Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **53**, 3 (2002), 265–276

Arithmetically Gorenstein curves on arithmetically Cohen-Macaulay surfaces

Alberto Dolcetti

Dipartimento di Matematica "U. Dini", viale Morgagni 67/a, I-50134, Firenze (Italy)

E-mail: dolcetti@math.unifi.it

Received July 17, 2001. Revised April 9, 2002

Abstract

Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected arithmetically Cohen-Macaulay surface. Then there are at most finitely many complete linear systems on Σ, not of the type $|kH - K|$ (H hyperplane section and K canonical divisor on Σ), containing integral arithmetically Gorenstein curves.

Introduction

The aim of this note is to prove the following.

Theorem

Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected arithmetically Cohen-Macaulay surface. Then there are at most finitely many complete linear systems on Σ, not of the type $|kH - K|$ (H hyperplane section and K canonical divisor on Σ), containing integral arithmetically Gorenstein curves.

This result is proved in §1 by two fundamental steps. The first one (Lemma 1.4) translates the existence of subcanonical curves on a smooth connected arithmetically Cohen-Macaulay surface in terms of certain divisors called *lone* and *minimal* (see 1.2 for definitions). This is similar to results obtained in case of subcanonical surfaces in $\mathbb{P}^N$ (see [3; 1.2], [6; 1.8], [5; 3]). The second step (Lemma 1.5) consists in proving a result of finitedness (up to linear equivalence) for some particular minimal divisors on a regular surface: this is closely related for ideas and methods to a result proved in case of arithmetically Cohen-Macaulay surfaces in [9; 8.6].

Keywords: Arithmetically Cohen-Macaulay, subcanonical, arithmetically Gorenstein curves and surfaces; lone and minimal divisors.

MSC2000: 14M05, 14C20, 14H50, 14J26, 14J28.

In §2 we consider the cases of rational and *K3* arithmetically Gorenstein surfaces as examples of explicit description of the possible arithmetically Gorenstein curves on an arithmetically Cohen-Macaulay surface (see 2.6 and 2.10). In both cases the goal is fulfilled by studying the lone divisors on them.

This note was initially written for rational and $K3$ arithmetically Gorenstein surfaces. The referee observed that, by means of [9; 8.6], it would have been possible to extend the proof to all arithmetically Gorenstein surfaces and suggested to indagate the phenomenon in the more general case of arithmetically Cohen-Macaulay ones. This is what I have tried to do. Furthermore, after finishing the present version, I was informed of [2; Theorem 3.2], where similar facts are considered in an algebraic context: compared to those the present methods are new and of geometric type.

Acknowledgement. I thank the referee of this note for some useful suggestions.

§1. Proof of the Theorem

In this paper we work over a fixed algebraically closed field K of characteristic 0. All schemes are locally Cohen-Macaulay without embedded components.

1.1. Basic notions. Let V be a closed subscheme of dimension $v \geq 1$ in $\mathbb{P}^N$, I_V be its sheaf of ideals and $M^i(V) := \oplus_{t\in\mathbb{Z}} H^i(I_V(t))$. V is said to be

- *arithmetically Cohen-Macaulay* (**aCM** for short): if the length of a minimal free resolution of
$$S_V := \frac{K[X_0, \ ..., \ X_N]}{M^0(V)}$$
as an $K[X_0, \ ..., \ X_N]$-module is $N - v$; equivalently $M^i(V) = 0$ for any $1 \leq i \leq v$ (see for instance [10; 1.2.2 and 1.2.3]);
- *arithmetically Gorenstein* (**aG** for short): if it is **aCM** and the last free module of a minimal free resolution of S_V has rank 1;

Any **aG** scheme V is subcanonical, i.e. $\omega_V \simeq O_V(\gamma)$ for some integer γ, equivalently $K \sim \gamma H$, with $K = K_V$, $H = H_V$ the canonical divisor and the hyperplane divisor respectively (see for instance [10; 4.1.5]). To point out the integer γ we say that V is γ-*subcanonical* or γ-*arithmetically Gorenstein* (γ-**aG** for short).

1.2. Lone and minimal divisors. An effective divisor $Y \neq 0$ on a smooth connected projective surface S is said to be

- *lone*: if $h^0(O_\Sigma(Y)) = 1$
- *minimal with respect to an ample effective divisor* $\mathcal{H}$:

$$\text{if } Y \sim \mathcal{H} \text{ or } h^0\big(O_\Sigma(Y - \mathcal{H})\big) = 0.$$

Fixed an ample effective divisor $\mathcal{H}$ on S, for any effective divisor $X \neq 0$ there is a pair (k, Y) with $k \geq 0$ and Y minimal with respect to $\mathcal{H}$ such that $X \sim Y + k\mathcal{H}$. It is easy to see that if (k', Y') is another pair with the same property, then $k = k'$ and $Y \sim Y'$, hence Y is uniquely determined by X up to linear equivalence. We refer to Y as *the minimal divisor, associated to* X, *with respect to* $\mathcal{H}$.

Suppose now $S \subset \mathbb{P}^N$. We say that a divisor is *minimal*, if it is minimal with respect to the hyperplane section of S. Any lone divisor is also minimal.

