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Translational averaging for completeness, characterization and oversampling of wavelets

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Abstract

The single underlying method of "averaging the wavelet functional over translates" yields first a new completeness criterion for orthonormal wavelet systems, and then a unified treatment of known results on characterization of wavelets on the Fourier transform side, on preservation of frame bounds by oversampling, and on the equivalence of affine and quasiaffine frames. The method applies to multiwavelet systems in all dimensions, to dilation matrices that are in some cases not expanding, and to dual frame pairs.

The completeness criterion we establish is precisely the discrete Calderón condition. In the single wavelet case this means we take invertible matrices a and b and a function $\psi \in L^2(\mathbb{R}^d)$, and assume either a is expanding or else a is amplifying for ψ. We prove that the system $\{|\det a|^{j/2}\psi(a^j x - bk) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if it is orthonormal and $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\xi a^j)|^2 = |\det b|$ for almost every row vector $\xi \in \mathbb{R}^d$.

1. Introduction

We aim to obtain a number of new and recent results on wavelets within a common framework, by averaging the wavelet functional for ψ over suitable translates of $f \in L^2$.

To illustrate in one dimension, let $\psi_{j,z}(x) = a^{j/2}\psi(a^j x - z)$ for $j \in \mathbb{Z}, z \in \mathbb{R}$, where $\psi \in L^2(\mathbb{R})$ and $a > 1$ are fixed. Write

$$C(f) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\langle f, \psi_{j,z}\rangle|^2 \, dz \qquad \text{and} \qquad D(f) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2$$

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for $f \in L^2(\mathbb{R})$, with $\langle\,,\rangle$ denoting the complex inner product on $L^2(\mathbb{R})$. The letters C and D refer to the continuous and discrete translations that are employed in these functionals, respectively. Let $f_z(x) = f(x-z)$ denote the translate of f by $z \in \mathbb{R}$.

The distinguishing and appealing feature of our approach is that the usual technical estimates involving $\hat{f}$ and $\hat{\psi}$ can be relegated to the background (in Section 9). The foreground is dominated instead by an expansion for the wavelet functional of a translate of f:

$$(1) \qquad D(f_z) = \sum_u c_u(f) \exp(2\pi i u z), \qquad z \in \mathbb{R},$$

where this almost periodic, absolutely convergent sum is taken over a certain countable set of $u \in \mathbb{R}$ (see (5) below). The expansion holds for a dense set of $f \in L^2(\mathbb{R})$.

As we show in Lemma 9.2, $C(f)$ equals the zero-th coefficient $c_0(f)$. And because this constant term $c_0(f)$ equals the large-scale average of the sum in (1), we deduce our first averaging formula,

$$C(f) = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} D(f_z)\, dz.$$

Now assume the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is orthonormal, so that $D(f) \leq \|f\|_2^2$. Call ψ an *orthonormal wavelet* if this system is complete in $L^2(\mathbb{R})$, which is equivalent to having $D(f) = \|f\|_2^2$ for all f. By the above large-scale averaging formula, this implies $C(f) = \|f\|_2^2$ for all f, which is known (see Appendix A) to be equivalent to the Calderón condition $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\xi a^j)|^2 = 1$ for almost every $\xi \in \mathbb{R}$. Conversely, if $C(f) = \|f\|_2^2$ for all f then from the averaging formula together with the orthonormality of the $\psi_{j,k}$ and the almost periodicity of the function $z \mapsto D(f_z)$, one deduces (in §3) that $D(f_z) = \|f\|_2^2$ for all z. Putting $z = 0$ shows ψ is an orthonormal wavelet.

The above account shows that one can characterize completeness of orthonormal wavelet systems by suitably averaging the wavelet functional (and employing some other facts). We carry this out in full in Sections 2 and 3 below, in all dimensions. More precisely, in Section 2 we specify the allowable dilation matrices, which need not map the lattice of translations into itself and need not be expanding in all directions, and then we state the completeness criterion to be proved. Section 3 derives this criterion as a corollary of results on multiwavelets with continuous (rather than discrete) families of translations. In Section 4 we handle continuous families of dilations.

Then in Section 5 we use the almost periodic function $z \mapsto D(f_z)$ to quickly reprove a tight frame characterization that is due in its initial form to Gripenberg and Wang, and which was developed to its most general form by many authors (as detailed in §5). Our method again involves averaging, this time in the guise of the elementary Lemma D.2 that allows us to equate coefficients of almost periodic trigonometric sums.

In Section 6 we prove the Second Oversampling Theorem, giving preservation of frame bounds for oversampled affine systems, in a form due to Ron and Shen that extends the original oversampling results of Chui and Shi. Our method explicitly expresses the oversampled functional $D_s(f)$ as an average of $D(f_z)$ over a countable set of z-translates (with these translates being determined by the oversampling matrix s). Ultimately, the proof involves averaging the exponentials in expansion (1) by means of the elementary Lemma D.3, which in one dimension reduces just to the geometric sum.

Next, Section 7 re-derives Ron and Shen's theorem on equivalence of affine and quasiaffine frames by explicitly expressing the quasiaffine functional $D^q(f)$ as the average of $D(f_z)$ over a countable set of z-translates, and expressing $D(f)$ as a limit of $D^q(\cdot)$ over dilates of f.

We extend the completeness and oversampling results to cover pairs of dual frames in Section 8. The wavelet functional is complex-valued, in the dual frame situation, and the oversampling result says that convex sets of values in the complex plane are preserved by oversampling.

Finally, Section 9 contains the technical estimates needed for all these proofs, with the main task being to show that the coefficients in the expansion of $z \mapsto D(f_z)$ are absolutely convergent, in other words $\sum_u |c_u(f)| < \infty$ in the terminology above.

Notes. This paper extends the author's earlier work [25], which proved the completeness criterion for single wavelet systems in one dimension. But [25] did not treat tight frame characterizations, quasiaffine frames or dual frames, as we do here, and in one dimension the 1×1 dilation matrices are automatically expanding in all directions, unlike in this paper.

Many of the characterization results in Sections 2, 3 and 5 have been extended to cover both wavelet and Gabor systems simultaneously, in a forthcoming work of Hernández, Labate and Weiss [19]. More precise references are given below.

2. Notation, and characterization of completeness

Throughout the paper we fix a positive integer d and write $L^2 = L^2(\mathbb{R}^d)$. We fix $m \in \mathbb{N}$ and take "dilation" matrices $a_1 \dots, a_m \in GL(d, \mathbb{R})$ with $|\det a_l| \neq 1$ for each l, and also "translation" matrices $b_1, \dots b_m \in GL(d, \mathbb{R})$. We take functions $\psi_1, \dots, \psi_m \in L^2$. For each $l = 1, \dots, m$ we assume either (i) or (ii) below. The reader might want to concentrate on the "expanding" case (i), when first reading the paper, and ignore the more technical "amplifying" case (ii).

(i) a_l is an *expanding* matrix, meaning there exists $0 < \kappa \leq 1 < \gamma$ such that

$$|\xi a_l^j| \geq \kappa \gamma^j |\xi| \quad \text{and} \quad |\xi a_l^{-j}| \leq \kappa^{-1} \gamma^{-j} |\xi|$$

for all row vectors $\xi \in \mathbb{R}^d$ and all integers $j \geq 0$. (Of course, either one of these two inequalities implies the other.) Recall a_l is expanding if all its eigenvalues satisfy $|\lambda| > 1$; see [6, Remark 2.2].

(ii) a_l is *amplifying for* ψ_l, defined as follows. Call an expanding sequence of open sets $\{A_l(r) : r = 1, 2, 3, \dots\}$ in $\mathbb{R}^d$ an *exhaustion of* $\mathbb{R}^d$ if the union $\cup_{r=1}^{\infty} \mathcal{A}_l(r)$ has full measure in $\mathbb{R}^d$ and $\mathcal{A}_l(r)$ is contained in the ball $B(r)$, for all r. We say a_l is *amplifying for* ψ_l if there exists an exhaustion of $\mathbb{R}^d$ such that for each $r \in \mathbb{N}$,

$$\operatorname{spt}(\widehat{\psi_l}) \cap (\mathcal{A}_l(r) a_l^j) = \emptyset \qquad \text{whenever } |j| \text{ is sufficiently large.}$$

The role of the expanding and amplifying assumptions, technically speaking, is to guarantee the absolute convergence of the series for $z \mapsto D(f_z)$, for example in equation (1). More precisely, they guarantee that Lemma 9.3 holds.

Note that any expanding matrix is automatically amplifying for ψ_l provided $\widehat{\psi_l}$ vanishes near the origin and infinity, as one sees by taking $\mathcal{A}_l(r) = \{\xi \in \mathbb{R}^d : r^{-1} < |\xi| < r\}$. For an example in which the non-expanding matrix $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is amplifying, with a particularly instructive exhaustion, see Remark 3 in Section 3.

The point of including the amplifying criterion (ii) in this paper is to demonstrate that certain non-expanding dilation matrices can indeed be handled. But I hope future authors will develop conditions that do not depend on the wavelets ψ_l. (Such a condition is included in the work of Hernández, Labate and Weiss [19, §5].)

Our first aim is to characterize the completeness of the system

$$\Psi = \{\psi_{j,k,l} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \dots, m\},$$

where

$$\psi_{j,k,l}(x) = |\det a_l|^{j/2} \psi_l(a_l^j x - b_l k), \qquad x \in \mathbb{R}^d.$$

Theorem 2.1

If Ψ is an orthonormal system in L^2, then it is complete in L^2 if and only if

$$\sum_{j \in \mathbb{Z}} \sum_{l=1}^{m} \frac{|\widehat{\psi_l}(\xi a_l^j)|^2}{|\det b_l|} = 1 \quad \textit{for almost every row vector} \xi \in \mathbb{R}^d. \tag{2}$$

That is, $\{\psi_1, \dots, \psi_m\}$ is a "multiwavelet family" with respect to the dilation matrices $a_1, \dots, a_m$ and translation matrices $b_1, \dots, b_m$ if and only if the the $\psi_{j,k,l}$ are orthonormal and the "discrete Calderón condition" (2) holds. The necessity of the Calderón condition was already known (at least when the matrices a_l are all expanding), and so the interesting part of the theorem is the claim of sufficiency. We prove the theorem in Section 3.

Theorem 2.1 was conjectured by Guido Weiss in 1999 (unpublished) in the wavelet case $m = 1$. The theorem was proved by Bownik [2] and Rzeszotnik [30] in the special case of equal dilation matrices $a_1 = \cdots = a_m$ that preserve the integer lattice, meaning in one dimension that the dilation factor is an integer. Their methods involved quasiaffine frames and dual Gramians (Bownik), and shift invariant subspaces and spectral functions (Rzeszotnik). A very different proof involving almost periodic functions was then found by the author in [25] for *arbitrary* dilations in one dimension, for the wavelet case $m = 1$. Interestingly, Bownik has recently extended his methods in one dimension to also cover arbitrary dilations [3, Corollary 4.6], and in higher dimensions to cover many (but not all) expanding dilation matrices [3, Theorem 4.2]; in particular his work applies to *rational* expanding dilation matrices. Our Theorem 2.1 applies to all expanding dilation matrices.

Both Theorem 2.1 and its generalization Theorem 3.1 (see below) have very recently been extended to reproducing systems generated by a finite family, by Hernández, Labate and Weiss [19, §4]. This lovely extension covers Gabor and wavelet systems at a single stroke. The proofs are at heart the same as in this paper, using the almost periodicity of $z \mapsto D(f_z)$.

Note that in general the multiwavelet case $m > 1$ is worth treating, in addition to the single wavelet case $m = 1$. For example, "MRA" (multi-resolution analysis) wavelets exist for the dyadic dilation $a = 2I$ precisely when $m = 2^d - 1$ (see [32, §4]); and if $d > 1$ then $m > 1$.

3. Continuous and discrete multiwavelets

DEFINITION. Call $\{\psi_1, \ldots, \psi_m\}$ a *(discrete) multiwavelet*, with respect to dilations by $a_1, \ldots, a_m$ and translations by $b_1, \ldots, b_m$, if Ψ is an orthonormal basis for L^2. Equivalently, $\{\psi_1, \ldots, \psi_m\}$ is a multiwavelet if Ψ is orthonormal and

$$\text{(D)} \qquad \|f\|_2^2 = D(f) \qquad \forall f \in L^2,$$

where

$$D(f) := \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{m} |\langle f, \psi_{j,k,l}\rangle|^2,$$

with $\langle\,,\rangle$ denoting the complex inner product on L^2.

Call $\{\psi_1, \ldots, \psi_m\}$ a *continuous multiwavelet*, with respect to dilations by $a_1, \ldots, a_m$ and translations by $b_1, \ldots, b_m$, if

$$\text{(C)} \qquad \|f\|_2^2 = C(f) \qquad \forall f \in L^2,$$

where

$$C(f) := \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \sum_{l=1}^{m} |\langle f, \psi_{j,z,l}\rangle|^2 \, dz.$$

Here we use

$$\psi_{j,z,l}(x) = |\det a_l|^{j/2} \psi_l(a_l^j x - b_l z), \qquad x \in \mathbb{R}^d, j \in \mathbb{Z}, z \in \mathbb{R}^d, l \in \{1, \ldots, m\}.$$

The terms "discrete" and "continuous" refer to the translations employed in $D(f)$ and $C(f)$, which are respectively discrete (integer lattice points $k \in \mathbb{Z}^d$) and continuous (real vectors $z \in \mathbb{R}^d$). To be clear: "continuous" multiwavelet functions $\psi_{j,z,l}(x)$ need not be continuous as functions of x.

Continuous wavelets have been extensively investigated by earlier authors, especially by S. Mallat and collaborators, c.f. [28] and [7, Chapter 3], who called the map $z \mapsto \langle f, \psi_{j,z}\rangle$ the "dyadic wavelet transform".

In this section we will establish a precise connection between discrete and continuous multiwavelets. But first some remarks.

Remarks.

1. If

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{m} \langle f, \psi_{j,k,l} \rangle \psi_{j,k,l} \tag{3}$$

with unconditional convergence in L^2, then taking the inner product against f obviously implies $\|f\|_2^2 = D(f)$. Conversely, if $D(f) = \|f\|_2^2$ for all $f \in L^2$ then (3) holds with unconditional convergence for all $f \in L^2$, by [20, Theorem 7.1.7]. Thus to show that all L^2-functions can be reconstructed using the $\psi_{j,k,l}$, as in (3), it is enough to show $D(f) = \|f\|_2^2$ for all $f \in L^2$, which is precisely condition (D).

2. For arbitrary expanding matrices a, single wavelets have been shown to exist by Dai, Larson and Speegle [14]. Their wavelets are by construction minimally supported frequency (MSF) wavelets, with $|\hat{\psi}|$ being a normalized characteristic function.

3. There do exist wavelets to which the "amplifying" assumption (ii) is applicable. Indeed, for the non-expanding matrix $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, Speegle and Gu (see [4]) have constructed an example of a discrete MSF wavelet ψ for which a is amplifying. In their example, the support of $\hat{\psi}$ intersects each deleted horizontal strip $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \neq 0, |\xi_2| \leq \text{const}\}$ in a compact set, and so the exhausting set $\mathcal{A}(r)$ can be taken as the intersection of the double vertical strip $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : r^{-1} < |\xi_1| < r\}$ with $B(r)$.

