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Self-equivalences of dihedral spheres

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Abstract

Let G be a finite group. The group of homotopy self-equivalences $\mathcal{E}_G(X)$ of an orthogonal G-sphere X is related to the Burnside ring $A(G)$ of G via the stabilization map $I\colon \mathcal{E}_G(X) \subset [X,X]_G \to \{X,X\}_G = A(G)$ from the set of G-homotopy classes of self-equivalences of X to the ring of stable G-homotopy classes of self-maps of X (that is, the 0-th dimensional G-homotopy group of S^0, which is isomorphic to the Burnside ring). As a consequence of the properties of I, $\mathcal{E}_G(X)$ is equal to an extension of a subgroup of the group of units in $A(G)$ with the kernel of I. The aim of the paper is to give examples of a family of equivariant (dihedral) spheres with the property that the kernel of I is a non-abelian torsion-free group with many generators, and to give estimates on the structure of $\mathcal{E}_G(X)$ itself.

1. Introduction

Let S^n, with $n > 0$, denote the sphere of dimension n. The canonical stabilization map I sending homotopy classes of self-maps to stable homotopy classes is a bijection from $[S^n, S^n]$ to $\{S^n, S^n\}$. Moreover, the group of self homotopy equivalences $\mathcal{E}(S^n)$ of S^n is the finite cyclic group of order 2. On the other hand, if we consider equivariant spheres in the category of G-spaces and G-maps (with G finite group) these properties do not hold. If S is a G-sphere, it can be that the canonical stabilization map $I\colon [S,S]_G \to$

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$\{S,S\}_G = A(G)$ from the monoid of equivariant homotopy classes of self-maps of S to the ring of stable homotopy classes of self-maps (which is isomorphic to the 0-th dimensional equivariant homotopy group of S^0 and to the Burnside ring of the group G) is not a bijection. It is trivial to see that it can be not onto, and it was first found by Rubinsztein [5] that it can be not mono.

This idea can be applied to the problem of homotopy self-equivalences of equivariant spheres, in the framework of the classical results relating equivariant stable homotopy theory to ordinary stable homotopy theory (dating back to the works of G. Segal, A. Kosinski, T. tom Dieck and H. Hauschild): it is easy to see that I sends the group of equivariant homotopy self-equivalences $\mathcal{E}_G(S) \subset [S,S]_G$ to the group of units $A(G)^*$ of the Burnside ring (which is a finite abelian 2-group). As before, it is trivial to see that the image $I\mathcal{E}_G(S)$ can be properly contained in $A(G)^*$, and that I can be non-injective even if restricted to $\mathcal{E}_G(S)$. The purpose of this paper is to provide some examples of equivariant spheres with the property that the kernel of $I\colon \mathcal{E}_G(S) \to A(G)^*$ is an infinite non-abelian group. Actually, we produce a family of spheres X_k, with $k = 2 \dots \infty$, equivariant with respect to a suitable group G_k depending upon k, such that the following theorem holds:

Theorem 1

The group of homotopy self-equivalences $\mathcal{E}_{G_k}(X_k)$ is an extension of the kernel of the stabilization map I by a subgroup of order 2^{2^k-1} and index 4 in the group of units of the Burnside ring $A(G)^*$:

$$I\colon \ker I \rightarrowtail \mathcal{E}_{G_k}(X_k) \twoheadrightarrow \mathbb{Z}_2^{2^k-1} \subset A(G_k)^* = \mathbb{Z}_2^{2^k+1}. \tag{1}$$

The kernel $\ker I$ *is a non-abelian torsion-free solvable group of Hirsch length* $h(\ker I) = 3^k - 2^k - k$ *and derived length* $l(\ker I)$ *bounded by*

$$\frac{k}{2} \le l(G_k) \le k-1. \tag{2}$$

The interesting fact is that therefore $\ker I$ is never trivial, for $k \ge 2$, and it is always non-abelian, for $k \ge 3$.

In the last decades many results have been proved about homotopy self-equivalences: for general surveys and the state-of-the-art we refer to the well-known references [6, 4, 3]. In [3] one can find a complete and up-to-date bibliography.

I sincerely wish to express my thanks to the referee, who improved significantly the results of the paper with some generous comments.

2. Preliminaries

If G is a group acting on a space X, then let G_x be the isotropy group of x (that is, the stabilizer of x, or the fixer of x), $G_x = \{g \in G \mid gx = x\}$. Then the space of the points fixed by a subgroup $K \subset G$ is denoted by $X^K = \{x \in X \mid Kx = x\}$; the singular set of X^K is $X^K_s = \{x \in X^K \mid G_x \neq K\}$. If $f\colon X \to X$ is a G-equivariant map and K a subgroup of G, the restriction of f to X^K is denoted by $f^K\colon X^K \to X^K$. The isotropy type of an isotropy group $K \subset G$ is the conjugacy class of K and is denoted with (K). The normalizer of the subgroup K of G in G is denoted by $N_G K$ and is equal to $\{g \in G \mid gKg^{-1} = K\}$. The Weyl group $W_G K$ is equal to the quotient $N_G K/K$. The monoid (with respect to composition of maps) of free G-homotopy classes of self-maps of X is denoted by $[X,X]_G$; the ring of stable G-homotopy classes of self-maps of X is denoted by $\{X,X\}_G$. If X is a sphere, then $\{X,X\}_G$ is isomorphic to the Burnside ring $A(G)$ (e.g. via the equivariant degree homomorphism). The equivariant degree homomorphism is defined as follows: for every $f\colon X \to X$ and for every conjugacy class (K), let $d_G(f)(K) = \deg(f^K)$. If all the subgroups of G are of isotropy, then this defines a homomorphism $\deg_G\colon [X,X]_G \to \prod_{(K)} \mathbb{Z}$ inducing a homomorphism $\deg_G\colon \{X,X\}_G \to \prod_{(K)} \mathbb{Z}$. Now, if $\dim X^G \geq 1$ then the image $\deg_G\{X,X\}_G \subset \prod_{(K)} \mathbb{Z}$ coincides with the image of the Burnside ring $A(G)$ in its ghost ring $\prod_{(K)} \mathbb{Z}$ (see e.g. the last chapter of [7] for more details), and therefore $A(G)$ is isomorphic to $\{X,X\}_G$ in a canonical way. This isomorphism actually can be seen as the definition itself of $A(G)$ in case G is a compact Lie group.