Minimal divisors on **aCM** surfaces in $\mathbb{P}^N$ are studied in [9; Chapter 8]; lone divisors (sometimes called *fixed* or *unique*) on subcanonical surfaces in $\mathbb{P}^N$ are discussed in [3], [6], [5] in relation to subcanonical curves.

1.3. Some known facts about divisors on aCM surfaces. a) Any effective divisor $D \neq 0$ on a smooth connected surface $\Sigma \subset \mathbb{P}^N$ can be viewed as a subscheme both of Σ and $\mathbb{P}^N$, the ideal sheaf $I_{D,\Sigma}$ of D in Σ is isomorphic to $O_\Sigma(-D)$ and, for any integer t, we have the exact sequence:

$$0 \to I_\Sigma(t) \to I_D(t) \to O_\Sigma(tH - D) \to 0$$

If Σ is **aCM**, then we get

$$H^1\big(I_D(t)\big) \simeq H^1\big(O_\Sigma(tH - D)\big)$$

for any integer t and if $D' \sim D + rH$ is another effective non-zero divisor, then

$$H^1\big(I_{D'}(t)\big) \simeq H^1\big(I_D(t - r)\big)$$

for any t.

b) Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected **aCM** surface then any non-zero divisor $C \in |\alpha H - K|$ is α-**aG** (see [9; 5.4, 5.5, 5.6] and [10; 4.2.8]). In particular it is known that $|\alpha H - K|$ on a Castelnuovo surface and on a Bordiga surface in $\mathbb{P}^4$ contain smooth connected **aG** curves for all $\alpha \geq 0$ and all $\alpha \geq 1$ respectively. The corresponding minimal divisors are a plane smooth cubic in case of Castelnuovo surface and a curve of degree 8 genus 5 in case of Bordiga surface (see [9; Chapter 8] p. 73 (xxxvi) and p. 74 (xxvi) respectively). In both cases $|\alpha H - K|$, $\alpha \geq 0$ or $\alpha \geq 1$, give infinitely many complete linear systems of integral **aG** curves. Finally note that if, moreover, Σ is β-**aG**, then $|\alpha H - K_\Sigma| = |(\alpha - \beta)H|$, hence the linear systems above are multiple of the hyperplane one, and give, for $\alpha > \beta$, infinitely many complete linear systems of integral **aG** curves, with minimal curve H.

Lemma 1.4

Let C be an α-subcanonical integral curve on a smooth connected **aCM** *surface* $\Sigma \subset \mathbb{P}^N$. *We have:*

i) if $\alpha \deg(C) < C \cdot K$, then C is a lone divisor on Σ;

*ii) if $\alpha \deg(C) = C \cdot K$, then $1 \leq h^0(O_\Sigma(C)) \leq 2$, in particular C is minimal on Σ (and not lone if moreover Σ is α-*__aG__*);*

iii) if $\alpha \deg(C) > C \cdot K$, then either $C \sim \alpha H - K$ or $C \sim Y + \alpha H - K$, where Y is a lone non-zero divisor on Σ with $C \cdot Y = 0$ and $\alpha = (K \cdot Y - Y^2)/\deg(Y)$.

Proof. By adjunction,

$$O_\Sigma(C) \otimes O_C \simeq \omega_C \otimes O_\Sigma(-K) \simeq O_\Sigma(\alpha H - K) \otimes O_C,$$

hence we have the exact sequence:

$$0 \to O_\Sigma \to O_\Sigma(C) \to O_\Sigma(\alpha H - K) \otimes O_C \to 0$$

Note that $h^1(O_\Sigma) = h^2(I_\Sigma) = 0$. Now the degree of $O_\Sigma(\alpha H - K) \otimes O_C$ is: $C \cdot (\alpha H - K) = \alpha\deg(C) - C \cdot K$. In case (i) this is negative, hence $h^0(O_\Sigma(C)) = 1$, i.e. C is lone. The degree is 0 in case (ii). This implies $h^0(O_\Sigma(C)) \leq 2$, with equality if $K \sim \alpha H$. In any case C is minimal, otherwise $C \sim X + H$, with X an effective or zero divisor, therefore $h^0(O_\Sigma(C)) > 2$.

Suppose now $\alpha\deg(C) > C \cdot K$. This implies in particular K not linearly equivalent to αH (i.e. Σ not α-subcanonical). The sequence above gives:

$$0 \to O_\Sigma(K - \alpha H) \to O_\Sigma(C + K - \alpha H) \to O_C \to 0$$

By Serre duality

$$h^1\big(O_\Sigma(K - \alpha H)\big) = h^1\big(O_\Sigma(\alpha)\big) = h^2\big(I_\Sigma(\alpha)\big) = 0$$

