Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **53**, 3 (2002), 287–302

Integral representation of functions on sectors, functional calculus and norm estimates

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Received October 17, 2001. Revised April 23, 2002

Abstract

We find an explicit integral representation for bounded holomorphic functions $f(z)$ on sectors $|\mathrm{Arg}(z)| < \psi$ in terms of the kernel $z(z+\lambda)^{-2}$ and present some applications to operator theory. Namely, given a sectorial operator A we define the functional calculus $A \to f(A)$ and find pointwise estimates and moment type inequalities for $\|f(A)x\|$. We show that sectorial operators have a bounded H^∞-functional calculus on a dense subspace. We also find exact estimates for the norm $\|e^{-\lambda A}\|$ of analytic semigroups.

1. Introduction

The classical Riesz-Dunford functional calculus for unbounded closed operators A is based on the formula (see [10])

$$f(A) = \frac{1}{2\pi} \int_{\partial G} \frac{f(\lambda)}{\lambda - A} d\lambda \,, \tag{1.1}$$

where $f(z)$ is an appropriate holomorphic function on a domain G and $\mathrm{Sp}(A) \subset G$. When f is bounded and we want to find a necessary and sufficient condition on A under which

$$\|f(A)\| \le C\|f\|_\infty, \tag{1.2}$$

the representation (1.1) is not very convenient. Work with sectorial operators [6], [7] has shown that a convenient necessary and sufficient condition can be given in terms of the expression $A(A+\lambda)^{-2}$ which corresponds to the function $z \to z(z+\lambda)^{-2}$.

Keywords: Sectorial operator, moment inequality.

MSC2000: Primary: 47A60, Secondary: 47B44, 47D06, 30E20.

We find here an explicit integral representation for bounded holomorphic functions $f(z)$ on sectors $|\mathrm{Arg}(z)| < \psi$ by means of the kernel $z(z+\lambda)^{-2}$ and the boundary values of $f(z)$ (see (2.2)) and present some applications to operator theory. Using the operator kernel $A(A+\lambda)^{-2}$ one can define a bounded H^∞-functional calculus $A \to f(A)$ for sectorial operators A of type $\theta < \psi$ and find sharp estimates of the form (1.2). Also, we prove the moment type pointwise estimates:

$$\|f(A)x\| \leq K(\psi, \alpha, \beta) \, \|A^\alpha x\|^{\beta/(\alpha+\beta)} \, \|A^{-\beta} x\|^{\alpha/(\alpha+\beta)} \, \|f\|_\infty$$

for every $x \in D(A^\alpha) \cap D(A^{-\beta}), 0 < \alpha, \beta \leq 1$ (see (4.1)). In particular, this shows that the calculus exists at least on the dense subspace $D(A) \cap D(A^{-1})$.

It is known that sectorial operators A of type $\theta < \pi/2$ generate holomorphic semigroups $e^{-\lambda A}$. We prove an inequality of the form

$$\|f(A)\| \leq C\|f'\|_\infty$$

which provides an exact estimate for the norm $\|e^{-\lambda A}\|$ (see 5.3).

Holomorphic functions on sectors can be represented in terms of different kernels. For instance, the representation (7.2) below is based on a Poisson-like kernel and was used in the functional calculus in [4], [5]. This kernel, however, contains fractional powers which is inconvenient for computations and for the estimates we need. Because of its pure resolvent form, the kernel $A(A+\lambda)^{-2}$ seems to be the natural choice for constructing a functional calculus based on resolvent properties.

Bounded H^∞-functional calculus for Hilbert space operators of type θ was constructed by McIntosh [16], [18]. An alternative approach was presented in [4], [17]. The Banach space case was first addressed in [6]. Further developments can be found in [1], [2], [4], [5], [7], [8], [9], [11], [14], [15], [23].

The paper is organized as follows: Section 2 presents the integral representation and its proof is given in Section 7. In Section 3 we describe the functional calculus. Section 4 is dedicated to the mentioned above moment inequality and in Section 5 one finds the norm estimate for bounded holomorphic semigroups. Section 6 deals with a functional calculus for two resolvent commuting operators.

2. The integral representation

Throughout, given $0 < \psi \leq \pi$ we define the angular sector

$$S_\psi = \{z \in \mathbb{C} \colon z \neq 0, \ |\mathrm{Arg}(z)| < \psi\}$$

with closure $\overline{S}_\psi$. Let $H^\infty(S_\psi)$ be the Banach algebra of all bounded holomorphic functions on S_ψ with the "sup" norm $\|f\|_\infty$.