We recall that the derived length $l(G)$ of a (solvable) group G is the length of the shortest abelian series in G (i.e. the number of its non-trivial factors). A group has derived length 0 if it is trivial, 1 if it is abelian. If the length is at least 2, the group is non-abelian. The Hirsch length $h(G)$, also known as torsion-free rank, of a group G is the number of infinite cyclic factors in a polycyclic decomposition of G, that is the number of infinite factors in a series of finite length with factors which are either torsion or infinite cyclic groups.

3. Dihedral spheres

Let $n \geq 2$ be an integer; let ζ_n and h be the transformations of the complex plane $\mathbb{C}$ defined by $\zeta_n(z) = e^{(2\pi i)/n} z$ and $h(z) = \overline{z}$, for every $z \in \mathbb{C}$. With an abuse of notation it is possible to write $\zeta_n = e^{(2\pi i)/n}$. With the symbol D_{2n} we denote the dihedral group of order $2n$, generated by ζ_n and h. Let G be equal to D_{2n}, $N \subset G$ the normal subgroup of G generated by ζ_n and H the subgroup generated by h.

For any integer j let $V(j)$ be the (2-dimensional real) linear representation of D_{2n} given by the action $\zeta_n \cdot z = \zeta_n^j z$ and $h \cdot z = \overline{z}$. Therefore the space fixed by the element $\zeta_n^\alpha \in G$, with $\alpha = 0 \dots n-1$, is

$$(V(j))^{\zeta_n^\alpha} = \begin{cases} V(j) & \text{if } \alpha j \equiv 0 \mod n \\ 0 & \text{if } \alpha j \not\equiv 0 \mod n \end{cases},$$

while the space fixed by the element $\zeta_n^\alpha h \in G$, with $\alpha = 0 \dots n-1$ is $\big(V(j)\big)^{\zeta_n^\alpha h} = \, < e^{\alpha j \pi/ni} >$, where $< z >$ denotes the one-dimensional subspace of the vector space $V(j)$ over $\mathbb{R}$ generated by the element z.

If $j_1, j_2, \ldots, j_k$ are integers, with $k \geq 2$, then let $D(j_1, j_2, \ldots, j_k)$ denote the unit sphere in $V(j_1) \oplus V(j_2) \oplus \ldots \oplus V(j_k)$, endowed with the D_{2n}-action. Assume that $n = p_1 p_2 \ldots p_k$ is the product of the k first odd primes p_i. We denote the corresponding dihedral group with G_k. For every $i = 1 \ldots k$, let $j_i = n/p_i$, and X_k the unit sphere $D(j_1, j_2, \ldots, j_k)$ for such a choice of j_i.

Some properties of X and G are the following:

(i) Two subgroups of G_k are conjugated if and only if they have the same order.

(ii) The minimal non-trivial subgroups of G_k (up to conjugacy) are the cyclic normal groups of order p_i generated by $\zeta_n^{j_i}$ and $H =< h >$.

(iii) The normal proper subgroups of G_k have odd order.

(iv) All the subgroups of G_k are isotropy subgroups, except G_k and the normal subgroup of index 2.

The first three statements are simple. The fourth is a consequence of the following computations. The space fixed by H is a $(k-1)$-dimensional sphere; it is the unit sphere in the vector space of the real parts of the $V(j_i)$, with $i = 1 \ldots k$. The Weyl group $W_G H$ is trivial. The singular set X_s^H in X^H is given by the points in X^H with coordinates $(z_1, z_2, \ldots, z_k)$ such that $\prod_{i=1}^k z_i = 0$ (that is, such that at least one coordinate is zero; it is the union of the intersections of the hyperplanes $z_i = 0$ with X^H). Actually, the isotropy group of a point $(z_1, z_2, \ldots, z_k)$ is the intersection of the isotropy groups of $z_i \in V(j_i)$, for $i = 1 \ldots k$. The principal isotropy type in X is trivial. Therefore the trivial subgroup of G is isotropy. Moreover, the singular set X_s is the union of the points fixed by the minimal cyclic subgroups of G_k. That is, the n conjugated copies of H fix $(k-1)$-spheres, while the normal cyclic groups $< \zeta_n^{j_i} >$ fix the $(2k-3)$-sphere of equation $z_i = 0$. The Weyl groups act on such spheres, and they are G'-homeomorphic to equivariant G'-spheres defined as $D(j'_1, \ldots, j'_{k-1})$ for suitable j'_i and a suitable homomorphic image G' of G (the Weyl groups themselves). More precisely, there are two kind of subgroups in G: normal and non-normal (i.e. odd order or even order): the space fixed by a normal subgroup $N \triangleleft G$ is, regarded as a G/N-space, equivariantly homeomorphic to a G/N-sphere $D(j'_1, \ldots, j'_{k'})$ for suitable j'_i and $k' < k$; k' is equal to the number of minimal odd-order cyclic subgroups G not contained in N. On the other hand, the space fixed by a non-normal subgroup K (which has always trivial Weyl group) is equal to $D(j'_1, \ldots, j'_{k'})^{H'}$ for suitable j'_i and $k' < k$, where H' denotes the minimal subgroup of order 2 in G/N_0 and N_0 denotes the maximal subgroup of K normal in G. That is, it is again equal to a sphere of dimension $k' - 1$ with as singular set the coordinates hyperplanes (if $k' > 0$). Like before, k' is equal to the number of minimal cyclic odd-order subgroups not in N_0. If $k' = 1$, then X^K is a zero-dimensional sphere. If $k' = 0$, that is if N_0 has order n, then clearly $K = G$ and hence the fixed point set is empty.