and we have also $h^0(O_\Sigma(K - \alpha H)) = 0$. Indeed, if not, $|K - \alpha H|$ would contain an effective divisor X, with $X \neq 0$ (since otherwise $K \sim \alpha H$) and $X \cdot C \geq 0$: for this inequality it suffices to write $X = tC + X'$, with $t \geq 0$ and X' effective, not containing C, remembering that $C^2 = \alpha\deg(C) - C \cdot K > 0$. The condition $X \cdot C \geq 0$ gives $C \cdot K - \alpha\deg(C) \geq 0$: a contradiction, therefore $h^0(O_\Sigma(K - \alpha H)) = 0$ and so $h^0(O_\Sigma(C + K - \alpha H)) = 1$. Hence either $C \sim \alpha H - K$ or $C \sim Y + \alpha H - K$, where Y is a lone divisor: the unique effective divisor in $|C + K - \alpha H|$. Finally for $C \sim Y + \alpha H - K$ we have $C^2 = Y \cdot C + \alpha\deg(C) - C \cdot K$, but $C^2 = \alpha\deg(C) - C \cdot K$ by adjunction, therefore $Y \cdot C = 0$. Hence we get

$$0 = Y \cdot C = Y \cdot (Y + \alpha H - K) = Y^2 + \alpha\deg(Y) - Y \cdot K,$$

i.e.

$$\alpha = \frac{K \cdot Y - Y^2}{\deg(Y)}. \quad \square$$

Lemma 1.5

Let S be a smooth connected regular (i.e. $h^1(O_S) = 0$) projective surface and fix an ample effective divisor $\mathcal{H}$. Then for every divisor D there exist at most finitely many divisors Y, up to linear equivalence, which are minimal with respect to $\mathcal{H}$ and such that $h^1(O_S(t\mathcal{H} - Y - D)) = 0$ for any $t \in \mathbb{Z}$.

Proof. We follow the proof [9; 8.6], where it is essentially proved that on a smooth connected **aCM** surface $\Sigma \subset \mathbb{P}^N$, fixed a graded $K[X_0, ..., X_N]$-module M of finite length, there are only finitely many minimal curves Y, up to linear equivalence, with $M^1(Y) \simeq M$ up to shift. Note that the result of [9] with $M = 0$ is the result of this Lemma in case of an **aCM** surface $S = \Sigma \subset \mathbb{P}^N$, with $\mathcal{H} = H$, $D = 0$ and Y a minimal curve. Indeed Σ is regular and, as in 1.3 (a), $h^1(I_Y(t)) = h^1(O_\Sigma(tH - Y))$.

Since S is regular, linear equivalence and algebraic equivalence generate the same relation. Moreover since the quotient group of divisors, which are numerically equivalent to zero, modulo algebraic equivalence is finite, it suffices to prove the finitedness of the number of such Y's up to numerical equivalence. Finally recall that the group $\mathrm{Num}(S)$ of divisors up to numerical equivalence is a finitely generated free $\mathbb{Z}$-module (for all this standard material see [7; Chapter 19, in particular 19.3.1]). The numerical equivalence will be denoted by "$\equiv$". We recall the following fact on the generation of $\mathrm{Num}(S)$.

Claim. There are $r := rk(\mathrm{Num}(S))$ effective non-zero divisors $L_1, ...L_r$ such that $\{L_1, ...L_r\}$ gives a base of the $\mathbb{Q}$-vector space $V_S := \mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the image of the canonical injection $\mathrm{Num}(S) \to V_S$ is contained in $\bigoplus_{j \geq 1}(1/\beta)\mathbb{Z}L_j$ for some integer $\beta \neq 0$.

Proof of the Claim. Let $D_1, ...D_r$ be divisors such that $\mathrm{Num}(S) = \bigoplus_{j\geq 1} \mathbb{Z}D_j$. If all D_j are effective, it suffices to put $L_j = D_j$ and $\beta = 1$. Otherwise choose an integer $e \geq 2$ such that every $|D_j + e\mathcal{H}|$ contains effective non-zero divisors and fix $L_j \in |D_j + e\mathcal{H}|$. $\{L_1, ...L_r\}$ gives a base of V_S: it suffices to prove that $L_1, ...L_r$ are linearly independent. If, in V_S, $\sum_{j\geq 1} p_j L_j = 0$, $p_j \in \mathbb{Q}$ not all zero, then we get a relation of the form $\sum_{j\geq 1} z_j L_j \equiv 0$, with $z_j \in \mathbb{Z}$ not all zero, therefore

$$0 \equiv \sum_{j\geq 1} z_j(D_j + e\mathcal{H}) = \sum_{j\geq 1} z_j D_j + ez\mathcal{H},$$

where $z = \sum_{j\geq 1} z_j$. Now $\mathcal{H} \equiv \sum_{j\geq 1} h_j D_j$ for some $h_j \in \mathbb{Z}$, so:

$$\sum_{j\geq 1} z_j D_j + ez \sum_{j\geq 1} h_j D_j \equiv 0$$

and this implies $z_j = -ezh_j$ for all j, therefore $z \neq 0$, because $z_i \neq 0$ for some i. Hence the relation $\sum_{j\geq 1} z_j L_j \equiv 0$ gives $ez\sum_{j\geq 1} h_j L_j \equiv 0$ and therefore $\sum_{j\geq 1} h_j L_j \equiv 0$. Now

$$0 \equiv \sum_{j\geq 1} h_j L_j \equiv \sum_{j\geq 1} h_j(D_j + e\mathcal{H}) \equiv \sum_{j\geq 1} h_j D_j + eh\mathcal{H}$$

with $h = \sum_{j\geq 1} h_j$. Therefore $\mathcal{H} + eh\mathcal{H} \equiv 0$, i.e. $(1 + eh)\mathcal{H} \equiv 0$, so $eh = -1$, but this is impossible because $e, h \in \mathbb{Z}$ and $e \geq 2$.