We use also the standard convolution for two functions on the line,

$$(f * g)(r) = \int_{-\infty}^{+\infty} f(u) g(r-u) du \, .$$

Theorem 2.1

Suppose $f(z)$ is a bounded holomorphic function on the sector $S_\psi, 0 < \psi \le \pi$. Then $f(z)$ has the representation

$$f(z) = \int_{\mathbb{R}} \tilde{W}_\psi(z,u)\big(f_\psi * k_\psi\big)(u)du \tag{2.1}$$

where

$$\tilde{W}_\psi(z,u) = \frac{1}{2}\left(\frac{ze^{u+i(\pi-\psi)}}{(z+e^{u+i(\pi-\psi)})^2} + \frac{ze^{u-i(\pi-\psi)}}{(z+e^{u-i(\pi-\psi)})^2}\right).$$

Also, here

$$f_\psi(r) = \frac{1}{2}\big(f(e^{r+i\psi}) + f(e^{r-i\psi})\big)$$

is the boundary value of the function on the sides of the sector and

$$k_\psi(r) = \frac{1}{\pi^2}\ln\left|\coth\Big(\frac{\pi r}{4\psi}\Big)\coth\Big(\frac{\pi r}{4(\pi-\psi)}\Big)\right| \qquad (\psi < \pi)$$

$$k_\pi(r) = \frac{1}{\pi^2}\ln\left|\coth\Big(\frac{r}{4}\Big)\right| \qquad (\psi = \pi)$$

is an even function on the real line with

$$k_\psi(r) \ge 0, \|k_\psi\|_1 = \int_{\mathbb{R}} k_\psi(r)dr = 1\,.$$

The proof is given in Section 7. Note that

$$\|f_\psi * k_\psi\|_\infty \le \|f_\psi\|_\infty \|k_\psi\|_1 = \|f\|_\infty\,.$$

The substitution $e^u = t$ brings (2.1) to a form more convenient for some applications

$$f(z) = \int_0^{+\infty} W_\psi(z,t)\big(f_\psi * k_\psi\big)(\ln t)dt \tag{2.2}$$

or

$$f(z) = \int_0^{+\infty} W_\psi(z,t)F_\psi(t)dt$$

with

$$W_\psi(z,t) = \frac{1}{2}\left(\frac{ze^{i(\pi-\psi)}}{(z+te^{i(\pi-\psi)})^2} + \frac{ze^{-i(\pi-\psi)}}{(z+te^{-i(\pi-\psi)})^2}\right),$$

and

$$\|F_\psi\|_\infty \le \|f\|_\infty\,.$$

Here

$$F_\psi(t) = (f_\psi * k_\psi)(\ln t) = \int_0^\infty \frac{1}{2}\big(f(xe^{i\psi}) + f(xe^{-i\psi})\big)k_\psi\Big(\ln\frac{t}{x}\Big)\frac{dx}{x}\,.$$

One can also write

$$W_\psi(z,t) = \frac{-z}{2}\left(\frac{e^{-i\psi}}{(z-te^{-i\psi})^2} + \frac{e^{i\psi}}{(z-te^{\psi})^2}\right),$$

and in particular,

$$\begin{aligned} W_\pi(z,t) &= \frac{z}{(z+t)^2}\,, \\ W_{\pi/2}(z,t) &= \frac{2z^2t}{(z^2+t^2)^2}\,. \end{aligned} \tag{2.3}$$

EXAMPLE 2.2: If $f(z)$ is a bounded holomorphic function on $S_\pi = \mathbb{C}\backslash(-\infty, 0]$, then

$$f(z) = \int_0^{+\infty} \frac{z}{(z+t)^2}(f_\pi * k_\pi)(\ln t)dt\,.$$

When $\psi = \pi/2$ and $f \in H^\infty(S_{\pi/2})$:

$$f(z) = \int_0^{+\infty} \frac{2z^2t}{(z^2+t^2)^2} F_{\pi/2}(t)dt\,,$$

where

$$F_{\pi/2}(t) = \big(f_{\pi/2} * k_{\pi/2}\big)(\ln t) = \frac{1}{\pi^2}\int_0^\infty \big(f(ix) + f(-ix)\big)\ln\left|\frac{t+x}{t-x}\right|\frac{dx}{x}\,.$$

This follows from

$$k_{\pi/2}(r) = \frac{2}{\pi^2}\ln\left|\coth\!\left(\frac{r}{2}\right)\right|\,,\; k_{\pi/2}\left(\ln\left(\frac{t}{x}\right)\right) = \frac{1}{\pi^2}\ln\left|\frac{t+x}{t-x}\right|\,.$$

Lemma 2.3

Suppose $0 < \psi \le \pi$ and $z = re^{i\theta} \in S_\psi$, i.e $|\theta| < \psi$. Then

$$w(\psi,\theta) \equiv \int_{\mathbb{R}} \big|\tilde{W}_\psi(z,u)\big|du = \int_0^\infty \big|W_\psi(z,t)\big|dt < \infty\,, \tag{2.4}$$

and

$$w(\psi,\theta) \le \frac{1}{2}\left(\frac{|\pi-(\psi-\theta)|}{|\sin(\psi-\theta)|} + \frac{|\pi-(\psi+\theta)|}{|\sin(\psi+\theta)|}\right),\quad w\left(\frac{\pi}{2},\theta\right) = \frac{2\theta}{\sin 2\theta}\,.$$

Proof. Set $\alpha = \theta \pm \psi$. Then $-2\pi < \alpha < 2\pi, \alpha \neq 0$. The lemma follows from the evaluation

$$\int_0^\infty \frac{r dt}{|t - re^{i\alpha}|^2} = \frac{|\pi - |\alpha|\,|}{|\sin\alpha|},$$

(assuming that the right hand side is 1 when $\alpha = \pm\pi$). The case $\psi = \pi/2$ is done separately by using (2.3). □

3. Functional calculus

Let X be a complex Banach space with dual X'.

