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Extrapolation theory for the real interpolation method*

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Abstract

We develop an abstract extrapolation theory for the real interpolation method that covers and improves the most recent versions of the celebrated theorems of Yano and Zygmund. As a consequence of our method, we give new endpoint estimates of the embedding Sobolev theorem for an arbitrary domain Ω.

1. Introduction

In 1951, Yano (see [14], [15]) using the ideas of Titchmarsh in [13], proved that for every sublinear operator T satisfying that there exists $C > 0$ such that, for every $1 < p \leq 2$,

$$\left(\int_{\mathcal{N}} |Tf(x)|^p \, d\nu(x) \right)^{1/p} \leq \frac{C}{p-1} \left(\int_{\mathcal{M}} |f(x)|^p \, d\mu(x) \right)^{1/p},$$

where $(\mathcal{N}, \nu)$ and $(\mathcal{M}, \mu)$ are two finite measure spaces, $T : L \log L(\mu) \longrightarrow L^1(\nu)$ is bounded. If the measures involved are not finite, then an easy modification of the above results shows that $T : L \log L(\mu) \longrightarrow L^1_{\mathrm{loc}}(\nu)$ and, in fact, $T : L \log L(\mu) \longrightarrow L^1(\nu) + L^\infty(\nu)$.

Quite recently, it has been proved (see [4] and [5]) that under a weaker condition on the operator T, namely that

$$\left(\int_{\mathcal{N}} |T\chi_A(x)|^p \, d\nu(x) \right)^{1/p} \leq \frac{C}{p-1} \mu(A)^{1/p},$$

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for every measurable set $A \subset \mathcal{M}$ and every $1 < p \le p_0$, with C independent of A and p, it holds that

$$T : L\log L(\mu) \longrightarrow M(\varphi),$$

is bounded, where $M(\varphi)$ is the maximal Lorentz space associated to the function $\varphi(t) = t/(1+\log^+ t)$; that is,

$$\|f\|_{M(\varphi)} = \sup_{t>0} \frac{tf_\nu^{**}(t)}{1+\log^+ t}\cdot$$

It turns out that this space $M(\varphi)$ is strictly embedded in $L^1(\nu) + L^{p,1}(\nu)$, for every $1 < p < \infty$, (see [6]) and therefore, Yano's theorem was improved.

Also, in [15], p. 119, it is proved that if T is a linear operator satisfying

$$\|Tf\|_{L^p(\nu)} \le Cp\|f\|_{L^p(\mu)}, \tag{1}$$

for every p near ∞ and, μ and ν are finite measures, then

$$T : L^\infty(\mu) \longrightarrow L_{\exp}(\nu)\cdot$$

Using a duality argument, this result was recently extended, in [5], to the case of general measures, proving that, if T is a linear operator satisfying (1), then

$$\sup_{t>0} \frac{(Tf)_\nu^{**}(t)}{(1+\log^+(1/t))} \le C\biggl(\int_1^\infty f_\mu^{**}(t)\frac{dt}{t} + \|f\|_\infty\biggr)\cdot$$

In the 90's, the extrapolation theory was extended to the setting of compatible couples of Banach spaces in real interpolation, with the works of Jawerth and Milman (see [9], [10]) and Milman (see [11]), (see Section 5).

The purpose of this work is to present an alternative extrapolation theory for the real interpolation method, that has the advantage, among others, of obtaining a better range space in the case of Yano's type result and a better domain in the case of Zygmund's type results than the ones known up to now. Moreover, with our theory, the identification of the extrapolation spaces is immediate and hence the consequences and applications are obtained in a very easy way.

The paper is organized as follows: in Section 2, we present the main results of the theory. In particular, these results are presented in detail, in Section 3, for the particular case that covers the real interpolation theory. To illustrate our method, we have applied our results, in Section 4, to obtain end-point estimates for the Riesz Potentials and for the Sobolev embedding theorem for an arbitrary domain Ω. We also present in this section, the improvement of Yano's and Zygmund's theorems.

Finally, in the last section of this paper, we study the relationship of our method with the Σ- and Δ-method of Jawerth and Milman.

As usual, the symbol $f \approx g$ will indicate the existence of an universal positive constant C (independent of all parameters involved) so that $(1/C)f \le g \le Cf$, while the symbol $f \preceq g$ means that $f \le Cg$. Given $(\mathcal{N}, \nu)$ a σ–finite measure space, we shall write $\|g\|_p$ to denote $\|g\|_{L^p(\nu)}$, $\lambda_g^\nu(y) = \nu(\{x \in \mathcal{N};\ |g(x)| > y\})$ is the distribution function of g with respect to the measure ν, $g_\nu^*(t) = \inf\{s; \lambda_g^\nu(s) \le t\}$ is the decreasing rearrangement and $g_\nu^{**}(t) = (1/t)\int_0^t g_\nu^*(s)\,ds$ (see [1]). In what follows we shall omit the indices ν whenever it is clear the measure we are working with. Also, for a measurable set E in $(0,\infty)$, we shall denote by $|E|$ the Lebesgue measure of the set, and whenever it is not specified the underlying measure in $(0,\infty)$ will be this one.