4. Our standing assumption that $|\det a_l| \neq 1$ for all l is necessary for $\{\psi_1, \ldots, \psi_m\}$ to be a continuous multiwavelet (at least when $a_1 = \cdots = a_m$), as one can show by adapting [26, Proposition 2.1], which treats the wavelet case ($m = 1$) in the context of more general dilation groups.

5. In the interests of brevity, we omit from this paper certain other discussions and references to related work that appear already in the one-dimensional paper [25].

In the next theorem, we say Ψ is a *Bessel family with constant* 1 if $D(f) \leq \|f\|_2^2$ for all $f \in L^2$. This certainly holds if Ψ is orthonormal.

Theorem 3.1

(a) (D)⇒(C).
(b) (C)⇒(D), if Ψ is a Bessel family with constant 1.

This implies Theorem 2.1, as follows.

Proof of Theorem 2.1. It is known that $\{\psi_1, \ldots, \psi_m\}$ is a continuous multiwavelet if and only if the Calderón condition (2) holds (see Appendix A). Thus Theorem 2.1 simply says that $\{\psi_1, \ldots, \psi_m\}$ is a discrete multiwavelet if and only if it is a continuous multiwavelet and Ψ is an orthonormal system. This clearly follows from Theorem 3.1. □

Proof of Theorem 3.1. The main tool is Proposition 3.2 below, which says we can express $C(f)$ as an average of $D(\cdot)$ over translates of f, at least for f in a certain dense subset $\mathcal{F}$ of L^2. This subset $\mathcal{F}$ will be defined in Section 9.

Proposition 3.2

For all $f \in \mathcal{F}$,

$$C(f) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} D(f_z)\, dz$$

where $Q(R) = [-R, R]^d$ is the cube of side $2R$ in $\mathbb{R}^d$ centered at the origin.

Proof of Proposition 3.2. Define nonnegative functions

$$g_{j,l}(z) = \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{j,k-z,l}\rangle|^2, \qquad z \in \mathbb{R}^d, \quad j \in \mathbb{Z}, \quad l \in \{1, \dots, m\}.$$

The $g_{j,l}$ are obviously 1-periodic in every coordinate direction, and they are continuous and equal their Fourier series pointwise, by Lemma 9.4 below. Also note

$$\langle f_z, \psi_{j,k,l}\rangle = \langle f, \psi_{j,k-b_l^{-1}a_l^j z,l}\rangle \tag{4}$$

by a simple change of variable, so that

$$\begin{aligned} D(f_z) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{k\in\mathbb{Z}^d} |\langle f_z, \psi_{j,k,l}\rangle|^2 &= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} g_{j,l}(b_l^{-1}a_l^j z) \\ &= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \exp(2\pi i n b_l^{-1} a_l^j z), \end{aligned} \tag{5}$$

where n is a row vector and z is a column vector.

The Fourier coefficients satisfy

$$\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |\widehat{g_{j,l}}(n)| < \infty$$

by Lemma 9.3. For the "missing" $n = 0$ terms of this last sum, we observe $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \widehat{g_{j,l}}(0) = C(f)$ by Lemma 9.2 and note that $\widehat{g_{j,l}}(0) = |\widehat{g_{j,l}}(0)|$ by the nonnegativity of the $g_{j,l}$. Hence

$$\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} |\widehat{g_{j,l}}(n)| < \infty \qquad \text{when } C(f) < \infty. \tag{6}$$

When $C(f) < \infty$ we deduce from Lemma D.1 (which is applicable because of the absolute convergence in (6)) that the large-scale average of $D(f_z)$ equals the sum of the constant terms in (5), namely the terms with $n = 0$:

$$\lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} D(f_z)\, dz = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \widehat{g_{j,l}}(0) = C(f)$$

as desired. When $C(f) = \infty$, we define $C_J(f) = \sum_{|j|\le J} \sum_{l=1}^m \widehat{g_{j,l}}(0) < \infty$ for each $J \in \mathbb{N}$ and sum only over $|j| \le J$ in the above argument, finding

$$\liminf_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} D(f_z)\, dz \ge C_J(f).$$

Letting $J \to \infty$ completes the proof, since $C_J(f) \to C(f) = \infty$. □

Now we prove Theorem 3.1.

Part (a): (D)⇒(C). If (D) holds then $D(f_z) = \|f_z\|_2^2 = \|f\|_2^2$ for all z, and hence Proposition 3.2 gives $C(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$. Then $C(f) = \|f\|_2^2$ for all $f \in L^2$ by Lemma B.2, because $\mathcal{F}$ is dense in L^2.

Part (b): (C)⇒(D). Assume (C) holds, so that $C(f)$ is certainly finite for all $f \in L^2$, and suppose Ψ is a Bessel family with constant 1. To prove (D) we need only show $\|f\|_2^2 = D(f)$ for all f in the dense class $\mathcal{F}$, by Lemma B.2.

First, $D(f_z)$ is an almost periodic function of z by Lemma C.1 and (5), in view of the absolute convergence in (6). And because Ψ is a Bessel family with constant 1, $D(f_z) \le \|f_z\|_2^2 = \|f\|_2^2$ for all z. Thus the function $h(z) := \|f\|_2^2 - D(f_z)$ is nonnegative and almost periodic, and has "mean"

$$M(h) := \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(z)\, dz = \|f\|_2^2 - C(f)$$

by Proposition 3.2 (valid since $f \in \mathcal{F}$). But $\|f\|_2^2 = C(f)$ by (C), and so $M(h) = 0$. Since h is almost periodic and nonnegative with zero mean, Proposition C.2(b) implies $h \equiv 0$. But $h(0) = 0$ means $\|f\|_2^2 = D(f)$, as desired. □

4. Extensions: continuous families of dilations and translations

We now generalize Theorem 3.1 so as to cover continuous families of *dilations*. For this we assume the dilation matrices can be written as exponentials, with

$$a_1 = e^{\alpha_1}, \dots, a_m = e^{\alpha_m} \text{ for some } d \times d \text{ real matrices } \alpha_1, \dots, \alpha_m.$$

(The matrix a_l equals such an exponential precisely when it has an even number of Jordan λ-blocks for each negative eigenvalue λ, by [22, p. 132]. The logarithm matrix α_l might not be unique.)

Under this exponential assumption, we have the continuous family of dilations $a_l^y := e^{\alpha_l y}$ for $y \in \mathbb{R}$. We continue to assume $|\det a_l| \ne 1$.

The generalizations of assumptions (i) and (ii) to the continuous setting are:

(i′) a_l is *continuously expanding*, meaning there exists $0 < \kappa \le 1 < \gamma$ such that

$$|\xi a_l^y| \ge \kappa \gamma^y |\xi| \quad \text{and} \quad |\xi a_l^{-y}| \le \kappa^{-1} \gamma^{-y} |\xi|$$

for all row vectors $\xi \in \mathbb{R}^d$ and all $y \ge 0$;

(ii′) a_l is *continuously amplifying for* ψ_l, meaning that there exists an exhaustion $\{\mathcal{A}_l(r) : r \in \mathbb{N}\}$ of $\mathbb{R}^d$ such that for each $r \in \mathbb{N}$,

$$\operatorname{spt}(\widehat{\psi_l}) \cap \mathcal{A}_l(r) a_l^y = \emptyset \qquad \text{whenever } |y| \text{ is sufficiently large.}$$

It is easy to see (i′) is equivalent to (i), and obviously (ii′) implies (ii). But (ii) holding does not imply that (ii′) holds using the same exhaustion sets. We do not explain this in detail: the essential point is just that a^y might rotate in an unhelpful fashion when $y \notin \mathbb{Z}$. For example with $\alpha = \begin{pmatrix} 1 & -\pi \\ \pi & 1 \end{pmatrix}$ we see that $a := e^\alpha = -eI$ simply stretches radially, while $a^{1/2} = e^{\alpha/2} = e^{1/2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ performs also a rotation by $90°$.

For this section we assume either (i′) or (ii′) holds, for each l. We further assume the dilation matrices are all equal:

$$a_1 = \cdots = a_m = a, \text{ say.}$$

Analogous to our definitions of $C(f)$ and $D(f)$, we let

$$E(f) := \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^m |\langle f, \psi_{y,k,l} \rangle|^2 \, dy \quad \text{and} \quad F(f) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \sum_{l=1}^m |\langle f, \psi_{y,z,l} \rangle|^2 \, dz dy$$

for $f \in L^2$, where

$$\psi_{y,z,l}(x) = |\det a|^{y/2} \psi_l(a^y x - b_l z), \qquad x, z \in \mathbb{R}^d, \ y \in \mathbb{R}, \ l \in \{1, \dots, m\}.$$

We also define corresponding "tight frame" conditions (with constant 1):

(E) $\qquad \|f\|_2^2 = E(f) \qquad \forall f \in L^2,$

(F) $\qquad \|f\|_2^2 = F(f) \qquad \forall f \in L^2.$

One can think of (C), (D) and (F) as relating to the *continuous*, *discrete* and *fully* continuous cases, respectively.

Theorem 4.1

(a) With the above assumptions, the following implications hold:

$$\begin{array}{ccc} (D) & \Longrightarrow & (C) \\ \Downarrow & & \Downarrow \\ (E) & \Longrightarrow & (F) \end{array}$$

(b) If in addition Ψ is a Bessel family with constant 1 then all the implications can be reversed:

$$\begin{array}{ccc} (D) & \Longleftarrow & (C) \\ \Uparrow & & \Uparrow \\ (E) & \Longleftarrow & (F) \end{array}$$

Theorem 3.1 already gives that (D)$\Rightarrow$(C), and that if Ψ is a Bessel family with constant 1 then (D)$\Leftarrow$(C).

Bownik [2] and Rzeszotnik [30] have shown in one dimension for integer dilation factors $a \geq 2$ that (D)$\Rightarrow$(F) holds, and that if Ψ is a Bessel family with constant 1 (Bownik) or is orthonormal (Rzeszotnik) then the reverse implication (D)$\Leftarrow$(F) also holds. In [25], I proved that (D)$\Leftarrow$(F) for all $a > 1$, assuming Ψ is a Bessel family. Bownik does the same by his methods in [3, Corollary 4.6], assuming orthonormality.

Theorem 4.1 extends my ideas from [25] to higher dimensions, for arbitrary expanding or amplifying dilation matrices. Bownik generalizes to higher dimensions in a different direction, in [2, Theorem 2.5] and [3, Theorem 4.1], by integrating against the reciprocal of a quasinorm for his expanding dilation matrix (which he assumes is rational), instead of considering a^y and integrating with respect to y, as we do.

The standing assumption $|\det a| \neq 1$ is necessary for (F) to hold, as one sees by adapting [26, Proposition 2.1].

Proof of Theorem 4.1. We use four averaging formulas:

$$E(f) = \int_0^1 D(f_{y,0})\,dy, \qquad f \in L^2, \tag{7}$$

$$F(f) = \int_0^1 C(f_{y,0})\,dy, \qquad f \in L^2, \tag{8}$$

$$C(f) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} D(f_{0,z})\,dz, \qquad f \in \mathcal{F}, \tag{9}$$

$$F(f) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} E(f_{0,z})\,dz, \qquad f \in \mathcal{F}, \tag{10}$$

where

$$f_{y,z}(x) = |\det a|^{y/2} f(a^y x - z), \qquad x, z \in \mathbb{R}^d,\ y \in \mathbb{R}.$$

These formulas are all proved like in the one dimensional case [25, pp. 462–463], but using Proposition 3.2 and Section 9 of this paper instead of Lemma 2 and Appendix A of [25].

Now parts (a) and (b) of the theorem can be proved like in [25, pp. 463–464]. Note that in part (b), the direction (D)$\Leftarrow$(C) is known already from Theorem 3.1(b). $\square$

Incidentally, without the assumption that the a_l are all equal we find that the averaging formulas (9) and (10) and their proofs still hold, and thus the implications (D)$\Rightarrow$(C) and (E)$\Rightarrow$(F) in part (a), and (D)$\Leftarrow$(C) in part (b), still hold without the assumption that the a_l are all equal. All other implications in the theorem seem likely to hold also, but our methods do not seem capable of showing this.

5. A known characterization of tight frames

We call Ψ a *frame* (or affine frame) if there exist constants $0 \leq A \leq B < \infty$ such that $A\|f\|_2^2 \leq D(f) \leq B\|f\|_2^2$ for all $f \in L^2$. (In fact we should require $A > 0$, before calling Ψ a frame, with the case $A = 0$ being called just a Bessel family. But for the purposes of our later results on oversampling and equivalence of affine and quasiaffine frames, such a distinction would be artificial.)

The frame is *tight* if $A = B$, so that in particular, Ψ is a tight frame with constant 1 if (D) holds, meaning $D(f) = \|f\|_2^2$ for all $f \in L^2$. Thus Theorem 3.1 characterizes tight frames with constant 1 in terms of Ψ being a Bessel family with constant 1 and $\{\psi_1, \ldots, \psi_m\}$ being a continuous multiwavelet.

A different characterization of tight frames, due to Gripenberg [16] and Wang [31], says in the dyadic wavelet case in one dimension ($a = 2, b = d = m = 1$) that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is a tight frame with constant 1 if and only if

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(\xi 2^j)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R},$$

$$\sum_{j=0}^{\infty} \hat{\psi}(\xi 2^j)\overline{\hat{\psi}((\xi - q)2^j)} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R} \text{ and all } q \in 2\mathbb{Z} + 1.$$

This characterization was extended to higher dimensions for integer expanding dilation matrices by Frazier *et al.* [15], Han [18], Ron and Shen [29], Calogero [6] and Bownik [1, 2], then to arbitrary real dilations in one dimension by Chui and Shi [12], and to arbitrary expanding dilation matrices in higher dimensions by Chui, Czaja, Maggioni and Weiss [8, Corollary 2.6].

Below we give a simpler proof of this characterization in higher dimensions. The same line of proof has been developed independently by Hernández, Labate and Weiss [19, §2], in a more general setting that encompasses Gabor systems as well.

Theorem 5.1 ([8, Corollary 2.6])

Assume either (i) or (ii) holds, for each l.

Then Ψ is a tight frame with constant 1 if and only if

(11)

$$t_\nu(\xi) := \sum_{l=1}^{m} \sum_{j\in J_l(\nu)} \frac{\hat{\psi}_l(\xi a_l^{-j})\overline{\hat{\psi}_l((\xi-\nu)a_l^{-j})}}{|\det b_l|} = \delta_{\nu,0} \quad \textit{for a.e. } \xi \in \mathbb{R}^d \textit{ and all } \nu \in \mathbb{R}^d,$$

where $J_l(\nu) = \{j \in \mathbb{Z} : \nu a_l^{-j} b_l \in \mathbb{Z}^d\}$ and the series in (11) converges absolutely.

If in addition $\|\psi_l\|_2 \geq 1$ for all l, then Ψ is an orthonormal basis for L^2.

Remarks.

1. Notice (11) is trivial unless ν belongs to the countable set $\mathcal{N} = \cup_{l=1}^{m} \cup_{j\in\mathbb{Z}} (\mathbb{Z}^d b_l^{-1} a_l^j)$, because the $J_l(\nu)$ are all empty when $\nu \notin \mathcal{N}$.

