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On the relationship between quasi-affine systems and the à trous algorithm

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Abstract

We seek to demonstrate a connection between refinable quasi-affine systems and the discrete wavelet transform known as the à trous algorithm. We begin with an introduction of the bracket product, which is the major tool in our analysis. Using multiresolution operators, we then proceed to reinvestigate the equivalence of the duality of refinable affine frames and their quasi-affine counterparts associated with a fairly general class of scaling functions that includes the class of compactly supported scaling functions. Our methods show that for negative scales only one of the generalized Smith-Barnwell equations is actually needed to establish the additivity property of the quasi-affine multiresolution operators. This fact is then identified with the à trous algorithm thereby illustrating the connection with quasi-affine systems. We then introduce the notion of a generalized quasi-affine (GQA) system, in which separate generating wavelets are used for non-negative and negative dilations. Sufficient conditions are described for two GQA systems to constitute dual frames, providing a means for the construction of frames from appropriate à trous systems. We conclude with a brief discussion of examples of GQA frames associated with two different biorthogonal wavelet systems. The novelty of this work is the connection established between the à trous algorithm and refinable quasi-affine systems together with the notion of GQA systems, which are introduced to exploit this connection.

1. Introduction

Throughout this analysis the dilation matrix M will be a fixed $n\times n$ matrix with integer entries such that each eigenvalue λ of M satisfies $|\lambda| > 1$, i.e. M is an expanding lattice-preserving $n\times n$ matrix. The unitary dilation operator on $L^2(\mathbb{R}^n)$ induced by M will be

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denoted D and is defined by $Df(x) := |\det M|^{1/2} f(Mx)$ for $f \in L^2(\mathbb{R}^n)$. Similarly, we use M to define an alternative dilation operator Δ, by $\Delta f(x) := f\big((M^T)^{-1}x\big)$, where M^T denotes the transpose of M. We are also interested in the translation operator, T_u, $u \in \mathbb{R}^n$, defined by $T_u f(x) := f(x-u)$. Lastly, we will adopt the following definition for the Fourier transform, $\hat{f}$, of $f \in L^2(\mathbb{R}^n)$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x\rangle}\, dx.$$

We now recall the definition of an affine system.

Definition 1. The *affine system generated by* $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$, denoted $X(\Psi)$, is the collection

$$X(\Psi) = \{\psi_{\ell;j,k} : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where $\psi_{\ell;j,k} := D^j T_k \psi_\ell$.

Examples of affine systems are numerous. We are interested here in affine systems that are based on the notion of a multiresolution analysis (MRA). In one-dimension, we have the 2-band orthonormal MRA wavelets as in [12] and [8] as well as the 2-band biorthogonal MRA wavelets found in [5]. M-band biorthogonal MRA wavelets have also been studied in one-dimension [17], [11]. In n-dimensions, orthonormal [3, 4] and tight-frame [2] MRA wavelets relative to expanding, lattice-preserving dilations have been described. By means of separable products the one-dimensional methods also provide examples of refinable affine systems in n-dimensions.

Closely related to affine systems are the quasi-affine systems introduced by Ron and Shen in [16] as a means for applying the theory of shift-invariant spaces to the characterization of affine frames. We have the following definition.

Definition 2. The *quasi-affine system generated by* $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$, denoted $X^{\mathrm{q}}(\Psi)$, is the collection

$$X^{\mathrm{q}}(\Psi) := \{\psi^{\mathrm{q}}_{\ell;j,k} : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where

$$\psi^{\mathrm{q}}_{\ell;j,k} := \begin{cases} D^j T_k \psi_\ell, & j \ge 0 \\ |\det M|^{j/2} T_k D^j \psi_\ell, & j < 0. \end{cases}$$

The reader should note the use of the superscript q in Definition 2. We will apply this notational tool to other objects below in order to distinguish between the quasi-affine and affine dilation structures. We now state the characterization of affine systems achieved by Ron and Shen [16] under a weak decay assumption that was later overcome in the work of Chui, Shi, and Stöeckler [7].

Theorem 1

Let $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$.

(a) $X(\Psi)$ *is a Bessel system if and only if* $X^{\mathrm{q}}(\Psi)$ *is a Bessel system. Moreover, the Bessel bounds are the same in either case.*

(b) $X(\Psi)$ *is a frame if and only if* $X^{\mathrm{q}}(\Psi)$ *is a frame. Moreover, the frame bounds are the same in either case.*

For completeness we review the definitions of Bessel systems and frames for a Hilbert space, $\mathbb{H}$.

DEFINITION 3. The collection $\{h_j\}_{j\in J} \subset \mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathbb{H}$

$$A\|f\|_{\mathbb{H}}^2 \leq \sum_{j\in J} \left|\langle f, h_j\rangle_{\mathbb{H}}\right|^2 \leq B\|f\|_{\mathbb{H}}^2. \tag{1}$$

The constants A and B are referred to as the lower and upper frame bounds, respectively. In the case that $A = B$ the frame is said to be *tight.* If only the right inequality of (1) holds, the system is called a *Bessel system* and in this case B is referred to as the Bessel bound. We say two frames for $\mathbb{H}$, $\{h_j\}_{j\in J}$ and $\{\tilde{h}_j\}_{j\in J}$, are *dual* if for each $f \in \mathbb{H}$ we have

$$f = \sum_{j\in J} \langle f, \tilde{h}_j\rangle h_j. \tag{2}$$

Finally, let us give a qualitative description of the à trous algorithm introduced in the work [10]. Simply put, the à trous algorithm is a variation on the discrete wavelet transform (DWT) that results in an integer-shift invariant representation of a discrete signal. The deviation from the ordinary DWT may be explained in two equivalent ways. First, one can view the à trous algorithm as a DWT in which the downsampling and upsampling stages are removed. Alternatively, one can realize the à trous algorithm as the analog of the DWT for the system obtained by reversing the order of the dilation and translation operators. This latter formulation is in accord with the quasi-affine scenario. The difference between à trous wavelets and quasi-affine systems lies in the fact that the originators of the à trous algorithm make no mention of positive scales because they were interested only in applications, where the scale $j = 0$ corresponds to the resolution of discrete signals. Besides [10], one may find a treatment of the à trous algorithm in [13] as well as an interesting application of the à trous algorithm to edge-detection in [14].

2. The bracket product

In this section we shall introduce the bracket product, developing basic facts relevant to our study of refinable affine and quasi-affine systems. Most, if not all, of this material can be found elsewhere in the literature, see e.g. [15] or [16], but we include it here for completeness. We have the following definition.

DEFINITION 4. The *bracket product* of f and g, $f, g \in L^2(\mathbb{R}^n)$, is denoted $[f, g]$ and is defined for a.e. $x \in \mathbb{R}^n$ by

$$[f, g](x) = \sum_{k\in\mathbb{Z}^n} f(x + 2\pi k)\overline{g(x + 2\pi k)}. \tag{3}$$

Notice that the bracket product is $2\pi\mathbb{Z}^n$ periodic. We present some elementary properties of the bracket product.

Lemma 2

Let $f, g \in L^2(\mathbb{R}^n)$.

(a) $\big|[\hat{f}, \hat{g}]\big| \le [\hat{f}, \hat{f}]^{1/2}[\hat{g}, \hat{g}]^{1/2}$.

(b) $\displaystyle\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}[\hat{f}, \hat{g}](\xi)d\xi = \langle f, g\rangle$.

(c) $[\hat{f}, \hat{g}] \in L^1(\mathbb{T}^n)$, *but in general,* $[\hat{f}, \hat{g}] \notin L^2(\mathbb{T}^n)$.

(d) $\displaystyle\langle f, T_k g\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}[\hat{f}, \hat{g}](\xi)e^{i[k,\xi]}d\xi$, *i.e.* $\displaystyle\sum_{k\in\mathbb{Z}^n}\langle f, T_k g\rangle e^{-i\langle k,\xi\rangle}$ *is the Fourier series of* $[\hat{f}, \hat{g}]$.

(e) $[\mu\hat{f}, \hat{g}] = \mu[\hat{f}, \hat{g}] = [\hat{f}, \overline{\mu}\hat{g}]$ *when μ is $2\pi\mathbb{Z}^n$ periodic.*

Lemma 3

Let $\psi, \varphi \in \mathcal{C} := \{f \in L^2(\mathbb{R}^n) : [\hat{f}, \hat{f}] \in L^\infty(\mathbb{T}^n)\}$.

(a) $[\hat{\psi}, \hat{\varphi}] \in L^p(\mathbb{T}^n)$, $1 \le p \le \infty$. *If, in addition, ψ and φ have compact support then $[\hat{\psi}, \hat{\varphi}]$ is a trigonometric polynomial.*

(b) *For all $f \in L^2(\mathbb{R})$, $[\hat{f}, \hat{\varphi}] \in L^2(\mathbb{T}^n)$ with*

$$\big\|[\hat{f}, \hat{\varphi}]\big\|_{L^2(\mathbb{T}^n)} \le \big\|[\hat{\varphi}, \hat{\varphi}]\big\|_\infty^{1/2}\|f\|. \tag{4}$$

Proof. (a) The first claim follows from Lemma 2 (a) and the definition of $\mathcal{C}$. We observe by Lemma 2 (d) that since ψ and φ are of compact support only finitely many Fourier coefficients of $[\hat{\psi}, \hat{\varphi}]$ will be non-zero and, hence, $[\hat{\psi}, \hat{\varphi}]$ is a trigonometric polynomial.