Definition. A closed, densely defined operator A on X is called an *operator of type* $\theta, 0 \leq \theta < \pi$, if $\sigma(A) \subseteq \overline{S}_\theta$ and

$$\|\lambda(A+\lambda)^{-1}\| \leq M_\phi, \quad \forall \lambda \in \overline{S}_{\pi-\phi}\,, \forall \phi\colon \theta < \phi \leq \pi \tag{3.1}$$

(with the agreement that $\overline{S}_0 = [0, \infty)$).

One operator A is of type $\theta < \pi/2$ exactly when $-A$ generates a bounded holomorphic semigroup of angle $\pi/2 - \theta$ [20]. If A is of type θ and A^{-1} is densely defined, then A^{-1} is also of type θ with constant $1 + M_\phi$ in (3.1) [3].

Notation. We denote the set of all injective type θ operators with dense range (i.e. A^{-1} is densely defined) by T_θ. Thus $A \in T_\theta$ if and only if $A^{-1} \in T_\theta$.

For operators A of type θ and every $\psi, \theta < \psi \leq \pi$, one can define the functional calculus $f \to f(A), \forall f \in H^\infty(S_\psi)$ by plugging A in the place of z in $W_\psi(z,t)$ (see (2.2))

$$A \to W_\psi(A,t) = \frac{1}{2}\left(\frac{Ae^{i(\pi-\psi)}}{(A + te^{i(\pi-\psi)})^2} + \frac{Ae^{-i(\pi-\psi)}}{(A + te^{-i(\pi-\psi)})^2}\right)$$

(Operator fractions $1/B$ stand for B^{-1}.) Then we define $f(A)$ by the formula

$$\langle f(A)x, y\rangle = \int_0^\infty \langle W_\psi(A,t)x, y\rangle (f_\psi * k_\psi)(\ln t) dt \tag{3.2}$$

$(x \in X,\ y \in X')$.

Under the condition $P_\psi = P_\psi(A)$,

$$P_\psi\colon \int_0^\infty \big|\langle W_\psi(A,t)x, y\rangle\big| dt \leq C_\psi \|x\| \ \|y\|\,, \quad \forall x \in X, \quad \forall y \in X', \tag{3.3}$$

the integral in (3.2) is absolutely convergent.

Theorem 3.1

Let A be an injective operator of type $\theta, 0 \leq \theta < \pi$, with a dense range (i.e. $A \in\ T_\theta$). Then:

(I) *If P_ψ holds for some $\theta < \psi \le \pi$, then the mapping $f \to f(A)$ is a bounded $H^\infty(S_\psi)$-functional calculus with $(\lambda + z)^{-1} \to (\lambda + A)^{-1}$ for all $0 \ne \lambda \in S_{\pi-\psi}$ and $f(z) = 1 \to f(A) = I$.*
Moreover,

$$\|f(A)\| \le C_\psi \|f\|_\infty \tag{3.4}$$

(II) *If A has a bounded $H^\infty(S_\psi)$ calculus as above for some $\psi, \theta < \psi < \pi$, then P_ϕ holds for all $\phi, \psi < \phi \le \pi$ with constant $w(\phi,\psi)C_\psi$, (the constant $w(\psi,\phi)$ is defined in (2.4)).*

This streamlines Theorem 5.1 from [6]. In part (I), condition P_ψ provides the convergence in (3.2) and (3.4) readily follows. We need A to be injective with a dense range in order to prove the properties of the calculus. It is easy to see that A^{-1} exists and is densely defined if and only if (cf. [24])

$$s - \lim_{\lambda \to 0} \lambda(\lambda + A)^{-1} = 0, \quad \lambda \in S_{\pi-\theta}\,. \tag{3.5}$$

The proof of Theorem 2.1 in Section 7 shows that our calculus can be reduced to the one defined in [5] (as we can substitute A for z in (7.2), (7.3), (7.5)). It also agrees with the calculus discussed in [7]. We want to demonstrate here how (3.5) guarantees the property $1 \to I$. Define

$$V_\psi(A,t) = \frac{1}{2}\left(\frac{te^{i(\pi-\psi)}}{A + te^{i(\pi-\psi)}} + \frac{te^{-i(\pi-\psi)}}{A + te^{-i(\pi-\psi)}}\right) \tag{3.6}$$

so that

$$\frac{d}{dt}V_\psi = W_\psi.$$

Also, $s - \lim_{t\to\infty} V(A,t) = I$ and $s - \lim_{t\to 0} V(A,t) = 0$, from (3.5). Now if $f(z) = 1$, then

$$f(A) = \int_0^\infty W_\psi(A,t)dt = V_\psi(A,t)|_0^\infty = I.$$

The homomorphic property of the calculus is proved either following the ideas from [5], [6] or the general scheme presented in [17]. One can reason also this way: since the representation (2.1) is, in fact, a modification of the Cauchy integral formula (1.1), one can refer to the classical functional calculus for closed (unbounded) operators developed by Dunford and Schwartz [10]. Part (II) of the theorem was proved in [6], but without the explicit estimate. For convenience we give it here. Let $x \in X, y \in X'$. The calculus $f \to f(A)$, $\forall f \in H^\infty(S_\psi)$ defines a bounded linear functional $f \to \langle f(A)x, y\rangle$ on $H^\infty(S_\psi)$ which has an integral representation

$$\langle f(A)x, y\rangle = \int_{S_\psi} f(z)d\mu_{x,y}(z)\,, \quad \|\mu_{x,y}\| \le C_\psi \|x\|\;\|y\|\,.$$