2. When $\nu = 0$, (11) simply says $t_0(\xi) = \sum_{l=1}^{m} \sum_{j\in\mathbb{Z}} |\hat{\psi}_l(\xi a_l^{-j})|^2 / |\det b_l| = 1$ for a.e. ξ, which is the Calderón condition (2).

3. The absolute convergence of the series for t_ν in (11) is automatic for all ν once it is known for $\nu = 0$. For suppose (11) holds with $\nu = 0$. Then $t_0(\xi - \nu) = 1$ for a.e. ξ and all $\nu \in \mathcal{N}$. Hence for a.e. ξ and all $\nu \in \mathcal{N}$, applying the elementary inequality

$|xy| \leq (|x|^2 + |y|^2)/2$ to (11) shows that the series for $t_\nu(\xi)$ converges absolutely with all its partial sums bounded by $[t_0(\xi) + t_0(\xi - \nu)]/2 = 1$.

4. The proof has two steps, First, we apply the wavelet functional to translates of f, expressing $z \mapsto D(f_z)$ as an almost periodic trigonometric sum. The tight frame condition implies this sum is independent of z, equalling $\|f_z\|_2^2 = \|f\|_2^2$ for all z, and thus we can equate coefficients in the trigonometric sum. The second step is to apply standard choices of f (following earlier authors on this topic) in order to deduce (11) from the coefficient formulas. If one accepts that all series converge absolutely, then this proof goes very quickly indeed.

Note that the first step in the proof follows the approach of Janssen [23, Proposition A] for characterizing Gabor systems (that is, for proving the Wexler–Raz result). In the Gabor case, the trigonometric sum is periodic, not merely almost periodic.

Proof of Theorem 5.1. First assume Ψ is a tight frame with constant 1. That is, $\|f\|_2^2 = D(f)$ for all $f \in L^2$. Then for all $f \in \mathcal{F}$ (where the dense subset $\mathcal{F}$ of L^2 is defined in Section 9) and for all column vectors $z \in \mathbb{R}^d$,

$$\|f\|_2^2 = \|f_z\|_2^2 = D(f_z) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \exp(2\pi i n b_l^{-1} a_l^j z)$$

by (5).

The last sum converges absolutely by Lemmas 9.2 and 9.3 (noting that $C(f) < \infty$ by (38), because $D(f) = \|f\|_2^2 < \infty$). Rearranging the sum and writing $\nu = nb_l^{-1}a_l^j \in \mathcal{N}$, we obtain

$$\|f\|_2^2 = \sum_{\nu\in\mathcal{N}} \left[\sum_{l=1}^{m} \sum_{j\in J_l(\nu)} \widehat{g_{j,l}}(\nu a_l^{-j} b_l) \right] e^{2\pi i \nu z}, \qquad z \in \mathbb{R}^d. \tag{12}$$

By equating coefficients in (12), in other words by invoking Lemma D.2, we deduce

$$\delta_{\nu,0}\|f\|_2^2 = \sum_{l=1}^{m} \sum_{j\in J_l(\nu)} \widehat{g_{j,l}}(\nu a_l^{-j} b_l).$$

Then by substituting formula (42) for the Fourier coefficient $\widehat{g_{j,l}}(n)$, with $n = \nu a_l^{-j} b_l$, and by changing variable with $\xi \mapsto \xi a_l^{-j}$ in the resulting expression, we find

$$\delta_{\nu,0}\|f\|_2^2 = \sum_{l=1}^{m} \sum_{j\in J_l(\nu)} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \hat{f}(\xi - \nu) \frac{\widehat{\psi_l}(\xi a_l^{-j}) \overline{\widehat{\psi_l}((\xi - \nu) a_l^{-j})}}{|\det b_l|} \, d\xi. \tag{13}$$

When $\nu = 0$ we can certainly interchange the sums and integral to obtain

$$\|f\|_2^2 = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 t_0(\xi) \, d\xi, \quad f \in \mathcal{F}. \tag{14}$$

Let ξ^* belong to the open set $\cup_{r=1}^{\infty} \cap_{l=1}^{m} \mathcal{A}_l(r)$ of full measure in $\mathbb{R}^d$ (this set arises in the definition of $\mathcal{F}$, in Section 9, and in the case where each a_l is expanding, the set simply equals $\mathbb{R}^d \setminus \{0\}$). Then by taking $f \in \mathcal{F}$ such that $|\hat{f}|^2$ is the characteristic function of a small ball centered at ξ^*, we see from (14) that t_0 is integrable near ξ^*. Further, if we let the radius of the small ball tend to zero in (14) then the Lebesgue differentiation theorem shows $t_0(\xi^*) = 1$ for a.e. ξ^*, hence $t_0(\xi) = 1$ for a.e. $\xi \in \mathbb{R}^d$, which is (11) with $\nu = 0$. Then for a.e. ξ and all $\nu \in \mathcal{N}$, the series for $t_\nu(\xi)$ converges absolutely and its partial sums are all bounded by 1, as explained in the third remark after the theorem. This means we can interchange the sums and integral in (13), obtaining

$$
0 = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \hat{f}(\xi - \nu) t_\nu(\xi)\, d\xi, \qquad \nu \in \mathcal{N} \setminus \{0\}, \quad f \in \mathcal{F}. \tag{15}
$$

Now let ξ^* be a Lebesgue point of $t_\nu(\xi)$ and suppose ξ^* and $\xi^* + \nu$ belong to the open set $\cup_{r=1}^{\infty} \cap_{l=1}^{m} \mathcal{A}_l(r)$ of full measure, for all $\nu \in \mathcal{N} \setminus \{0\}$. The set of such ξ^* has full measure. Given $\nu \in \mathcal{N} \setminus \{0\}$, we define $f \in \mathcal{F}$ by

$$
\hat{f} = \frac{\chi_{B(\xi^*,\rho)} + \chi_{B(\xi^*-\nu,\rho)}}{\sqrt{2|B(\rho)|}},
$$

where ρ is taken so small that $\rho < |\nu|/2$ and $\hat{f}$ is supported in $\cap_{l=1}^{m} \mathcal{A}_l(r)$ for some r (thus ensuring $f \in \mathcal{F}$). Putting $\hat{f}$ into (15) and then letting $\rho \to 0$ yields by the Lebesgue differentiation theorem that $0 = t_\nu(\xi^*)/2$, which proves (11).

For the other direction, suppose now that (11) holds. As explained in the third remark after the theorem, this means that the series for t_ν converges absolutely with all its partial sums bounded by 1, for a.e. ξ and all $\nu \in \mathcal{N}$. Given $f \in \mathcal{F}$, we multiply (11) by $\overline{\hat{f}(\xi)} \hat{f}(\xi - \nu)$ and integrate to obtain (13), hence (12), hence (5). (In reversing these steps we need that $C(f) = \|f\|_2^2 < \infty$ by Appendix A, noting that (11) with $\nu = 0$ is exactly the Calderón condition (2).) Putting $z = 0$ in (5) gives $\|f\|_2^2 = D(f)$ for all f in the dense subset $\mathcal{F}$ of L^2, which implies $\|f\|_2^2 = D(f)$ for all $f \in L^2$ by Lemma B.2. That is, Ψ is a tight frame with constant 1.

Finally, the last statement of the theorem is a well known (and easily proved) fact about Hilbert spaces [20, Theorem 7.1.8]. □

6. Oversampling

In this section we establish conditions under which oversampling preserves the bounds on an affine frame. Suppose s is an invertible integer matrix (that is, $s \in GL(d, \mathbb{R})$ and s has integer entries). For $f \in L^2$, define

$$
D_s(f) = \frac{1}{|\det s|} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{m} |\langle f, \psi_{j,s^{-1}k,l} \rangle|^2.
$$

That is, D_s *oversamples* the translates of the ψ_l by a factor of s. Obviously $D_I = D$, and indeed $D_s = D$ whenever $|\det s| = 1$, because $s^{-1}\mathbb{Z}^d = \mathbb{Z}^d$ in that case. So we might as well assume $|\det s| > 1$.

We further assume that the a_l are all equal, with $a_1 = \cdots = a_m = a$ say, that the b_l are also all equal, with $b_1 = \cdots = b_m = b$, and that

$$\tilde{a} := b^{-1}ab$$

is an integer matrix.

The next theorem involves two conditions. The first is that $s\tilde{a}s^{-1}$ is an integer matrix, in other words, $\mathbb{Z}^d s\tilde{a} \subset \mathbb{Z}^d s$. The second is that $\tilde{a}$ *is prime relative to* s, meaning that if $n_1\tilde{a} = n_2 s$ for some row vectors $n_1, n_2 \in \mathbb{Z}^d$ then $n_1 = n_3 s$ for some $n_3 \in \mathbb{Z}^d$. In other words, $(\mathbb{Z}^d\tilde{a} \cap \mathbb{Z}^d s) \subset \mathbb{Z}^d s\tilde{a}$. Thus the condition in the next theorem that $\tilde{a}$ is prime relative to s and $s\tilde{a}s^{-1}$ is an integer matrix is equivalent to

$$\mathbb{Z}^d\tilde{a} \cap \mathbb{Z}^d s \subset \mathbb{Z}^d s\tilde{a} \subset \mathbb{Z}^d s. \tag{16}$$

Theorem 6.1 ("Second Oversampling Theorem")

Let $a, b \in GL(d, \mathbb{R})$ with $|\det a| > 1$, and suppose either a is expanding or else a is amplifying for ψ_l for each l. Take s to be an invertible integer matrix with $|\det s| > 1$. Assume $\tilde{a}$ and $s\tilde{a}s^{-1}$ are integer matrices, and that $\tilde{a}$ is prime relative to s. Fix $0 \le A \le B < \infty$.

If $\{\psi_{j,k,l} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, l \in \{1, \ldots, m\}\}$ is an affine frame with bounds A and B, then so is $\{|\det s|^{-1/2}\psi_{j,s^{-1}k,l} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, l \in \{1, \ldots, m\}\}$. That is, if

$$A\|f\|_2^2 \le D(f) \le B\|f\|_2^2 \qquad \text{for all } f \in L^2 \tag{17}$$

then

$$A\|f\|_2^2 \le D_s(f) \le B\|f\|_2^2 \qquad \text{for all } f \in L^2, \tag{18}$$

so that oversampling by s preserves the frame bounds.

In one dimension the hypotheses of the theorem simply say that a and s are relatively prime integers with $|a| \ge 2, |s| \ge 2$, and $b \ne 0$. The theorem is then precisely the Second Oversampling Theorem of Chui and Shi [9, Theorem 4] (proved earlier by them for $a = 2$ and s odd in [10]; see also [25, Theorem 4]).

In higher dimensions, Theorem 6.1 improves on a theorem of Chui and Shi [9, Theorem 8], which assumes a is a multiple of the identity and s and b are diagonal. It also improves on [11, Proposition 3], which assumes $s = \sigma I$ and $b = \beta I$ are multiples of the identity. (Note that [11, Proposition 3] in addition assumes $\gcd(\sigma, |\det \tilde{a}|) = 1$, which implies our assumption that $\tilde{a}$ is prime relative to s, in view of [11, Lemma 4].) Later in the section we give an example where a, s and b satisfy the hypotheses of Theorem 6.1 but not the hypotheses of [9] or [11].

Ron and Shen [29, Theorem 4.19] proved the Second Oversampling Theorem under the assumption [29, (4.18)], which says

$$\mathbb{Z}^d\tilde{a}^j \cap \mathbb{Z}^d s = \mathbb{Z}^d s\tilde{a}^j \qquad \forall j \ge 0. \tag{19}$$

(Their dilation matrices were expanding, and they imposed a decay condition on the ψ_l.) When $j = 1$, this assumption (19) is equivalent to our hypothesis (16), and so our hypothesis might seem weaker. But by a pleasant exercise, (16) also implies (19) and so the two are equivalent. Thus Theorem 6.1 is a restatement of Ron and Shen's result.

Lastly, we show at the end of the section that in the special case of *tight* frames (that is, $A = B > 0$), when a is expanding, Theorem 6.1 follows from work of Chui, Czaja, Maggioni and Weiss in [8]. That paper *only* treats tight frames, but the dilation and oversampling matrices considered there are more general than we can treat in Theorem 6.1; $\tilde{a}$ and s need not be integer matrices, for example.

To prove Theorem 6.1, we will express $D_s(f)$ as an average of $D(\cdot)$ over translates of f. Similar averaging ideas were used, though in a different expression, by Chui and Shi [9, 10, 11] and Ron and Shen [29] in their oversampling work.

Proposition 6.2

Let $a, b \in GL(d, \mathbb{R})$ with $|\det a| > 1$, and suppose either a is expanding or else a is amplifying for ψ_l for each l. Take s to be an invertible integer matrix with $|\det s| > 1$. Assume $\tilde{a} = b^{-1}ab$ and $s\tilde{a}s^{-1}$ are integer matrices, and that $\tilde{a}$ is prime relative to s.

Let V be a maximal set of distinct column vectors in the group $(s^{-1}\mathbb{Z}^d)/\mathbb{Z}^d$ (so that V contains $|V| = |\det s|$ vectors). Then for all $f \in \mathcal{F}$, with $\mathcal{F}$ as in Section 9,

$$D_s(f) = \lim_{J\to\infty} \frac{1}{|V|} \sum_{v\in V} D(f_{a^J bv}) \tag{20}$$

where $f_z(\cdot) = f(\cdot - z)$ for $z \in \mathbb{R}^d$.

In one dimension, we can simply take $V = \{0, 1/s, \dots, (|s| - 1)/s\}$.

Equation (20) is a discrete analogue of the averaging formula proved in Proposition 3.2.

Proof of Theorem 6.1. It suffices to prove (18) for $f \in \mathcal{F}$, in view of Lemma B.2.

Because $\|f_z\|_2^2 = \|f\|_2^2$ for all $z \in \mathbb{R}^d$, (17) implies $A\|f\|_2^2 \le D(f_{a^J bv}) \le B\|f\|_2^2$ for all J, v. Then Proposition 6.2 gives $A\|f\|_2^2 \le D_s(f) \le B\|f\|_2^2$, which is (18). □

Proof of Proposition 6.2. We prove the lemma initially under the supposition $\sum_{j\in\mathbb{Z}} \sum_{l=1}^m |\widehat{g_{j,l}}(0)| < \infty$, which implies as in (39) that the sum of all the $\widehat{g_{j,l}}(n)$ converges absolutely.

To begin with we show that when $j + J > 0$ and $n \in \mathbb{Z}^d$,

$$\frac{1}{|V|} \sum_{v\in V} \exp(2\pi i n \tilde{a}^{j+J} v) = \begin{cases} 1 & \text{if } n \in \mathbb{Z}^d s \\ 0 & \text{if } n \notin \mathbb{Z}^d s \end{cases}. \tag{21}$$

Here $n\tilde{a}^{j+J}v$ is a 1×1 matrix, which we regard as a number.