(b) We simply compute the norm of $[\hat{f}, \hat{\varphi}]$, using Lemma 2 (a)

$$\begin{aligned}\big\|[\hat{f}, \hat{\varphi}]\big\|^2_{L^2(\mathbb{T}^n)} &\le \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}[\hat{f}, \hat{f}](\xi)\ [\hat{\varphi}, \hat{\varphi}](\xi)d\xi \\ &\le \big\|[\hat{\varphi}, \hat{\varphi}]\big\|_\infty \|f\|^2. \ \square\end{aligned}$$

For $\tilde{\varphi}, \varphi \in \mathcal{C}$, let $R_{\tilde{\varphi},\varphi}$ be the operator mapping $f \in L^2(\mathbb{R})$ to

$$R_{\tilde{\varphi},\varphi}f = \sum_{k\in\mathbb{Z}^n}\langle f, T_k\tilde{\varphi}\rangle T_k\varphi. \tag{5}$$

We will see shortly that $R_{\tilde{\varphi},\varphi}$ is bounded on $L^2(\mathbb{R}^n)$ and we will obtain a characterization of $R_{\tilde{\varphi},\varphi}$ in terms of the Fourier transforms of φ and $\tilde{\varphi}$ that will play an important role in the next section.

Proposition 4

Let $\tilde{\varphi}, \varphi \in \mathcal{C}$ and $f \in L^2(\mathbb{R}^n)$.

(a) $\widehat{\big(R_{\tilde{\varphi},\varphi}f\big)} = [\hat{f}, \hat{\tilde{\varphi}}]\hat{\varphi}$.

(b) $\|R_{\tilde{\varphi},\varphi}f\|_{L^2(\mathbb{R}^n)} \le \big\|[\hat{\varphi}, \hat{\varphi}]\big\|_\infty^{1/2}\big\|[\hat{f}, \hat{\tilde{\varphi}}]\big\|_{L^2(\mathbb{T}^n)}$.

(c) *Consequently, $R_{\tilde{\varphi},\varphi}$ is a bounded operator on $L^2(\mathbb{R}^n)$ with* $\|R_{\tilde{\varphi},\varphi}\| \le \big\|[\hat{\tilde{\varphi}}, \hat{\tilde{\varphi}}]\big\|_\infty^{1/2}\big\|[\hat{\varphi}, \hat{\varphi}]\big\|_\infty^{1/2}$.

Proof. We first show that $[\hat{f},\hat{\tilde{\varphi}}]\hat{\varphi} \in L^2(\mathbb{R}^n)$. By Lemma 3 (b) we have

$$\begin{aligned}\big\|[\hat{f},\hat{\tilde{\varphi}}]\hat{\varphi}\big\|^2 &= \int_{\mathbb{T}^n} \big|[\hat{f},\hat{\tilde{\varphi}}](\xi)\big|^2 \big|[\hat{\varphi},\hat{\varphi}](\xi)\big| d\xi \\ &\le (2\pi)^n \big\|[\hat{\tilde{\varphi}},\hat{\tilde{\varphi}}]\big\|_\infty \big\|[\hat{f},\hat{\tilde{\varphi}}]\big\|^2_{L^2(\mathbb{T}^n)} \\ &\le (2\pi)^n \big\|[\hat{\varphi},\hat{\varphi}]\big\|_\infty \big\|[\hat{\tilde{\varphi}},\hat{\tilde{\varphi}}]\big\|_\infty \|f\|^2.\end{aligned}$$

Thus, $[\hat{f},\hat{\tilde{\varphi}}]\hat{\varphi} \in L^2(\mathbb{R}^n)$. (a) follows once we have established the fact that $\big(\widehat{R_{\tilde{\varphi},\varphi} f}\big) = [\hat{f},\hat{\tilde{\varphi}}]\hat{\varphi}$. We have for each $g \in L^2(\mathbb{R}^n)$

$$\begin{aligned}\frac{1}{(2\pi)^n}\big\langle [\hat{f},\hat{\tilde{\varphi}}]\hat{\varphi},\hat{g}\big\rangle &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} [\hat{f},\hat{\tilde{\varphi}}](\xi)\,\hat{\varphi}(\xi)\overline{\hat{g}(\xi)} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} [\hat{f},\hat{\tilde{\varphi}}](\xi)\,[\hat{\varphi},\hat{g}](\xi) d\xi \\ &= \big\langle [\hat{f},\hat{\tilde{\varphi}}], [\hat{g},\hat{\varphi}]\big\rangle_{L^2(\mathbb{T}^n)} \\ &= \sum_{k\in\mathbb{Z}^n} \langle f, T_k\tilde{\varphi}\rangle\,\overline{\langle g, T_k\varphi\rangle} \\ &= \big\langle R_{\tilde{\varphi},\varphi} f, g\big\rangle\end{aligned}$$

With (a) proven, (b) and (c) follow from the above observations. □

3. Another look at affine and quasi-affine systems

Due to the relative importance of refinable wavelet systems in applications we offer a separate examination of the affine, quasi-affine phenomenon in this context. The results of Ron and Shen in [16] and Chui, Shi, and Stöeckler in [7], such as Theorem 1, suggest that affine and quasi-affine systems function in the same way. We would like to understand, at least in the context of refinable systems, if there are any significant differences in the behavior of the two systems and, if so, whether they can be exploited in any useful way. We will introduce and study multiresolution operators for both the affine and quasi-affine systems along the lines of many previous works on refinable systems. Our methods will differ mainly in that we will be applying multiresolution operators to quasi-affine systems as well as affine systems, but also in the use of the bracket product as a tool for efficiently describing each of the multiresolution operators involved.

Let us fix scaling functions φ, $\tilde{\varphi} \in \mathcal{C}$ with associated $2\pi\mathbb{Z}^n$ periodic low-pass filters m_0 and $\tilde{m}_0$ such that

$$\hat{\varphi}(M^T\xi) = m_0(\xi)\hat{\varphi}(\xi) \text{ and } \hat{\tilde{\varphi}}(M^T\xi) = \tilde{m}_0(\xi)\hat{\tilde{\varphi}}(\xi) \tag{6}$$

for a.e. $\xi \in \mathbb{R}^n$. Let $m_1,\dots,m_L$ and $\tilde{m}_1,\dots,\tilde{m}_L$ be two sets of $2\pi\mathbb{Z}^n$ periodic high-pass filters and define $\Psi := \{\psi_1,\dots,\psi_L\}$, $\tilde{\Psi} := \{\tilde{\psi}_1,\dots,\tilde{\psi}_L\}$ by the refinement identities

$$\hat{\psi}_\ell(M^T\xi) = m_\ell(\xi)\hat{\varphi}(\xi) \text{ and } \hat{\tilde{\psi}}_\ell(M^T\xi) = \tilde{m}_\ell(\xi)\hat{\tilde{\varphi}}(\xi) \tag{7}$$

for $1 \le \ell \le L$. For notational convenience, let us define $\psi_0 = \varphi$ and $\tilde{\psi}_0 = \tilde{\varphi}$. We will assume hereafter that the filters satisfy the generalized Smith-Barnwell equations for the dilation M, namely for $0 \le p \le m-1$ we have

$$\sum_{\ell=0}^{L} \overline{m_\ell(\xi)}\tilde{m}_\ell(\xi + 2\pi(M^T)^{-1}\vartheta_p) = \delta_{0,p} \quad \text{a.e. } \xi \in \mathbb{T}^n, \tag{8}$$

where $\{\vartheta_p\}_{p=0}^{m-1}$ is a complete set of distinct coset representatives of $\mathbb{Z}^n/M^T\mathbb{Z}^n$, $m := |\det M|$, and $\delta_{0,p}$ is the Kronecker delta. We assume $\vartheta_0 = 0$. The following lemma, which is proven in [9], facilitates the derivation of the generalized Smith-Barnwell equations and will also aid our analysis of multiresolution operators below.

Lemma 5

Let M be an expanding, lattice-preserving $n \times n$ matrix and let $\{\vartheta_p\}_{p=0}^{m-1}$ be a complete set of distinct coset representatives of $\mathbb{Z}^n/M^T\mathbb{Z}^n$, where $m = |\det M|$. For each $k \in \mathbb{Z}^n$, we have

$$\sum_{p=1}^{m} e^{-2\pi i\left\langle (M^T)^{-1}\vartheta_p, k\right\rangle} = \begin{cases} m, & k \in M\mathbb{Z}^n \\ 0, & k \notin M\mathbb{Z}^n. \end{cases} \tag{9}$$

At this point, a few observations regarding the filters and the wavelets are in order. If we restrict our filters to the class $L^\infty(\mathbb{T}^n)$, then the wavelets $\psi_\ell, \tilde{\psi}_\ell$ will belong to $\mathcal{C}$. To see this, we observe

$$\begin{aligned} [\hat{\psi}_\ell, \hat{\psi}_\ell](\xi) &= \sum_{k\in\mathbb{Z}^n} \left|\hat{\psi}_\ell(\xi + 2\pi k)\right|^2 \\ &= \sum_{k\in\mathbb{Z}^n} \left|m_\ell\big((M^T)^{-1}(\xi + 2\pi k)\big)\right|^2 \left|\hat{\varphi}\big((M^T)^{-1}(\xi + 2\pi k)\big)\right|^2 \\ &= \sum_{p=0}^{m-1} \left|m_\ell\big((M^T)^{-1}(\xi + 2\pi\vartheta_p)\big)\right|^2 [\hat{\varphi}, \hat{\varphi}]\big((M^T)^{-1}(\xi + 2\pi\vartheta_p)\big), \end{aligned}$$

from which we conclude $\left\|[\hat{\psi}_\ell, \hat{\psi}_\ell]\right\|_\infty \le m\|m_\ell\|_\infty^2 \left\|[\hat{\varphi}, \hat{\varphi}]\right\|_\infty$. Similar reasoning leads to the conclusion that if $f \in \mathcal{C}$ then $D^j f \in \mathcal{C}$ for each $j \in \mathbb{Z}$. These classes of scaling functions and filters are sufficiently general, since in practice it is likely that we would restrict the filters to the subclass of $L^\infty(\mathbb{T}^n)$ consisting of trigonometric polynomials, in which case the scaling functions and wavelets would belong to the subclass of $\mathcal{C}$ corresponding to the compactly supported functions in $L^2(\mathbb{R}^n)$.