4. Some results on equivariant spheres

Let G be a finite group and X be an orthogonal G-sphere. Consider the stabilization map $I\colon \mathcal{E}_G(X) \to A(G)^*$. If K is an isotropy group in G for X, then let γ_K denote the number of positive-dimensional components of the space $X_{(K)}/G$ (which is the subspace of the orbit space X/G of the elements with isotropy type equal to (K)). It is not difficult to see that it is homeomorphic to X_K/W_GK. These numbers will play a key role in the computation of the kernel of I.

Proposition 2

The kernel $\ker I$ *of* $I\colon \mathcal{E}_G(X) \to A(G)^*$ *is a torsion-free solvable group, with Hirsch length equal to*

$$h(G) = \sum_{(K)} (\gamma_K - 1),$$

where γ_K *as above is equal to* $|\pi_0(X_K/W_GK)|$ *and* (K) *ranges over the set of isotropy classes such that* $\dim X^K \geq 1$.

Proof. As a consequence of Theorem 1.1 of [2], there is an abelian series

$$1 = \ker I_N \subset \ker I_{N-1} \ldots \subset \ker I_1 \subset \ker I_0 = \ker I$$

such that

$$\frac{\ker I_{j-1}}{\ker I_j} \cong \begin{cases} \mathbb{Z}^{(\gamma_{K_j}-1)} & \text{if } \dim X^{K_j} > 0 \\ 0 & \text{if } \dim X^{K_j} = 0, \end{cases}$$

where N is the number of isotropy types in X. This implies the proposition. □

Proposition 3

The kernel $\ker I$ *of* $I : \mathcal{E}_G(X) \to A(G)^*$ *has derived length* $l(\ker I) \leq d$, *where* d *denotes the maximum dimension of spheres in* X *fixed by subgroups* K *such that* $\gamma_K \geq 2$ *and* $\dim X^K \geq 1$.

Proof. Let X^d denote the union in X of all the spheres of dimension at most d fixed by some non-trivial elements in G, with $0 \leq d$. It is a G-invariant subspace, and therefore there is a well-defined homomorphism (with respect to composition of maps) given by restriction $r_i : \ker I \to [X^i, X^i]^{\mathrm{str}}$, where the latter symbol denotes the homotopy classes of maps preserving the stratification $X^i \supset X^{i-1} \supset \ldots X^0$. A homotopy class f is in $\ker r_i$ if and only if it is G-homotopic to a map which is the identity on X^i. The kernels $\ker r_i$ are a chain of subgroups of $\ker I$ such that

$$1 \triangleleft \ker r_d \triangleleft \ker r_{d-1} \triangleleft \ldots \triangleleft \ker r_0 \triangleleft \ker I$$

and $\ker r_i / \ker r_{i+1}$ can be embedded in the product $\prod_K [X^K, X^K]^*_{X^K_s}$ of groups of the self-equivalences of X^K which are the identity on X^K_s, modulo relative homotopy and with $\dim X^K = i+1$. But this is a product of abelian groups, thus the series $\ker r_i$

has abelian factors. On the other hand it is not difficult to see that $\ker r_d = 1$ (this is a consequence of the fact that if K is the principal isotropy type in X then $\gamma_K = 1$) and that $\ker I = \ker r_0$, because there are no self-equivalences in spheres of dimension 0 which are not detected by the degree. Thus the abelian series is actually of length d. □

The following corollary is an immediate consequence of Proposition 3. Note that if the proof of Proposition 3 had been carried out directly on $\mathcal{E}_G(X)$, then the upper bound would have been $d+2$ instead of $d+1$.

Corollary 4

The group of homotopy self-equivalences $\mathcal{E}_G(X)$ is a solvable group with derived length at most $d+1$ and at least $l(\ker I)$.

5. Homotopy self-equivalences of X_k

Now let $k \geq 2$ and go back to the equivariant sphere $X = X_k$ defined in Section 3. All the proper subgroups (other than the normal subgroup of index 2) of G_k are of isotropy, and there is just one conjugacy class for each order. Let H be the isotropy subgroup generated by $h \in G_k$ as above. The complement of the singular part X_s^H in X^H has 2^k components. Because the Weyl group $W_G H$ is trivial, $\gamma_H = 2^k$. On the other hand, since the minimal cyclic subgroups N of odd order are normal, it is $\gamma_N = 1$. Now, because of the recursive property sketched at the end of Section 2, this is what happens in general to the subgroups: if K is a normal subgroup of G them $\gamma_K = 1$; otherwise, K is isomorphic to $N_0 \rtimes \mathbb{Z}_2$ for a suitable subgroup N_0 normal in G, and $W_G K = 1$. Moreover, in this case the space X^K is equal to a sphere of dimension $k' - 1$, and the singular set X_s^K to the intersection with the coordinates hyperplanes. Thus, again $\gamma_K = 2^{k'}$.

