^{\Psi_1}(B)},\end{aligned} \tag{12}$$

where $\|1\|_{L^{\Psi_1}(B)} < \infty$. The set $B' = \{x \in B : F(x) \neq 0\}$ is open and hence there exist $c > 0$ and a ball $B_0 \subset B'$ so that $|F(x)| > c$ for all $x \in B_0$. In particular, for $a \leq c$ and $x \in B_0$, we have $|\log\Phi_a \circ F(x)| < \log\log(1/c)$ by (ii). As $n \geq 3$, we have $n < np/(n-p)$ with $p = n-1$ and Lemma 8 implies

$$\|\log\Phi_a \circ F\|_{L^n(B)} \leq C\big(\|\nabla(\log\Phi_a \circ F)\|_{L^{n-1}(B)} + \|\log\Phi_a \circ F\|_{L^{n-1}(B_0)}\big). \tag{13}$$

The second term on the right-hand side does not exceed $\log\log(1/c)|B_0|^{1/(n-1)}$ and the first term is estimated as in Lemma 4 from [13]. More precisely, let $\tilde\eta \in C_0^\infty(B(x_0,2r))$ be a nonnegative function such that $\tilde\eta = 1$ on B and $|\nabla\tilde\eta| \leq 2/r$. Then by the Hölder inequality and (8) we have

$$\begin{aligned}&\|\nabla(\log\Phi_a \circ F)\|_{L^{n-1}(B)}\\ &\quad\leq \Big(\int_{B(x,2r)} |\nabla(\log\Phi_a \circ F)(x)|^n\frac{\tilde\eta(x)^n}{K(x)}\,dx\Big)^{1/n}\|K\|^{1/n}_{L^{n-1}(B(x,2r))}\\ &\quad\leq \frac{C}{r}\|K\|_{L^{n-1}(B(x,2r))}.\end{aligned} \tag{14}$$

Putting together (9)–(14) gives

$$\|\nabla V_a\|_{L^{\Psi_3}(B)} \leq \frac{C}{\log\log(1/a)}$$

and letting $a \to 0+$ yields $\operatorname{cap}_{\Psi_3}(K, B(x_0, 2r)) = 0$ for every compact subset K of $F^{-1}(0) \cap B(x_0, r)$. Theorem 10 then shows that $H^1(K) = 0$, provided that

$$\int_1^\infty \left(\frac{t}{\Psi_3(t)}\right)^{1/(n-2)} dt < \infty.$$

As $\Psi_2(t) \le C\Psi_3(t)$ by Lemma 6, this follows from the following lemma and finishes the proof of the theorem. □

Lemma 12

Let Ψ be a Young function satisfying (1) *such that $\Psi(t)/t^{n-1}$ is nondecreasing. Let Ψ_1 and Ψ_2 be as in the proof of Theorem* 1. *Then*

$$\int_1^\infty \left(\frac{t}{\Psi_2(t)}\right)^{1/(n-2)} dt < \infty. \tag{15}$$

Proof. By Lemma 6, we can assume that $\Psi \in C^1(0, \infty)$ and hence also $\Psi_1, \Psi_2 \in C^1(0, \infty)$. We have, using integration by parts,

$$\int_1^R \left(\frac{t}{\Psi_2(t)}\right)^{1/(n-2)} dt = \left[\frac{n-2}{n-1}\left(\frac{t^{n-1}}{\Psi_2(t)}\right)^{1/(n-2)}\right]_1^R + \frac{1}{n-1}\int_1^R \left(\frac{t}{\Psi_2(t)}\right)^{(n-1)/(n-2)} \Psi_2'(t)\, dt. \tag{16}$$

Here, the definitions of Ψ_1 and Ψ_2 imply

$$\Psi_2^{-1}(s) = \bigl(s\Psi^{-1}(s)\bigr)^{1/n} \tag{17}$$

and hence

$$\frac{t^{n-1}}{\Psi_2(t)} = \frac{\Psi_2^{-1}(s)^{n-1}}{s} = \frac{\Psi^{-1}(s)^{1-1/n}}{s^{1/n}} = \left(\frac{u^{n-1}}{\Psi(u)}\right)^{1/n},$$

where $s = \Psi_2(t)$ and $u = \Psi^{-1}(\Psi_2(t)) \to \infty$, as $t \to \infty$. As $\Psi(t)/t^{n-1}$ is nondecreasing, we see that the first term on the right-hand side in (16) remains bounded as $R \to \infty$.