We are now equipped to introduce multiresolution operators associated with the proposed dual affine and quasi-affine systems generated by Ψ and $\tilde{\Psi}$. At each scale j we will have one operator that essentially approximates a given function by incorporating information from all scales coarser than j and another operator that captures the variation of the function at the scale j. In the affine orthonormal wavelet setting these soon to be defined operators become orthogonal projections, but in general this is not true although the inspired intuition remains useful. The affine approximation and detail operators at the scale $j \in \mathbb{Z}$, P_j and Q_j, respectively, act on $f \in L^2(\mathbb{R}^n)$ by

$$P_j f := \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} \text{ and } Q_j f := \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\psi}_{\ell;j,k}\rangle \psi_{\ell;j,k}, \tag{10}$$

whereas the quasi-affine approximation and detail operators at the scale j, P_j^{q} and Q_j^{q}, respectively, are defined similarly by

$$P_j^{\mathrm{q}} f := \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}^{\mathrm{q}}_{j,k}\rangle \varphi^{\mathrm{q}}_{j,k} \text{ and } Q_j^{\mathrm{q}} f := \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\psi}^{\mathrm{q}}_{\ell;j,k}\rangle \psi^{\mathrm{q}}_{\ell;j,k}. \tag{11}$$

By definition, $P_j^{\mathrm{q}} = P_j$ and $Q_j^{\mathrm{q}} = Q_j$ for each $j \geq 0$. Note that we have again used the superscript q to distinguish the quasi-affine objects from their affine counterparts.

Proposition 6

Let $\psi_\ell, \tilde{\psi}_\ell \in \mathcal{C}$ for $0 \leq \ell \leq L$. For each $j \in \mathbb{Z}$, the operators P_j, Q_j, P_j^{q}, and Q_j^{q} are bounded on $L^2(\mathbb{R}^n)$ and we have

(a) $P_j = D^j R_{\tilde{\psi}_0,\psi_0} D^{-j}$, $j \in \mathbb{Z}$,

(b) $Q_j = \sum_{\ell=1}^{L} D^j R_{\tilde{\psi}_\ell,\psi_\ell} D^{-j}$, $j \in \mathbb{Z}$,

(c) $P_j^{\mathrm{q}} = |\det M|^j R_{D^j\tilde{\psi}_0, D^j\psi_0}$, $j < 0$,

(d) $Q_j^{\mathrm{q}} = \sum_{\ell=1}^{L} |\det M|^j R_{D^j\tilde{\psi}_\ell, D^j\psi_\ell}$, $j < 0$.

Proof. The boundedness of the operators follows from Proposition 4 (a) and the above remarks once we establish the claimed formulas for P_j, Q_j, P_j^{q}, and Q_j^{q}. We will demonstrate only the characterizations of P_j and P_j^{q} as the other two follow by analogy. Fix $j \in \mathbb{Z}$ and let $f \in L^2(\mathbb{R}^n)$. We have

$$\begin{aligned} P_j f &= \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k}\rangle \varphi_{j,k} \\ &= \sum_{k\in\mathbb{Z}^n} \langle f, D^j T_k \tilde{\varphi}\rangle D^j T_k \varphi \\ &= D^j \Big(\sum_{k\in\mathbb{Z}^n} \langle D^{-j} f, T_k \tilde{\varphi}\rangle T_k \varphi \Big) \\ &= D^j R_{\tilde{\psi}_0,\psi_0} D^{-j} f. \end{aligned}$$

We now perform a similar calculation for P_j^{q}, $j < 0$.

$$\begin{aligned} P_j^{\mathrm{q}} f &= \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}^{\mathrm{q}}_{j,k}\rangle \varphi^{\mathrm{q}}_{j,k} \\ &= \sum_{k\in\mathbb{Z}^n} \langle f, T_k D^j \tilde{\varphi}\rangle T_k D^j \varphi \\ &= |\det M|^j R_{D^j\tilde{\psi}_0, D^j\psi_0} f. \ \square \end{aligned}$$

Recall the operator Δ from the first section, where $\Delta\hat{f}(\xi) = \hat{f}\big((M^T)^{-1}\xi\big)$. Notice that $\Delta^j\hat{f} = |\det M|^{j/2}(\widehat{D^j f})$ for each $j \in \mathbb{Z}$. It follows that for any linear operator R,

$$(\widehat{D^j R D^{-j} f}) = \Delta^j \hat{R} \Delta^{-j} \hat{f}, \tag{12}$$

where $\hat{R}$ is defined by $\hat{R}\hat{f} := (\widehat{Rf})$. Together with Proposition 6 this observation puts us in the position to describe the multiresolution operators via the characterization of Proposition 4 (c). We pause for an elementary lemma.

Lemma 7

For all $f, g \in L^2(\mathbb{R}^n)$ and $j \in \mathbb{Z}$,

$$[\hat{f}, \hat{g}] = \sum_{p=0}^{m-1} T_{2\pi\vartheta_p} \Delta[\Delta^{-1}\hat{f}, \Delta^{-1}\hat{g}]. \tag{13}$$

Proof. It will be helpful to note the following elementary identity,

$$T_u \Delta^j = \Delta^j T_{(M^T)^{-j}u},$$

where $u \in \mathbb{R}^n$. Applying the definition of the bracket product we have

$$\begin{aligned}
[\hat{f}, \hat{g}] &= \sum_{k\in\mathbb{Z}^n} T_{2\pi k}\hat{f} T_{2\pi k}\overline{\hat{g}} \\
&= \Delta\Big(\sum_{k\in\mathbb{Z}^n} T_{2\pi(M^T)^{-1}k}\Delta^{-1}\hat{f}\, T_{2\pi(M^T)^{-1}k}\Delta^{-1}\overline{\hat{g}}\Big) \\
&= \Delta\Big(\sum_{p=0}^{m-1}\sum_{k\in\mathbb{Z}^n} T_{2\pi(M^T)^{-1}\vartheta_p+2\pi k}\Delta^{-1}\hat{f}\, T_{2\pi(M^T)^{-1}\vartheta_p+2\pi k}\Delta^{-1}\overline{\hat{g}}\Big) \\
&= \sum_{p=0}^{m-1} \Delta T_{2\pi(M^T)^{-1}\vartheta_p}[\Delta^{-1}\hat{f}, \Delta^{-1}\hat{g}] \\
&= \sum_{p=0}^{m-1} T_{2\pi\vartheta_p}\Delta[\Delta^{-1}\hat{f}, \Delta^{-1}\hat{g}]. \; \square
\end{aligned}$$

Proposition 8

For each $j \in \mathbb{Z}$, we have

(a) $P_j + Q_j = P_{j+1}$,

(b) $P_j^{\mathrm{q}} + Q_j^{\mathrm{q}} = P_{j+1}^{\mathrm{q}}$.

Proof. (a) Let $f \in L^2(\mathbb{R}^n)$. The remarks preceding Lemma 7 combined with Propositions 4 and 6 lead us to

$$\widehat{(P_j f)} = \Delta^j [\Delta^{-j} \hat{f}, \hat{\tilde{\varphi}}] \hat{\varphi}, \tag{14}$$

with a similar formula for $\widehat{(Q_j f)}$. We observe that the scaling equations (7) can be written as

$$\Delta^{-1} \hat{\psi}_\ell = m_\ell \hat{\varphi} \text{ and } \Delta^{-1} \hat{\tilde{\psi}}_\ell = \tilde{m}_\ell \hat{\tilde{\varphi}} \tag{15}$$

for $0 \le \ell \le L$. Now we compute $\widehat{(P_j f)} + \widehat{(Q_j f)}$ incorporating the (14), (15), Lemma 7, and the filter equations (8).

$$\begin{aligned}
\widehat{(P_j f)} + \widehat{(Q_j f)} &= \sum_{\ell=0}^{L} \Delta^j [\Delta^{-j} \hat{f}, \hat{\tilde{\psi}}_\ell]\ \Delta^j \hat{\psi}_\ell \\
&= \sum_{\ell=0}^{L} \Delta^j \Big(\sum_{p=0}^{m-1} T_{2\pi\vartheta_p} \Delta [\Delta^{-(j+1)} \hat{f}, \tilde{m}_\ell \hat{\tilde{\varphi}}] \Big)\ \Delta^{j+1} \big(m_\ell \hat{\varphi} \big) \\
&= \Delta^j \Big(\sum_{p=0}^{m-1} T_{2\pi\vartheta_p} \Delta [\Delta^{-(j+1)} \hat{f}, \hat{\tilde{\varphi}}] \Big)\ \Delta^{j+1} \hat{\varphi}\ \Delta^{j+1} \Big(\sum_{\ell=0}^{L} T_{2\pi (M^T)^{-1} \vartheta_p} \overline{\tilde{m}_\ell} m_\ell \Big) \\
&= \Delta^{j+1} [\Delta^{-(j+1)} \hat{f}, \hat{\tilde{\varphi}}]\ \Delta^{j+1} \hat{\varphi} \\
&= \widehat{(P_{j+1} f)}.
\end{aligned}$$