Now we must prove the statement on the image of $\mathrm{Num}(S)$. We have

$$\mathcal{H} \equiv \sum_{j\geq 1} h_j D_j \equiv \sum_{j\geq 1} h_j(L_j - e\mathcal{H}) \equiv \sum_{j\geq 1} h_j L_j - eh\mathcal{H},$$

therefore $(1+eh)\mathcal{H} \equiv \sum_{j\geq 1} h_j L_j$, as above $1+eh \neq 0$ and $\mathcal{H} = \sum_{j\geq 1}[h_j/(1+eh)]L_j$ in V_S. Put $\beta := 1+eh$, so for any divisor $X \equiv \sum_{j\geq 1} x_j D_j$, with $x_j \in \mathbb{Z}$, we get $X \equiv \sum_{j\geq 1} x_j(L_j - e\mathcal{H})$ and therefore in V_S:

$$X = \sum_{j\geq 1} x_j L_j - ex\sum_{j\geq 1}\frac{h_j}{\beta}L_j = \sum_{j\geq 1}\Big(x_j - ex\frac{h_j}{\beta}\Big)L_j,$$

where $x := \sum_{j\geq 1} x_j$ and this concludes the Claim.

From now on we fix a set $L_1, ...L_r$ as in the Claim and put $\Delta = (\delta_{ij})$, where $\delta_{ij} = L_i \cdot L_j$. Note that $rk(\Delta) = r$, because the intersection pairing is non-degenerate on $\text{Num}(S)$. Suppose that such a divisor Y exists and that $Y \notin |\mathcal{H}|$. We distinguish two cases.

1$^{\text{st}}$ case: there is an index i such that $L_i - D - Y$ is linearly equivalent to an effective or zero divisor. We have $(L_i - D - Y)\cdot\mathcal{H} \geq 0$, therefore $Y\cdot\mathcal{H} \leq L_i\cdot\mathcal{H} - D\cdot\mathcal{H}$ and we conclude because the set of effective divisors on S with assigned intersection product with $\mathcal{H}$ is a finite set up to numerical equivalence (see for instance [8; V, ex 1.11]).

2$^{\text{nd}}$ case: for all i, $L_i - Y - D$ is not effective and not zero. We write $Y = \sum_{j\geq 1} y_j L_j$ in V_S, where $y_j \in (1/\beta)\mathbb{Z}$ for some integer β. It suffices to prove that the n-tuple $(y_1, ..., y_r)$ belongs to a bounded subset $\mathbb{B} \subset \mathbb{Q}^r$: indeed in this case $(1/\beta)\mathbb{Z}^r \cap \mathbb{B}$ is finite. Look at the exact sequence:

$$0 \to O_S(-Y-D) \to O_S(L_i - Y - D) \to O_{L_i}(L_i - Y - D) \to 0.$$

This implies

$$h^0\big(O_{L_i}(L_i - Y - D)\big) \leq h^1\big(O_S(-Y-D)\big) = 0.$$

Therefore

$$L_i\cdot(L_i - Y - D) + 1 - p_a(L_i) = \chi\big(O_{L_i}(L_i - Y - D)\big) \leq h^0\big(O_{L_i}(L_i - Y - D)\big) = 0,$$

hence

$$L_i^2 - L_i\cdot Y - D\cdot L_i + 1 - p_a(L_i) \leq 0.$$

Now remembering that the intersection product extends to divisors with rational coefficients:

$$(*) \qquad \sum_{j\geq 1}\delta_{ij}y_j \geq k_i := \delta_{ii} - D\cdot L_i + 1 - p_a(L_i)$$

for any i, $1 \leq i \leq r$.

On the other hand for any integer i, $1 \leq i \leq r$, we define

$$q_i := \min\{t \in \mathbb{Z} \;/\; -L_i - K - D + (t-1)\mathcal{H} \text{ is effective}\}$$

and $q := \max\{q_i \;/\; 1 \leq i \leq r\}$ (note that q does not depend on Y). Now $Y + D - q\mathcal{H} + K + L_i$ is not effective, indeed, otherwise, since $-L_i - K - D + (q-1)\mathcal{H}$ is effective by definition of q, the divisor $Y + D - q\mathcal{H} + K + L_i - L_i - K - D + (q-1)\mathcal{H} = Y - \mathcal{H}$ should be effective, but this is impossible, because $Y \notin |\mathcal{H}|$ and it is minimal with respect to $\mathcal{H}$. This gives

$$0 = h^0\big(O_S(Y + D - q\mathcal{H} + K + L_i)\big) = h^2\big(O_S(-Y - D + q\mathcal{H} - L_i)\big).$$

Now look at the exact sequence:

$$0 \to O_S(q\mathcal{H} - Y - D - L_i) \to O_S(q\mathcal{H} - Y - D) \to O_{L_i}(q\mathcal{H} - Y - D) \to 0.$$