Proposition 5

The kernel $\ker I$ *of the stabilization map*

$$I\colon \mathcal{E}_G(X_k) \to A(G)^*$$

is a torsion-free solvable group of Hirsch length

$$h(G_k) = 3^k - 2^k - k.$$

Proof. Because of Proposition 2, $\ker I$ is a torsion-free solvable group of Hirsch rank

$$\sum_{(K)} (\gamma_K - 1)\,.$$

The sum ranges over all the isotropy types of G such that $\dim X^K \geq 1$; if K is normal then $\gamma_K = 1$, therefore the sum ranges over all the conjugacy classes of subgroups in G that are non-normal, and such that $\dim X^K \geq 1$. These consist of all the even-order proper subgroups of G. The number conjugacy classes of such subgroups fixing a sphere of dimension $d-1$ is equal, up to conjugacy, to $\binom{k}{d}$. Thus the rank of $\ker I$ is equal to

$$\mathrm{rk} = \sum_{\mathrm{d}=2}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{d}} (2^{\mathrm{d}} - 1) = 3^{\mathrm{k}} - 2^{\mathrm{k}} - \mathrm{k}.$$

This completes the proof. □

Proposition 6

The image of $\mathcal{E}_{G_k}(X_k)$ in $A(G)^*$ under the stabilization map I is a subgroup of index 4 in $A(G)^*$.

Proof. The double suspension of the self-join $\Sigma^2 X_k * X_k$, with diagonal action on X_k has the property that all the subgroups of G_k are isotropy (G_k and the maximal normal subgroup N are the only subgroups of G that are isotropy for $\Sigma^2 X_k * X_k$ but not for X_k) and that

$$[\Sigma^2 X_k * X_k, \Sigma^2 X_k * X_k]_{G_k} = \{X_k, X_k\}_{G_k} = A(G_k)$$

(see [5]); hence the index of the image of the stabilization map in $A(G_k)^*$

$$I\colon \mathcal{E}_{G_k}(X_k) \to A(G_k)^* = \mathcal{E}_{G_k}(\Sigma^2 X_k * X_k)$$

is equal to the index of the image of $\mathcal{E}_{G_k}(X_k)$ in $\mathcal{E}_{G_k}(\Sigma^2 X_k * X_k)$, under the inclusion given by $f \mapsto \Sigma^2 f * 1$. It is not difficult to show that this has index 4, by the method given in [2], so the proof follows. Compare with the proof of the following proposition, where the method is applied directly. □

Proposition 7

The group of units in the Burnside ring $A(G_k)^*$ is an elementary abelian 2-group of order 2^{2^k+1}.

Proof. Consider again $\mathcal{E}_{G_k}(\Sigma^2 X_k * X_k)$: the degree homomorphism

$$d_G\colon \mathcal{E}_{G_k}(\Sigma^2 X_k * X_k) \to \prod_{(K)} \mathbb{Z}^* \subset \prod_{(K)} \mathbb{Z}$$

is mono, and hence, by applying again Theorem 1.1 of [2], it is obtained that $A(G)^*$ is an elementary abelian 2-group with at most $\kappa(\Sigma^2 X_k * X_k)$ generators, where κ denotes the number of isotropy groups with Weyl group of order at most 2. In our case, there are exactly 2^k conjugacy classes of subgroups with trivial isotropy, and one conjugacy class with index 2 in G_k (the maximal normal subgroup of G_k). Therefore, $\kappa = 2^k + 1$, and therefore the order of $A(G_k)^*$ is at most 2^{2^k+1}. To show that this is actually an equality, we proceed by using the fact that the order of $A(G)^*$ is the number of solutions of the equation $y \cdot T \in \{0,2\}^{2^{k+1}}$, where 2^{k+1} is the number of conjugacy classes of subgroups of G_k, T is the table of marks matrix of G_k, that is the $2^{k+1} \times 2^{k+1}$ lower triangular matrix with entries equal to $\left|\left(\frac{G}{K}\right)^H\right|$ (the cardinality of the fixed point set of the action of H on the set of left K-cosets in G; see [2] for further details); and y is a 2^{k+1}-dimensional row vector of integers. Let y_K denote the entry corresponding to the conjugacy class of subgroups (K). We can rewrite the same system of equations as

$$\forall (H) \leq (G)\colon \quad \sum_{(K) \geq (H)} y_K \left|\left(\frac{G}{K}\right)^H\right| \in \{0,2\}. \tag{3}$$

Consider now first the subgroups $K \subset G$ belonging to the following class $\mathcal{I}$: either they are self-normalizing, or of index 2 in G. These subgroups consist of the non-normal subgroups, plus G and the maximal normal subgroup N. If H belongs to $\mathcal{I}$ then whenever $(K) \geq (H)$ necessarily $K \in \mathcal{I}$. This means that in the equations (3) corresponding to a subgroup H in $\mathcal{I}$ the only non-zero coefficients of the y_K's are those $K \in \mathcal{I}$. That is, the integers y_K with $K \in \mathcal{I}$ are completely determined once they solve the sub-system of equations

$$\forall (H) \leq (G), H \in \mathcal{I}\colon \quad \sum_{\substack{(K) \geq (H)\\ K \in \mathcal{I}}} y_K \left|\left(\frac{G}{K}\right)^H\right| \in \{0,2\}. \tag{4}$$

But because $\left|\left(\frac{G}{H}\right)^H\right| = |W_G H|$, on the diagonal of this sub-matrix there are only 1's and one 2 (corresponding to N). Thus the matrix can be inverted, and the number of solutions in the y_K is exactly $2^k + 1$, i.e. the number of conjugacy classes in $\mathcal{I}$.