First, if $n \in \mathbb{Z}^d s$ then $u := n\tilde{a}^{j+J} \in \mathbb{Z}^d s$, because $us^{-1} = n\tilde{a}^{j+J}s^{-1} = (ns^{-1})(s\tilde{a}s^{-1})^{j+J} \in \mathbb{Z}^d$, using that $s\tilde{a}s^{-1}$ is an integer matrix and $j+J>0$. Second, if $n \notin \mathbb{Z}^d s$ then $n\tilde{a} \notin \mathbb{Z}^d s$ (indeed, the contrapositive implication follows from the primeness of $\tilde{a}$ relative to s); repeating this argument $j+J$ times shows that $u = n\tilde{a}^{j+J} \notin \mathbb{Z}^d s$. Now (21) follows from Lemma D.3.

Next we compute

$$
\begin{aligned}
\frac{1}{|V|}\sum_{v\in V} D(f_{a^J bv}) &= \frac{1}{|V|}\sum_{v\in V}\sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}^d}\widehat{g_{j,l}}(n)\exp(2\pi i n b^{-1}a^j a^J bv) \qquad \text{by (5)}\\
&= \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}^d}\widehat{g_{j,l}}(n)\frac{1}{|V|}\sum_{v\in V}\exp(2\pi i n\tilde{a}^{j+J}v)\\
&= \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}^d}\widehat{g_{j,l}}(n)\begin{Bmatrix}1 & \text{if } n\in\mathbb{Z}^d s\\ 0 & \text{otherwise}\end{Bmatrix} + o(1) \qquad \text{as } J\to\infty, \tag{22}
\end{aligned}
$$

by applying (21) to the terms with $j > -J$, and using the absolute convergence in (39) to show that the sum of all terms with $j \le -J$ is $o(1)$.

But the quantity in (22) equals $D_s(f)$, just by summing over j and l in Lemma 9.5 and using the definition of $D_s(f)$. This proves (20).

Finally we prove the lemma supposing that $\sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}|\widehat{g_{j,l}}(0)| = \infty$. We argue like above to show that for all $I\in\mathbb{N}$,

$$
\begin{aligned}
\liminf_{J\to\infty}\frac{1}{|V|}\sum_{v\in V} D(f_{a^J bv}) &\ge \sum_{|j|\le I}\sum_{l=1}^{m}\sum_{n\in\mathbb{Z}^d}\widehat{g_{j,l}}(n)\begin{Bmatrix}1 & \text{if } n\in\mathbb{Z}^d s\\ 0 & \text{otherwise}\end{Bmatrix}\\
&= \sum_{|j|\le I}\sum_{l=1}^{m}\widehat{g_{j,l}}(0) + O(1) \qquad \text{as } I\to\infty,
\end{aligned}
$$

using the absolute convergence of the sum of the $\widehat{g_{j,l}}(n)$ with $n \neq 0$ (see Lemma 9.3). Letting $I\to\infty$ now shows the left hand side of (20) equals ∞ (using that $\widehat{g_{j,l}}(0) = |\widehat{g_{j,l}}(0)|$, since $g_{j,l}$ is nonnegative). By summing over j and l in Lemma 9.5 and using the definition of $D_s(f)$, we similarly deduce $D_s(f)=\infty$. $\square$

EXAMPLE: Let us construct a nontrivial oversampling example to which Theorem 6.1 applies. Working in two dimensions, we take

$$
a = \begin{pmatrix}3 & 0\\ 1 & 2\end{pmatrix}, \qquad b = \sqrt{5}a, \qquad s = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},
$$

so that s rotates the plane by 45° and dilates it by a factor of $\sqrt{2}$. The "oversampling" lattice $s^{-1}\mathbb{Z}^2$ then consists of $\mathbb{Z}^2$ together with $\mathbb{Z}^2 + (\frac{1}{2},\frac{1}{2})$.

Obviously $\tilde{a} = a$, and a and s have integer entries. It is easy to check they are invertible, with a being expanding (satisfying assumption (i)) and with $|\det a| > 1, |\det s| > 1$. Further, $b \in GL(2,\mathbb{R})$ and sas^{-1} is an integer matrix. To show a

is prime relative to s, suppose $n_1 a = n_2 s$ for some row vectors $n_1 = (n_{11}, n_{12})$ and $n_2 = (n_{21}, n_{22})$ in $\mathbb{Z}^2$. The equation $n_1 a = n_2 s$ implies $n_{11} = \frac{1}{2}n_{21} + \frac{1}{6}n_{22}$ and $n_{12} = -\frac{1}{2}n_{21} + \frac{1}{2}n_{22}$, so that $n_{22} = 3(2n_{11} - n_{21})$ is divisible by 3. Now, $n_1 = n_3 s$ where

$$n_3 = \left(n_{11} - \frac{1}{3}n_{22}, \frac{1}{3}n_{22}\right) \in \mathbb{Z}^2,$$

and so a is prime relative to s. (Incidentally, s is not prime relative to a, since with $n_1 = (1,1)$ and $n_2 = (1,3)$ we have $n_1 a = n_2 s$ but $n_2 \neq n_3 a$ for all $n_3 \in \mathbb{Z}^2$.)

Remark. The above example is not covered by [9, Theorem 8], which assumes a is a multiple of the identity and s and b are diagonal, and is also not covered by [11, Proposition 3], which assumes $s = \sigma I$ and $b = \beta I$ are multiples of the identity.

One can always multiply s on the left by an integer matrix with determinant ± 1, since this does not change the oversampling lattice $s^{-1}\mathbb{Z}^2$. But in the example above, it is an easy exercise to show s cannot be so multiplied to yield a diagonal matrix, and thus Theorem 6.1 seems to be genuinely stronger than the results of [9, 11].

Relation to the work of Chui *et al.* on tight affine frames. On the other hand, in the special case when the affine frame is tight ($A = B$) and a is expanding, Theorem 6.1 follows from [8, Theorem 4.1(i)].

For suppose a is expanding and a, b and s satisfy the hypotheses of Theorem 6.1, and write $c = bs^{-1}b^{-1}$. The matrix c is relevant because in our work the oversampling is achieved by multiplying k on the left by s^{-1}, resulting in translation vectors of the form $bs^{-1}k$, while in [8] the authors multiply b on the left by c, resulting in translation vectors of the form $cbk = bs^{-1}k$.

Using the notation of [8, Definition 3.1] we show $a \in E_1(b) \cap E_1(cb)$ with $\gamma_b = 1$ and $\gamma_{cb} = 1$. Indeed, $\tilde{a}$ is an integer matrix and so $\mathbb{Z}^d b^{-1} a = \mathbb{Z}^d \tilde{a} b^{-1} \subset \mathbb{Z}^d b^{-1}$, so that $a \in E_1(b)$ with $\gamma_b = 1$. And similarly

$$\mathbb{Z}^d (cb)^{-1} a = \mathbb{Z}^d (s\tilde{a}s^{-1})(cb)^{-1} \subset \mathbb{Z}^d (cb)^{-1},$$

using that $cb = bs^{-1}$ and $s\tilde{a}s^{-1}$ is an integer matrix. Hence $a \in E_1(cb)$ with $\gamma_{cb} = 1$.

Furthermore, the hypothesis (4.2) in [8] is equivalent to our assumption that $\tilde{a}$ is prime relative to s, by the following reasoning. Condition (4.2) in [8] reads

$$(\mathbb{Z}^d b^{-1} c^{-1} \setminus \mathbb{Z}^d b^{-1} c^{-1} a) \subset (\mathbb{Z}^d b^{-1} \setminus \mathbb{Z}^d b^{-1} a).$$

Multiplying this condition on the right by b and using the definition of $c = bs^{-1}b^{-1}$, we reduce to just $\mathbb{Z}^d s \setminus \mathbb{Z}^d s\tilde{a} \subset \mathbb{Z}^d \setminus \mathbb{Z}^d \tilde{a}$, or in other words

$$(\mathbb{Z}^d \tilde{a} \cap \mathbb{Z}^d s) \subset \mathbb{Z}^d s\tilde{a}$$

since $\tilde{a}$ and s are integer matrices. Hence $\tilde{a}$ is prime relative to s.

Thus all the hypotheses of [8, Theorem 4.1(i)] are satisfied, including the assumption that a is expanding. The conclusion of that theorem is then precisely the same as the conclusion of our Theorem 6.1, in the "tight" case $A = B$.

Remark. It would be interesting to find a proof of all of the tight frame oversampling result [8, Theorem 4.1] by means of averaging ideas similar to Proposition 6.2.

7. Quasiaffine frames

Ron and Shen [29] created shift invariant "quasiaffine" frames by introducing the functions

$$\begin{aligned}\psi^q_{j,k,l}(x) &= \begin{cases} |\det a_l|^{j/2}\psi_l(a_l^j x - b_l k), & \text{if } j \geq 0, \\ |\det a_l|^{j}\psi_l(a_l^j (x - b_l k)), & \text{if } j < 0, \end{cases} \\ &= \begin{cases} \psi_{j,k,l}(x), & \text{if } j \geq 0, \\ |\det \tilde{a}_l|^{j/2}\psi_{j,\tilde{a}_l{}^j k,l}(x), & \text{if } j < 0, \end{cases}\end{aligned} \tag{23}$$

where $\tilde{a}_l = b_l^{-1} a_l b_l$. (That is, when $j < 0$ one oversamples the affine system using a matrix $\tilde{a}_l{}^{-j}$ adapted to j.) Define

$$D^q(f) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{m} |\langle f, \psi^q_{j,k,l}\rangle|^2,$$

and call the collection

$$\Psi^q = \{\psi^q_{j,k,l} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \ldots, m\}$$

a *quasiaffine* frame with constants A and B if $A\|f\|_2^2 \leq D^q(f) \leq B\|f\|_2^2$ for all $f \in L^2$.

We assume in this section that the a_l are all equal, with $a_1 = \cdots = a_m = a$, and that the b_l are also all equal, with $b_1 = \cdots = b_m = b$.

Ron and Shen proved the following theorem (in the expanding case, under a decay assumption on the ψ_l that was removed by Chui, Shi and Stöckler in [13, Theorem 2]).

Theorem 7.1 ([29, Theorem 5.5])

Let $a, b \in GL(d,\mathbb{R})$ with $|\det a| > 1$, and suppose either a is expanding or else a is amplifying for ψ_l for each l. Assume $\tilde{a} = b^{-1}ab$ is an integer matrix, and fix $0 \leq A \leq B < \infty$.

Then Ψ is an affine frame with bounds A and B if and only if Ψ^q is a quasiaffine frame with bounds A and B. That is,

$$A\|f\|_2^2 \leq D(f) \leq B\|f\|_2^2 \qquad \text{for all } f \in L^2 \tag{24}$$

if and only if

$$A\|f\|_2^2 \leq D^q(f) \leq B\|f\|_2^2 \qquad \text{for all } f \in L^2. \tag{25}$$

I do not know whether the theorem is true without the restriction that the a_l are all equal and the b_l are all equal.

In order to prove Theorem 7.1, we first express $D^q(f)$ as an average of $D(\cdot)$ over translates of f, and express $D(f)$ as a limit of $D^q(\cdot)$ over dilates of f.

Proposition 7.2

Assume the hypotheses of Theorem 7.1 hold. For each $J \in \mathbb{N}$, let W_J be a maximal set of distinct column vectors in the group $\mathbb{Z}^d/(\tilde{a}^J\mathbb{Z}^d)$.

Then for all $f \in \mathcal{F}$,

$$D^q(f) = \lim_{J\to\infty} \frac{1}{|W_J|} \sum_{w\in W_J} D(f_{0,bw}) \tag{26}$$

and

$$D(f) = \lim_{J\to\infty} D^q(f_{J,0}), \tag{27}$$

where $f_{J,z}(x) = |\det a|^{J/2} f(a^J x - z)$ for $x, z \in \mathbb{R}^d$, $J \in \mathbb{Z}$.

In one dimension we can simply take $W_J = \{1, 2, \ldots, |a|^J\}$.

The translational and dilational limits in Proposition 7.2 are distilled from Ron and Shen's original proof (see page 427 and Theorem 4.11 of [29], respectively). Then when Chui, Shi and Stöckler [13] proved Theorem 7.1, they established all the ingredients for proving Proposition 7.2 (albeit for a different dense class of $f \in L^2$). Their paper, though, did not use these ingredients to explicitly state the limiting relations (26) or (27).

We will now prove the theorem and proposition. Then we explain our debt to Chui, Shi and Stöckler in more detail, and remark on a few ways in which our approach differs from theirs.

Proof of Theorem 7.1. It suffices to consider $f \in \mathcal{F}$, in view of Lemma B.2.

Because $\|f_{0,z}\|_2^2 = \|f\|_2^2$ for all $z \in \mathbb{R}^d$, (24) implies $A\|f\|_2^2 \le D(f_{0,bw}) \le B\|f\|_2^2$ for all $w \in W_J$. Then (26) gives $A\|f\|_2^2 \le D^q(f) \le B\|f\|_2^2$, which is (25).

In the other direction, because $\|f_{J,0}\|_2^2 = \|f\|_2^2$ for all J, formula (25) implies $A\|f\|_2^2 \le D^q(f_{J,0}) \le B\|f\|_2^2$ for all J. Then (27) gives $A\|f\|_2^2 \le D(f) \le B\|f\|_2^2$, which is (24). □

Proof of Proposition 7.2. We prove the lemma initially under the supposition $\sum_{j\in\mathbb{Z}} \sum_{l=1}^m |\widehat{g_{j,l}}(0)| < \infty$, which implies as in (39) that the sum of all the $\widehat{g_{j,l}}(n)$ converges absolutely.

First we compute

$$\begin{aligned}
\frac{1}{|W_J|} \sum_{w\in W_J} D(f_{bw}) &= \frac{1}{|W_J|} \sum_{w\in W_J} \sum_{j\in\mathbb{Z}} \sum_{l=1}^m \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \exp(2\pi i n b^{-1} a^j b w) \qquad \text{by (5)}, \\
&= \sum_{j\in\mathbb{Z}} \sum_{l=1}^m \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \frac{1}{|W_J|} \sum_{w\in W_J} \exp(2\pi i n \tilde{a}^j w) \\
&= \sum_{j\in\mathbb{Z}} \sum_{l=1}^m \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \left\{ \begin{matrix} 1 & \text{if } n\tilde{a}^j \in \mathbb{Z}^d \\ 0 & \text{otherwise} \end{matrix} \right\} + o(1) \qquad \text{as } J \to \infty,
\end{aligned} \tag{28}$$

by applying Lemma D.4 with $s = \tilde{a}^J$ to the terms with $j > -J$, and using the absolute convergence in (39) to show that the sum of all terms with $j \le -J$ is $o(1)$.

When $j \ge 0$ it is immediate that $n\tilde{a}^j \in \mathbb{Z}^d$, in which case the $(j,l)^{\text{th}}$ term in (28) equals

$$\sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) = g_{j,l}(0) = \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{j,k,l}\rangle|^2 = \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^q_{j,k,l}\rangle|^2.$$

When $j < 0$ we apply Lemma 9.5 with $s = \tilde{a}^{-j}$ to find that the $(j,l)^{\text{th}}$ term equals

$$\sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n) \left\{ \begin{matrix} 1 & \text{if } n\tilde{a}^j \in \mathbb{Z}^d \\ 0 & \text{otherwise} \end{matrix} \right\} = |\det \tilde{a}|^j \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{j,\tilde{a}^j k,l}\rangle|^2 = \sum_{k\in\mathbb{Z}^d} |\langle f, \psi^q_{j,k,l}\rangle|^2,$$

by the definition (23) of $\psi^q_{j,k,l}$. By summing these last two formulas over j and l and then substituting into (28), we obtain $D^q(f) + o(1)$. Letting $J \to \infty$ gives (26).