(b) First we note that when $j \ge 0$ (b) follows from (a) since the quasi-affine multiresolution operators agree with the corresponding affine operators at these scales. Fix $j < 0$ and let $f \in L^2(\mathbb{R}^n)$. We follow the proof of (a), but with less sleight of hand,

$$\begin{aligned}
\widehat{(P^{\mathrm{q}}_j f)} + \widehat{(Q^{\mathrm{q}}_j f)} &= \sum_{\ell=0}^{L} [\hat{f}, \Delta^j \hat{\tilde{\psi}}_\ell]\ \Delta^j \hat{\psi}_\ell \\
&= \sum_{\ell=0}^{L} [\hat{f}, \Delta^{j+1} \big(\tilde{m}_\ell \hat{\tilde{\varphi}} \big)]\ \Delta^{j+1} \big(m_\ell \hat{\varphi} \big) \\
&= [\hat{f}, \Delta^{j+1} \hat{\tilde{\varphi}}]\ \Delta^{j+1} \hat{\varphi}\ \Delta^{j+1} \Big(\sum_{\ell=0}^{L} \overline{\tilde{m}_\ell} m_\ell \Big) \\
&= [\hat{f}, \Delta^{j+1} \hat{\tilde{\varphi}}]\ \Delta^{j+1} \hat{\varphi} \\
&= \widehat{(P^{\mathrm{q}}_{j+1} f)}. \ \square
\end{aligned}$$

Proposition 8 uncovers an important feature of the behavior of the quasi-affine multiresolution operators for scales $j < 0$. In the proof of (b), we only made use of the $p = 0$ case of the generalized Smith-Barnwell equations (7). We will see in the next section how this is reminiscent of the à trous algorithm and in Section 5 we will offer a generalized version of quasi-affine systems that takes advantage of this fact.

In light of Proposition 8, for each $f \in L^2(\mathbb{R}^n)$ and $J \ge 0$ we have

$$P_{-J} f + \sum_{j=-J}^{J-1} Q_j f = P_J f = P^{\mathrm{q}}_J f = P^{\mathrm{q}}_{-J} f + \sum_{j=-J}^{J-1} Q^{\mathrm{q}}_j f. \tag{16}$$

We will now examine the behavior of the approximation operators acting on $L^2(\mathbb{R}^n)$ functions as the scale j tends to $-\infty$.

Proposition 9

Let $\tilde{\varphi}, \varphi \in \mathcal{C}$. Then for each $f \in L^2(\mathbb{R}^n)$ $\|P_j f\| \to 0$ and $\|P_j^{\mathrm{q}} f\| \to 0$ as $j \to -\infty$.

Proof. By an approximation argument it is sufficient to prove each result for f a characteristic function of some compact set $K \subset \mathbb{R}^n$. We begin with the affine approximation operator, P_j. We have by (14) and Proposition 4 (b)

$$\begin{aligned}\|(\widehat{P_j f})\|^2 &= \left\|\Delta^j\left([\Delta^{-j}\hat{f}, \hat{\tilde{\varphi}}]\hat{\varphi}\right)\right\|^2 \\ &= |\det M|^j \left\|[\Delta^{-j}\hat{f}, \hat{\tilde{\varphi}}]\hat{\varphi}\right\|^2 \\ &= \left\|[\widehat{D^{-j}f}, \hat{\tilde{\varphi}}]\hat{\varphi}\right\|^2 \\ &\le (2\pi)^n \left\|[\widehat{D^{-j}f}, \hat{\tilde{\varphi}}]\right\|^2_{L^2(\mathbb{T}^n)} \left\|[\hat{\varphi}, \hat{\varphi}]\right\|_\infty.\end{aligned}$$

Now, as a consequence of Lemma 2 (d) we see that

$$\left\|[\widehat{D^{-j}f}, \hat{\tilde{\varphi}}]\right\|^2_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,k}\rangle\right|^2,$$

but

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} \left|\langle f, \varphi_{j,k}\rangle\right|^2 &\le C_K \sum_{k\in\mathbb{Z}^n} \int_K |\det M|^j |\tilde{\varphi}(M^j x - k)|^2 dx \\ &= C_K \sum_{k\in\mathbb{Z}^n} \int_{M^j K} |\tilde{\varphi}(x-k)|^2 dx \\ &= C_K \sum_{k\in\mathbb{Z}^n} \int_{M^j K - k} |\tilde{\varphi}(x)|^2 dx.\end{aligned}$$

Since $\tilde{\varphi} \in L^2(\mathbb{R}^n)$, this expression tends to 0 as $j \to -\infty$ by the dominated convergence theorem and we conclude that $\|P_j f\| \to 0$ as $j \to -\infty$.

The quasi-affine approximation operator will be handled in a similar fashion. Recall that $(\widehat{P_j^{\mathrm{q}} f}) = [\hat{f}, \Delta^j \hat{\tilde{\varphi}}]\Delta^j \hat{\varphi}$. Our next step is to apply Proposition 4 (b), but we first note that when $j < 0$, $\left\|[\Delta^j \hat{\varphi}, \Delta^j \hat{\varphi}]\right\|_\infty \le \left\|[\hat{\varphi}, \hat{\varphi}]\right\|_\infty$. Hence,

$$\left\|(\widehat{P_j^{\mathrm{q}} f})\right\|^2 \le (2\pi)^n \left\|[\hat{\varphi}, \hat{\varphi}]\right\|_\infty \left\|[\hat{f}, \Delta^j \hat{\tilde{\varphi}}]\right\|^2_{L^2(\mathbb{T}^n)},$$

but Lemma 2 (d) reveals the fact that

$$\begin{aligned}\left\|[\hat{f}, \Delta^j \hat{\tilde{\varphi}}]\right\|^2_{L^2(\mathbb{T}^n)} &= \left\|[\hat{f}, |\det M|^{j/2} (\widehat{D^j \tilde{\varphi}})]\right\|^2_{L^2(\mathbb{T}^n)} \\ &= \sum_{k\in\mathbb{Z}^n} \left|\langle f, |\det M|^{j/2} T_k D^j \tilde{\varphi}\rangle\right|^2 \\ &= \sum_{k\in\mathbb{Z}^n} \left|\langle f, \tilde{\varphi}^{\mathrm{q}}_{j,k}\rangle\right|^2.\end{aligned}$$

Finally, we estimate the squared sum of the sequence of inner products,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}\left|\langle f,\tilde{\varphi}^{\mathrm{q}}_{j,k}\rangle\right|^2 &\le C_K\sum_{k\in\mathbb{Z}^n}\int_K|\det M|^{2j}\left|\tilde{\varphi}\big(M^j(x-k)\big)\right|^2dx\\ &= C_K\sum_{k\in\mathbb{Z}^n}\int_{M^j(K-k)}|\det M|^j\left|\tilde{\varphi}(x)\right|^2dx\\ &= C_K\int_{\mathbb{R}^n}g_j(x)dx,\end{aligned}$$

where $g_j(x)$ is given by

$$g_j(x)=\sum_{k\in\mathbb{Z}^n}|\det M|^j\left|\tilde{\varphi}(x)\right|^2\chi_{S_{j,k}}(x)$$

and $S_{j,k}$ is defined to be

$$S_{j,k}:=M^j(K-k).$$

Since K is compact there exists some $N>0$, independent of j, such that S_{j,k_0} intersects at most N of the sets $S_{j,k}$, $k\in\mathbb{Z}^n$. This implies that $|g_j|\le N|\tilde{\varphi}|^2$, providing a dominating function for the collection $\{g_j\}_{j<0}$. Since $g_j\to 0$ a.e. as $j\to-\infty$ we conclude by the dominated convergence theorem that $\|P^{\mathrm{q}}_j f\|\to 0$ as $j\to-\infty$. □

With the help of Theorem 1 (a) we obtain an equivalence between dual refinable affine and quasi-affine frames. We should note that the following result is a special case of those given in [7].

Theorem 10

Let $\tilde{\varphi},\varphi\in\mathcal{C}$ and $m_\ell,\tilde{m}_\ell\in L^\infty(\mathbb{T}^n)$, $0\le\ell\le L$, such that (6) and (8) hold and suppose that $\tilde{\Psi}=\{\tilde{\psi}_1,\dots,\tilde{\psi}_L\}$ and $\Psi=\{\psi_1,\dots,\psi_L\}$ are defined by (7). Then $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames for $L^2(\mathbb{R}^n)$ if and only if $X^{\mathrm{q}}(\Psi)$ and $X^{\mathrm{q}}(\tilde{\Psi})$ are dual frames for $L^2(\mathbb{R}^n)$.