Now we show that every such $(2^k + 1)$-tuple can be extended to a solution $y \in \mathbb{Z}^{2^{k+1}}$, i.e. that given a solution of the restricted system (4) it is possible to define integers y_K also for $K \notin \mathcal{I}$ such that the global equations (3) hold. If K is a subgroup not in $\mathcal{I}$, then its order is odd. Moreover, for each K there is a unique conjugacy class (K') of subgroups of order $2|K|$. Define $y_K = -\frac{1}{2} y_{K'}$. By assumption, if H is in $\mathcal{I}$ then (3) is automatically satisfied. Otherwise, the equations can be written as follows:

$$\sum_{\substack{(K) \geq (H)\\ K \notin \mathcal{I}}} y_K \left[\left|\left(\frac{G}{K}\right)^H\right| + y'_K \left|\left(\frac{G}{K'}\right)^H\right|\right] + y_N \left|\frac{G}{N}\right| + y_G \in \{0,2\}. \tag{5}$$

Now, because $H \triangleleft G$ we have for all K

$$\left|\left(\frac{G}{K}\right)^H\right| = \left|\frac{G}{K}\right| = 2\left|\frac{G}{K'}\right|$$

thus equation (5) can be written as

$$\sum_{\substack{(K) \geq (H)\\ K \notin \mathcal{I}}} \left[\Big(-\frac{1}{2} y'_K\Big) 2\left|\frac{G}{K'}\right| + y'_K \left|\frac{G}{K'}\right|\right] + y_N \left|\frac{G}{N}\right| + y_G \in \{0,2\}, \tag{6}$$

that is $2y_N + Y_G \in \{0,2\}$, and this is certainly true because it is equation (4) with $H = N$.

As a consequence, the number of generators of the Burnside ring is exactly $2^k + 1$, i.e. the claim. $\square$

6. The derived length

The collection of the previous propositions can easily be seen as a proof of the first part of Theorem 1. There is only to give the lower bound of the derived length $l(\ker I)$. So consider once more the abelian series $\ker r_i$ in $\ker I$ defined in the proof of Proposition 3. Let d denote the maximum dimension of a sphere fixed by a subgroup K with $\gamma_K \geq 2$. In our case, $d = k - 1$. If $k = 2$, then $\ker I = \mathbb{Z}^3$ and $l(\ker I) = 1$, being abelian. So from now on, assume $k \geq 3$.

Proposition 8

For every $i = 1 \dots d-1$ the group $\ker r_{i-1}/\ker r_{i+1}$ is non-abelian.

Proof. The group $\ker r_{i-1}/\ker r_{i+1}$ by definition is isomorphic to the image of $\ker r_{i-1}$ under the restriction homomorphism r_{i+1}; that is, its elements can be seen as homotopy classes of isotropy stratification preserving equivariant (with respect to the Weyl group action) self-equivalences defined on fixed spheres of dimension $i + 1$, that are the identity whenever restricted to spheres of dimension $i-1$ and such that the restriction to any single sphere of dimension $i + 1$ or i has degree 1. The group $\ker r_{i-1}/\ker r_i$ can be embedded in $\ker r_{i-1}/\ker r_{i+1}$ as follows: an element is, as above, a class of orientation-preserving self-equivalences on X^i that fix X^{i-1} and that have degree 1 whenever restricted to single spheres of dimension i. Let f_0 denote such a map. We will extend f_0 to a map f on X^{i+1}.

Let X^K be a sphere of dimension $i+1$ in X^d; the singular set X^K_s is equal to the intersection of X^K with X^i, therefore f^K_s is defined as the restriction of f_0. If $\gamma_K = 1$, then there is a single map with the appropriate degree, and therefore f can be defined on X^K. Otherwise, X^K_s is the union of $i + 2$ spheres of dimension i in X^K, which divide X^K into $\gamma_K = 2^{i+2}$ components.

Every component of X_K is homeomorphic to the interior of a $i+1$-simplex Δ_{i+1}, and the interiors of its $i + 2$ i-dimensional faces are components of $X^K \cap X^i \setminus X^{i-1}$. The total amount of such faces for all the components of X_K is therefore $(i + 2)2^{i+1}$.

So let Δ_{i+1} be one of the components of X_K and $\Delta_{i+1} \times I$ the cartesian product with the unit interval. We first define a map f_c on $\Delta_{i+1} \times \{0\} \cup \partial\Delta_{i+1} \times I$, and then by composition with the cylinder map

$$\Delta_{i+1} \to \Delta_{i+1} \times \{0\} \cup \partial\Delta_{i+1} \times I$$

we obtain a map on Δ_{i+1}. The map f_c is defined by $f_c(x, t) = x$ if $x \in \Delta_{i+1} \cap X^{i-1}$ or $t = 0$; else, $f_c(x, 1) = f^K_s$ if $x \in \partial\Delta_{i+1} = \Delta_{i+1} \cap X^K_s$. The side-faces of the cylinder are of the form $\Delta_i \times I$, where Δ_i is a i-simplex in Δ_{i+1}. The map on the boundary of $\Delta_i \times I$ is thus defined, and attains its values outside inside the sphere S^i in X^K containing Δ_i. It is therefore possible to extend f_c in a unique (up to homotopy) way such that the image of $\Delta_i \times I$ does not contain any point of the component of the complement of S^i containing Δ_{i+1}. Doing this for all the faces of Δ_{i+1}, and for all the possible components Δ_{i+1} in X_K gives a well-defined map f^K. The degree in Δ_{i+1} of f^K relative to a regular point $x \in X_K$ is, by construction, equal to