Let us now start our presentation by briefly recalling some classical results about real interpolation theory. Our main references will be [1], [2] and [3] (and the references quoted therein) where we refer the reader for further information.

We consider **compatible pairs** of Banach spaces $\bar{A} = \big(A_0, A_1\big)$, that is, we assume that there is a topological vector space $\mathcal{V}$ such that $A_i \subset \mathcal{V}$, $i = 0,1$, continuously. Usually we drop the terms "compatible" and "Banach" and refer to a compatible Banach pair simply as a "pair".

The Peetre $K-$**functional** associated with a pair $\bar{A}$ is defined, for each $a \in A_0 + A_1$ and $t > 0$, by

$$K(a,t) = K(a,t;\bar{A}) = \inf\left\{ \|a_0\|_{A_0} + t\,\|a_1\|_{A_1}\,;\ a = a_0 + a_1,\ a_i \in A_i \right\}\cdot$$

It is easy to see that $K(t,a)$ is a nonnegative and concave function of $t > 0$, (and thus also continuous). Therefore

$$K(a,t;\bar{A}) = K(a,0^+;\bar{A}) + \int_0^t k(a,s;\bar{A})\,ds,$$

where the $k-$**functional**, $k(a,s;\bar{A})$, is a uniquely defined, nonnegative, decreasing and right-continuous function of $s > 0$.

Example 1.1: The pair $\bar{A} = \big(L^1(\nu), L^\infty(\nu)\big)$ satisfies that $k(a,s;\bar{A}) = f^*(s)$, that is

$$K(f,t) = \int_0^t f^*(s)\,ds.$$

Let us also recall that given a positive concave function φ on $(0,\infty)$, the maximal Lorentz space $M(\varphi)$ is defined (see [1]) as the set of measurable functions such that

$$\|f\|_{M(\varphi)} = \sup_{t>0}\Big(f^{**}(t)\varphi(t) \Big) < \infty,$$

and the minimal Lorentz space $\Lambda(\varphi)$ is defined by the condition

$$\|f\|_{\Lambda(\varphi)} = \int_0^\infty f^*(s)\,d\varphi(s) < \infty.$$

Obviously, if $\varphi(0+) = 0$, we have that

$$\|f\|_{\Lambda(\varphi)} = \int_0^\infty \varphi\big(\lambda_f(y)\big)\,dy < \infty.$$

By analogy, we extend the definition of minimal and maximal Lorentz space to the case of general pairs as follows (see [3]):

DEFINITION 1.1. Given a pair $\bar{A} = \big(A_0, A_1\big)$ and a concave function φ, the minimal Lorentz space, $\Lambda(\varphi; \bar{A})$, is the set of elements $a \in A_0 + A_1$ such that $K(a, 0^+; \bar{A}) = 0$ and

$$\|a\|_{\Lambda(\varphi;\bar{A})} = \int_0^\infty k(a, s; \bar{A})\, d\varphi(s) < \infty,$$

and the maximal Lorentz space $M(\varphi; \bar{A})$ is the set of elements $a \in A_0 + A_1$ such that

$$\|a\|_{M(\varphi;\bar{A})} = \sup_{t>0} \left(\frac{K(a, t; \bar{A})}{t} \varphi(t) \right) < \infty \cdot$$

Remark 1.1. i) Notice that in the case that $\bar{A} = \big(L^1(\nu), L^\infty(\nu)\big)$, we obtain the "classical" Lorentz spaces.

ii) Standard arguments (see [1]) show that $\Lambda(\varphi; \bar{A})$ and $M(\varphi; \bar{A})$ are Banach spaces such that

$$\Lambda(\varphi; \bar{A}) \subset M(\varphi; \bar{A}) \subset A_0 + A_1$$

continuously.

iii) If we consider the function $\varphi_\theta(t) = t^{1-\theta}$ with $0 < \theta < 1$, then

$$\Lambda(\varphi_\theta; \bar{A}) = \bar{A}_{\theta,1} \subset M(\varphi_\theta; \bar{A}) = \bar{A}_{\theta,\infty},$$

where the constant of the above embedding is 1, and the spaces $\bar{A}_{\theta,q}$ are the classical real interpolation spaces defined as the set of elements $a \in A_0 + A_1$ such that the following quantity is finite

$$\|a\|_{\bar{A}_{\theta,q}} = \begin{cases} \left(\theta(1-\theta)q \displaystyle\int_0^\infty \big(t^{-\theta} K(a, t)\big)^q \frac{dt}{t} \right)^{1/q}, & 0 < \theta < 1,\ 1 \le q < \infty, \\ \displaystyle\sup_{t>0} t^{-\theta} K(a, t), & 0 < \theta < 1,\ q = \infty. \end{cases}$$

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2. Main results

To prove our main result, we need a technical decomposition lemma, which is fundamental in our theory.