We get

$$h^1(O_{L_i}\big(q\mathcal{H} - Y - D)\big) \leq h^1\big(O_S(q\mathcal{H} - Y - D)\big) = 0,$$

hence $h^1(O_{L_i}(q\mathcal{H} - Y - D)) = 0$, so:

$$\begin{aligned} L_i \cdot (-Y - D + q\mathcal{H}) + 1 - p_a(L_i) &= \chi\big(O_{L_i}(q\mathcal{H} - Y - D)\big) \\ &= h^0\big(O_{L_i}(q\mathcal{H} - Y - D)\big) \geq 0, \end{aligned}$$

i.e., $-Y \cdot L_i - D \cdot L_i + q\mathcal{H} \cdot L_i + 1 - p_a(L_i) \geq 0$, and this gives

$$(**) \qquad \sum_{j \geq 1} \delta_{ij} y_j \leq k'_i := -D \cdot L_i + q\mathcal{H} \cdot L_i + 1 - p_a(L_i)$$

for any i, $1 \leq i \leq r$.

Now the $2r$ bounds $(*)$ and $(**)$, together with the fact that Δ has rank r, imply that $(y_1, ..., y_r)$ belongs to a bounded set of $\mathbb{Q}^r$. □

1.6. Final step of the Proof of the Theorem. Let C be an integral **aG** curve on Σ and suppose that $C \notin |\alpha H - K|$. If C is minimal, then, since $h^1(O_\Sigma(tH - C)) = h^1(I_C(t)) = 0$ for any $t \in \mathbb{Z}$ (remember 1.3 (a)), we conclude by means of 1.5 with $\mathcal{H} = H$ and $D = 0$ (or with [9; 8.6] with $M = 0$). If C is not minimal, then by 1.4 we have: $C \sim Y + \big(\frac{K \cdot Y - Y^2}{\deg(Y)}\big)H - K$, with Y a lone divisor on Σ. Again by 1.3 (a), for any $t \in \mathbb{Z}$ we have

$$0 = h^1\big(I_C(t)\big) = h^1\big(O_\Sigma(tH - C)\big) = h^1\Big(O_\Sigma\big((t - \frac{K \cdot Y - Y^2}{\deg(Y)})H - Y + K\big)\Big),$$

and the result follows from 1.5 with $\mathcal{H} = H$ and $D = -K$.

§2. The cases of rational and *K3* Gorenstein surfaces

As examples of explicit description of the possible **aG** curves on an **aCM** surface we consider the cases of rational and K-3 **aG** surfaces, i.e. γ-**aG** surfaces with $\gamma \leq 0$. In both cases the goal is fulfilled by means of the characterization of lone divisors on them. First of all we note that, in case of an **aG** surface the Theorem and Lemma 1.4 can be easily restated respectively as follows

Proposition 2.1

Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected **aG** *surface. Then there are at most finitely many complete linear systems on Σ, not of the type $|tH|$, containing integral* **aG** *curves.*

Lemma 2.2

*Let C be an α-subcanonical integral curve on a smooth connected β-**aG** surface $\Sigma \subset \mathbb{P}^N$. We have:*

i) if $\alpha < \beta$, then C is a lone divisor on Σ;
ii) if $\alpha = \beta$, then $h^0(O_\Sigma(C)) = 2$ and C is minimal, not lone on Σ;
iii) if $\alpha > \beta$, then either $C \sim (\alpha - \beta)H$ or $C \sim Y + (\alpha - \beta)H$, where Y is a lone non-zero divisor on Σ with $C \cdot Y = 0$ and $\alpha = -Y^2/(\deg(Y)) + \beta$.

2.3. Planes and quadrics. Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected β-subcanonical surface. By Castelnuovo's theorem (see for instance [1; V.1]) Σ is rational if and only if $\beta \le -1$. Moreover it is easy to see that $\beta \ge -3$ with $\beta = -3$ if and only if Σ is a plane and $\beta = -2$ if and only if Σ is a quadric surface.

Any curve C on a plane Π is linearly equivalent to $\deg(C)L$, where L is a line (i.e. a hyperplane section of Π) and C is complete intersection in $\mathbb{P}^N$ of type $(d, \underbrace{1, ..., 1}_{N-2})$.

Suppose now that $\Sigma \subset \mathbb{P}^N$ is a smooth connected quadric surface (of course it is non-degenerate in some projective 3-subspace of $\mathbb{P}^N$). Σ contains no lone divisor, while, up to linear equivalence, it contains exactly three minimal divisors: H and two skew lines L_1, L_2. Therefore, for any integral curve C on Σ, from 2.2 we get that C is subcanonical if and only if it is complete intersection if and only if either C is a line (complete intersection of type $(\underbrace{1, ..., 1}_{N-1})$) or $C \sim kH,\ k \ge 1$ (complete intersection of type $(2, k, \underbrace{1, ..., 1}_{N-3})$). Hence only the case $\beta = -1$ is not trivial.