$$\deg(f^K|\Delta_{i+1};x) = \begin{cases} 1 & \text{if } x \in \Delta_{i+1} \\ -\sum_{\Delta_i \in \partial\Delta_{i+1}} w_f(\Delta_i)\cdot \chi_{\Delta_{i+1},\Delta_i}(x) & \text{if } x \in X_K \setminus \Delta_{i+1}, \end{cases} \tag{7}$$

where $\chi_{\Delta_{i+1},\Delta_i}(x)$ is defined to be, 0 or 1 if x respectively is or is not in the same component of $X^K \setminus S^i$ as Δ_{i+1}, and S^i as above is the i-sphere containing Δ_i; f_{Δ_i} denotes the map f_0 restricted to the sphere S^i; furthermore, $w_f(\Delta_i)$ denotes the degree $\deg(f_{\Delta_i}|\Delta_i; x' \in S^i \setminus \Delta_i)$. Then by construction

$$\sum_{\Delta_i \in S^i} -w_f(\Delta_i) = \deg(f_{\Delta_i}) - 1 = 0,$$

and summing the equations in (7) allows to conclude that $\deg(f^K) = 1$.

We have proved that this construction gives an embedding, not a homomorphism. In fact, as seen in Remark 1, it is not a homomorphism.

Now let S^i one of the i-spheres in X^K for a K, and let $\overline{\Delta}_i$ and $\overline{\Delta}'_i$ two consecutive i-faces in S^i. Let $\varphi : S^i \to S^i$ be the map which is the identity whenever restricted to $S^i \cap X^{i-1}$, such that the degree of φ in $\overline{\Delta}_i$ relative to a point in $\overline{\Delta}'_i$ is 1, and the degree in $\overline{\Delta}'_i$ relative to a point in $\overline{\Delta}_i$ is -1. It is a stratified self-equivalence of S^i, with as inverse the map constructed in the same way but exchanging the roles of $\overline{\Delta}_i$ and $\overline{\Delta}'_i$. Extend it then to f^K_s, being it the identity on the other spheres in X^K_s, and to a map $f^K\colon X^K \to X^K$ with the procedure above described. Let φ' the map $X^K \to X^K$ obtained in the same fashion, by starting with the inverse of the equivalence φ in S^i, instead. To show that actually φ' is the inverse of φ (as stratified maps of X^K), it is sufficient to show that the composition $\varphi\varphi'$ is homotopic (relative to the stratification) to the identity. We again use the local degrees:

$$\begin{aligned} &\deg(\varphi\varphi'|\Delta'_{i+1};\Delta''_{i+1}) \\ &\quad = \sum_{\Delta_{i+1}} \deg(\varphi|\Delta_{i+1};\Delta''_{i+1})\cdot\deg(\varphi'|\Delta'_{i+1};\Delta_{i+1}) \\ &\quad = \sum_{\substack{\Delta_{i+1}\neq\Delta'_{i+1}\\ \Delta_{i+1}\neq\Delta''_{i+1}}} \sum_{\substack{\Delta_i\in\partial\Delta_{i+1}\\ \Delta'_i\in\partial\Delta'_{i+1}}} w_\varphi(\Delta_i) w_{\varphi'}(\Delta'_i)\chi_{\Delta_{i+1},\Delta_i}(\Delta''_{i+1})\chi_{\Delta'_{i+1},\Delta'_i}(\Delta_{i+1}) \\ &\qquad + \deg(\varphi|\Delta''_{i+1};\Delta''_{i+1})\deg(\varphi'|\Delta'_{i+1};\Delta''_{i+1}) \\ &\qquad \big(+\deg(\varphi|\Delta'_{i+1};\Delta''_{i+1})\deg(\varphi'|\Delta'_{i+1};\Delta'_{i+1})\big). \end{aligned}$$

The last term, of course, occurs if and only if $\Delta'_{i+1} \neq \Delta''_{i+1}$. In this case, the sum of the two last terms vanishes; but the first sum can be written as

$$\sum_{\Delta'_i\in\partial\Delta'_{i+1}} w_{\varphi'}(\Delta'_i)\Bigg(\sum_{\substack{\Delta_{i+1}\neq\Delta'_{i+1}\\ \Delta_{i+1}\neq\Delta''_{i+1}}} \sum_{\Delta_i\in\partial\Delta_{i+1}} w_\varphi(\Delta_i) \times \chi_{\Delta_{i+1},\Delta_i}(\Delta''_{i+1})\chi_{\Delta'_{i+1},\Delta'_i}(\Delta_{i+1})\Bigg);$$

moreover, the product $w_\varphi(\Delta_i)w_{\varphi'}(\Delta_i')$ is different than zero if and only if both Δ_i and Δ_i' belong to the set $\{\overline{\Delta}_i, \overline{\Delta}_i'\}$; thus, for such terms in the sum the product

$$\chi_{\Delta_{i+1},\Delta_i}(\Delta''_{i+1})\chi_{\Delta'_{i+1},\Delta'_i}(\Delta_{i+1})$$

is equal to one if and only if Δ'_{i+1} and Δ''_{i+1} both belong to the same component of $X^K \setminus S^i$ (where S^i is the sphere containing Δ_i. This immediately implies that if Δ'_{i+1} and Δ''_{i+1} do not belong to the same component, then $\deg(\varphi\varphi'|\Delta'_{i+1};\Delta''_{i+1}) = 0$. Otherwise, there are just two non-zero terms in the sum, i.e. those corresponding to the two components Δ_{i+1} such that $\overline{\Delta}_i \in \partial\Delta_{i+1}$ or $\overline{\Delta}_i' \in \partial\Delta_{i+1}$. But by definition the sum $w_\varphi(\Delta_i) + w_\varphi(\Delta_i')$ vanishes, and hence $\deg(\varphi\varphi'|\Delta'_{i+1};\Delta''_{i+1}) = 0$. With a similar argument it is easy to show that $\deg(\varphi\varphi'|\Delta'_{i+1};\Delta'_{i+1}) = 1$, that is, φ' is the (stratified homotopical) inverse of φ.