Lemma 2.1

Given an element $a \in A_0 + A_1$ such that $K(a, 0^+; \bar{A}) = 0$, there exists a constant γ (depending only on $\bar{A}$) and a collection of elements $(a_i)_{i\in\mathbb{Z}}$ such that

$$a = \sum_{i\in\mathbb{Z}} 2^i a_i \ \text{(convergence in } A_0 + A_1),$$

and

$$K(a_i, t) \le \gamma \min\big(t, |E_i|\big)$$

where $E_i = \big\{s \in (0, \infty);\ k(a, s; \bar{A}) > 2^i\big\}$.

Proof. Let $a \in A_0 + A_1$ such that $K(a, 0^+; \bar{A}) = 0$. Since $k(t) = k(a, t; \bar{A})$, is a decreasing function of t, we have that

$$\sum_{i \in \mathbb{Z}} 2^i \chi_{A_i}(t) \leq k(t) \leq \sum_{i \in \mathbb{Z}} 2^{i+1} \chi_{A_i}(t),$$

where $A_i = \big\{t \in [0, \infty); 2^i < k(t) \leq 2^{i+1}\big\}$, and hence, if we consider the interval E_i, we get that

$$\frac{1}{2} \sum_{i \in \mathbb{Z}} 2^i \chi_{E_i}(t) \leq k(t) \leq 2 \sum_{i \in \mathbb{Z}} 2^i \chi_{E_i}(t).$$

Since

$$K(a, t; \bar{A}) = \int_0^t k(a, s; \bar{A})\, ds \leq 2 \sum_{i \in \mathbb{Z}} 2^i \int_0^t \chi_{E_i}(s)\, ds$$

and each term appearing in the above series is a concave function of t, satisfying that

$$2 \sum_{i \in \mathbb{Z}} 2^i \int_0^1 \chi_{E_i}(s)\, ds \leq 4K(a, 1; \bar{A}) < \infty,$$

we can apply the $K-$divisibility theorem (see [3], Theorem 3.2.7) to obtain a sequence $(a_i)_{i \in \mathbb{Z}}$ in $A_0 + A_1$ such that

$$a = \sum_{i \in \mathbb{Z}} 2^i a_i \text{ (convergence in } A_0 + A_1)$$

and such that the elements a_i satisfy that

$$K(a_i, t) \leq \gamma \int_0^t \chi_{E_i}(s)\, ds,$$

where γ is a positive constant depending only on the pair $\bar{A}$. □

We shall say that $a = \sum_{i=-\infty}^{\infty} 2^i a_i$ is a **diadyc decomposition** of a and each term a_i will be called a **characteristic element**.

Definition 2.1. Given an element $a \in A_0 + A_1$ and a decreasing set $E = (0, r)$ in $(0, \infty)$, we say that the pair $\big(a, E\big)$ is a **characteristic pair** of $\bar{A}$, if it satisfies that

$$K(t, a; \bar{A}) \leq \gamma \min\big(t, |E|\big),$$

where γ is a universal constant depending only on $\bar{A}$.

An element $a \in A_0 + A_1$ is said to be **simple** if there exist $N \in \mathbb{N}$ and a finite collection of characteristic elements $(a_i)_i$ such that

$$a = \sum_{i=-N}^{N} 2^i a_i.$$

As a consequence of Lemma 2.1, we obtain the following results which will be very useful for our purpose:

Proposition 2.1

If φ is a concave function such that $\varphi(0^+) = 0$, then

$$\|a\|_{\Lambda(\varphi; \bar{A})} = \int_0^\infty \varphi\Big(\lambda_{k(a,.;\bar{A})}(y)\Big)\, dy \approx \sum_{i \in \mathbb{Z}} 2^i \varphi\Big(|E_i|\Big),$$

where $E_i = \{t;\ k(a, t; \bar{A}) > 2^i\}$ and the constant in the above equivalence does not depend on φ.

Proposition 2.2

Let φ be a positive concave function such that $\varphi(0^+) = 0$. Then the following conditions hold.

a) If (a, E) is a characteristic pair of $\bar{A}$,

$$\|a\|_{\Lambda(\varphi;\bar{A})} \leq \gamma\varphi(|E|).$$

b) If $a = \sum_{i\in\mathbb{Z}} 2^i a_i$ is a diadyc decomposition of an element $a \in \Lambda(\varphi; \bar{A})$, then the sequence $\left(\sum_{i=-N}^{N} 2^i a_i\right)_N$ converges to a in $\Lambda(\varphi; \bar{A})$. Consequently, the set of simple elements of $\bar{A}$ is dense in $\Lambda(\varphi; \bar{A})$.