Next we prove (27). The idea is to dilate f repeatedly in order to bring it up towards the scales $j \ge 0$, where the quasiaffine system is the same as the affine one. We have

$$\begin{aligned} D(f) &= \lim_{J\to\infty} \sum_{j\ge -J} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^m |\langle f, \psi_{j,k,l}\rangle|^2 \\ &= \lim_{J\to\infty} \sum_{j\ge 0} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^m |\langle f, \psi_{j-J,k,l}\rangle|^2 && \text{by } j \mapsto j - J \\ &= \lim_{J\to\infty} \sum_{j\ge 0} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^m |\langle f_{J,0}, \psi_{j,k,l}\rangle|^2 && \text{by a change of variable} \\ &= \lim_{J\to\infty} \sum_{j\ge 0} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^m |\langle f_{J,0}, \psi^q_{j,k,l}\rangle|^2 && \text{since } \psi_{j,k,l} = \psi^q_{j,k,l} \text{ when } j \ge 0. \end{aligned}$$

Thus to obtain (27) we need only show that the corresponding sum over $j < 0$ vanishes in the limit:

$$\lim_{J\to\infty} \sum_{j<0} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^m |\langle f_{J,0}, \psi^q_{j,k,l}\rangle|^2 = 0. \tag{29}$$

Write V_j for a maximal set of distinct column vectors in the group $(\tilde{a}^j\mathbb{Z}^d)/\mathbb{Z}^d$, when $j < 0$, and notice $|V_j| = |\det \tilde{a}|^{-j}$. The left hand side of (29) equals

$$\begin{aligned} &\lim_{J\to\infty} \sum_{j<0} \sum_{l=1}^m |\det \tilde{a}|^j \sum_{k\in\mathbb{Z}^d} |\langle f_{J,0}, \psi_{j,\tilde{a}^j k,l}\rangle|^2 && \text{by the definition (23) of } \psi^q_{j,k,l} \\ &= \lim_{J\to\infty} \sum_{j<0} \sum_{l=1}^m \frac{1}{|V_j|} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{j-J,\tilde{a}^j k,l}\rangle|^2 && \text{by a change of variable} \\ &= \lim_{J\to\infty} \sum_{j<0} \sum_{l=1}^m \frac{1}{|V_j|} \sum_{v\in V_j} \sum_{k^*\in\mathbb{Z}^d} |\langle f, \psi_{j-J,k^*-v,l}\rangle|^2 && \text{since } \tilde{a}^j\mathbb{Z}^d = \mathbb{Z}^d - V_j \end{aligned}$$

$$
\begin{aligned}
&= \lim_{J\to\infty} \sum_{j<0} \sum_{l=1}^{m} \frac{1}{|V_j|} \sum_{v\in V_j} g_{j-J,l}(v) && \text{by definition of } g_{j,l} \\
&= \lim_{J\to\infty} \sum_{j<0} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} \widehat{g_{j-J,l}}(n) \frac{1}{|V_j|} \sum_{v\in V_j} \exp(2\pi i n v) && \text{by Lemma 9.4} \\
&= 0
\end{aligned}
$$

by the absolute convergence in in (39), since we are summing only over $j < 0$ and so $j - J < -J \to -\infty$. This proves (29) as desired.

Finally, when $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} |\widehat{g_{j,l}}(0)| = \infty$ the proof can be completed by arguments similar to the end of the proof of Proposition 6.2. □

Remarks.

1. As we explained before the proof, Chui, Shi and Stöckler [13] established all the ingredients for proving Proposition 7.2, for a different dense class of $f \in L^2$. We use several of these ingredients in the proof above, as we now explain.

In our proof of (26), for the $j > -J$ terms we use a variant of [13, Lemma 3] (the variant being that we expand in the Fourier series before performing the translational averaging). For the $j \leq -J$ terms, though, we proceed differently from [13], using the absolute convergence of the Fourier coefficients to show the sum of all terms with $j \leq -J$ is $o(1)$, rather than using direct estimates as in [13, Lemma 4].

Then in our proof of (27) we reduce to a sum over $j \geq 0$ by a dilation argument that appeared in [13, p. 10]. But to handle the $j < 0$ terms, that is to prove (29), we again use the absolute convergence of the Fourier coefficients, whereas in [13, Lemma 4] the $j < 0$ terms are estimated directly.

2. Note that Bownik [3] has extended the definition of quasiaffine systems to *rational* dilation matrices, and has proved the equivalence of affine and quasiaffine frames in this case. His proof is mostly in the spirit of Ron and Shen.

8. Dual frame pairs

Here we extend some of the preceding theorems to the setting of dual frame pairs. The key idea is to polarize with respect to the ψ_l. Then the earlier averaging formula for ordinary frames yields an averaging formula for the dual functionals.

So in addition to the functions $\psi_1, \ldots, \psi_m \in L^2$ already considered, fix $\tilde{\psi}_1, \ldots, \tilde{\psi}_m \in L^2$. For each $l = 1, \ldots, m$ we assume either (i) holds (a_l is expanding) or else (ii) holds for both ψ_l and $\tilde{\psi}_l$, that is, a_l is amplifying for both ψ_l and $\tilde{\psi}_l$ using the same exhaustion. Let $\eta, \tilde{\eta} \in \mathbb{C}$. For the polarization step, we will later want to consider the linear combinations $\eta\psi_l + \tilde{\eta}\tilde{\psi}_l$ for $l = 1, \ldots, m$. Notice that for each l, either (i) holds or else (ii) holds for $\eta\psi_l + \tilde{\eta}\tilde{\psi}_l$ using the same exhaustion as for ψ_l and $\tilde{\psi}_l$; the reason is simply that $\mathrm{spt}\,(\eta\widehat{\psi_l} + \tilde{\eta}\widehat{\tilde{\psi}_l}) \subset \mathrm{spt}\,(\widehat{\psi_l}) \cup \mathrm{spt}\,(\widehat{\tilde{\psi}_l})$. The dense set "$\mathcal{F}$"$\subset L^2$ corresponding to the $\eta\psi_l + \tilde{\eta}\tilde{\psi}_l$ depends only on the exhaustion (as in Section 9), and therefore it equals the original set $\mathcal{F}$ corresponding to the ψ_l.

Let $f \in L^2$ and write $C(f; \eta\psi + \tilde{\eta}\tilde{\psi})$ for the quantity analogous to $C(f)$ except using the functions $\eta\psi_l + \tilde{\eta}\tilde{\psi}_l$ instead of the ψ_l. For example with $\eta = 1$ and $\tilde{\eta} = 0$ we have $C(f) = C(f; \psi)$. As a special case, write $\tilde{C}(f)$ for $C(f; \tilde{\psi})$. It is easy to see that if both $C(f)$ and $\tilde{C}(f)$ are finite, then so is $C(f; \eta\psi + \tilde{\eta}\tilde{\psi})$.

Assuming $C(f)$ and $\tilde{C}(f)$ are finite, we can define the continuous wavelet functional for the dual frame pair to be

$$C^*(f) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \sum_{l=1}^{m} \langle f, \psi_{j,z,l} \rangle \langle \tilde{\psi}_{j,z,l}, f \rangle \, dz,$$

where the multiple series/integral converges absolutely by Cauchy–Schwarz. Obviously if $\tilde{\psi}_l = \psi_l$ for all l, then $C^* = C$.

Similarly when $D(f)$ and $\tilde{D}(f)$ are finite we can define

$$D^*(f) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{m} \langle f, \psi_{j,k,l} \rangle \langle \tilde{\psi}_{j,k,l}, f \rangle, \tag{30}$$

where the multiple series converges absolutely.

To motivate the use of D^*, we note that it arises when one tries to use the functions $\{\psi_1, \dots, \psi_m\}$ for analyzing and $\{\tilde{\psi}_1, \dots, \tilde{\psi}_m\}$ for reconstructing (or vice versa). Indeed if

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{m} \langle f, \psi_{j,k,l} \rangle \tilde{\psi}_{j,k,l} \qquad \text{or} \qquad f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{m} \langle f, \tilde{\psi}_{j,k,l} \rangle \psi_{j,k,l} \tag{31}$$

with unconditional convergence in L^2, then taking the inner product with f implies $\|f\|_2^2 = D^*(f)$. Conversely, if for some $B > 0$ we have $D(f), \tilde{D}(f) \le B\|f\|_2^2$ and $D^*(f) = \|f\|_2^2$, for all $f \in L^2$, then (31) holds for all $f \in L^2$ with unconditional convergence by [15, Lemma 4.20]. In this case we call $\{\psi_1, \dots, \psi_m\}$ and $\{\tilde{\psi}_1, \dots, \tilde{\psi}_m\}$ a "dual tight frame pair".

Next we state a "dual pair" analogue of Theorem 2.1, involving the two conditions:

$$\text{(C}^*\text{)} \qquad \|f\|_2^2 = C^*(f) \qquad \forall f \in L^2,$$

$$\text{(D}^*\text{)} \qquad \|f\|_2^2 = D^*(f) \qquad \forall f \in L^2.$$

Theorem 8.1

If Ψ and $\tilde{\Psi}$ are Bessel families with constant 1, then (D)⇔(C*).*

To prove Theorem 8.1, we first establish an analogue of Theorem 3.1:

Theorem 8.2

(a) Suppose $C(f), \tilde{C}(f) \le B\|f\|_2^2$ for all $f \in L^2$, for some $B > 0$. Then (D)⇒(C*).*
(b) Suppose $D(f)$, and $\tilde{D}(f) \le \|f\|_2^2$ for all $f \in L^2$. Then (C)⇒(D*).*

We show in the proof of part (a) that for all $f \in \mathcal{F}$, both $D(f)$ and $\tilde{D}(f)$ are finite and thus $D^*(f)$ certainly exists. Then we show that if $D^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$, then (C*) holds. Thus we do not actually use anything about $D^*(f)$ for $f \notin \mathcal{F}$. Similarly in part (b) we need only assume $C^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$; but in fact this is equivalent to (C*) by the continuity of $f \mapsto C^*(f)$ that we establish in the proof.

See [8, 15] for a great deal more information on dual pairs, and a characterization of (D*) on the Fourier side. A Fourier characterization of (C*), though, is relatively easy to obtain. To ensure $C^*(f)$ exists, one should first make sure $C(f)$ and $\tilde{C}(f)$ are finite for all $f \in L^2$, by assuming $\Delta(\xi)$ and $\tilde{\Delta}(\xi)$ are bounded functions (see Appendix A). Then the condition (C*) is characterized by the equality

$$\Delta^*(\xi) := \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \frac{\overline{\widehat{\psi}_l(\xi a_l^j)}\widehat{\tilde{\psi}}_l(\xi a_l^j)}{|\det b_l|} = 1 \qquad \text{for almost every row vector } \xi \in \mathbb{R}^d,$$

which one can see by using the identity $C(f) = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \Delta(\xi)\, d\xi$ proved in Appendix A and the formula (32) below for $C^*(f)$. Similarly, the condition that $C(f) \leq B\|f\|_2^2$ for all $f \in L^2$, which occurs in Theorem 8.2(a), holds if and only if $\Delta(\xi) \leq B$ a.e., where

$$\Delta(\xi) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \frac{|\hat{\psi}_l(\xi a_l^j)|^2}{|\det b_l|}.$$

Proof of Theorem 8.2. Several times in the proof we will use the "polarization" decompositions

$$C^*(f) = \frac{1}{4} \sum_{p=1}^{4} i^p C(f; i^p\psi + \tilde{\psi}), \tag{32}$$

$$D^*(f) = \frac{1}{4} \sum_{p=1}^{4} i^p D(f; i^p\psi + \tilde{\psi}), \tag{33}$$

valid whenever $C(f), \tilde{C}(f)$ or $D(f), \tilde{D}(f)$ are finite, respectively.

Part (a). $D^*(f)$ definitely exists when $f \in \mathcal{F}$, because $C(f)$ and $\tilde{C}(f)$ are finite and thus so are $D(f)$ and $\tilde{D}(f)$ by (38). And since $C(f)$ and $\tilde{C}(f)$ are bounded by a multiple of $\|f\|_2^2$, so is $C(f; i^p\psi + \tilde{\psi})$, and hence by Lemma B.1 this quantity is continuous as a function of $f \in L^2$. Then by (32) we conclude the map $f \mapsto C^*(f)$ is continuous on L^2. Thus we need only prove $C^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$, because $\mathcal{F}$ is dense in L^2.

Now for $f \in \mathcal{F}$, Proposition 3.2 and formulas (32) and (33) give the averaging formula

$$C^*(f) = \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} D^*(f_z)\, dz. \tag{34}$$

Also $D^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$ by hypothesis, and so $D^*(f_z) = \|f_z\|_2^2 = \|f\|_2^2$ for all z. Thus $C^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$, by (34).

Part (b). Since $D(f)$ and $\tilde{D}(f)$ are bounded by a multiple of $\|f\|_2^2$ for all $f \in L^2$, so are $C(f)$ and $\tilde{C}(f)$ for all $f \in \mathcal{F}$ by Proposition 3.2. Hence $D(f; i^p\psi + \tilde{\psi})$ and $C(f; i^p\psi+\tilde{\psi})$ are also bounded by a multiple of $\|f\|_2^2$ for $f \in \mathcal{F}$, so that by Lemma B.1, all these quantities are continuous as functions of $f \in L^2$. Hence so are $C^*(f)$ and $D^*(f)$, by (32) and (33). And the averaging formula (34) again holds for all $f \in \mathcal{F}$.

We need only prove $D^*(f) = \|f\|_2^2$ for all $f \in \mathcal{F}$, because $\mathcal{F}$ is dense in L^2. Take $f \in \mathcal{F}$. First, $h^*(z) := \|f\|_2^2 - \text{Re } D^*(f_z)$ is an almost periodic function, in view of (33) and the fact that $z \mapsto D(f_z; i^p\psi+\tilde{\psi})$ is almost periodic by the proof of Theorem 3.1(b) in Section 3. And h^* is nonnegative since

$$|D^*(f_z)| \leq D(f_z)^{1/2}\tilde{D}(f_z)^{1/2} \leq \|f_z\|_2^2 = \|f\|_2^2,$$

by using Cauchy–Schwarz on (30) then the assumption that $D(\cdot), \tilde{D}(\cdot) \leq \|\cdot\|_2^2$. Further,

$$M(h^*) := \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h^*(z)\, dz = \|f\|_2^2 - \text{Re } C^*(f)$$

by (34). But $C^*(f) = \|f\|_2^2$ by hypothesis, and so $M(h^*) = 0$. Since h^* is almost periodic and nonnegative with zero mean, Proposition C.2 implies $h^* \equiv 0$. But $h^*(0) = 0$ gives Re $D^*(f) = \|f\|_2^2$, which combines with $|D^*(f)| \leq \|f\|_2^2$ to imply $D^*(f) = \|f\|_2^2$ as desired. □

Proof of Theorem 8.1. If $D(f) \leq \|f\|_2^2$ and $\tilde{D}(f) \leq \|f\|_2^2$ for all $f \in L^2$ then $C^*(f)$ and $D^*(f)$ do exist, as shown in the proof of Theorem 8.2(b), and $C(f), \tilde{C}(f) \leq \|f\|_2^2$ for all $f \in L^2$ by Proposition 3.2 and Lemma B.1. Now Theorem 8.1 follows from Theorem 8.2. □

Our result on continuous dilation families, Theorem 4.1, also has a “dual frame” analogue, but we leave this to the interested reader to pursue.