Now, consider a self-equivalence of degree one $m\colon X^K \to X^K$ which extends the identity on X_s^K; let $M(\Delta_{i+1},\Delta'_{i+1})$ denote the degree of m restricted to a component Δ_{i+1} relative to a point in Δ'_{i+1}. As a matrix, M is of the form $I + M_0$, where M_0 is a matrix such that $M_0(\Delta_{i+1},\Delta'_{i+1})$ does not depend upon Δ'_{i+1} and with trace equal to 0. Every matrix M_0' with integer coefficients, constant columns and trace 0 can occur as a degrees matrix $I + M_0'$ for some self-equivalence.

We want to show that $\varphi m \not\sim m\varphi$. To do that, it is better to simplify the procedures used above (see also Remark 1 below). Let K denote the matrix of degrees of φ and M the matrix of degrees of m. Then $M = I + M_0$ for a matrix M_0 with constant columns and trace 0, and $K = I + K_0$, where K_0 is a matrix with just 4 columns different than zero (the ones corresponding to the components Δ_{i+1} such that $\partial\Delta_{i+1} \cap \{\overline{\Delta}_i, \overline{\Delta}_i'\}$ is not empty); the values in the column relative to a component Δ_{i+1} containing e.g. $\overline{\Delta}_i$ are all 0 for those rows corresponding to a Δ''_{i+1} on the same component in $X^K \setminus S^i$ of Δ_{i+1}, otherwise equal to $-w_\varphi(\overline{\Delta}_i)$. Looking at rows, it is easy to see that in any row of K_0 there are exactly 2 non-zero values, one $+1$ and one -1 (see for example the matrices occurring in Remark 1).

Now, because of the property of rows of K_0 and columns of M_0, $K_0M_0 = 0$. On the other hand M_0K_0 is the zero matrix if and only if the trace of the matrix M_0', obtained by deleting all the rows and columns corresponding to components Δ_{i+1} belonging to one of the two components of $X^K \setminus S^i$, is zero. But it's clear that it is possible to select M_0 in a way that this does not occur: for example, M_0 might have just two columns, one full of 1's, corresponding to a Δ_{i+1}, and one full of -1's, corresponding to a Δ'_{i+1} on the different component in $X^K \setminus S^i$. Thus $fm \sim mf$ if and only if

$$(I + K_0)(I + M_0) - (I + M_0)(I + K_0) = K_0M_0 - M_0K_0 = 0,$$

but $K_0M_0 - M_0K_0 = M_0K_0 \neq 0$, and hence $fm \not\sim mf$.

To complete the proof, it suffices to extend such maps to the whole X^{i+1} as the identity outside X^K: for the same reason, it occurs that for all i the group $\ker r_{i-1}/\ker r_{i+1}$ is not commutative. $\square$

We are now in the position to finish the proof of Theorem 1. Applying Proposition 8, it is seen that the abelian series

$$1 = \ker r_d \triangleleft \ker r_{d-1} \triangleleft \ldots \triangleleft r_0 = \ker I$$

has non-abelian consecutive factors $\ker r_{i-1}/\ker r_{i+1}$. Therefore its derived length must be at least $(d+1)/2$. Being $d = k-1$, the derived length $l(\ker I)$ is thus at least $k/2$. The proof of the theorem is finished.

7. Remarks

Remark 1. The extension procedure used in the proof of Proposition 8 actually does not yield an homomorphism. To see the reason, consider two maps φ_1 and φ_2 defined in the same way as φ above, with the faces $\overline{\Delta}_i$ and $\overline{\Delta}'_i$ used for φ_1 different than the ones used for the definition of φ_2. For example, in a 2-sphere X^K, let $1 \ldots 8$ denote the components of $X^K \cap X^1$, such that $\{1,2,3,4\}$ is the upper hemisphere, $\{1,2,5,6\}$ the East hemisphere and $\{2,3,6,7\}$ the North hemisphere (assume the East-Nord plane to be horizontal). Then φ_1, defined using the faces $\{1,4\}$ and $\{5,8\}$ (i.e. the faces belonging to the boundaries of 1 and 4, or 5 and 8), has matrix form $I + K_1$, with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the faces $\{2,3\}$ and $\{6,7\}$ instead, the matrix form is $I + K_2$, with

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Since $K_1K_2 - K_2K_1$ is a matrix with 8 rows equal to the vector

$$(1, -1, 1, -1, -1, 1, -1, 1),$$

$K_1K_2 - K_2K_1 \neq 0$, and so the composition $(I + K_1)(I + K_2)$ cannot be the map obtained using the procedure of above, starting from the restriction of the composition