Proof. a) Since $\|a\|_{\Lambda(\varphi;\bar{A})} = \|k(a, .; \bar{A})\|_{\Lambda(\varphi)}$, and $\Lambda(\varphi)$ is a rearrangement invariant function space, it follows by Theorem 4.6, p. 61 of [1], that if $K(a, t; \bar{A}) = \int_0^t k(a, s; \bar{A})(s)\, ds \leq \int_0^t h(s)\, ds$, for some decreasing function h, then $\|a\|_{\Lambda(\varphi;\bar{A})} \leq \|h\|_{\Lambda(\varphi)}$, from which the result follows. To prove b), let $a \in \Lambda(\varphi; \bar{A})$ and let $a = \sum_{i=-\infty}^{\infty} 2^i a_i$ be a diadyc decomposition of a. Let $a^N = \sum_{i=-N}^{N} 2^i a_i$. Then, for every $N < M$, we have, using a), that

$$\|a^N - a^M\|_{\Lambda(\varphi;\bar{A})} \leq \sum_{N<|i|\leq M} 2^i \|a_i\|_{\Lambda(\varphi;\bar{A})} \leq \gamma \sum_{N<|i|\leq M} 2^i \varphi(|E_i|),$$

and, by Proposition 2.1, we obtain that the last expression tends to zero whenever N and M tend to infinity. Hence, $(a^N)_N$ is a Cauchy sequence in $\Lambda(\varphi; \bar{A})$ and by completeness, there exists an element $b \in \Lambda(\varphi; \bar{A})$ such that $(a^N)_N$ converges to b in $\Lambda(\varphi; \bar{A})$. Finally, since, by Remark 1.1 ii), we have that $\Lambda(\varphi; \bar{A}) \subset A_0 + A_1$ and $(a^N)_N$ tends to a in $A_0 + A_1$, we get that $b = a$ as we wanted to see. □

Proposition 2.3

Let B be any Banach space embedded in $A_0 + A_1$ and let φ be a positive concave function such that $\varphi(0^+) = 0$. Then

$$\Lambda(\varphi; \bar{A}) \subset B$$

with constant K, if for every characteristic pair (a, E) of $\bar{A}$, it holds that $\|a\|_B \leq \frac{K}{2}\varphi(|E|)$.

Proof. Let $a \in \Lambda(\varphi; \bar{A})$ and let $a = \sum_{i\in\mathbb{Z}} 2^i a_i$ be a diadyc decomposition of a. Then, we have that,

$$\Big\| \sum_{i=-N}^{N} 2^i a_i \Big\|_B \leq \sum_{i=-N}^{N} 2^i \|a_i\|_B \leq \frac{K}{2} \sum_{i=-N}^{N} 2^i \varphi(|E_i|) \leq K\|a\|_{\Lambda(\varphi;\bar{A})}.$$

Since B is complete, it is embedded in $A_0 + A_1$ and $\left(\sum_{i=-N}^{N} 2^i a_i\right)_N$ converges to a in $A_0 + A_1$, we conclude the result by standard arguments. □

Let us present now the **General Setting** of our method:

Definition 2.2. Let $\bar{A}$ and $\bar{B}$ two pairs. Let Θ be an arbitrary set of parameters and set

$$F_\Theta = \left\{(\varphi_\theta, \psi_\theta);\ \theta \in \Theta\right\},$$

where φ_θ and ψ_θ are positive concave functions on $(0, \infty)$ such that, for every $\theta \in \Theta$, $\varphi_\theta(0^+) = 0$. Then, we say that a linear operator T is a **minimal-maximal extrapolation** operator associated to the **triple** $\bar{\Theta} = (F_\Theta, \bar{A}, \bar{B})$ if, for every $\theta \in \Theta$,

$$T : \Lambda(\varphi_\theta; \bar{A}) \longrightarrow M(\psi_\theta; \bar{B})$$

is bounded.

Definition 2.3. Given a triple $\bar{\Theta} = (F_\Theta, \bar{A}, \bar{B})$ and a minimal-maximal extrapolation operator T associated to $\bar{\Theta}$, we consider the function

$$h(t,s) = h_{\bar{\Theta},T}(t,s) = \inf_{\theta \in \bar{\Theta}} \left\{ \frac{\varphi_\theta(s) t \|T\|_\theta}{\psi_\theta(t)} \right\},$$

where $\|T\|_\theta$ is the norm of the operator $T : \Lambda(\varphi_\theta; \bar{A}) \longrightarrow M(\psi_\theta; \bar{B})$.

From now on we shall omit the indexes $\bar{\Theta}$ and T in the function h and we shall simply write $h(t,s)$.

Observe that, for every t fixed, the function $h(t,\cdot)$ is a positive concave function so that $h(t, 0^+) = 0$ and, hence, we can consider the measure $dh(t,s)$.