We next want to state an analogue of the Second Oversampling Theorem for dual frames. When $D_s(f)$ and $\tilde{D}_s(f) = D_s(f; \tilde{\psi})$ are finite we can define

$$D_s^*(f) = \frac{1}{|\det s|} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{l=1}^{m} \langle f, \psi_{j,s^{-1}k,l}\rangle \langle \tilde{\psi}_{j,s^{-1}k,l}, f\rangle,$$

with the multiple series converging absolutely.

We assume for the next theorem that the a_l are all equal, with $a_1 = \cdots = a_m = a$ say, and that the b_l are also all equal, with $b_1 = \cdots = b_m = b$.

Theorem 8.3 (“Second Oversampling Theorem for dual frames”)

Let $a, b \in GL(d, \mathbb{R})$ with $|\det a| > 1$, and suppose either a is expanding or else a is amplifying for both ψ_l and $\tilde{\psi}_l$ (using the same exhaustion) for each l. Take s to be an invertible integer matrix with $|\det s| > 1$. Assume $\tilde{a} = b^{-1}ab$ and $s\tilde{a}s^{-1}$ are integer matrices, and that $\tilde{a}$ is prime relative to s. Suppose $D(f), \tilde{D}(f) \leq \tilde{B}\|f\|_2^2$ for all $f \in L^2$, for some constant $\tilde{B} > 0$. Let χ be a closed, convex set in the complex plane.

If $D^*(f) \in \|f\|_2^2\chi$ for all $f \in L^2$, then $D_s^*(f) \in \|f\|_2^2\chi$ for all $f \in L^2$.

Recall convexity of χ means that χ contains all finite linear combinations of the form $\sum_q \eta_q \zeta_q$ with $\zeta_q \in \chi$ and $0 < \eta_q < 1, \sum_q \eta_q = 1$.

For example, one might take χ to be the interval $[A, B]$ for some positive numbers A and B, in which case Theorem 8.3 says the frame bounds A and B are preserved by the oversampling. This corollary of the theorem was proved in the "tight" case $A = B$ by Chui and Shi [9, Theorem 8], assuming also a is a multiple of the identity and s and b are diagonal.

Proof of Theorem 8.3. First note that $D(f)$ and $\tilde{D}(f)$ are finite, always, and so certainly $D^*(f)$ exists for all $f \in L^2$. And because $D(f)$ and $\tilde{D}(f)$ are bounded by a multiple of $\|f\|_2^2$, so are $D_s(f)$ and $\tilde{D}_s(f)$ for all $f \in \mathcal{F}$, by Proposition 6.2. Therefore $D_s(f; i^p\psi + \tilde{\psi})$ is also bounded by a multiple of $\|f\|_2^2$ for all $f \in \mathcal{F}$, and so all these quantities are continuous as functions of $f \in L^2$, by Lemma B.1.

This continuity, together with the easily-proved polarization identity

$$D_s^*(f) = \frac{1}{4} \sum_{p=1}^{4} i^p D_s(f; i^p\psi + \tilde{\psi}), \tag{35}$$

means that the map $f \mapsto D_s^*(f)$ is continuous on L^2. Therefore we need only prove $D_s^*(f) \in \|f\|_2^2 \chi$ for all $f \in \mathcal{F}$, since $\mathcal{F}$ is dense in L^2 and χ is closed in the complex plane.

Now for $f \in \mathcal{F}$, formula (35) and Proposition 6.2 give the averaging formula

$$D_s^*(f) = \lim_{J \to \infty} \frac{1}{|V|} \sum_{v \in V} D^*(f_{a^J bv}), \tag{36}$$

where V is a maximal set of distinct column vectors in the group $(s^{-1}\mathbb{Z}^d)/\mathbb{Z}^d$. And we assume in this theorem that $D^*(f) \in \|f\|_2^2 \chi$ for all $f \in L^2$, and so $D^*(f_z) \in \|f_z\|_2^2 \chi = \|f\|_2^2 \chi$ for all z. Using this in (36) gives $D_s^*(f) \in \|f\|_2^2 \chi$, by the convexity and closedness of the set $\|f\|_2^2 \chi$. This completes the proof. □

9. The Fourier series of $g_{y,l}$

In this section we prove certain facts about the functions $g_{j,l}$ and $g_{y,l}$ that were used repeatedly earlier in the paper. We need not assume the dilation matrices are all equal, but we do assume they are exponentials: $a_1 = e^{\alpha_1}, \ldots, a_m = e^{\alpha_m}$ for some $d \times d$ real matrices $\alpha_1, \ldots, \alpha_m$. (Thus it makes sense to raise each matrix a_l to a real power, by $a_l^y := e^{\alpha_l y}$ for $y \in \mathbb{R}$.) And we assume either (i$'$) or (ii$'$) holds for each l, as in Section 4. But the results and proofs in this section are also valid without the exponential assumption and assuming only (i) or (ii) for each l, provided we consider only integer values of y, in other words if $y = j \in \mathbb{Z}$. In particular the results from Lemmas 9.2 and 9.3 below, that that $C(f) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \widehat{g_{j,l}}(0)$ and that $\sum_{j\in\mathbb{Z}} \sum_{n\neq 0} |\widehat{g_{j,l}}(n)| < \infty$, remain valid without the exponential assumption. (In the proofs, simply take $y = 0$ wherever the expression $j + y$ occurs.)

Our first task is to define the dense subset $\mathcal{F}$ of L^2. For each l, if (i′) holds (a_l is continuously expanding) then for all $r \in \mathbb{N}$ we define $\mathcal{A}_l(r) = \{\xi \in \mathbb{R}^d : r^{-1} < |\xi| < r\}$, so that $\cup_{r=1}^{\infty} \mathcal{A}_l(r) = \mathbb{R}^d \setminus \{0\}$ has full measure. If (i′) fails then (ii′) holds (a_l is continuously amplifying for ψ_l), in which case the sets $\mathcal{A}_l(r)$ are already defined by the exhaustion and $\cup_{r=1}^{\infty} \mathcal{A}_l(r)$ has full measure by definition. Now we can define

$$\mathcal{F} = \left\{ f \in L^2 : \hat{f} \in L^\infty(\mathbb{R}^d), \text{ and } \operatorname{spt}(\hat{f}) \subset \cap_{l=1}^{m} \mathcal{A}_l(r) \text{ for some } r \in \mathbb{N} \right\},$$

which is dense in L^2 because the set $\cup_{r=1}^{\infty} \cap_{l=1}^{m} \mathcal{A}_l(r) = \cap_{l=1}^{m} \cup_{r=1}^{\infty} \mathcal{A}_l(r)$ has full measure.

Notice that if $f \in \mathcal{F}$ then $\hat{f}$ has compact support, because $\mathcal{A}_l(r) \subset B(r)$. Also, if f belongs to $\mathcal{F}$ then so do its translate functions $f_z(\cdot) = f(\cdot - z), z \in \mathbb{R}^d$, since $\widehat{f_z}(\xi) = e^{-2\pi i \xi z} \hat{f}(\xi)$ has the same support as $\hat{f}$.

In the five lemmas that follow, we fix $f \in \mathcal{F}$ and consider $y \in \mathbb{R}, l \in \{1, \ldots, m\}$. We study the nonnegative function

$$g_{y,l}(z) = \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{y,k-z,l} \rangle|^2, \qquad z \in \mathbb{R}^d,$$

which is clearly 1-periodic in each coordinate direction. The following lemmas investigate the Fourier coefficients $\widehat{g_{y,l}}(n) = \int_{[0,1]^d} g_{y,l}(z) e^{-2\pi i n z}\, dz$, where $n \in \mathbb{Z}^d$ is always a row vector and z is a column vector.

Lemma 9.1

$g_{y,l} \in L^1([0,1]^d)$ and

$$0 \le \widehat{g_{y,l}}(0) = \int_{[0,1]^d} g_{y,l}(z)\, dz \le \frac{|\det a_l|^y}{|\det b_l|} \|\hat{f}\|_\infty^2 \|\psi_l\|_2^2. \tag{37}$$

Lemma 9.2

$$C(f) = \sum_{j \in \mathbb{Z}} \sum_{l=1}^{m} \widehat{g_{j,l}}(0) \quad \textit{and} \quad F(f) = \int_{\mathbb{R}} \sum_{l=1}^{m} \widehat{g_{y,l}}(0)\, dy.$$

Lemma 9.3

$$\sum_{j \in \mathbb{Z}} \sum_{n \ne 0} |\widehat{g_{j,l}}(n)| < \infty \quad \textit{and} \quad \int_{\mathbb{R}} \sum_{n \ne 0} |\widehat{g_{y,l}}(n)|\, dy < \infty.$$

Lemma 9.4

$g_{y,l}$ is continuous and equals its Fourier series at every point. This Fourier series converges absolutely, and so the order of summation is unimportant.

Lastly, we relate the Fourier coefficients to oversampling.

Lemma 9.5

Suppose $s \in GL(d,\mathbb{R})$ is an integer matrix. Then

$$\frac{1}{|\det s|} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{y,s^{-1}k,l}\rangle|^2 = \sum_{n\in\mathbb{Z}^d} \widehat{g_{y,l}}(n) \begin{Bmatrix} 1 & \text{if } n \in \mathbb{Z}^d s \\ 0 & \text{otherwise} \end{Bmatrix}.$$

Remarks.

1. Clearly

$$D(f) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} g_{j,l}(0) = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} \widehat{g_{j,l}}(n)$$

for $f \in \mathcal{F}$, by Lemma 9.4, and thus Lemmas 9.2 and 9.3 show for $f \in \mathcal{F}$ that

(38) $$C(f) < \infty \quad \text{if and only if} \quad D(f) < \infty.$$

2. The next observation is used several times elsewhere: for $f \in \mathcal{F}$, if

$$\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} |\widehat{g_{j,l}}(0)| < \infty$$

then

(39) $$\sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \sum_{n\in\mathbb{Z}^d} |\widehat{g_{j,l}}(n)| < \infty,$$

by Lemma 9.3.

3. Many elements of the proofs of the above lemmas are taken from the literature, although the formulation of these ideas into the lemmas presented here seems to be new, as is the treatment of continuously expanding dilations. The simpler one dimensional case is treated in [25].

Proof of Lemma 9.1. For $z \in \mathbb{R}^d$, define

$$\begin{aligned} G_{y,l}(z) = \langle f, \psi_{y,z,l}\rangle &= \int_{\mathbb{R}^d} f(x)\overline{|\det a_l|^{y/2}\psi_l(a_l^y x - b_l z)}\, dx \\ &= |\det a_l|^{-y/2} \int_{\mathbb{R}^d} f(a_l^{-y}x)\overline{\psi_l(x - b_l z)}\, dx && \text{by } x \mapsto a_l^{-y}x \\ &= |\det a_l|^{y/2} \int_{\mathbb{R}^d} \hat{f}(\xi a_l^y)\overline{\widehat{\psi_l}(\xi)} e^{2\pi i \xi b_l z}\, d\xi && \text{by Parseval} \\ &= |\det a_l|^{y/2} \left[\hat{f}(\cdot a_l^y)\overline{\widehat{\psi_l}(\cdot)}\ \right]\hat{}\,(-b_l z) && (40) \\ &\in L^2, \end{aligned}$$

where we observe that $\hat{f}(\cdot a_l^y)\overline{\widehat{\psi_l}(\cdot)} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ because $\hat{f}$ is bounded with compact support (since $f \in \mathcal{F}$). Hence by Plancherel,

$$\|G_{y,l}\|_2^2 = \frac{|\det a_l|^y}{|\det b_l|} \int_{\mathbb{R}^d} \left|\hat{f}(\xi a_l^y)\overline{\widehat{\psi_l}(\xi)}\right|^2 d\xi \le \frac{|\det a_l|^y}{|\det b_l|} \|\hat{f}\|_\infty^2 \|\widehat{\psi_l}\|_2^2 < \infty.$$

But

$$\int_{[0,1]^d} g_{y,l}(z)\,dz = \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{y,k-z,l}\rangle|^2\,dz$$

$$= \int_{\mathbb{R}^d} |\langle f, \psi_{y,z,l}\rangle|^2\,dz = \|G_{y,l}\|_2^2, \tag{41}$$

and so $g_{y,l}$ is integrable and $\widehat{g_{y,l}}(0) = \int_{[0,1]^d} g_{y,l}(z)\,dz$ satisfies the desired estimate (37). □

Proof of Lemma 9.2. Using (41),

$$C(f) = \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m} \int_{\mathbb{R}^d} |\langle f, \psi_{j,z,l}\rangle|^2\,dz = \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m} \int_{[0,1]^d} g_{j,l}(z)\,dz = \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m} \widehat{g_{j,l}}(0),$$

$$F(f) = \int_{\mathbb{R}} \sum_{l=1}^{m} \int_{\mathbb{R}^d} |\langle f, \psi_{y,z,l}\rangle|^2\,dzdy = \int_{\mathbb{R}} \sum_{l=1}^{m} \int_{[0,1]^d} g_{y,l}(z)\,dzdy$$

$$= \int_{\mathbb{R}} \sum_{l=1}^{m} \widehat{g_{y,l}}(0)\,dy. \ □$$

Proof of Lemma 9.3. Lemma 9.1 tells us $g_{y,l}$ is integrable on $[0,1]^d$, and so its Fourier coefficients are well defined. We estimate those coefficients as follows. We first show

$$\widehat{g_{y,l}}(n) = \frac{|\det a_l|^y}{|\det b_l|} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi a_l^y)}\hat{f}((\xi - nb_l^{-1})a_l^y)\widehat{\psi_l}(\xi)\overline{\widehat{\psi_l}(\xi - nb_l^{-1})}\,d\xi, \qquad n\in\mathbb{Z}^d, \tag{42}$$

where we regard both ξ and n as row vectors.

For simplicity we will write $a = a_l, b = b_l$ and $\psi = \psi_l$ in most of the rest of the proof. To deduce (42), note that

$$\widehat{g_{y,l}}(n) = \int_{[0,1]^d} g_{y,l}(z)e^{-2\pi inz}\,dz$$

$$= \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{y,k-z,l}\rangle|^2 e^{2\pi in(k-z)}\,dz$$

$$= \int_{\mathbb{R}^d} |\langle f, \psi_{y,z,l}\rangle|^2 e^{2\pi inz}\,dz$$

$$= \int_{\mathbb{R}^d} G_{y,l}(-b^{-1}z)e^{-2\pi inb^{-1}z}\overline{G_{y,l}(-b^{-1}z)}\,dz/|\det b|,$$

by definition of $G_{y,l}$ and changing $z\mapsto -b^{-1}z$. Using (40) and Parseval, we now quickly obtain (42).