Then, according to Lemma 2.3:

$$\begin{aligned}\int_0^\infty |\langle W_\phi(A,t)x,y\rangle| dt &= \int_0^\infty \left| \int_{S_\psi} W_\phi(z,t) d\mu_{x,y}(z) \right| dt \\ &\leq \int_{S_\psi} \left(\int_0^\infty |W_\phi(z,t)| dt \right) d|\mu_{x,y}| \\ &= \int_{S_\psi} w(\phi,\psi) d|\mu_{x,y}| \leq w(\phi,\psi) C_\psi \|x\| \, \|y\| \, .\end{aligned}$$

Corollary 3.2

When the operator $A \in T_\theta$ has a bounded $H^\infty(S_\psi)$-functional calculus, so does the operator A^{-1}.

Proof. Using the substitution $t \to 1/t$ we see that $P_\psi(A) = P_\psi(A^{-1})$. □

4. Interpolation inequalities

In this section we shall use the classical definition of fractional powers of operators [17]:

$$A^\alpha = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} A(A+t)^{-1} dt, \quad 0 < \operatorname{Re}\alpha < 1 \, .$$

A proof of the following estimate can be found in [19, Lemma 2].

Lemma 4.1

Let A be a type θ operator. Then for every $\psi, \theta < \psi \leq \pi$, and every $\alpha, 0 \leq \alpha \leq 1$, there is a constant $C(\psi,\alpha)$ such that $\forall \lambda \in \overline{S}_{\pi-\psi}, \lambda \neq 0$.

$$\|A^\alpha (A+\lambda)^{-1}\| \leq C(\psi,\alpha) \, |\lambda|^{\alpha-1}$$

where $C(\psi,0) = M_\psi, C(\psi,1) = 1 + M_\psi$ and for $0 < \alpha < 1$:

$$C(\psi,\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha(1-\alpha)} M_\psi (1 + M_\psi).$$

(Recall that M_ψ is introduced in (3.1).)

The following theorem provides a moment type inequality.

Theorem 4.2

Suppose $A \in T_\theta$ and $x \in D(A^\alpha) \cap (A^{-\beta})$ where $0 < \alpha, \beta \leq 1$. Then the integral

$$f(A)x = \int_0^\infty \big(W_\psi(A,t)x\big) F_\psi(t) dt$$

converges absolutely whenever $f \in H^\infty(S_\psi)$, $\psi > \theta$, *and*

$$\|f(A)x\| \le K(\psi,\alpha,\beta)\,\|A^\alpha x\|^{\beta/(\alpha+\beta)}\,\|A^{-\beta}x\|^{\beta/(\alpha+\beta)}\,\|f\|_\infty. \tag{4.1}$$

The constant $K(\psi,\alpha,\beta)$ is given explicitly below in (4.3).

Proof. With $\lambda \in \overline{S}_{\pi-\psi}, \lambda \ne 0$ one can write the estimate

$$(1)\quad \left\|\frac{A}{(A+\lambda)^2}x\right\| = \left\|\Big(\frac{A}{A+\lambda}\Big)\Big(\frac{A^\beta}{A+\lambda}\Big)A^{-\beta}x\right\| \le (1+M_\psi)\,C(\psi,\beta)\,\|A^{-\beta}x\|\;|\lambda|^{\beta-1}$$

which we shall use when λ is close to zero, and also the estimate

$$(2)\quad \begin{aligned}\left\|\frac{A}{(A+\lambda)^2}x\right\| &= \left\|\Big(\frac{A^{1-\alpha}}{A+\lambda}\Big)\Big(\frac{\lambda}{A+\lambda}\Big)\Big(\frac{1}{\lambda}\Big)A^\alpha x\right\| \\ &\le C(\psi,1-\alpha)\,M_\psi\|A^\alpha x\|\;|\lambda|^{-\alpha-1}\end{aligned}$$

to be used for large $|\lambda|$.

For every $p > 0$ one has:

$$\begin{aligned}\int_0^\infty \|W_\psi(A,t)x\|dt = \int_0^p + \int_p^\infty &\le a\int_0^p t^{\beta-1}dt \\ &+ b\int_p^\infty t^{-\alpha-1}dt = \frac{a}{\beta}p^\beta + \frac{b}{\alpha}p^{-\alpha}\end{aligned}$$

where

$$a = (1+M_\psi)\,C(\psi,\beta)\,\|A^{-\beta}x\|\,, \quad b = M_\psi\,C(\psi,1-\alpha)\,\|A^\alpha x\|\,.$$