2.4. Del Pezzo surfaces. Let $\Sigma \subset \mathbb{P}^N$ be a smooth connected (-1)-subcanonical surface, then it is a del Pezzo surface of degree $n \ge 3$, non-degenerate in some projective n-space contained in $\mathbb{P}^N$ or some of its isomorphic projections in some projective n'-space $(5 \le n' < n)$. We recall that a del Pezzo surface of degree $n \ge 3$, non-degenerate in $\mathbb{P}^n$, is one of the following 8 types (see for instance [8; V.4.7.1]):

- one of the surfaces $S_n,\ 3 \le n \le 9$, obtained by blowing up $9-n$ general points in $\mathbb{P}^2$ and embedded in $\mathbb{P}^n$ by the complete linear system corresponding to plane cubics through such points
- S'_8 of degree 8, obtained by the 2-tuple embedding of a smooth quadric surface in $\mathbb{P}^3$.

All the above surfaces are **aCM** (see [4; p. 63]); of course they are complete intersections if and only if $3 \le n \le 4$.

As in case of a smooth quadric surface, S'_8 contains no lone divisor, while, up to linear equivalence, it contains exactly three minimal divisors: H, C_1, C_2, where C_j is the smooth plane conic image of the line L_j through the 2-tuple embedding. Hence a smooth connected curve C on S'_8 is subcanonical if and only if it is **aG** if and only if either C is a plane conic (complete intersection of type $(2, 1, 1, 1, 1, 1, 1)$) or $C \sim kH,\ k \ge 1$.

Lemma 2.5

Let $\Sigma \subset \mathbb{P}^N$ be a del Pezzo surface of degree $n \geq 3$.

i) If T is an integral lone divisor on Σ, then it is a line.

ii) If $C \sim kH + Y$ is an integral **aG** *curve on Σ with $\alpha \geq 0$, $k \geq 1$ and Y lone, then Y is a line and $k = 1$, in particular $\deg(C) = n + 1$, $p_a(C) = 1$.*

Proof. i) From the exact sequence $0 \to O_\Sigma(-T) \to O_\Sigma \to O_T \to 0$, we get $h^1(O_\Sigma(-T)) = 0$, hence, by Serre duality, $h^1(O_\Sigma(T - H)) = 0$. The exact sequence $0 \to O_\Sigma(T - H) \to O_\Sigma(T) \to O_H(T) \to 0$ implies $1 = h^0(O_\Sigma(T)) = h^0(O_H(T)) = T \cdot H = \deg(T)$, indeed $g(H) = 1$ and $h^1(O_H(T)) = 0$.

ii) Any integral component of Y is also lone, therefore it is a line by (i). Suppose that T_1, T_2 are 2 distinct integral components of Y, then either T_1 and T_2 are skew lines (so $T_1 \cdot T_2 = 0$) or they intersect in only one point (so $T_1 \cdot T_2 = 1$). We claim: $T_1 \cdot T_2 = 0$. Indeed otherwise we would have $O_{T_1}(T_1) = O_{T_1}(-1)$ and $O_{T_2}(T_1 + T_2) = O_{T_2}$, because $T_1^2 = -1$ and $T_2 \cdot (T_1 + T_2) = 0$. Then from the exact sequence $0 \to O_\Sigma \to O_\Sigma(T_1) \to O_{T_1}(-1) \to 0$, we would get $h^1(O_\Sigma(T_1)) = 0$, hence from the exact sequence $0 \to O_\Sigma(T_1) \to O_\Sigma(T_1+T_2) \to O_{T_2} \to 0$, we would get $h^0(O_\Sigma(T_1+T_2)) = 2$, a contradiction: indeed $T_1 + T_2$ is lone because it is contained in Y. Hence $C \sim kH + Y$ with $Y = h_1T_1 + ... + h_rT_r$, with $T_1, ..., T_r$ mutually skew lines, but C is **aCM**, so from 1.3 (a) $H^1(I_Y) = 0$, hence $h^0(O_Y) = 1$, which gives $r = 1$. Therefore we can write $C \sim kH + hL$, with L a line and $h \geq 1$. If $h \geq 2$, then the exact sequence $0 \to O_\Sigma(-hL) \to O_\Sigma[(1 - h)L] \to O_L[(1 - h)L] \to 0$ would imply $h^1(O_\Sigma(-hL)) \neq 0$, indeed $h^0(O_\Sigma[(1-h)L]) = 0$ and $h^0(O_L[(1-h)L]) \neq 0$. Hence, from the exact sequence $0 \to I_\Sigma(k) \to I_C(k) \to O_\Sigma(-hL) \to 0$ $(kH - C \sim -hL)$, we would get $h^1(I_C(k)) \neq 0$, so C could not be **aCM**. Therefore $C \sim kH + L$. We conclude that $k = 1$, because, by 2.2 (iii), $k = -L^2/(\deg(\mathrm{L})) = 1$. The degree and the arithmetic genus follows from standard computations. □

Remembering that S_9 does not contain lines, the following result summarizes the previous 2.3, 2.4, 2.5:

Proposition 2.6

*If C is an integral α-***aG** *curve, not linearly equivalent to tH, on a rational smooth connected* **aG** *surface $\Sigma \subset \mathbb{P}^N$, then only one of the following cases can occur:*

a) $\alpha = -2$, $\deg(C) = 1$, $p_a(C) = 0$ and C is a line;

b) $\alpha = -1$, $\deg(C) = 2$, $p_a(C) = 0$ and C is a smooth conic;

c) Σ is a del Pezzo surface of degree n, $3 \leq n \leq 8$, non-degenerate in some projective n-space contained in $\mathbb{P}^N$, $\alpha = 0$, $3 \leq \deg(C) \leq n + 1$, $p_a(C) = 1$ and $C \sim H + L$ with L a line.