Next, let again $u = \Psi^{-1}(\Psi_2(t))$, i.e. $\Psi_2'(t)\, dt = \Psi'(u)\, du$. The formula (17) implies $u = t^n/\Psi_2(t)$ and hence the change of variables $u = \Psi^{-1}(\Psi_2(t))$ yields that the integral on the right-hand side in (16) is equal to

$$\int_{\Psi^{-1}(\Psi_2(1))}^{\Psi^{-1}(\Psi_2(R))} \left(\frac{u}{\Psi(u)^{n-1}}\right)^{(n-1)/n(n-2)} \Psi'(u)\, du.$$

Finally, another integration by parts shows that the last integral is equal to

$$\left[-n(n-2)\left(\frac{u^{n-1}}{\Psi(u)}\right)^{1/n(n-2)}\right]_{\Psi^{-1}(\Psi_2(1))}^{\Psi^{-1}(\Psi_2(R))} + (n-1)\int_{\Psi^{-1}(\Psi_2(1))}^{\Psi^{-1}(\Psi_2(R))} \left(\frac{u^{n-1}}{\Psi(u)}\right)^{1/n(n-2)} \frac{du}{u}.$$

As before, the first term remains bounded as $R \to \infty$ and can be disregarded and the second term remains bounded as $R \to \infty$, by (1). □

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References

1. J. Ball, Global invertibility of Sobolev functions and the interpenetration of matter, *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 315–328.
2. J. Heinonen and P. Koskela, Sobolev mappings with integrable dilatations, *Arch. Rational Mech. Anal.* **125** (1993), 81–97.
3. S. Hencl and J. Malý, Mappings of dinite distortion: Hausdorff measure of zero sets, Preprint 2001.
4. T. Iwaniec and V. Šverák, On mappings with integrable dilatation, *Proc. Amer. Math. Soc.* **118** (1993), 181–188.
5. T. Iwaniec, P. Koskela, and J. Onninen, Mappings of finite distortion: monotonicity and continuity, *Invent. Math.* **144** (2001), 507–531.
6. J. Kauhanen, P. Koskela, and J. Malý, Mappings of finite distortion: discreteness and openness, *Arch. Ration. Mech. Anal.* **160** (2001), 135–151.
7. R. O'Neil, Fractional integration in Orlicz spaces, I, *Trans. Amer. Math. Soc.* **115** (1965), 300–328.
8. J. Onninen, Orlicz capacities and Hausdorff measures in metric spaces, (in preparation).
9. Yu.G. Reshetnyak, Space mappings with bounded distortion, *Sibirsk. Mat. Zh.* **8** (1967), 629–658 (in Russian), *Siberian Math. J.* **3** (1967), 466–486 (English transl.).
10. Yu.G. Reshetnyak, *Spatial mappings with bounded distortion*, Nauka, Moscow, 1982 (in Russian), Amer. Math. Soc., Providence, RI, 1989 (English transl.).
11. S. Rickman, *Quasiregular mappings,* Springer-Verlag, Berlin, 1993.
12. C.J. Titus and G.S. Young, The extension of interiority, with some applications, *Trans. Amer. Math. Soc.* **103** (1962), 329–340.
13. E. Villamor and J.J. Manfredi, An extension of Reshetnyak's theorem, *Indiana Univ. Math. J.* **47** (1998), 1131–1145.
14. W.P. Ziemer, *Weakly differentiable functions, Sobolev spaces and functions of bounded variation,* Springer-Verlag, New York, 1989.