We are using (1) and (2) for the first and second integrals correspondingly. Minimizing the right hand side for $p > 0$ (with $p = (b/a)^{1/(\alpha+\beta)}$) one comes to

$$\int_0^\infty \|W_\psi(A,t)x\|dt \le K(\psi,\alpha,\beta)\|A^\alpha x\|^{\beta/(\alpha+\beta)}\|A^{-\beta}x\|^{\alpha/(\alpha+\beta)}, \tag{4.2}$$

which yields (4.1). Here

$$K(\psi,\alpha,\beta) = \big(C(\psi,1-\alpha)M_\psi\big)^{\beta/(\alpha+\beta)}\big(C(\psi,\beta)(1+M_\psi)\big)^{\alpha/(\alpha+\beta)}\Big(\frac{1}{\alpha}+\frac{1}{\beta}\Big) \tag{4.3}$$

Taking $f(z) = z^{is}$, $s \in \mathbb{R}$, in (4.1) defines locally bounded imaginary powers $A^{is}x$ with

$$\|A^{is}x\| \le K(\psi,\alpha,\beta)\,\|A^\alpha x\|^{\beta/(\alpha+\beta)}\,\|A^{-\beta}x\|^{\alpha/(\alpha+\beta)}e^{\psi|s|} \tag{4.4}$$

for all $\psi > \theta$.

The theorem is also true for complex α,β in which case we replace α,β in (4.3) by $\operatorname{Re}\alpha, \operatorname{Re}\beta$. When $\alpha,\beta = 1$, inequality (4.1) reduces to:

$$\|f(A)x\| \le 2M_\psi(1+M_\psi)\big(\|Ax\|\;\|A^{-1}x\|\big)^{1/2}\|f\|_\infty \tag{4.5}$$

($\forall\psi > 0$, $\forall f \in H^\infty(S_\psi)$, $\forall x \in D(A)\cap D(A^{-1})$). $\square$

Remark 4.3. Note that when A is injective with dense range, the subspace $D(A) \cap D(A^{-1})$ is dense in X (see [22, p. 431]). Because of (4.2), all representations in Section 7 with A in the place of z will converge on a dense set. This also provides legitimacy to the representations in the next section.

Interpolation results of a different nature were obtained by Dore [8, 9].

5. Estimates for analytic semigroups

Condition P_ψ can be replaced by restrictions on the function f ensuring the convergence of the integral in (3.2). This way we can define $f(A)$ for certain functions in $H^\infty(S_\psi)$. Set

$$\|f'_\psi\|_1 = \frac{1}{2}\int_0^\infty \left|e^{i\psi} f'(te^{i\psi}) + e^{-i\psi} f'(te^{-i\psi})\right| dt\,.$$

Proposition 5.1

Let $A \in T_\theta$ and let the function $f \in H^\infty(S_\psi), \psi > \theta$, be analytic in a larger sector and such that

$$\|f'_\psi\|_1 < \infty,\;\; f_\psi(\infty) = 0\,. \tag{5.1}$$

Then the representation

$$f(A) = \int_0^\infty W_\psi(A,t)(f_\psi * k_\psi)(\ln t)dt$$

is weakly convergent and we have

$$\|f(A)\| \le M_\psi \|f'_\psi\|_1\,. \tag{5.2}$$

Proof. We integrate by parts to get

$$f(A) = \int_0^{+\infty} W_\psi(A,t)(f_\psi * k_\psi)(\ln t)dt = -\int_0^{+\infty} V_\psi(A,t)(f'_\psi * k_\psi)(\ln t)\frac{dt}{t}$$

where V_ψ is defined above in (3.6). The intermediate term is zero, because the Lebesgue dominated convergence theorem and (5.1) imply:

$$\lim_{r\to\infty} \big(f_\psi * k_\psi\big)(r) = \lim_{r\to\infty} f_\psi(r) = f_\psi(\infty) = 0\,.$$

According to Remark 4.3, the representations above are understood in a week sense on a dense subspace. We also have

$$\|V_\psi(A,t)\| \le M_\psi$$

and

$$\int_0^{+\infty} \left|(f'_\psi * k_\psi)(\ln t)\right|\frac{dt}{t} = \int_{\mathbb{R}} \left|(f'_\psi * k_\psi)(u)\right| du \le \|f'_\psi\|_1.$$

Our estimate follows immediately. □

It is known that when $\theta < \pi/2$, the operator $-A$ generates a holomorphic semigroup with angle $\pi/2-\theta$. (This follows also from the next corollary.) We shall find an estimate for the norm of this semigroup in terms of the constant M_ψ from the original estimate on the resolvent (3.1).

Corollary 5.2

Let A be an injective operator of type $\theta < \pi/2$ with a dense range. Let $\lambda \in S_{\pi/2-\theta}$ and set $\mathrm{Arg}\lambda = \arctan(\mathrm{Im}\lambda/\mathrm{Re}\lambda)$. *Then for every ψ: $\theta < \psi < \pi/2 - |\mathrm{Arg}\lambda|$ we have:*

$$\|e^{-\lambda A}\| \leq \frac{1}{2} M_\psi \left(\frac{1}{\cos(\mathrm{Arg}(\lambda)+\psi)} + \frac{1}{\cos(\mathrm{Arg}(\lambda)-\psi)} \right). \tag{5.3}$$

Here the constant M_ψ, according to the maximum principle, can be defined by

$$M_\psi = \sup_{t>0} \|t(A+te^{\pm(\pi-\psi)})^{-1}\|.$$

Proof. Apply Proposition 5.1 to the function $f_\lambda(z) = e^{-\lambda z}$. One has

$$\begin{aligned}
\|f'_\psi\|_1 &= \frac{1}{2}\int_0^\infty \left|e^{i\psi}\lambda e^{-\lambda t e^{i\psi}} + e^{-i\psi}\lambda e^{-\lambda t e^{-i\psi}}\right| dt \\
&\leq \frac{1}{2}\left(\frac{1}{\cos(\mathrm{Arg}(\lambda)+\psi)} + \frac{1}{\cos(\arg(\lambda)-\psi)} \right).
\end{aligned}$$

The estimate (5.3) follows immediately from (5.2).