Using the elementary inequality

$$|\hat{\psi}(\xi)\hat{\psi}(\xi - nb^{-1})| \le \frac{1}{2}|\hat{\psi}(\xi)|^2 + \frac{1}{2}|\hat{\psi}(\xi - nb^{-1})|^2,$$

we see from (42) that

$$\sum_{n\neq 0} |\widehat{g_{y,l}}(n)| \leq \frac{1}{2}\frac{|\det a|^y}{|\det b|}\sum_{n\neq 0}\int_{\mathbb{R}^d}\left|\hat{f}(\xi a^y)\hat{f}((\xi-nb^{-1})a^y)\right||\hat{\psi}(\xi)|^2\,d\xi$$
$$+\frac{1}{2}\frac{|\det a|^y}{|\det b|}\sum_{n\neq 0}\int_{\mathbb{R}^d}\left|\hat{f}(\xi a^y)\hat{f}((\xi-nb^{-1})a^y)\right||\hat{\psi}(\xi-nb^{-1})|^2\,d\xi. \tag{43}$$

The second term equals the first term, after making the substitutions $\xi \mapsto \xi + nb^{-1}$ and $n \mapsto -n$.

We want to show

$$\sum_{j\in\mathbb{Z}}\sum_{n\neq 0}|\widehat{g_{j,l}}(n)| \quad \text{and} \quad \int_{\mathbb{R}}\sum_{n\neq 0}|\widehat{g_{y,l}}(n)|\,dy$$

are finite, which we will accomplish by showing $y \mapsto \sum_{j\in\mathbb{Z}}\sum_{n\neq 0}|\widehat{g_{j+y,l}}(n)|$ is bounded for $y \in [0,1]$. In view of the preceding paragraph, we actually need only show the integral $\int_{\mathbb{R}^d}\sigma(y,\xi)|\hat{\psi}(\xi)|^2\,d\xi$ is bounded for $y \in [0,1]$, where

$$\sigma(y,\xi) = \sum_{j\in\mathbb{Z}}\frac{|\det a|^{j+y}}{|\det b|}\sum_{n\neq 0}\left|\hat{f}(\xi a^{j+y})\hat{f}\big((\xi-nb^{-1})a^{j+y}\big)\right|. \tag{44}$$

And since $|\hat{\psi}|^2$ is integrable, it suffices to show σ is bounded for $\xi \in \operatorname{spt}(\hat{\psi})$ and $y \in [0,1]$. We prove the boundedness of σ in two cases.

Case (i′): a_l is continuously expanding. In this case there exist numbers $0 < \kappa \leq 1 < \gamma$ such that

$$|\xi a^y| \geq \kappa\gamma^y|\xi| \quad \text{and} \quad |\xi a^{-y}| \leq \kappa^{-1}\gamma^{-y}|\xi|$$

for all row vectors $\xi \in \mathbb{R}^d$ and all $y \geq 0$.

Fix $\xi \in \mathbb{R}^d$ and $0 \leq y \leq 1$. Because $f \in \mathcal{F}$, there exists $r \in \mathbb{N}$ such that $\operatorname{spt}(\hat{f}) \subset \{\zeta \in \mathbb{R}^d : r^{-1} < |\zeta| < r\}$. Clearly then

$$\sigma(y,\xi) \leq |\det b|^{-1}\sum_{\{j:r^{-1}<|\xi a^{j+y}|<r\}}\|\hat{f}\|_\infty^2 \cdot \#\{n\neq 0 : |nb^{-1}a^{j+y}| < 2r\}\cdot|\det a|^{j+y}.$$

The number of j-values such that $r^{-1} < |\xi a^{j+y}| < r$ is at most $j_0 := 1 + \lfloor\log_\gamma\frac{r^2}{\kappa}\rfloor$, because for all $\zeta \in \mathbb{R}^d \setminus \{0\}$ and all $j \geq j_0$ we have $|\zeta a^j| \geq \kappa\gamma^j|\zeta| > r^2|\zeta|$.

Thus it suffices to show there exists $N = N(a,b,d,r)$ such that

$$\#\{n\neq 0 : |nb^{-1}a^{j+y}| < 2r\}\cdot|\det a|^{j+y} \leq N \qquad \text{for all } j \in \mathbb{Z}, \tag{45}$$

because then $\sigma(y,\xi) \leq |\det b|^{-1}j_0\|\hat{f}\|_\infty^2 \cdot N$, so that σ is bounded.

For (45), we first take j_1 to be the smallest positive integer greater than $\log_\gamma(2r\|b\|/\kappa)$, and observe that if $j \geq j_1$ and $n \neq 0$ then

$$|nb^{-1}a^{j+y}| \geq \kappa\gamma^{j+y}|nb^{-1}| \geq \kappa\gamma^{j_1}|n|/\|b\| > 2r$$

by choice of j_1. Thus when proving (45), we can suppose $j < j_1$.

We now make another reduction: write

$$\tilde{a} = b^{-1}ab \qquad \text{and} \qquad \tilde{r} = 2r \max_{0\leq y\leq 1} \|a^{-(j_1+y)}b\|,$$

and observe that if $|nb^{-1}a^{j+y}| < 2r$ then straightforwardly $|n\tilde{a}^{j-j_1}| < \tilde{r}$. Writing $J = j_1 - j > 0$ we see that (45) will follow once we prove

$$\#\{n \in \mathbb{Z}^d : |n\tilde{a}^{-J}| < \tilde{r}\} \leq N|\det \tilde{a}|^J \qquad \text{for all } J \in \mathbb{N}. \tag{46}$$

Notice $\tilde{a}$ is expanding, with the same value "γ" as a, but possibly with a smaller "κ", say $\tilde{\kappa} = \kappa/\|b\|\|b^{-1}\|$. Fix $j_2 > 0$ so large that $\tilde{\kappa}^{-1}\gamma^{-j_2}(\tilde{r}+\sqrt{d}) < \tilde{r}$. Let $J \in \mathbb{N}$ and suppose n satisfies $|n\tilde{a}^{-J}| < \tilde{r}$. Then $n + [0,1]^d \subset B(\tilde{r})\tilde{a}^{J+j_2}$, because for $\zeta \in [0,1]^d$ we have

$$|(n+\zeta)\tilde{a}^{-J-j_2}| \leq (\tilde{\kappa}^{-1}\gamma^{-j_2}|n\tilde{a}^{-J}| + \tilde{\kappa}^{-1}\gamma^{-J-j_2}|\zeta|) < \tilde{\kappa}^{-1}\gamma^{-j_2}(\tilde{r}+\sqrt{d}) < \tilde{r}$$

by choice of j_2. Now (46) follows immediately by comparing volumes (in fact one obtains $N = |B(\tilde{r})||\det \tilde{a}|^{j_2}$).

Note. The above approach is by now standard in the wavelet literature, at least in the case of discrete dilation families (cf. [20, Lemma 7.1.16], [6, §3] and [1, §2]). The amplifying case below seems somewhat new, though.

Case (ii′): a_l is continuously amplifying for ψ_l. If (i′) does not apply then (ii′) must hold. Thus for each $r \in \mathbb{N}$ we have $\operatorname{spt}(\widehat{\psi_l}) \cap \mathcal{A}_l(r)a_l^y = \emptyset$ whenever $|y|$ is sufficiently large. But $\mathcal{A}_l(r) \supset \operatorname{spt}(\hat{f})$ for some r because $f \in \mathcal{F}$, and so

$$\operatorname{spt}(\hat{f}) \cap \operatorname{spt}(\widehat{\psi_l})a_l^{j+y} = \emptyset \qquad \text{for all } y \in [0,1], \text{ whenever } |j| \text{ is sufficiently large.}$$

That is, there exists $J \in \mathbb{N}$ such that

$$\xi \in \operatorname{spt}(\widehat{\psi_l}),\ \xi a^{j+y} \in \operatorname{spt}(\hat{f}) \quad \Longrightarrow \quad |j| \leq J.$$

Hence the definition (44) of σ implies that for $\xi \in \operatorname{spt}(\widehat{\psi_l})$ and $y \in [0,1]$,

$$\sigma(y,\xi) \leq \sum_{|j|\leq J} \frac{|\det a|^{j+y}}{|\det b|}\|\hat{f}\|_\infty^2 \cdot \#\{n \in \mathbb{Z}^d : nb^{-1}a^{j+y} \in B(2r)\},$$

since if ξa^{j+y} and $(\xi - nb^{-1})a^{j+y}$ both belong to $\operatorname{spt}(\hat{f}) \subset \mathcal{A}_l(r) \subset B(r)$, then $nb^{-1}a^{j+y} \in B(2r)$. The last displayed formula shows that σ is bounded for $\xi \in \operatorname{spt}(\widehat{\psi_l})$ and $y \in [0,1]$. $\square$

Proof of Lemma 9.4. For simplicity we again write $a = a_l, b = b_l$ and $\psi = \psi_l$.

Recall from the definition of $\mathcal{F}$ that $\hat{f}$ is supported in the ball $B(r)$ for some $r \in \mathbb{N}$. Hence from (43) and the comment following it we find that

$$\begin{aligned}\sum_{n\neq 0} |\widehat{g_{y,l}}(n)| &\leq \frac{|\det a|^y}{|\det b|} \sum_{n\neq 0} \int_{\mathbb{R}^d} \left| \hat{f}(\xi a^y) \hat{f}\big((\xi - nb^{-1})a^y\big) \right| |\widehat{\psi}(\xi)|^2 \, d\xi \\ &\leq \frac{|\det a|^y}{|\det b|} N_{y,l} \|\hat{f}\|_\infty^2 \|\psi\|_2^2,\end{aligned}$$

where $N_{y,l}$ is the number of lattice points n in the set $B(2r)a^{-y}b$. Also

$$|\widehat{g_{y,l}}(0)| \leq \frac{|\det a|^y}{|\det b|} \|\hat{f}\|_\infty^2 \|\psi\|_2^2$$

by Lemma 9.1, and so

$$\sum_{n\in\mathbb{Z}^d} |\widehat{g_{y,l}}(n)| \leq \frac{|\det a|^y}{|\det b|} (1 + N_{y,l}) \|\hat{f}\|_\infty^2 \|\psi\|_2^2 < \infty. \tag{47}$$

Thus the Fourier coefficients of $g_{y,l}$ belong to $\ell^1(\mathbb{Z}^d)$, and so the Fourier series of $g_{y,l}$ converges absolutely and uniformly to a continuous function we call $Sg_{y,l}$. The uniform convergence implies $Sg_{y,l}$ and $g_{y,l}$ have the same Fourier coefficients, and so they agree almost everywhere. We aim to show they agree *everywhere*, by showing $g_{y,l}$ is continuous.

Note that $g_{y,l}(z)$ is lower semicontinuous, because it is the sum (over k) of the nonnegative continuous functions $z \mapsto |\langle f, \psi_{y,k-z,l}\rangle|^2$. Lower semicontinuity together with the almost everywhere equality of $g_{y,l}$ and $Sg_{y,l}$ implies that $0 \leq g_{y,l} \leq Sg_{y,l} < \infty$ at every point.

Define a sequence $c(f) := \{\langle f, \psi_{y,k,l}\rangle\}_{k\in\mathbb{Z}^d}$. By (4) we find $\|c(f_z)\|_{\ell^2(\mathbb{Z}^d)}^2 = g_{y,l}(b^{-1}a^y z) < \infty$, and in particular $\|c(f)\|_{\ell^2(\mathbb{Z}^d)}^2 = g_{y,l}(0)$. Thus $c(f_z) \in \ell^2(\mathbb{Z}^d)$, for all $z \in \mathbb{R}^d$. The triangle inequality in ℓ^2 now gives for all $z, z^* \in \mathbb{R}^d$ that

$$\begin{aligned}\left| g_{y,l}(b^{-1}a^y z)^{1/2} - g_{y,l}(b^{-1}a^y z^*)^{1/2} \right| &= \big| \|c(f_z)\|_{\ell^2} - \|c(f_{z^*})\|_{\ell^2} \big| \\ &\leq \|c(f_z) - c(f_{z^*})\|_{\ell^2} = \|c(f_z - f_{z^*})\|_{\ell^2},\end{aligned}$$

and the square of this last quantity equals

$$g_{y,l}(0; f_z - f_{z^*}) \leq Sg_{y,l}(0; f_z - f_{z^*}) \leq \frac{|\det a|^y}{|\det b|} (1 + N_{y,l}) \|\,(f_z - f_{z^*})\hat{}\,\|_\infty^2 \|\psi\|_2^2$$

by (47) applied to the function $f_z - f_{z^*} \in \mathcal{F}$. Since $(f_z - f_{z^*})\hat{}\,(\xi) = (e^{-2\pi i\xi z} - e^{-2\pi i \xi z^*})\hat{f}(\xi)$, and since $(e^{-2\pi i\xi z} - e^{-2\pi i\xi z^*})$ converges uniformly to zero for ξ in the support of $\hat{f}$, as $z \to z^*$, we conclude that $\|\,(f_z - f_{z^*})\hat{}\,\|_\infty \to 0$ as $z \to z^*$. The continuity of $g_{y,l}^{1/2}$ follows, proving the lemma. $\square$

Proof of Lemma 9.5. Let V be a maximal set of distinct column vectors in the group $(s^{-1}\mathbb{Z}^d)/\mathbb{Z}^d$, so that $|V| = |\det s|$. Then

$$\begin{aligned}
\frac{1}{|\det s|}\sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{y,s^{-1}k,l}\rangle|^2 &= \frac{1}{|V|}\sum_{v\in V}\sum_{k^*\in\mathbb{Z}^d} |\langle f, \psi_{y,k^*-v,l}\rangle|^2 && \text{since } s^{-1}\mathbb{Z}^d = \mathbb{Z}^d - V \\
&= \frac{1}{|V|}\sum_{v\in V} g_{y,l}(v) && \text{by definition of } g_{y,l} \\
&= \frac{1}{|V|}\sum_{v\in V}\sum_{n\in\mathbb{Z}^d} \widehat{g_{y,l}}(n)\exp(2\pi i n v) && \text{by Lemma 9.4} \\
&= \sum_{n\in\mathbb{Z}^d} \widehat{g_{y,l}}(n) \begin{Bmatrix} 1 & \text{if } n \in \mathbb{Z}^d s \\ 0 & \text{otherwise} \end{Bmatrix} && \text{by Lemma D.3 with } u = n,
\end{aligned}$$

where the above series converge absolutely by Lemma 9.4. □

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Appendix A. Fourier characterization of continuous multiwavelets

Discrete dilations: proof that (C)⇔(2) In Section 3 we remarked that (C) holds (in other words $\{\psi_1, \ldots, \psi_m\}$ is a continuous multiwavelet) if and only if the Calderón condition (2) holds, which is

$$\Delta(\xi) := \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m} \frac{|\hat{\psi}_l(\xi a_l^j)|^2}{|\det b_l|} = 1 \quad \text{for almost every row vector } \xi \in \mathbb{R}^d. \tag{48}$$

The following proof that (C)⇔(2) is well known in outline, but we provide detail since we are dealing with multiwavelets having differing dilation and translation matrices.