Remark. Subcanonical, arithmetically Gorenstein and complete intersection curves on del Pezzo surfaces are classified in [5; 10 and 14].

2.7. $K3$ **surfaces.** i) A $K3$ surface is an algebraic smooth connected regular 0-subcanonical surface. For general facts on such surfaces we refer, for instance, to [1; VIII], [11; Chapter 3].

Suppose now $\Sigma \subset \mathbb{P}^N,\ N \geq 3$. We have $h^2(I_\Sigma(t)) = h^1(O_\Sigma(t)) = 0$ for any $t \in \mathbb{Z}$ (for $t = 0$ since Σ is regular and for $t \neq 0$ by Kodaira vanishing). If Σ is a $K3$ **aCM** surface, then $\deg(\Sigma) = 2n - 2$, where n, $3 \leq n \leq N$, is the minimum dimension of a projective subspace of $\mathbb{P}^N$ containing Σ. This follows from the fact that the general hyperplane section of Σ is a canonical curve (see [11; 3.2, 3.3]). Note that for any $n \geq 3$ there exists a $K3$ surface Σ, non degenerate in $\mathbb{P}^n$ of degree $2n - 2$ (see [1; VIII.15]) and that this is **aCM**. Indeed it suffices to prove that $h^1(I_\Sigma(t)) = 0$ for any $t \in \mathbb{Z}$ by induction on t, by means of the exact sequence $0 \to I_\Sigma(t-1) \to I_\Sigma(t) \to I_{H,\ \mathbb{P}}(t) \to 0$, where $\mathbb{P}$ is a general hyperplane of $\mathbb{P}^n$, by remembering that H is again a canonical curve.

ii) Let $\Sigma \subset \mathbb{P}^N,\ N \geq 3$, be a $K3$ surface and $C \subset \Sigma$ be an integral α-subcanonical curve and look at 2.2.

If $\alpha < 0$, then either $\alpha = -2$ and C is a line or $\alpha = -1$ and C is a smooth conic. In both cases C is lone on Σ.

If $\alpha = 0$, then $h^0(O_\Sigma(C)) = 2$ and C is minimal, but not lone. Now if C is **aCM**, then from Riemann-Roch and from the fact that $h^0(O_C(1)) \leq h^0(O_\Sigma(1))$ we get $\deg(C) \leq n + 1$, where n, $3 \leq n \leq N$, is the minimum dimension of a projective subspace of $\mathbb{P}^N$ containing Σ.

If $\alpha > 0$, then either $C \sim \alpha H$ or $C \sim Y - [Y^2/(\deg(Y))]H$, where Y is a lone divisor on Σ.

Lemma 2.8

Let Y be a lone divisor on a $K3$ surface S. Then $Y = n_1T_1 + ... + n_kT_k$, where $n_i > 0$ and $T_1, ..., T_k$ are distinct smooth connected rational curves such that $0 \leq T_i \cdot T_j \leq 1$ for any $i \neq j$.

Proof. First we prove the following

Claim. Let Y be as above and suppose that Y is integral. Then $h^1(O_S(Y)) = h^1(O_S(-Y)) = 0$ and Y is smooth rational.

Indeed, arguing as in the proof of 2.4 (i), we get $h^1(O_S(-Y)) = 0$, hence $h^1(O_S(Y)) = 0$ by Serre duality. Now, by Riemann-Roch, $1 = h^0(O_S(Y)) = \chi(O_S(Y)) = Y^2/2 + 2$, i.e. $Y^2 = -2$. On the other hand $Y^2 = 2p_a(Y) - 2$, hence $p_a(Y) = 0$, so Y is smooth rational.

Now let T, T' be distinct components of a lone divisor Y on S. We have $T \cdot T' \geq 0$, because they are distinct. Suppose that $T \cdot T' > 0$. Both T and T' are lone, hence smooth rational, in particular $T^2 = -2$ and $h^1(O_S(T')) = 0$. Since $T + T'$ is lone because it is contained in Y, from the exact sequence $0 \to O_S(T') \to O_S(T+T') \to O_T(T+T') \to 0$, we get $h^0(O_T(T+T')) = 0$. Since

$$\deg_{s_T}(T+T') = T \cdot (T+T') = -2 + T \cdot T' > -2 = 2g(T) - 2,$$

Riemann-Roch theorem for curves implies $0 = T \cdot (T+T') + 1$, i.e. $T \cdot T' = 1$. □