Suppose that A is a positive Hilbert space operator. Then $\theta = 0$ and $M_\psi = 1,\ \forall \psi > 0$. For real $\lambda > 0$, (5.3) becomes

$$\|e^{-\lambda A}\| \leq \frac{1}{\cos\psi}.$$

Setting $\psi \to 0$ we come to the expected inequality $\|e^{-\lambda A}\| \leq 1$ which is the best possible for positive operators. In this sense (5.3) is exact. This inequality is of a different nature compared to the usual estimates for $\|e^{-\lambda A}\|$; it works well when $\mathrm{Arg}(\lambda) \pm \psi \approx 0$ and not so well when $\mathrm{Arg}(\lambda) + \psi \approx \pi/2$. □

6. Functions of two variables and a calculus for two commuting operators

Using integral representations analogous to (2.1) for bounded holomorphic functions of many variables, one can define joint functional calculus for two or more commuting operators. Suppose $f(z,w)$ is a bounded holomorphic function on the Cartesian product of two sectors: S_ψ and S_ϕ.

Recently Lancien et al [15] and Albrecht et al [1] defined the calculus $A, B \to f(A,B)$, where A, B are two (resolvent) commuting operators of types less than ψ and

ϕ correspondingly. Such a calculus can be constructed very easily when we use the integral representation

$$f(z,w) = \int_0^{+\infty} \int_0^{+\infty} W_\psi(z,t) W_\phi(w,s) F(t,s)\, dt\, ds$$

with

$$\|F\|_\infty \le \|f\|_\infty .$$

This integral representation is obtained in the same way as the (single integral) representation in Section 7 below. The function $F(t,s)$ is defined by the repeated convolution

$$F(e^u, e^v) = \int_{\mathbb{R}\times\mathbb{R}} \tilde{f}_{\psi,\phi}(p,q) k_\psi(u-p) k_\phi(v-q)\, dp\, dq$$

(Here $e^u = t,\ e^v = s$, and $\tilde{f}_{\psi,\phi}(p,q)$ represents the boundary values of $f(z,w)$ on the sides of the sectors S_ψ and S_ϕ.)

When the condition

$$P_{\psi,\phi} : \int_0^\infty \int_0^\infty \left|\langle W_\psi(A,t)\, W_\phi(B,s)x, y\rangle\right| dt\, ds \le C_{\psi,\phi}\, \|x\|\ \|y\|\,,$$

$(\forall x \in X,\ \forall y \in X')$ is satisfied, then the formula

$$\langle f(A,B)x, y\rangle = \int_0^{+\infty} \int_0^{+\infty} \langle W_\psi(A,t) W_\phi(B,s)x, y\rangle F(t,s)\, dt\, ds$$

defines the desired calculus with

$$\|f(A,B)\| \le C_{\psi,\phi} \|f\|_\infty\,.$$

One direct corollary is this:

Proposition 6.1

Suppose the operator A has a bounded $H^\infty(S_\psi)$-functional calculus (i.e. $P_\psi(A)$) holds - see (3.3)), and the operator B is separated from zero in S_ϕ: $\exists\, \epsilon > 0$,

$$\|(B+\lambda)^{-1}\| \le \frac{M(\phi,B)}{|\lambda| + \epsilon}\,,\ \forall \lambda \in \overline{S}_{\pi-\phi}\,.$$

Then $f(A,B)x$ is defined for every $x \in D(B)$ and

$$\|f(A,B)x\| \le \varepsilon^{-1} C_\psi M(\phi,B) \|Bx\|\ \|f\|_\infty\,.$$

When B is bounded,

$$\|f(A,B)\| \le \varepsilon^{-1} C_\psi\, M(\phi,B)\, \|B\|\ \|f\|_\infty\,.$$

Proof. For every $x \in D(B)$ and every $y \in X'$ one has

$$\int_0^\infty \int_0^\infty |\langle W_\psi(A,t)W_\phi(B,s)x, y\rangle|\, dt\, ds \leq C_\psi M(\phi, B)\, \|y\|\, \|Bx\| \int_0^\infty (s+\varepsilon)^{-2} ds\,,$$

which implies the above estimate. □

7. Proof of Theorem 2.1

We need some simple technical lemmas. The first one is a fact from Fourier transform theory.

Lemma 7.1

Let $a > 0$ and suppose that $f(z)$ is holomorphic and bounded on the strip $G = \{z | 0 \leq \operatorname{Im}(z) \leq a\}$ and zero at infinity. Then for every real t:

$$e^{at} \int_{-\infty}^{+\infty} e^{itx} f(x) dx = \int_{-\infty}^{+\infty} e^{it(x-ia)} f(x) dx = \int_{-\infty}^{+\infty} e^{itx} f(x+ia) dx\,.$$

Here and further we assume that all involved integrals exist. The proof follows from Cauchy's theorem applied to the function $e^{it(z-ia)} f(z)$ on G.