$\varphi_1\varphi_2$, for the simple reason that the restrictions of φ_i commute, while the maps φ_1 and φ_2 themselves not. In fact, these commute if and only if

$$(I + K_1)(I + K_2) - (I - K_1)(I - K_2) = 0 \iff K_1K_2 - K_2K_1 = 0.$$

Remark 2. The group of self-equivalences of S^1 fixing 4 distinct points (i.e. the first step in the inductive process of defining maps on X_k) is represented as follows: let B denote the set of all the 4×4 matrices with integer coefficients that can be written as $I + M$, where I is the identity matrix and M any 4×4–matrix with constant columns and trace equal to -2 or 0. Then, if M_1 and M_2 are such matrices, $M_1M_2 = \mathrm{Tr(M_1)M_2}$, and

$$(I + M_1)(I + M_2) = I + M_1 + M_2 + \mathrm{Tr(M_1)M_2}.$$

It is a group with respect to the product of matrices, isomorphic to $\mathbb{Z}^3 \rtimes \mathbb{Z}_2$. The action of $\mathbb{Z}_2$ on $\mathbb{Z}^3$ is non-trivial, as can be seen by the fact that, if $\mathrm{Tr(M)} = 0$ and $\mathrm{Tr(M_2)} = -2$, then $I + M_2$ has order 2 and acts as

$$(I + M_2)(I + M)(I + M_2) = I - M = (I + M)^{-1}.$$

But, taking local degrees, it is possible to see that B is actually isomorphic to the group of self-equivalences fixing 4 points. Thus, by restricting to the sphere $S^1 \subset X_k$ for some $k \geq 2$, the sequence $\mathbb{Z}^3 \to B \to \mathbb{Z}_2$ is a homomorphic image of the sequence $\ker I \to \mathcal{E}_{G_k}(X_k) \to A(G)^*$. That is, $A(G)^*$ cannot be a direct factor of $\mathcal{E}_{G_k}(X_k)$ (but the torsion group of $\mathcal{E}_{G_k}(X_k)$ is still unknown).

Remark 3. The quotient X_2/G_2 is a sphere. To see that, proceed as follows: the group G_2 has polycyclic decomposition as $\mathbb{Z}_3 \lhd \mathbb{Z}_{15} \lhd G_2$. $D(3,5)/\mathbb{Z}_3$ is a sphere (actually, it is a orbifold with a singular part of dimension 1 of cone points). Then $\mathbb{Z}_{15}$ acts on this quotient, again providing as a quotient a sphere with another circle of cone points. These two circles are linked in the resulting sphere, as well as their pre-images in $D(3,5)$. The last factor of the series is $\mathbb{Z}_2$ and the quotient of $D(3,5)/(\mathbb{Z}_{15})$ under this action is again a sphere, with a third circle of cone points, which intersect the other two described above.

Another more general way of proving the same result is the following: consider the map

$$\Lambda\colon D(j_1,\ldots,j_k) \to S^{2k-1}$$

defined by

$$\Lambda\colon (z_1,\ldots,z_k) \mapsto \left(z_1\frac{z_1^{p_1-1}}{|z_1|^{p_1-1}},\ldots,z_k\frac{z_k^{p_k-1}}{|z_k|^{p_k-1}} \right) \in S^{2k-1} \subset \mathbb{C}^k.$$

It is easy to see that $\Lambda(x) = \Lambda(y)$ if and only if there is a power ζ_n^α such that $y = \zeta_n^\alpha x$. Thus Λ provides a homeomorphism from $D(j_1,\ldots,j_k)/N$ to S^{2k-1}, where N denotes the maximal cyclic normal subgroup of G. Furthermore, the Λ is G_k-equivariant, if we define the action of G on S^{2k-1} as $h(z_1,z_2,\ldots,z_k) = (\overline{z_1},\ldots,\overline{z_k})$ and $\zeta_n z = z$. The quotient X_k/G_k is thus equal to the quotient S^{2k-1}/h, i.e. the join $\mathbb{R}P^{k-1} * S^{k-1}$; if $k=2$ then this is equal to the three-sphere $S^1 * S^1 = S^3$.

Remark 4. As we have seen in Section 2, there is a (unique) inclusion $G_k \hookrightarrow G_{k+1}$ and a (unique) projection $G_{k+1} \twoheadrightarrow G_k$, as long as an inclusion $X_k \hookrightarrow X_{k+1}$, which is G_{k+1}-equivariant (using the projection to let G_{k+1} act on X_k). Thus $\mathcal{E}_{G_k}(X_k) = \mathcal{E}_{G_{k+1}}(X_k)$, and so there is a well-defined restriction G_{k+1}-homomorphism

$$\mathcal{E}_{G_{k+1}}(X_{k+1}) \twoheadrightarrow \mathcal{E}_{G_k}(X_k).$$

It is possible to see that it is onto, and that its kernel is an solvable non-abelian group with Hirsch length $2\,3^k - 2^k - 1$.

The spheres X_k can be seen as the a direct system $X_k \subset X_{k+1}$ of G_∞-spheres, where G_∞ is the direct limit of the G_k's. Let X_∞ denote such limit. Using again the restriction homomorphism, there are surjective projections $\mathcal{E}_{G_\infty}(X_\infty) \twoheadrightarrow \mathcal{E}_{G_k}(X_k)$ for every k. Thus $\mathcal{E}_{G_\infty}(X_\infty)$ is a group with infinitely many generators.

Remark 5. It is also possible to consider the general case of the dihedral group $G = D_{2n}$. The 3-dimensional dihedral sphere $D(q_1, q_2)$ defined as above might be of some interest by itself: if q_1 and q_2 are arbitrary integers such that $(q_1, n) = (q_2, n) = 1$, then the quotient $D(q_1, q_2)/G$ is another (orbifold) sphere. The partial factorization

$$D(q_1, q_2) \to D(q_1, q_2)/N \to D(q_1, q_2)/G,$$

where $N =< \zeta_n >\subset G$ is the normal n-cyclic subgroup, gives rise to two maps of degrees n and 2 (the first is the covering)

$$S^3 \to L\big(n, q_1^{-1} q_2\big) \to S^3,$$

where $L(n, q_1^{-1} q_2)$ is the lens space and the inverse q_1^{-1} is meant as inverse mod$n$. Another feature of this equivariant sphere is that the singular set of points of isotropy of order 2 is a link of n circles, with linking numbers 1. In case of X_2 these circles are not disjoint.

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