Proof. We begin with the ($\Rightarrow$) implication, supposing that $X(\Psi)$ and $X(\tilde{\Psi})$ are dual frames for $L^2(\mathbb{R}^n)$. By definition, we have for each $f\in L^2(\mathbb{R}^n)$

$$f=\sum_{\ell=0}^{L}\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\langle f,\tilde{\psi}_{\ell;j,k}\rangle\psi_{\ell;j,k}=\sum_{j\in\mathbb{Z}}Q_jf.$$

Letting $J\to\infty$ in (16) we see that

$$f=\lim_{J\to\infty}\sum_{j=-J}^{J}Q^{\mathrm{q}}_jf.$$

By Theorem 1 (a), $X^{\mathrm{q}}(\tilde{\Psi})$ and $X^{\mathrm{q}}(\Psi)$ are Bessel systems and, thus, the convergence of the above sum is unconditional and we have the dual reproducing formula

$$f=\sum_{\ell=0}^{L}\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\langle f,\tilde{\psi}^{\mathrm{q}}_{\ell;j,k}\rangle\psi^{\mathrm{q}}_{\ell;j,k},$$

for all $f \in L^2(\mathbb{R}^n)$. It remains only to demonstrate the lower frame bounds for $X^{\mathrm{q}}(\tilde{\Psi})$ and $X^{\mathrm{q}}(\Psi)$ and for this we make use of a standard argument involving the reproducing formula and the Bessel bounds. We have

$$\begin{aligned} \|f\|^2 &= \sum_{\ell=0}^{L} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \tilde{\psi}^{\mathrm{q}}_{\ell;j,k}\rangle \, \langle \psi^{\mathrm{q}}_{\ell;j,k}, f\rangle \\ &\le \Big(\sum_{\ell=0}^{L} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} |\langle f, \tilde{\psi}^{\mathrm{q}}_{\ell;j,k}\rangle|^2\Big)^{1/2} \Big(\sum_{\ell=0}^{L} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} |\langle f, \psi^{\mathrm{q}}_{\ell;j,k}\rangle|^2\Big)^{1/2} \\ &\le C\|f\| \Big(\sum_{\ell=0}^{L} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} |\langle f, \psi^{\mathrm{q}}_{\ell;j,k}\rangle|^2\Big)^{1/2}, \end{aligned}$$

from which we conclude

$$\frac{1}{C^2}\|f\|^2 \le \sum_{\ell=0}^{L} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} |\langle f, \psi^{\mathrm{q}}_{\ell;j,k}\rangle|^2.$$

The lower frame bound for $X^{\mathrm{q}}(\tilde{\Psi})$ follows by analogy. Moreover, the reverse implication follows from a completely similar argument. $\square$

4. The à trous connection

In order to better illustrate the connection between the à trous algorithm and the multiresolution operators associated with quasi-affine systems we will take another look at Proposition 8 (b). We will provide another proof of the result in the case that $j < 0$ that is an adaptation of the usual à trous algorithm found in [10] and [13] to the case of n dimensions and a more general dilation matrix M, as described above.

Let us assign coefficient representations to the filters of the last section,

$$m_\ell(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_{\ell;k} e^{-i\langle \xi,k\rangle} \quad \text{and} \quad \tilde{m}_\ell(\xi) = \sum_{k\in\mathbb{Z}^n} \tilde{\alpha}_{\ell;k} e^{-i\langle \xi,k\rangle},$$

which for all intents and purposes we may think of as trigonometric polynomials. An elementary computation with the refinement relationships (6) and (7) reveals

$$|\det M|^{-1/2}\psi_{\ell;j-1,k} = \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\, \varphi_{j,r+Mk} \quad \text{and} \quad |\det M|^{-1/2}\tilde{\psi}_{\ell;j-1,k} = \sum_{r\in\mathbb{Z}^n} \tilde{\alpha}_{\ell;r}\, \tilde{\varphi}_{j,r+Mk},$$

for $0 \le \ell \le L$, where we recall the convention that $\psi_0 = \varphi$ and $\tilde{\psi}_0 = \tilde{\varphi}$. We would like to write these formulas in terms of the quasi-affine dilation structure. When $j \ge 0$ the difference is just the cosmetic addition of brackets in the subscripts, but for $j < 0$ we obtain something altogether different. Indeed, for $j < 0$, $k \in \mathbb{Z}^n$, and $0 \le \ell \le L$ we have

$$\begin{aligned}\psi^{\mathrm{q}}_{\ell;j-1,k} &= |\det M|^{(j-1)/2} T_k D^{j-1}\psi_\ell \\ &= |\det M|^{(j-1)/2} T_k \psi_{\ell;j-1,0} \\ &= |\det M|^{j/2} \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\, T_k \varphi_{j,r} \\ &= |\det M|^{j/2} \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\, T_k D^j T_r \varphi \\ &= \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;r}\, \varphi^{\mathrm{q}}_{j,M^{-j}r+k}(x).\end{aligned}$$

Notice that the sum in the last line of this calculation includes only the translates of the scaling function in the sub-lattice $M^{-j}\mathbb{Z}^n + k$ rather than $\mathbb{Z}^n$. It is this fact that connects quasi-affine systems with the à trous algorithm of Holschneider, Kronland-Martinet, Morlet, and Tchamitchian [10]. We follow their lead, defining for each scale $j < 0$ upsampled filter coefficients

$$\alpha_{\ell;j,r} := \begin{cases} \alpha_{\ell;M^j r} & r \in M^{-j}\mathbb{Z}^n \\ 0 & r \notin M^{-j}\mathbb{Z}^n, \end{cases}$$

where $0 \le \ell \le L$. Inserting the upsampled coefficients into the preceding formula, we achieve the desired replacements for the affine scaling equations in the quasi-affine theory,

$$\psi^{\mathrm{q}}_{\ell;j-1,k} = \sum_{r\in\mathbb{Z}^n} \alpha_{\ell;j,r}\, \varphi^{\mathrm{q}}_{j,r+k} \quad\text{and}\quad \tilde\psi^{\mathrm{q}}_{\ell;j-1,k} = \sum_{r\in\mathbb{Z}^n} \tilde\alpha_{\ell;j,r}\, \tilde\varphi^{\mathrm{q}}_{j,r+k},$$

where $j < 0$, $k \in \mathbb{Z}^n$, and $0 \le \ell \le L$. The last piece of information we require comes from the computation of the r^{th} Fourier coefficient ($r \in \mathbb{Z}^n$) of the filter equation (8) in the case that $p = 0$,

$$\begin{aligned}\delta_{0,r} &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \sum_{\ell=0}^{L} \overline{m_\ell(\xi)}\tilde m_\ell(\xi) e^{-i\langle\xi,r\rangle} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \sum_{\ell=0}^{L} \Big(\sum_{k\in\mathbb{Z}^n} \overline{\alpha_{\ell;k}} e^{i\langle\xi,k\rangle}\Big)\Big(\sum_{k'\in\mathbb{Z}^n} \tilde\alpha_{\ell;k'} e^{-i\langle\xi,k'\rangle}\Big) e^{-i\langle\xi,r\rangle} d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}^n} \overline{\alpha_{\ell;k}}\tilde\alpha_{\ell;k-r} d\xi \\ &= \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}^n} \overline{\alpha_{\ell;k}}\tilde\alpha_{\ell;k-r}.\end{aligned}$$

We now reexamine the result of Proposition 8 (b),

$$\big(P^{\mathrm{q}}_j + Q^{\mathrm{q}}_j\big) f = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}^n} \langle f, \tilde\psi^{\mathrm{q}}_{\ell;j,k}\rangle \psi^{\mathrm{q}}_{\ell;j,k}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{L} \sum_{k\in\mathbb{Z}^n} \sum_{r,s\in\mathbb{Z}^n} \alpha_{\ell;j+1,r}\overline{\tilde{\alpha}_{\ell;j+1,s}}\langle f, \tilde{\varphi}^{\mathrm{q}}_{j+1,s+k}\rangle \varphi^{\mathrm{q}}_{j+1,r+k}\\
&= \sum_{r,s\in\mathbb{Z}^n} \Big(\sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}^n} \alpha_{\ell;j+1,r-k}\overline{\tilde{\alpha}_{\ell;j+1,s-k}}\Big)\langle f, \tilde{\varphi}^{\mathrm{q}}_{j+1,s}\rangle \varphi^{\mathrm{q}}_{j+1,r}\\
&= \sum_{r,s\in\mathbb{Z}^n} C_{r,s}\langle f, \tilde{\varphi}^{\mathrm{q}}_{j+1,s}\rangle \varphi^{\mathrm{q}}_{j+1,r}.
\end{aligned}
$$

It suffices to prove $C_{r,s} = \delta_{r,s}$ for each $r, s \in \mathbb{Z}^n$. First, suppose that $r-s \in M^{-(j+1)}\mathbb{Z}^n$, in which case $s = r + M^{-(j+1)}u$ for some $u \in \mathbb{Z}^n$ and we have

$$
\begin{aligned}
C_{r,s} &= \sum_{\ell=0}^{L} \sum_{k\in M^{-(j+1)}\mathbb{Z}^n+r} \alpha_{\ell;j+1,r-k}\overline{\tilde{\alpha}_{\ell;j+1,r-k+M^{-j}u}}\\
&= \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}^n} \alpha_{\ell;k}\overline{\tilde{\alpha}_{\ell;k+u}}\\
&= \delta_{0,u} = \delta_{r,s}.
\end{aligned}
$$

Secondly, if $r - s \notin M^{-j+1}\mathbb{Z}^n$, then the supports of the coefficient sequences $\{\alpha_{\ell;j+1,r-k}\}_{k\in\mathbb{Z}^n}$ and $\{\overline{\tilde{\alpha}_{\ell;j+1,s-k}}\}_{k\in\mathbb{Z}^n}$ are disjoint and thus $C_{r,s} = \delta_{r,s} = 0$, completing the argument.

5. Generalized quasi-affine frames

In Section 3 we saw how the usual multiresolution operators can be extended to the quasi-affine scheme. Our study revealed the fact that only the $p = 0$ case of the perfect reconstruction equations (8) was needed to prove $P^{\mathrm{q}}_{j-1} + Q^{\mathrm{q}}_{j-1} = Q^{\mathrm{q}}_{j}$ for $j < 0$. In the last section we saw how this ties the quasi-affine multiresolution operators to the à trous algorithm. Here, we will introduce the notion of a generalized quasi-affine system in order to exploit this relationship, the basic idea being that we will relax the structure of the systems for negative scales.