Lemma 2.9

*Let $\Sigma \subset \mathbb{P}^n$, $n \geq 3$ be a non-degenerate $K3$ **aCM** surface and $C \sim Y + \alpha H$, $\alpha = -Y^2/(\deg(Y))$ be an integral α-subcanonical curve with Y lone. If $h^1(I_C(\alpha)) = 0$, then either $\alpha = 2$ and Y is a line or $\alpha = 1$ and Y is a reduced plane conic (i.e. a smooth connected plane conic or the union of two intersecting distinct lines). In particular only 2 pairs (degree, arithmetic genus) can occur for C: $(4n-3, 4n-2)$, $(2n, n+1)$.*

Proof. Look at the exact sequences

$$0 \to I_\Sigma(\alpha) \to I_C(\alpha) \to O_\Sigma(\alpha H - C) \to 0$$

$$0 \to I_\Sigma \to I_Y \to O_\Sigma(\alpha H - C) \to 0$$

Since Σ is **aCM**, we have $h^1(I_Y) = h^1(O_\Sigma(\alpha H - C)) = h^1(I_C(\alpha)) = 0$ and this implies $h^0(O_Y) = 1$. Therefore, arguing as in the proof of Claim in Lemma 2.8, we have $h^1(O_\Sigma(Y)) = h^1(O_\Sigma(-Y)) = 0$, hence Riemann-Roch gives $Y^2 = -2$, so $p_a(Y) = 0$. If Y is integral, then it is smooth rational and, from $-2 = Y^2 = -\alpha\deg(Y)$, either $\alpha = 2$ and Y is a line or $\alpha = 1$ and Y is a smooth plane conic. If Y is not integral, then, from 2.8, its integral components are smooth rational, so, from $\alpha\deg(Y) = 2$, we get that $\alpha = 1$, $Y = T_1 + T_2$, where $T_1 \neq T_2$ are lines verifying $T_1 \cdot T_2 = 1$ (since $h^0(O_Y) = 1$), or $Y = 2T$, where T is a line. We cannot have the last case, because $Y^2 = -2$ and $(2T)^2 = -8$. The assertion on degrees and genera is just a computation, via the fact that $\deg(\Sigma) = 2n - 2$ by 2.7. □

The following result summarizes the previous 2.7, 2.8, 2.9:

Proposition 2.10

*Let $\Sigma \subset \mathbb{P}^N$, $N \geq 3$, be a $K3$ **aCM** surface, non-degenerate in some n-dimensional projective subspace of $\mathbb{P}^N$. If C is an integral α-**aG** curve, not linearly equivalent to tH, on $\Sigma \subset \mathbb{P}^N$, then one of the following cases can occur:*

a) $\alpha = -2$, $\deg(C) = 1$, $p_a(C) = 0$ and C is a line;
b) $\alpha = -1$, $\deg(C) = 2$, $p_a(C) = 0$ and C is a smooth conic;
c) $\alpha = 0$, $3 \leq \deg(C) \leq n+1$, $p_a(C) = 1$;
d) $\alpha = 1$, $\deg(C) = 2n$, $p_a(C) = n+1$ and $C \sim \Delta + H$ with Δ reduced plane conic;
e) $\alpha = 2$, $\deg(C) = 4n-3$, $p_a(C) = 4n-2$ and $C \sim L + 2H$ with L a line. □

References

1. A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Lecture Notes Series **68**, Cambridge University Press, Cambridge, 1983.
2. M. Boij, Gorenstein Artin algebras and points in projective space, *Bull. Lond. Math. Soc.* **31**(1) (1999), 11–16 .
3. G. Casnati, A. Dolcetti, and Ph. Ellia, On subcanonical curves lying on smooth surfaces in $\mathbb{P}^3$, *Rev. Roumaine Math. Pures Appl.* **40**(3–4) (1995), 289–300.
4. M. Demazure, Surfaces de Del Pezzo II, III, IV, V, *Séminaire sur les Singularités des Surfaces, (French) [Seminar on the Singularities of Surfaces], Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977. Edited by Michel Demazure, Henry Charles Pinkham and Bernard Teissier*, 21–69, Lecture Notes in Math. **777**, Springer-Verlag, Berlin Heidelberg New York, 1980.
5. A. Dolcetti, Subcanonical, Gorenstein and complete intersection curves on del Pezzo surfaces, *Ann. Univ. Ferrara, Sez. VII - Sc. Mat.* **57**, (2001), 231–241.
6. Ph. Ellia and R. Hartshorne, Smooth specializations of space curves: questions and examples, *Commutative algebra and algebraic geometry (Ferrara)*, 53–79, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999.
7. W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihre Grenzgebiete 3. Folge · Band 2, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.
8. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York Heidelberg Berlin, 1977.
9. J.O. Kleppe, J.C. Migliore, R.M. Miró-Roig, U. Nagel, and Ch. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc.**154**, 2001.
10. J.C. Migliore, *Introduction to liaison theory and deficiency modules,* Progress in Mathematics **165**, Birkhäuser, Boston Basel Berlin, 1998.
11. M. Reid, Chapters on algebraic surfaces, *Complex algebraic geometry, (Park City, UT, 1993)*, 3–159, IAS/Park City Math. Ser. 3, Amer. Math. Soc., Providence, RI, 1997.