First let $f \in L^2$ and assume $C(f) < \infty$. We express $C(f)$ as an integral involving Δ:

$$\begin{aligned}
C(f) &= \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}\int_{\mathbb{R}^d} |\langle f, \psi_{j,z,l}\rangle|^2\, dz \\
&= \sum_{j\in\mathbb{Z}}\sum_{l=1}^{m}\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{\psi}_l(\xi a_l^{-j})}\exp(2\pi i \xi a_l^{-j} b_l z)\, d\xi\right|^2 dz\, |\det a_l|^{-j}
\end{aligned}$$

by Parseval. Now making the change of variable $z \mapsto -b_l^{-1} a_l^j z$, we find

$$
\begin{aligned}
C(f) &= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\widehat{\psi}_l(\xi a_l^{-j})} \exp(-2\pi i \xi z)\, d\xi \right|^2 dz\, |\det b_l|^{-1} \\
&= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \int_{\mathbb{R}^d} \left| \left[\hat{f}(\cdot) \overline{\widehat{\psi}_l(\cdot a_l^{-j})} \right]^{\wedge}(z) \right|^2 dz\, |\det b_l|^{-1} \qquad (49)\\
&= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{m} \int_{\mathbb{R}^d} \left| \hat{f}(\xi) \overline{\widehat{\psi}_l(\xi a_l^{-j})} \right|^2 d\xi\, |\det b_l|^{-1} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \Delta(\xi)\, d\xi.
\end{aligned}
$$

The finiteness of $C(f)$ was invoked implicitly at line (49) to deduce that $\left[\hat{f}(\cdot) \overline{\widehat{\psi}_l(\cdot a_l^{-j})} \right]^{\wedge}$ is square integrable, which then justifies the use of Plancherel's identity in the next line.

If (C) holds then certainly $C(f) = \|f\|_2^2 < \infty$, and so the above argument yields

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 [\Delta(\xi) - 1]\, d\xi = C(f) - \|f\|_2^2 = 0.$$

This holds for all $\hat{f} \in L^2$, and so $\Delta(\xi) = 1$ for almost every ξ, which is (48).

On the other hand, if $\Delta = 1$ a.e. then by reversing the above calculations we deduce $C(f) = \|\hat{f}\|_2^2 = \|f\|_2^2$, which is (C).

Continuous dilations. Condition (F) holds if and only if

$$\int_{\mathbb{R}} \sum_{l=1}^{m} \frac{|\widehat{\psi}_l(\xi a_l^y)|^2}{|\det b_l|} dy = 1 \quad \text{for almost every row vector } \xi \in \mathbb{R}^d. \qquad (50)$$

The proof is just the same as for (C)$\Leftrightarrow$(2) above, except that summation over j is replaced by integration over y, and we only know that $\left[\hat{f}(\cdot) \overline{\widehat{\psi}_l(\cdot a_l^{-y})} \right]^{\wedge}$ is square integrable for almost every $y \in \mathbb{R}$.

The characterization (50) of continuous wavelets having continuous dilations was considered (with $m = 1$) by Calderón [5, §34], and later by Grossman and Morlet [17]. For some later developments, see [32].

Appendix B. Continuity and approximation

Write "X" for any one of the letters C, D, E, F or D^q, D_s.

Lemma B.1

Let $B \geq 0$ and suppose $\mathcal{L}$ is a dense subset of L^2. If $X(f) \leq B\|f\|_2^2$ for all $f \in \mathcal{L}$ then the map $f \mapsto X(f)$ is continuous from L^2 to $\mathbb{R}$.

The proof is a trivial modification of the corresponding lemma in one dimension, [25, Lemma B.1]. An immediate corollary is:

Lemma B.2

Fix $0 \leq A \leq B < \infty$. *If* $A\|f\|_2^2 \leq X(f) \leq B\|f\|_2^2$ *for a dense set of* $f \in L^2$, *then* $A\|f\|_2^2 \leq X(f) \leq B\|f\|_2^2$ *for all* $f \in L^2$.

Appendix C. Almost periodic functions

Here we establish the properties of almost periodic functions used in Section 3. We borrow freely from the book of Loomis [27, pp. 165–166] and elsewhere, but we cover only those aspects of the theory employed earlier in this paper. Part of the goal of this appendix, in fact, is to demonstrate that the deeper aspects of almost periodic function theory are unnecessary for this paper.

Recall that a set in a metric space is *totally bounded* if for every $\epsilon > 0$ the set can be covered by finitely many balls of radius ϵ centered at points in the set. By general facts from metric topology [24, §§59–65], we know:

- every subset of a totally bounded set is totally bounded,
- a set is totally bounded if and only if its closure (in the metric space) is totally bounded,
- a set is totally bounded and complete if and only if it is compact.

We say a bounded continuous function $h(x)$ on $\mathbb{R}^d$ is *almost periodic* if its set of translates $S_h = \{h(\cdot + z) : z \in \mathbb{R}^d\}$ is totally bounded in the metric space $L^\infty = L^\infty(\mathbb{R}^d)$. Equivalently, h is almost periodic if and only if the closure $\overline{S_h}$ in L^∞ is compact, since $\overline{S_h}$ is a closed subset of the complete space L^∞ and hence is complete.

Clearly all constant functions are almost periodic. So are periodic functions of one coordinate, that is, functions of the form $h(x) = h_0(x_p)$ for some $p \in \{1, \ldots, d\}$ and some continuous periodic function h_0 of one variable; one can prove this almost periodicity of h using the uniform continuity of h_0 over its period. As a special case, we see $x \mapsto e^{2\pi i u_p x_p}$ is almost periodic, for each fixed $u_p \in \mathbb{R}$.

Now we can show the sum of two almost periodic functions is almost periodic. For suppose h_1 and h_2 are almost periodic. Using the topologies inherited from L^∞, the Cartesian product space $\overline{S_{h_1}} \times \overline{S_{h_2}}$ is compact and the additive map $(H_1, H_2) \mapsto H_1 + H_2$ from the product space into L^∞ is continuous. The image of this additive map is therefore compact and hence is totally bounded. Thus $S_{h_1+h_2}$ is also totally bounded, since it is a subset of the image of the additive map. Thus $h_1 + h_2$ is totally bounded.

Similarly the product of two almost periodic functions is almost periodic. Hence a finite linear combination of products of almost periodic functions is almost periodic. In particular, finite trigonometric sums of the form $\sum_u c_u \exp(2\pi i u x)$ are almost periodic, where $c_u \in \mathbb{C}$ and we sum over a finite set of row vectors $u \in \mathbb{R}^d$, and where x is a column vector.

Similar arguments also show that the real and imaginary parts of an almost periodic function are almost periodic.

Next, if a sequence of almost periodic functions $\{h_j\}_{j\in\mathbb{N}}$ converges in L^∞ to a (necessarily continuous and bounded) function h, then h is almost periodic. Indeed, given $\epsilon > 0$, we simply fix j so large that $\|h_j - h\|_\infty < \epsilon/3$ and cover the totally bounded set S_{h_j} with $\epsilon/3$-balls in L^∞ centered at the translates of h_j by $z_1, \ldots, z_q$, say. Then the triangle inequality shows S_h is covered by the ϵ-balls in L^∞ centered at the translates of h by $z_1, \ldots, z_q$. Thus h is almost periodic.

In particular, we obtain:

Lemma C.1

If $\sum_u |c_u| < \infty$ then the infinite trigonometric sum $h(x) = \sum_u c_u \exp(2\pi i u x)$ is almost periodic, where $c_u \in \mathbb{C}$ and we sum over a countable set of row vectors $u \in \mathbb{R}^d$.

(In fact, every almost periodic function can be written as such an infinite trigonometric sum [27, §41E].)

Next comes a well-known proposition used in Section 3.

Proposition C.2

(a) If h is almost periodic then it is uniformly continuous.
(b) Suppose h is almost periodic and nonnegative, and that the mean value

$$M(h) := \lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} h(x)\,dx$$

exists. Then $M(h) = 0$ if and only if $h \equiv 0$.

The mean value $M(h)$ definitely does exist [21, Theorem 18.10], and the conclusion that $M(h) = 0 \Leftrightarrow h \equiv 0$ is true in much greater generality [21, Theorem 18.8(ii)]. But the proposition as stated here gives us all we need for the current paper.

Proof of Proposition C.2. (a) Let $\epsilon > 0$ and take points $z_1, \ldots, z_q \in \mathbb{R}^d$ such that S_h is covered by $\epsilon/3$-balls in L^∞ centered at the translates of h by $z_1, \ldots, z_q$. Then for each $x \in \mathbb{R}^d$ there exists a z_p with $\|h(\cdot + x) - h(\cdot + z_p)\|_\infty < \epsilon/3$. Choosing $\delta > 0$ such that $|h(w + z_p) - h(z_p)| < \epsilon/3$ for all $w \in [-\delta, \delta]^d$ and $p = 1, \ldots, q$, we deduce

$$|h(w + x) - h(x)| \le |h(w + x) - h(w + z_p)| + |h(w + z_p) - h(z_p)| + |h(z_p) - h(x)| < \epsilon$$

whenever $x \in \mathbb{R}^d$ and $w \in [-\delta, \delta]^d$. Thus h is uniformly continuous.

(b) Our first step for (b) is to show that for every $\epsilon > 0$ there exists $r > 0$ and $\delta > 0$ such that for all $n \in \mathbb{Z}^d$, the cube $n + [-r, r]^d$ contains a subcube Q_n of sidelength 2δ with

$$\|h(\cdot + z) - h\|_\infty \le \epsilon \quad \text{for all } z \in Q_n. \tag{51}$$

That is, every point of Q_n is an "ϵ-almost period" for h.

To find r and δ with the desired properties, let $\epsilon > 0$ and use the uniform continuity in part (a) to yield $\delta \in (0, 1)$ such that

$$\|h(\cdot + z) - h(\cdot + w)\|_\infty < \epsilon/2 \qquad \text{whenever } z - w \in [-\delta, \delta]^d. \tag{52}$$

Then use total boundedness to cover S_h with $\epsilon/2$-balls in L^∞ centered at the translates of h by $z_1, \dots, z_q \in \mathbb{R}^d$, so that for each $n \in \mathbb{Z}^d$ there exists z_p with $\|h(\cdot + n) - h(\cdot + z_p)\|_\infty < \epsilon/2$. Write $w_n = n - z_p$, so that

$$\|h(\cdot + w_n) - h\|_\infty < \epsilon/2. \tag{53}$$

Fixing $r \geq 2$ so large that $z_1, \dots, z_q \in [-r/2, r/2]^d$, we see that

$$Q_n := w_n + [-\delta, \delta]^d \subset n + [-r, r]^d,$$

since $w_n \in n + [-r/2, r/2]^d$ and $\delta < 1 \leq r/2$. Furthermore for all $z \in Q_n$,

$$\|h(\cdot + z) - h\|_\infty \leq \|h(\cdot + z) - h(\cdot + w_n)\|_\infty + \|h(\cdot + w_n) - h\|_\infty < \epsilon$$

by (52) and (53).

Now we can prove part (b) of the lemma, by showing that $h \not\equiv 0 \Rightarrow M(h) > 0$. Assume $0 \leq h \not\equiv 0$, so that $h(x_0) > 0$ for some x_0. Write $h(x_0) = 2\epsilon$. Then $h(x_0 + z) \geq \epsilon$ whenever $z \in Q_n$, for all $n \in \mathbb{Z}^d$, by (51). One now easily deduces that the mean value $M(h)$ is positive because $x_0 + \cup_n Q_n$ has positive density in $\mathbb{R}^d$:

$$\limsup_{R \to \infty} \frac{|(x_0 + \cup_n Q_n) \cap Q(R)|}{|Q(R)|} > 0. \ \square$$

Appendix D. Averages of complex exponentials

Here we gather the elementary averaging results used throughout the paper.

The first lemma says the large-scale average of a sum of oscillating exponentials is simply equal to the constant term in the sum. In both this lemma and the next, we take coefficients $c_u \in \mathbb{C}$ and sum over a countable set of row vectors $u \in \mathbb{R}^d$, with 0 being one of the u-values. We write $Q(R) = [-R, R]^d$ for the cube of side $2R$ in the column space $\mathbb{R}^d$.

Lemma D.1

If $\sum_u |c_u| < \infty$ then

$$\lim_{R \to \infty} \frac{1}{|Q(R)|} \int_{Q(R)} \left[\sum_u c_u \exp(2\pi i u x) \right] dx = c_0.$$

Proof. The sum and integral can be interchanged, in view of the absolute convergence of $\sum_u |c_u|$, and then the sum and limit can be interchanged by dominated convergence. Then we need only observe that

$$\lim_{R\to\infty} \frac{1}{|Q(R)|} \int_{Q(R)} \exp(2\pi i u x)\, dx = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{otherwise,} \end{cases}$$

by splitting into an iterated integral over d copies of the interval $[-R, R]$.

Incidentally, the use of cubes in this averaging lemma is just for technical convenience. □

The next lemma concerns the uniqueness of coefficients.

Lemma D.2

If $\sum_u |c_u| < \infty$ and $\sum_u c_u \exp(2\pi i u x) = 0$ for all $x \in \mathbb{R}^d$ then $c_u = 0$ for all u.

Proof. For each fixed index u', we have $\sum_u c_u \exp(2\pi i(u - u')x) \equiv 0$. The constant term $c_{u'}$ in this sum must equal the large-scale average, by Lemma D.1, but of course this average is zero, giving $c_{u'} = 0$. □

The next two lemmas are *discrete* averaging results, and are special cases of [21, Lemma 23.19].

Lemma D.3

Let $s \in GL(d, \mathbb{R})$ be an integer matrix, with V a maximal set of distinct column vectors in the group $(s^{-1}\mathbb{Z}^d)/\mathbb{Z}^d$. Then

$$\frac{1}{|V|} \sum_{v \in V} \exp(2\pi i u v) = \begin{cases} 1 & \textit{if } u \in \mathbb{Z}^d s \\ 0 & \textit{if } u \in \mathbb{Z}^d \setminus \mathbb{Z}^d s. \end{cases}$$

Here u is a row vector, and $\mathbb{Z}^d \setminus \mathbb{Z}^d s$ means $\mathbb{Z}^d \cap (\mathbb{Z}^d s)^c$.

Lemma D.4

Let $s \in GL(d, \mathbb{R})$ be an integer matrix, with W a maximal set of distinct column vectors in the group $\mathbb{Z}^d/(s\mathbb{Z}^d)$. Then

$$\frac{1}{|W|} \sum_{w \in W} \exp(2\pi i u w) = \begin{cases} 1 & \textit{if } u \in \mathbb{Z}^d \\ 0 & \textit{if } u \in (\mathbb{Z}^d s^{-1}) \setminus \mathbb{Z}^d \end{cases}.$$

Taking $W = sV$ in the second lemma reduces it to the first.

In one dimension, for $s \in \mathbb{N}$, Lemma D.4 simply says that $s^{-1} \sum_{w=0}^{s-1} e^{2\pi i u w}$ equals 1 when u is an integer, and equals 0 when u is not an integer but is a multiple of $1/s$.

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