DEFINITION 5. The *generalized quasi-affine (GQA) system generated by*

$$\Psi = \{\psi_1, \dots, \psi_L\}, \Phi = \{\phi_1, \dots, \phi_{L'}\} \subset L^2(\mathbb{R}^n),$$

denoted $X^{\mathrm{gq}}(\Psi, \Phi)$, is the collection

$$X^{\mathrm{gq}}(\Psi,\Phi) = \{\psi_{\ell;j,k} : 1 \le \ell \le L, j \ge 0, k \in \mathbb{Z}^n\} \bigcup \{\phi^{\mathrm{q}}_{\ell;j,k} : 1 \le \ell' \le L', j < 0, k \in \mathbb{Z}^n\}.$$

We will refer to the collections, $\{\psi_{\ell;j,k} : 1 \le \ell \le L, j \ge 0, k \in \mathbb{Z}^n\}$ and $\{\phi^{\mathrm{q}}_{\ell;j,k} : 1 \le \ell' \le L', j < 0, k \in \mathbb{Z}^n\}$, respectively, as the *standard* and *à trous* components of the GQA system $X^{\mathrm{gq}}(\Psi, \Phi)$.

Throughout this section Ψ, $\tilde{\Psi}$, φ, and $\tilde{\varphi}$ along with the filters $m_0, \ldots, m_L$ and $\tilde{m}_0, \ldots, \tilde{m}_L$ remain as fixed in the Section 3, collectively defining the standard component of a GQA system. For the à trous component, let $\Phi = \{\phi_1, \ldots, \phi_{L'}\}$ and $\tilde{\Phi} = \{\tilde{\phi}_1, \ldots, \tilde{\phi}_{L'}\} \in L^2(\mathbb{R}^n)$ such that the refinement identities

$$\hat{\phi}_{\ell'}(M^T\xi) = \mu_{\ell'}(\xi)\hat{\varphi}(\xi) \text{ and } \hat{\tilde{\phi}}_{\ell'}(M^T\xi) = \tilde{\mu}_{\ell'}(\xi)\hat{\tilde{\varphi}}(\xi) \tag{17}$$

hold for $1 \leq \ell' \leq L'$ and a.e. $\xi \in \mathbb{R}^n$ with $\mu_{\ell'}, \tilde{\mu}_{\ell'} \in L^\infty(\mathbb{T}^n)$. In accordance with the remarks above, we assume that these filters satisfy only

$$\overline{m_0(\xi)}\tilde{m}_0(\xi) + \sum_{\ell'=1}^{L'} \overline{\mu_{\ell'}(\xi)}\tilde{\mu}_{\ell'}(\xi) = 1, \tag{18}$$

for a.e. $\xi \in \mathbb{T}^n$. Given this setup we would like to determine sufficient conditions that $X^{\text{gq}}(\Psi, \Phi)$ and $X^{\text{gq}}(\tilde{\Psi}, \tilde{\Phi})$ comprise dual frames for $L^2(\mathbb{R}^n)$.

Since we are using the scaling functions $\varphi, \tilde{\varphi}$ in the refinement of both the standard and à trous components it is reasonable to expect P_j^{q} to serve in the role of the approximation operator for the proposed dual GQA systems at the scale j. The detail operators will differ between non-negative and negative dilations. We will denote the GQA detail operator at the scale j by Q_j^{gq} and define it by its action on $f \in L^2(\mathbb{R}^n)$,

$$Q_j^{\text{gq}} f := \begin{cases} \displaystyle\sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{\ell;j,k} \rangle \psi_{\ell;j,k} & j \geq 0 \\ \displaystyle\sum_{\ell'=1}^{L'} \sum_{k \in \mathbb{Z}^n} \langle f, \tilde{\phi}^{\text{q}}_{\ell';j,k} \rangle \phi^{\text{q}}_{\ell';j,k} & j < 0. \end{cases}$$

We have already described the behavior of Q_j^{gq} for $j \geq 0$ and using the techniques of Section 3 we obtain a complete description of Q_j^{gq} for all scales $j \in \mathbb{Z}$.

Proposition 11

Suppose $\psi_\ell, \tilde{\psi}_\ell, \phi_{\ell'}, \tilde{\phi}_{\ell'} \in \mathcal{C}$ for $0 \leq \ell \leq L$, $1 \leq \ell' \leq L'$. Then Q_j^{gq} is a bounded operator on $L^2(\mathbb{R}^n)$ for each $j \in \mathbb{Z}$ and

(a) $Q_j^{\text{gq}} = \displaystyle\sum_{\ell'=1}^{L'} |\det M|^j R_{D^j \tilde{\phi}_\ell, D^j \phi_\ell}$, $j < 0$,

(b) $P_j^{\text{qq}} + Q_j^{\text{gq}} = P_{j+1}^{\text{qq}}$, $j \in \mathbb{Z}$.

Finally, we would like to answer the question of when the two GQA systems are dual frames for $L^2(\mathbb{R}^n)$. Again, we mimic the proof of Theorem 10 from the Section 3, but in order that the proof work out here we must additionally assume that the two GQA systems, $X^{\text{gq}}(\Psi, \Phi)$ and $X^{\text{gq}}(\tilde{\Psi}, \tilde{\Phi})$ are Bessel. This assumption is not overly restrictive as weak decay properties of the scaling functions would suffice to ensure that the two GQA systems are Bessel.

Theorem 12

Let $\tilde{\varphi}, \varphi \in \mathcal{C}$ and $m_\ell, \tilde{m}_\ell, \mu_{\ell'}, \tilde{\mu}_{\ell'} \in L^\infty(\mathbb{T}^n)$, $0 \le \ell \le L$, $1 \le \ell' \le L'$, such that (6), (8), and (18) hold. Suppose that $\tilde{\Psi} = \{\tilde{\psi}_1, \ldots, \tilde{\psi}_L\}$ and $\Psi = \{\psi_1, \ldots, \psi_L\}$ are defined by (7) and that $\tilde{\Phi} = \{\tilde{\phi}_1, \ldots, \tilde{\phi}_{L'}\}$ and $\Phi = \{\phi_1, \ldots, \phi_{L'}\}$ are defined by (17). Then, if $X(\Psi), X(\tilde{\Psi})$ are dual frames for $L^2(\mathbb{R}^n)$ and $X^{\mathrm{gq}}(\Psi, \Phi), X^{\mathrm{gq}}(\tilde{\Psi}, \tilde{\Phi})$ are Bessel systems then $X^{\mathrm{gq}}(\Psi, \Phi)$ and $X^{\mathrm{gq}}(\tilde{\Psi}, \tilde{\Phi})$ are dual frames for $L^2(\mathbb{R}^n)$.

6. Examples of generalized quasi-affine frames

In this section we will study examples of generalized quasi-affine frames produced from two separate 2-band one-dimensional biorthogonal wavelets. The first pair of examples will be based upon a Burt-Adelson biorthogonal system presented in [5] and arising from the well-known Burt-Adelson Laplacian Pyramid [1]. A second pair of examples will be given that stem from a piecewise linear spline biorthogonal system, also borrowed from [5]. We hope with these examples to demonstrate the flexibility present in the choice of high-pass filters for the negative dilations of the GQA system $X^{\mathrm{gq}}(\psi, \phi)$. We would hope that this flexibility will allow for better design of ϕ and $\tilde{\phi}$ in terms of support, vanishing moments, or symmetry than is possible in the affine case.

In this setting, the perfect reconstruction filter equations (8) reduce to

$$\overline{m_0(\xi)}\tilde{m}_0(\xi) + \overline{m_1(\xi)}\tilde{m}_1(\xi) = 1 \tag{19}$$

and

$$\overline{m_0(\xi)}\tilde{m}_0(\xi + \pi) + \overline{m_1(\xi)}\tilde{m}_1(\xi + \pi) = 0 \tag{20}$$

for a.e. $\xi \in \mathbb{T}^n$. As discussed above, the main difference between the standard and à trous components of a generalized quasi-affine system is the lack of downsampling on the à trous side. Accordingly, the à trous high-pass filters, μ_1, $\tilde{\mu}_1$, must satisfy only (18) which becomes

$$\overline{m_0(\xi)}\tilde{m}_0(\xi) + \overline{\mu_1(\xi)}\tilde{\mu}_1(\xi) = 1 \tag{21}$$

in this case. We will additionally constrain the high-pass filters to vanish at zero, in order that the wavelets have at least one vanishing moment. Rearranging (21), we see

$$\overline{\mu_1(\xi)}\tilde{\mu}_1(\xi) = 1 - \overline{m_0(\xi)}\tilde{m}_0(\xi), \tag{22}$$

which suggests that the family of choices for μ_1 and $\tilde{\mu}_1$ are characterized by a complete factorization of $1 - \overline{m_0(\xi)}\tilde{m}_0(\xi)$.

Before proceeding to construct examples we must first consider the question of which choices of filters will result in GQA systems with the Bessel property. The following lemma is borrowed from [6] and describes a sufficient condition on the scaling functions to guarantee the desired Bessel bounds. Similar techniques appear in other articles, e.g. [5].

Lemma 13

Suppose φ satisfies

$$\sup_{\xi\in\mathbb{R}}\sum_{k\in\mathbb{Z}}|\hat{\varphi}(\xi+2\pi k)|^{2-\sigma}<\infty$$

and

$$\sup_{\xi\in\mathbb{R}}(1+|\xi|)^{\sigma}|\hat{\varphi}(\xi)|<\infty$$

for some σ, $0<\sigma<2$. If $\hat{\psi}(2\xi)=m(\xi)\hat{\varphi}(\xi)$ with m a trigonometric polynomial satisfying $m(0)=0$ then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Bessel system for $L^2(\mathbb{R})$.

It is pointed out in [5] that both the Burt-Adelson and spline low-pass filters considered here do give rise to scaling functions satisfying the hypotheses of Lemma 13 and, thus, the systems considered below will be Bessel and Theorem 12 will apply. As a final preliminary remark, we note that the regularity of the à trous wavelets is inherited from the scaling functions because each wavelet is a finite linear combination of translates of the corresponding scaling function. With these comments, we now proceed to consider some examples.