Corollary 7.2

Suppose that $f(z)$ is holomorphic and bounded on the strip $\{|\operatorname{Im}(z)| \leq a\}$ and zero at infinity. Then for every real t:

$$\cosh(at) \int_{-\infty}^{+\infty} e^{itx} f(x) dx = \int_{-\infty}^{+\infty} e^{itx} \frac{1}{2}\big(f(x+ia) + f(x-ia)\big) dx\,.$$

Lemma 7.3

Let $f(z)$ be a bounded holomorphic function on the right half plane $S_{\pi/2}$. Then we have the representation

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(it) \frac{z}{z^2 + t^2} dt \tag{7.1}$$

where $f(it)$ is the boundary value of the function.

Proof. For real z, (7.1) is the well-known Poisson representation. It is true for all $z \in S_{\pi/2}$ too by the uniqueness theorem. □

Lemma 7.4

Suppose $f \in H^\infty(S_\psi)$, $0 < \psi \leq \pi$. Then

$$f(z) = \frac{1}{\psi} \int_{\mathbb{R}} f_\psi(r) \frac{z^{\pi/(2\psi)} e^{\pi r/(2\psi)}}{z^{\pi/\psi} + e^{\pi r/\psi}} dr \tag{7.2}$$

with $f_\psi(r) = \frac{1}{2}\big(f(e^{r+i\psi}) + f(e^{r-i\psi})\big)$, the boundary value of f.

Proof. If $z \in S_\psi$, then $w = z^{\pi/(2\psi)} \in S_{\pi/2}$ and the function $g(w) \equiv f(z) = f(w^{2\psi/\pi})$ belongs to $H^\infty(S_{\pi/2})$. By (7.1)

$$\begin{aligned} f(z) = g(w) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} g(it)\frac{w}{w^2+t^2}dt \\ &= \frac{1}{\pi}\int_0^\infty \big(g(it)+g(-it)\big)\frac{w}{w^2+t^2}\,dt \\ &= \frac{1}{\pi}\int_0^\infty \big(f(e^{i\psi}t^{2\psi/\pi})+f(e^{-i\psi}t^{2\psi/\pi})\big)\frac{w}{w^2+t^2}dt \\ &= \frac{2}{\pi}\int_0^\infty \tilde{f}_\psi(t^{2\psi/\pi})\frac{z^{\pi/(2\psi)}}{z^{\pi/\psi}+t^2}dt \end{aligned}$$

where

$$\tilde{f}_\psi(t^{2\psi/\pi}) = \frac{1}{2}\big(f(t^{2\psi/\pi}e^{i\psi}) + f(t^{2\psi/\pi}e^{-i\psi})\big)$$

is the boundary value of the function on the rays $re^{i\psi}, re^{-i\psi}, r > 0$. Substitute here

$$t^{2\psi/\pi} = e^r, t = e^{\pi r/(2\psi)},\ \ f_\psi(r) = \tilde{f}_\psi(e^r)$$

to get (7.2). □

Example 7.5: When $\psi = \pi$ we have

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f_\pi(r)\frac{z^{1/2}e^{r/2}}{z+e^r}\,dr$$

for any bounded holomorphic function on $S_\pi = \mathbb{C}\backslash(-\infty, 0]$.

The kernel

$$\frac{z^{\pi/(2\psi)}e^{\pi r/(2\psi)}}{z^{\pi/\psi}+e^{\pi r/\psi}}$$

has already been used for the functional calculus in [5] (see also [4]). We want to express this kernel in terms of $\tilde{W}_\psi(z, u)$. Consider the Fourier transforms (with $z \in S_\psi$):

$$\frac{z^{\pi/(2\psi)}e^{\pi r/(2\psi)}}{z^{\pi/\psi}+e^{\pi r/\psi}} = \frac{\psi}{2\pi}\int_{\mathbb{R}} e^{-irs}\frac{z^{is}}{\cosh(\psi s)}\,ds \tag{7.3}$$

and

$$z^{is} = \frac{\sinh(\pi s)}{\pi s}\int_{\mathbb{R}} e^{isu}\frac{ze^u}{(z+e^u)^2}\,du \tag{7.4}$$

The first one is the inverse Fourier transform of

$$z^{is} = \frac{\cosh(\psi s)}{\psi}\int_{-\infty}^{+\infty} e^{irs}\frac{z^{\pi(2\psi)}e^{\pi r/(2\psi)}}{z^{\pi/\psi}+e^{\pi r/\psi}}\,dr$$

which is (7.2) for $f(z) = z^{is}$.

The second one, (7.4), comes from the representation

$$z^\alpha = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}\frac{z}{z+t}\,dt\,,\quad 0<\mathrm{Re}\,(\alpha)<1$$

after integration by parts and setting $\alpha = is,\ t = e^u$ (cf. [3]).