In this context, our first main result can be formulated as follows:

Theorem 2.1

Given a triple $\bar{\Theta}$, a linear operator T is a minimal-maximal extrapolation operator associated to $\bar{\Theta}$ if and only if, for every $t > 0$ and every $a \in \cup_{\theta \in \Theta} \Lambda(\varphi_\theta; \bar{A})$,

$$K(Ta, t; \bar{B}) \preceq \int_0^\infty k(a, s; \bar{A})\, dh(t,s). \tag{2}$$

Proof. To prove the necessary condition, let (a, E) be a characteristic pair of $\bar{A}$. Then, by Proposition 2.2,

$$\begin{aligned} K(Ta, t; \bar{B}) &\le \frac{t}{\psi_\theta(t)} \|Ta\|_{M(\psi_\theta; \bar{B})} \le \frac{t\|T\|_\theta}{\psi_\theta(t)} \|a\|_{\Lambda(\varphi_\theta; \bar{A})} \\ &\preceq \frac{t\|T\|_\theta \varphi_\theta(|E|)}{\psi_\theta(t)}, \end{aligned}$$

and taking the infimum in $\theta \in \Theta$, we obtain that

$$K(Ta, t; \bar{B}) \preceq h(t, |E|).$$

Let now $a \in \cup_{\theta\in\Theta}\Lambda(\varphi_\theta;\bar{A})$ and let $a = \sum_{i\in\mathbb{Z}} 2^i a_i$ be a diadyc decomposition of a. Set $a^N = \sum_{i=-N}^{N} 2^i a_i$. Then, $T(a^N) = \sum_{i=-N}^{N} 2^i T(a_i)$ and we have that

$$\begin{aligned} K\Big(T(a^N),t;\bar{B}\Big) &= K\Big(\sum_{i=-N}^{N} 2^i T a_i, t;\bar{B}\Big) \\ &\preceq \sum_{i=-N}^{N} 2^i h(t,|E_i|) \le \sum_{i\in\mathbb{Z}} 2^i h(t,|E_i|) \\ &\approx \int_0^\infty h(t,\lambda_{k(a,\cdot;\bar{A})}(s))\,ds. \end{aligned}$$

Now, since $a \in \cup_{\theta\in\Theta}\Lambda(\varphi_\theta;\bar{A})$, we have that there exists $\theta_0 \in \Theta$ such that $a \in \Lambda(\varphi_{\theta_0};\bar{A})$ and hence, we can similarly prove that

$$\begin{aligned} K\Big(T(a^N)-T(a^M),t;\bar{B}\Big) &\preceq \sum_{N\le|i|\le M} 2^i h(t,|E_i|) \\ &\le \frac{t\|T\|_{\theta_0}}{\psi_{\theta_0}(t)} \sum_{N\le|i|\le M} 2^i \varphi_{\theta_0}(|E_i|), \end{aligned}$$

which, by Proposition 2.2, converges to zero, whenever N and M tend to infinity.

Therefore, $\big(T(a^N)\big)_N$ converges in B_0+B_1 to an element b such that

$$K(b,t;\bar{B}) \preceq \int_0^\infty h\big(t,\lambda_{k(a,\cdot;\bar{A})}(s)\big)\,ds.$$

But, since $a \in \Lambda(\varphi_{\theta_0},\bar{A})$, $(a^N)_N$ converges, by Proposition 2.2 b), to a in $\Lambda(\varphi_{\theta_0};\bar{A})$ and, since $T:\Lambda(\varphi_{\theta_0};\bar{A}) \to M(\psi_{\theta_0};\bar{B})$ is bounded, we have that $\big(T(a^N)\big)_N$ tends to Ta in $M(\varphi_{\theta_0};\bar{B})$. Therefore $\big(T(a^N)\big)_N$ tends to Ta in B_0+B_1 and thus, $Ta=b$, from which the result follows.

Conversely, if (2) holds, then, for every $\theta\in\Theta$,

$$\begin{aligned} \|Ta\|_{M(\Psi_\theta;\bar{B})} &= \sup_{t>0} \frac{K(Ta,t;\bar{B})\Psi_\theta(t)}{t} \\ &\preceq \int_0^\infty \frac{h\Big(\lambda_{k(a,\cdot;\bar{A})}(y),t\Big)\Psi_\theta(t)}{t}\,dy \\ &\le \|T\|_\theta \int_0^\infty \varphi_\theta\Big(\lambda_{k(a,\cdot;\bar{A})}(y)\Big)\,dy = \|T\|_\theta\|a\|_{\Lambda(\varphi_\theta;\bar{A})}. \;\square \end{aligned}$$

As a trivial consequence of Theorem 2.1, we get the following extrapolation result.

Theorem 2.2 (Extrapolation theorem)

Let T be a minimal-maximal extrapolation operator associated to a triple $\bar{\Theta}$. Then, if E and D are two positive concave functions such that $D(0^+)=0$ and

$$h(t,s) \le E(t)D(s), \tag{3}$$

T can be extended from $\cup_{\theta\in\Theta}\Lambda(\varphi_\theta,\bar{A})\cap\Lambda(D;\bar{A})$ to a bounded operator

$$T:\Lambda(D;\bar{A}) \longrightarrow M(R;\bar{B}),$$

where $R(t)=t/E(t)$.