We begin with the Burt-Adelson case, recalling the generic form of the Burt-Adelson low-pass filter with parameter a,

$$m_{0,a}(\xi)=\Big(\frac{1}{4}-\frac{a}{2}\Big)e^{i2\xi}+\frac{1}{4}e^{i\xi}+a+\frac{1}{4}e^{-i\xi}+\Big(\frac{1}{4}-\frac{a}{2}\Big)e^{-i2\xi}.$$

We will work with $a=\dfrac{3}{5}$, in which case the low-pass filter becomes

$$m_0(\xi)=-\frac{1}{20}e^{i2\xi}+\frac{1}{4}e^{i\xi}+\frac{3}{5}+\frac{1}{4}e^{-i\xi}-\frac{1}{20}e^{-i2\xi}.$$

The dual low-pass filter, $\tilde{m}_0$, associated to m_0 is given by

$$\tilde{m}_0(\xi)=-\frac{3}{280}e^{i3\xi}-\frac{3}{56}e^{i2\xi}+\frac{73}{280}e^{i\xi}+\frac{17}{28}+\frac{73}{280}e^{-i\xi}-\frac{3}{56}e^{-i2\xi}-\frac{3}{280}e^{-i3\xi}.$$

Associated to these dual low-pass filters are the high-pass filters, m_1 and $\tilde{m}_1$, defined in accord with (19) and (20) above as

$$m_1(\xi)=\frac{3}{280}e^{i2\xi}-\frac{3}{56}e^{i\xi}-\frac{73}{280}+\frac{17}{28}e^{-i\xi}-\frac{73}{280}e^{-i2\xi}-\frac{3}{56}e^{-i3\xi}+\frac{3}{280}e^{-i4\xi},$$

and

$$\tilde{m}_1(\xi)=-\frac{1}{20}e^{i\xi}-\frac{1}{4}+\frac{3}{5}e^{-i\xi}-\frac{1}{4}e^{-i2\xi}-\frac{1}{20}e^{-i3\xi}.$$

Together, these four filters define the scaling functions, φ and $\tilde{\varphi}$, as well as the dual biorthogonal wavelets, ψ and $\tilde{\psi}$, via the refinement equations (6) and (7), respectively. The graphs of these four functions are given in Figure 1. This completely describes the standard component of our dual GQA system and we now proceed to discuss the à trous component.

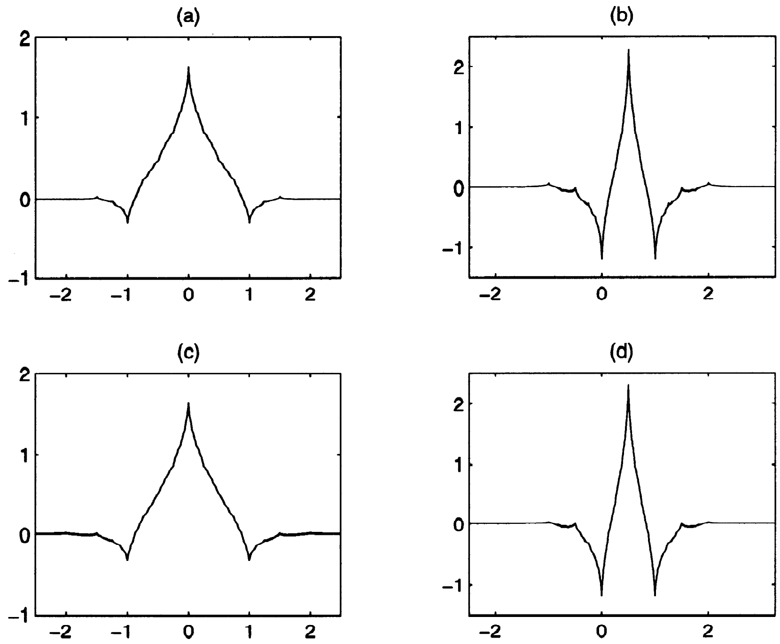

Figure 1: The $a = \frac{3}{5}$ Burt-Adelson biorthogonal system. (a): The scaling function, φ. (b): The wavelet, ψ. (c): The dual scaling function, $\tilde{\varphi}$. (d): The dual wavelet, $\tilde{\psi}$.

As remarked above, our à trous high-pass filters will result from a factorization of (22), which in terms of the Burt-Adelson filters becomes

$$\begin{aligned} &1 - \overline{m_0(\xi)}\tilde{m}_0(\xi) \\ &\quad = \frac{\left(e^{-i5\xi}(-1+e^{i\xi})^4(1+7e^{i\xi}+e^{i2\xi})(-3+9e^{i\xi}+94e^{i2\xi}+9e^{i3\xi}-3e^{i4\xi})\right)}{5600}. \end{aligned} \tag{23}$$

We are free to define μ_1 by choosing terms from the factorization (23), provided we include at least one factor of $(-1+e^{i\xi})$ while simultaneously leaving at least one such factor for $\tilde{\mu}_1$ to ensure that each à trous wavelet has a vanishing moment. Choosing any power of $e^{i\xi}$ from (23) for our high-pass filter will correspond to an unimportant translation of the associated wavelet; hence, excluding these unimodular delay terms and the factors $(-1+e^{i\xi})$ we have two remaining factors that may be distributed arbitrarily between μ_1 and $\tilde{\mu}_1$. Each of these factors possesses symmetric coefficients, thus the symmetry or anti-symmetry of the à trous wavelets will depend entirely on the distribution of the factors $(-1+e^{i\xi})$. In the original biorthogonal case, each of the high-pass filters has two vanishing moments which implies that the biorthogonal Burt-Adelson wavelets have a symmetry property. If we assign only one vanishing moment to either high-pass filter, the remaining filter will have three vanishing moments and each à trous wavelet will have an anti-symmetry property.

Example A: The simplest possible choice for μ_1 would be the high-pass filter of the Haar wavelet, which consists of a single vanishing moment. Thus, we define ϕ^A by means of the high-pass filter μ_1^A given by

$$\mu_1^A(\xi) = \frac{-1}{2}\big(1 - e^{-i\xi}\big).$$

This results in à trous wavelets with anti-symmetry properties and provides the shortest possible analysis filter. Three vanishing moments are left for the dual wavelet, $\tilde{\phi}^A$, defined by the corresponding dual à trous high-pass filter

$$\tilde{\mu}_1^A(\xi) = \frac{1}{2800}\big(- 3e^{i4\xi} - 3e^{i3\xi} + 181e^{i2\xi} + 181e^{i\xi} - 1400 + 1400e^{-i\xi} - 181e^{-i2\xi} - 181e^{-i3\xi} + 3e^{-i4\xi} + 3e^{-i5\xi}\big).$$

Each of the scaling functions, φ and $\tilde{\varphi}$, is symmetric about 0, which makes it easy to determine the symmetry properties of the functions ϕ^A, $\tilde{\phi}^A$. Recall that equation (17) implies that

$$\frac{1}{2}\phi^A\Big(\frac{x}{2}\Big) = \sum_{k\in\mathbb{Z}} \alpha_{1;k}\varphi(x-k),$$

where $\{\alpha_{1;k}\}_k$ are the coefficients of the filter μ_1^A. Thus, the anti-symmetry of μ_1^A with respect to $k = \frac{1}{2}$ implies that ϕ^A is antisymmetric about $x = \frac{1}{4}$. The same reasoning applies to $\tilde{\phi}^A$. The graphs of the two à trous wavelets, ϕ^A and $\tilde{\phi}^A$, are given in Figure 2, (a) and (b), respectively.

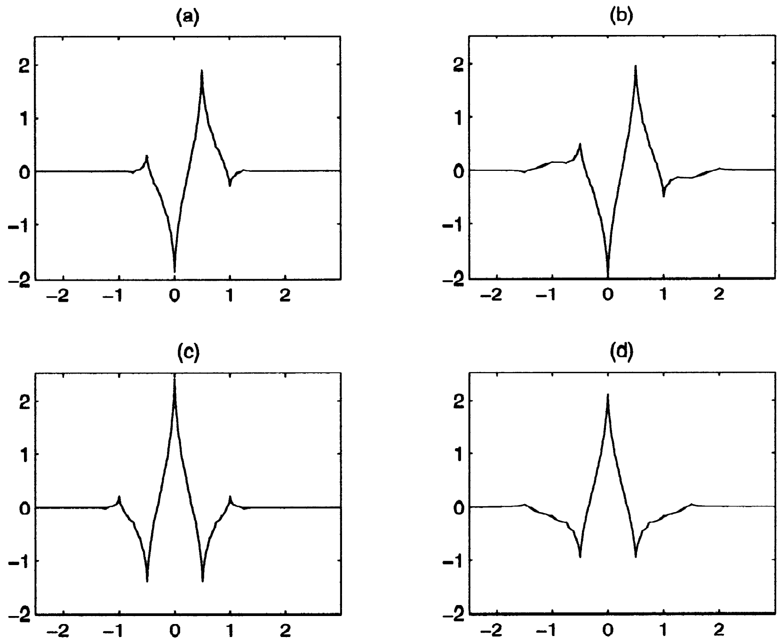

Figure 2: Examples of Burt-Adelson à trous wavelets. (a): The wavelet, ϕ^A. (b): The dual wavelet, $\tilde{\phi}^A$. (c): The wavelet, ϕ^B. (d): The dual wavelet, $\tilde{\phi}^B$.