We multiply and divide (7.4) by $\cosh((\pi-\psi)s)$. Applying Corollary 7.2 with $a = \pi - \psi$ we get

$$z^{is} = \frac{\sinh(\pi s)}{\pi s\,\cosh((\pi-\psi)s)}\int_{\mathbb{R}} e^{isu}\,\tilde{W}_\psi(z,u)du\,.$$

This substituted in (7.3) gives

$$\begin{aligned}\frac{z^{\pi/(2\psi)}e^{\pi r/(2\psi)}}{z^{\pi/\psi}+e^{\pi r/\psi}} &= \psi\int_{\mathbb{R}}\tilde{W}_\psi(z,u)\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{is(u-r)}\frac{\sinh(\pi s)}{\pi s\,\cosh(\psi s)\,\cosh((\pi-\psi)s)}ds\right)du\\ &= \psi\int_{\mathbb{R}}\tilde{W}_\psi(z,u)\,k_\psi(u-r)\,du\end{aligned}$$

where

$$k_\psi(r) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{isr}\frac{\sinh(\pi s)}{\pi s\,\cosh(\pi s)\ \cosh((\pi-\psi)s)}\,ds\,.$$

As we shall see, $k_\psi \in L^1(\mathbb{R})$. Also, $\tilde{W}_\psi(z,u) \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ as a function of u, according to (2.4). Therefore, their convolution is also in $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and is uniformly continuous on $\mathbb{R}$. The equality

$$\frac{z^{\pi/(2\psi)}e^{\pi r/(2\psi)}}{z^{\pi/\psi}+e^{\pi r/\psi}} = \psi\int_{\mathbb{R}}\tilde{W}_\psi(z,u)k_\psi(u-r)du \tag{7.5}$$

can be justified this way: both sides are continuous L^1 functions with the same Fourier transform:

$$\frac{\psi}{2\pi}\ \frac{z^{is}}{\cosh(\psi s)}\,.$$

Now (2.1) follows from (7.2) combined with (7.5). To complete the proof we need to check the properties of $k_\psi(r)$. Write

$$\sinh(\pi s) = \sinh\big((\pi-\psi+\psi)s\big) = \sinh\big((\pi-\psi)s\big)\cosh(\psi s) + \cosh\big((\pi-\psi)s\big)\sinh(\psi s).$$

This way

$$\begin{aligned}\frac{\sinh(\pi s)}{s\,\cosh((\pi-\psi)s)\cosh(\psi s)} &= \frac{\tanh((\pi-\psi)s)}{s}+\frac{\tanh(\psi s)}{s}\,,\\ \pi^2 k_\psi(r) &= \frac{1}{2}\int_{\mathbb{R}} e^{isr}\frac{\sinh(\pi s)}{s\,\cosh(\psi s)\ \cosh((\pi-\psi)s)}ds\\ &= \int_0^\infty\cos(rs)\frac{\tanh((\pi-\psi)s)}{s}ds + \int_0^\infty\cos(rs)\frac{\tanh(\psi s)}{s}ds\\ &= \ln\left|\coth\frac{r\pi}{4(\pi-\psi)}\right| + \ln\left|\coth\frac{r\pi}{4\psi}\right| = \ln\left|\coth\frac{r\pi}{4(\pi-\psi)}\ \coth\frac{r\pi}{4\psi}\right|\end{aligned}$$

(see [21, p. 470, n. 8], or [12, 1.9. (28)]).

Therefore

$$k_\psi(r) = \frac{1}{\pi^2} \ln \left| \coth\Big(\frac{\pi r}{4\psi}\Big) \coth\Big(\frac{\pi r}{4(\pi - \psi)}\Big) \right|.$$

This is obviously a non-negative even function defined on $\mathbb{R}\backslash\{0\}$ and absolutely integrable on $\mathbb{R}$. Also, $\|k_\psi\|_1 = 1$ according to [21, p. 535, n. 7]. When $\psi = \pi$, one has

$$k_\pi(r) = \frac{1}{\pi^2} \ln \left| \coth\Big(\frac{r}{4}\Big) \right|.$$

The proof is complete.

Comment 7.6. To understand better the function k_ψ, it is good to look at its graph. We present here the graph of $k_\psi(x)$, $-\infty < x < +\infty$, with $\psi = \pi/4$ (Figure 1).

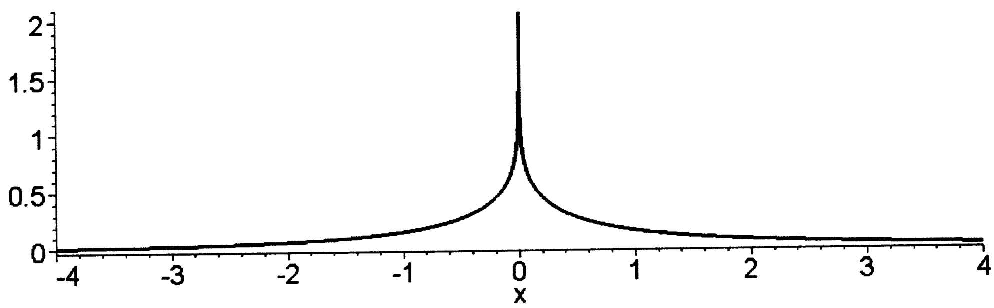

Figure 1

In conclusion, I want to express my gratitude to the referee whose competent remarks helped to improve the paper.

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