Proof. a) By (2), we have that, if $a \in \cup_{\theta\in\Theta}\Lambda(\varphi_\theta, \bar{A})$,

$$\begin{aligned}\|Ta\|_{M(R;\bar{B})} &= \sup_{t>0} \frac{K(Ta,t;\bar{B})R(t)}{t} = \sup_{t>0} \frac{K(Ta,t;\bar{B})}{E(t)} \\ &\preceq \int_0^\infty \frac{h\Big(\lambda_{k(a,\cdot;\bar{A})}(y),t\Big)}{E(t)}\,dy \le \int_0^\infty D\Big(\lambda_{k(a,\cdot;\bar{A})}(y)\Big)\,dy \\ &= \|a\|_{\Lambda(D;\bar{A})}.\end{aligned}$$

From this and the fact that by Proposition 2.2, $\cup_{\theta\in\Theta}\Lambda(\varphi_\theta, \bar{A}) \cap \Lambda(D;\bar{A})$ is dense in $\Lambda(D;\bar{A})$, we conclude the result. □

Remark 2.1. i) If h satisfies (3) and we fix D, we have that

$$F(t) := \sup_s \frac{h(t,s)}{D(s)} \le E(t),$$

and hence there exists the least concave majorant E_D of the function F. Since obviously $h(t,s) \le E_D(t)D(s)$, we conclude that

$$T : \Lambda(D;\bar{A}) \longrightarrow M(R_D;\bar{B}),$$

where $R_D(t) = t/E_D(t)$. Since $E_D \le E$, we have that $M(R_D;\bar{B}) \subset M(R;\bar{B})$ and hence, for D fixed, $M(R_D;\bar{B})$ is the least range space we can get with the method of Theorem 2.2.

Similarly, if h satisfies (3) and we fix E, then

$$T : \Lambda(D_R;\bar{A}) \longrightarrow M(R;\bar{B}),$$

where D_R is the least concave majorant of $\sup_t \Big(h(t,s)/E(t)\Big)$. Since $D_R \le D$, we have that $\Lambda(D;\bar{A}) \subset \Lambda(D_R;\bar{A})$ and therefore, for E fixed, $\Lambda(D_R;\bar{A})$ is the biggest domain space we can get with our method.

ii) If we have that, for $j = 1, 2$,

$$h(t,s) \le D_j(t)E_j(t),$$

with D_j and E_j concave functions and $D_1 \le D_2$, then $R_{D_1} \le R_{D_2}$ and hence

$$\Lambda(D_2;\bar{A}) \subset \Lambda(D_1;\bar{A}),$$

and

$$M(R_{D_2};\bar{B}) \subset M(R_{D_1};\bar{B}).$$

Now, if h satisfies (3), then $h(1,s) \le E(1)D(s)$ and hence if we denote by $D_\Sigma(s) = h(1,s)$, we get that for every D satisfying (3)

$$\Lambda(D;\bar{A}) \subset \Lambda(D_\Sigma;\bar{A}).$$

Similarly, if h satisfies (3), then $h(t,1) \le E(t)D(1)$, and hence, if we denote by $R_\Delta(t) = t/h(t,1)$, one can easily see that

$$M(R_\Delta;\bar{B}) \subset M(R;\bar{B}),$$

where $R(t) = t/E(t)$.

We shall see in the last section that the functions D_Σ and R_Σ are connected with the Σ-method of Jawerth and Milman, and D_Δ and R_Δ with the Δ-method.

3. Real extrapolation method

In this section, the set of parameters Θ will be a subset of the interval $(0,1)$, and, for every $\theta \in \Theta$,

$$\varphi_\theta(t) = \psi_\theta(t) = t^{1-\theta}.$$

We shall consider the two classical cases:

I) **Yano's case:** Here, $\Theta = (0,\theta_0]$ for some $\theta_0 < 1$ and T is a minimal-maximal extrapolation operator such that $\|T\|_\theta \le \theta^{-\alpha}$ for some $\alpha > 0$. Then, simple computations show that

$$h(t,s) \approx \begin{cases} s^{1-\theta_0}t^{\theta_0} & \text{if } t \le s \\ s\Big(1+\log\dfrac{t}{s}\Big)^\alpha & \text{if } t > s \end{cases} \tag{4}$$

and, therefore, inequality (2) reads

$$\begin{aligned} K(Ta,t;\bar{B}) &\preceq \int_0^t k(a,s)\Big(1+\log\frac{t}{s}\Big)^\alpha ds + (1-\theta_0)t^{\theta_0}\int_t^\infty \frac{k(a,s)}{s^{\theta_0}}\,ds \\ &\approx \int_0^t \frac{K(a,s)}{s}\Big(1+\log\frac{t}{s}\Big)^{\alpha-1} ds + (1-\theta_0)t^{\theta_0}\int_t^\infty \frac{K(a,s)}{s^{\theta_0+1}}\,ds, \end{aligned}$$

which is related to the so-called KJ-inequalities in [9].