EXAMPLE B: For our next example, we will choose filters such that each of the à trous wavelets possesses two vanishing moments, but such that the analysis filter, μ_1^B, is as short as possible. Let μ_1^B be given by

$$\mu_1^B(\xi) = \frac{-1}{25}\big(8e^{i\xi} - 16 + 8e^{-i\xi}\big),$$

with the corresponding dual filter

$$\begin{aligned}\tilde{\mu}_1^B(\xi) = \frac{-1}{1792}\big(&-3e^{i4\xi} - 6e^{i3\xi} + 175e^{i2\xi} + 356e^{i\xi}\\ &- 1044 + 356e^{-i\xi} + 175e^{-i2\xi} - 6e^{-i3\xi} - 3e^{-i4\xi}\big).\end{aligned}$$

In contrast to the last example, each filter here is even, implying that the dual à trous wavelets are symmetric about zero. The associated wavelets, ϕ^B and $\tilde{\phi}^B$, respectively, are displayed in Figure 2, (c) and (d).

We now turn to another base biorthogonal system based on the piecewise linear spline scaling function of [5]. In the terminology of [5] this system consists of the dual scaling functions φ_2, $\tilde{\varphi}_{2,4}$, $\psi_{2,4}$, and $\tilde{\psi}_{2,4}$, which are illustrated in Figure 3. We have the associated filters

$$\begin{aligned}m_0(\xi) &= \frac{1}{4}\big(e^{i\xi} + 2 + e^{-i\xi}\big),\\ \tilde{m}_0(\xi) &= \frac{1}{128}\big(3e^{i4\xi} - 6e^{i3\xi} - 16e^{i2\xi} + 38e^{i\xi}\\ &\qquad + 90 + 38e^{-i\xi} - 16e^{-i2\xi} - 6e^{-i3\xi} + 3e^{-i4\xi}\big),\\ m_1(\xi) &= \frac{1}{128}\big(3e^{i3\xi} + 6e^{i2\xi} - 16e^{i\xi} - 38\\ &\qquad + 90e^{-i\xi} - 38e^{-i2\xi} - 16e^{-i3\xi} + 6e^{-i4\xi} + 3e^{-i5\xi}\big),\end{aligned}$$

and

$$\tilde{m}_1(\xi) = \frac{-1}{4}\big(1 - 2e^{-i\xi} + 1e^{-i2\xi}\big).$$

The factorization of (22) in terms of these spline filters becomes

$$1 - \overline{m_0(\xi)}\tilde{m}_0(\xi) = \frac{\Big(-e^{-i5\xi}(-1+e^{i\xi})^6(3 + 18e^{i\xi} + 38e^{i2\xi} + 18e^{i3\xi} + 3e^{i4\xi})\Big)}{512}. \quad (24)$$

Notice that in this case besides the delay and moment factors we have just a single remaining factor with symmetric coefficients.

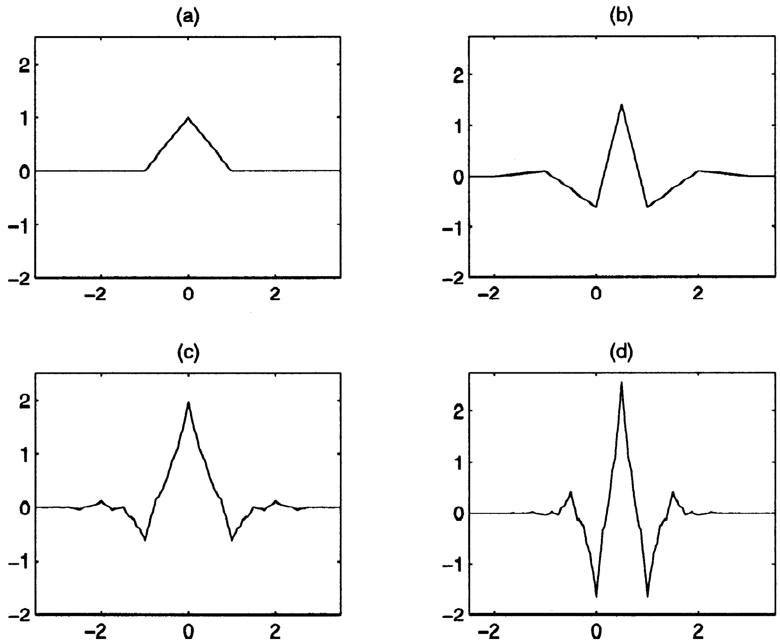

Figure 3: The piece-wise linear spline biorthogonal system. (a): The scaling function, φ. (b): The wavelet, ψ. (c): The dual scaling function, $\tilde{\varphi}$. (d): The dual wavelet, $\tilde{\psi}$.

Example C: We begin with the analog of Example A, corresponding to μ_1^C equal to a multiple of the Haar high-pass filter,

$$\mu_1^C(\xi) = \frac{-3}{4}\big(1 - e^{-i\xi}\big),$$

which forces

$$\tilde{\mu}_1^C(\xi) = \frac{1}{384}\big(- 3e^{i4\xi} - 3e^{i3\xi} + 22e^{i2\xi} + 22e^{i\xi} - 128 + 128e^{-i\xi} - 22e^{-i2\xi} - 22e^{-i3\xi} + 3e^{-i4\xi} + 3e^{-i5\xi}\big).$$

The resulting à trous wavelets, which possess one and five vanishing moments, respectively, are depicted in Figure 4, (a) and (b). Since each scaling function is symmetric about $x = 0$ we see that ϕ^C and $\tilde{\phi}^C$ are anti-symmetric about $x = \frac{1}{4}$.

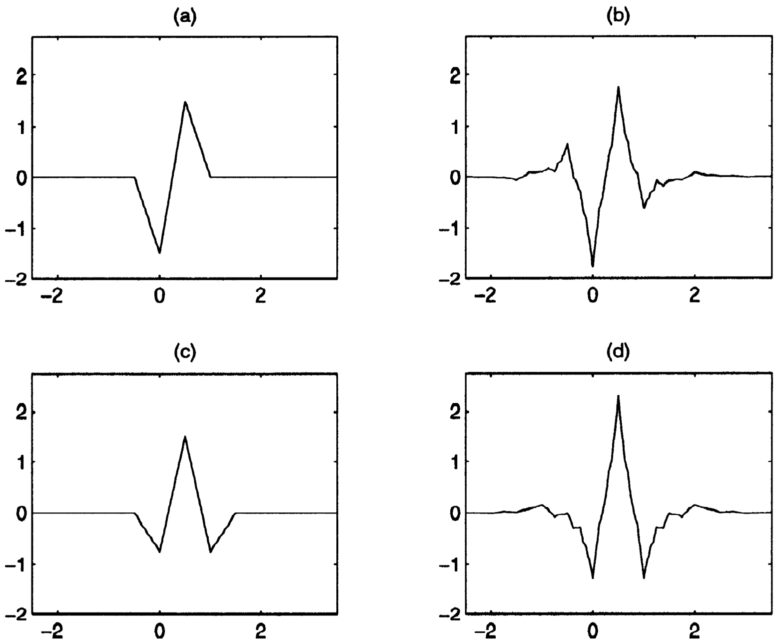

Figure 4: Examples of spline à trous wavelets. (a): The wavelet, ϕ^C. (b): The dual wavelet, $\tilde{\phi}^C$. (c): The wavelet, ϕ^D. (d): The dual wavelet, $\tilde{\phi}^D$.

EXAMPLE D: Finally, we consider another example in which the vanishing moments are not shared evenly between the two à trous wavelets, but with this example we will assign two vanishing moments for μ_1^D with the shortest possible filter. Letting

$$\mu_1^D(\xi) = \frac{1}{8}\big(-3 + 6e^{-i\xi} - 3e^{-i2\xi}\big),$$

and

$$\tilde{\mu}_1^D(\xi) = \frac{1}{192}\big(3e^{i3\xi} + 6e^{i2\xi} - 16e^{i\xi} - 38 + 90e^{-i\xi} - 38e^{-i2\xi} - 16e^{-i3\xi} + 6e^{-i4\xi} + 3e^{-i5\xi}\big),$$

we obtain the à trous wavelets, ϕ^D and $\tilde{\phi}^D$, pictured in Figure 3, (c) and (d). As in the Burt-Adelson case, by assigning even numbers of vanishing moments to the à trous wavelets we obtain wavelets with symmetry properties, in this case about $x = \frac{1}{2}$.

These four examples show the flexibility present in the design of à trous wavelets for use in the negative dilations of a generalized quasi-affine frame arising from a given biorthogonal system. This flexibility grants us total control over the number of vanishing moments in analysis and synthesis or the relative filter lengths and permits the creation of symmetry properties for the à trous wavelets not present in the original biorthogonal wavelets, ψ and $\tilde{\psi}$. We did notice, however, for our two biorthogonal systems that even numbers of vanishing moments leads to symmetry properties while odd numbers of vanishing moments yield anti-symmetry properties for the resulting à trous wavelets.

7. Conclusion

The main intent of this work is to establish a connection between the à trous algorithm and quasi-affine systems and to demonstrate how this connection can be used to obtain frames for $L^2(\mathbb{R}^n)$ beginning with appropriate à trous systems. Using techniques associated with the bracket product we extended the multiresolution operators common in analyses of affine systems to the quasi-affine setting. This analysis lead to the observation that the behavior of the quasi-affine multiresolution operators resembles the à trous algorithm and, motivated by this observation, we showed that for negative dilations the original generating wavelets could be replaced by appropriately chosen à trous wavelets and the resulting GQA systems would constitute dual frames for $L^2(\mathbb{R}^n)$. Lastly, we considered examples of GQA systems for $L^2(\mathbb{R})$ with primary focus on the design of the à trous wavelets in terms of a given 2-band biorthogonal wavelet system.

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