Obviously,

$$h(t,s) \le s\Big(1+\log^+\frac{t}{s}\Big)^\alpha \le s\Big(1+\log^+\frac{1}{s}\Big)^\alpha(1+\log^+ t)^\alpha,$$

and since both functions are quasi-concave (that is, equivalent to a concave function), we can take $D(s) = s\big(1+\log^+\frac{1}{s}\big)^\alpha$ and $E(t) \approx (1+\log^+ t)^\alpha$ in our Theorem 2.2 to conclude the following real extrapolation result.

Recall also that, as was mentioned in the introduction

$$\Lambda(\varphi_\theta;\bar{A}) = \bar{A}_{\theta,1} \qquad \text{and} \qquad M(\varphi_\theta;\bar{B}) = \bar{B}_{\theta,\infty}.$$

Theorem 3.1

Let $\bar{A} = (A_0,A_1)$ and $\bar{B} = (B_0,B_1)$ be two pairs and let T be a linear operator such that

$$T : \bar{A}_{\theta,1} \to \bar{B}_{\theta,\infty}$$

is bounded with $\|T\| \le \theta^{-\alpha}$, for every $0 < \theta \le \theta_0 \le 1$. Then, T can be extended continuously

$$T : \Lambda_0^\alpha(\bar{A}) \to M_0^\alpha(\bar{B}),$$

where $\Lambda_0^\alpha(\bar{A}) = \Lambda(\varphi;\bar{A})$ and $\varphi(t) = t(1+\log^+\frac{1}{t})^\alpha$, and $M_0^\alpha(\bar{B}) = M(\psi;\bar{B})$ with $\psi(t) = t(1+\log^+ t)^{-\alpha}$.

An easy computation shows that

$$\|a\|_{\Lambda_0^\alpha(\bar{A})} \approx \int_0^\infty k(t,a;\bar{A})\Big(1+\log^+\frac{1}{t}\Big)^\alpha\, dt,$$

and, by integration by parts, we get the following characterization of $\Lambda_0^\alpha(\bar{A})$ in terms of the K-functional:

Proposition 3.1

The space $\Lambda_0^\alpha(\bar{A})$ coincides with the elements in $A_0 + A_1$ such that

$$\sup_{t>0} K(t,a;\bar{A}) + \int_0^1 \frac{K(t,a;\bar{A})}{t}\Big(1+\log^+\frac{1}{t}\Big)^{\alpha-1}\, dt < \infty\,.$$

II) **Zygmund's case:** Here, $\Theta = [\theta_1, 1)$ for some $\theta_1 > 0$ and T is a minimal-maximal extrapolation operator such that $\|T\|_\theta \leq (1-\theta)^{-\alpha}$ for some $\alpha > 0$. Then, simple computations show that

$$h(t,s) \approx \begin{cases} t^{\theta_1} s^{1-\theta_1} & \text{if } \ s \leq t \\ t\Big(1+\log\dfrac{s}{t}\Big)^\alpha & \text{if } \ s > t \end{cases}$$

and, therefore, inequality (2) reads

$$\begin{aligned} K(Ta,t;\bar{B}) &\preceq (1-\theta_1)t^{\theta_1}\int_0^t \frac{k(a,s)}{s^{\theta_1}}\, ds + t\int_t^\infty \frac{k(a,s)}{s}\Big(1+\log\frac{s}{t}\Big)^{\alpha-1} ds \\ &\approx t^{\theta_1}\int_0^t \frac{K(a,s)}{s^{1+\theta_1}}\, ds + t\int_t^\infty \frac{K(a,s)}{s^2}\Big(1+\log\frac{s}{t}\Big)^{\alpha-1} ds \end{aligned}$$

Since

$$h(t,s) \leq t\Big(1+\log^+\frac{1}{t}\Big)^\alpha (1+\log^+ s)^\alpha,$$

we can apply Theorem 2.2 to obtain the following result.

Theorem 3.2

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two pairs and let T be a linear operator such that

$$T : \bar{A}_{\theta,1} \to \bar{B}_{\theta,\infty}$$

is bounded with $\|T\|_\theta \leq (1-\theta)^{-\alpha}$, for every $\theta_1 \leq \theta < 1$. Then, T can be extended continuously

$$T : \Lambda_1^\alpha(\bar{A}) \to M_1^\alpha(\bar{B}),$$

where $\Lambda_1^\alpha(\bar{A}) = \Lambda(\varphi;\bar{A})$ with $\varphi(t) \approx \big(1+\log^+ t\big)^\alpha$, and $M_1^\alpha(\bar{B}) = M(\psi;\bar{B})$ with $\psi(t) \approx \big(1+\log^+\frac{1}{t}\big)^{-\alpha}$.

Similar to Proposition 3.2, we can also give a description of $\Lambda_1^\alpha(\bar{A})$ in terms of the K-functional.

Proposition 3.2

The space $\Lambda_1^\alpha(\bar{A})$ coincides with the elements in $A_0 + A_1$ such that

$$\sup_{t>0} \frac{K(t,a;\bar{A})}{t} + \int_1^\infty \frac{K(t,a;\bar{A})}{t^2}(1+\log t)^{\alpha-1}\,dt < \infty\,.$$

4. Applications

First of all, let us mention that although the theory has been developed for the case of linear operators, one can very easily check that if $\bar{B} = (B_0, B_1)$ are lattices, it can be extended, with the trivial modifications, to the case of sublinear operators (see [5]).

Let us start with two simple lemmas.

Lemma 4.1

Let $1 \le p_0 < p$ and let us consider $\bar{A} = (L^{p_0}, L^\infty)$. Then, if $\theta = 1 - p_0/p$,

$$\bar{A}_{\theta,1} \subset L^p \subset \bar{A}_{\theta,\infty},$$

where the constants of the above embeddings do not blow up when p tends to p_0.

Proof. Recall that (see [2], Theorem 5.2.1)

$$K(t,f;\bar{A}) \approx \left\{ \int_0^{t^{p_0}} \big(f^*(s)\big)^{p_0}\,ds \right\}^{1/p_0} \tag{5}$$

where constants in the above equivalence only depend on p_0.

Thus, if (a,E) is a characteristic pair, we have that

$$\int_0^t \big(a^*(s)\big)^{p_0}\,ds \preceq \min\Big(t, |E|^{p_0}\Big). \tag{6}$$

Now, since $p/p_0 > 1$, inequality (6) implies (see [1], Theorem 4.6, p. 61) that

$$\|a\|_p \preceq |E|^{p_0/p}$$

and Proposition 2.3 applies. The second embedding follows from (5) and Hölder's inequality. □

Lemma 4.2

Let $1 \le p < p_1$ and let us consider $\bar{A} = (L^1, L^{p_1})$. Then, for $\theta = p_1'/p'$ (where q' is, as usual, the conjugate exponent of q),

$$\Lambda(\varphi_\theta;\bar{A}) \subset L^p \subset M(\varphi_\theta;\bar{A}),$$

where the constants of the above embeddings do not blow up when p tends to either 1 or p_1.

Proof. To show the first embedding, we only need to prove, by Proposition 2.3, that if (a, E) is a characteristic pair, then

$$\|a\|_p \leq C|E|^{1-p_1'/p'},$$

where C does not blow up when p tends to either 1 or p_1.

Now, it is known (see [1], p. 311) that

$$K(t, f; L^1, L^\infty) \leq CK(t^\delta, f; \bar{A}),$$

where $\delta = 1/p_1'$ and C is independent (obviously) of p. Then,

$$\int_0^t k(s, f; L^1, L^\infty)\, ds \leq C \int_0^t k(s^\delta, f; \bar{A}) s^{\delta-1}\, ds,$$

and hence

$$\begin{aligned}\|a\|_p &= \left(\int_0^\infty k(t, a; L^1, L^\infty)^p\, dt \right)^{1/p} \leq C \left(\int_0^\infty k(t^\delta, a; \bar{A})^p t^{p(\delta-1)}\, dt \right)^{1/p} \\ &= C \left(\int_0^\infty k(t, a; \bar{A})^p t^{p(\delta-1)/\delta+1/\delta-1} dt \right)^{1/p}.\end{aligned}$$

Since, $K(t, a; \bar{A}) \preceq \min(t, |E|)$ and $\frac{\delta-1}{\delta}p + \frac{1}{\delta} - 1 \leq 0$, we have (see [1], Theorem 4.6) that

$$\begin{aligned}\|a\|_p &\leq C \left(\int_0^{|E|} t^{p(\delta-1)/\delta+1/\delta-1}\, dt \right)^{1/p} = C \left(\frac{1}{p + \frac{1}{\delta}(1-p)} \right) |E|^{(\delta-1)/\delta+1/(\delta p)} \\ &= C \left(\frac{1}{p + p_1'(1-p)} \right) |E|^{1-p_1'/p'},\end{aligned}$$

from which the first result follows.

b) The second embedding is a trivial consequence of the fact that

$$K(t, f; \bar{A}) \approx t \left(\int_{t^{p_1'}}^\infty f^{**}(s)^{p_1}\, ds \right)^{1/p_1}, \tag{7}$$

and Hölder's inequality. □

Since the constants are very important in this theory, let us emphasize that in what follows the space $L^{p,1}$ is endowed with the norm

$$\|f\|_{L^{p,1}} = \frac{1}{p} \int_0^\infty f^*(t) t^{1/p} \frac{dt}{t},$$

and the space $L^{p,\infty}$ is endowed with the norm

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^{**}(t),$$

1) Yano's Operators

Our first application is an improvement of Yano's extrapolation theorem and follows immediately by taking $\bar{A} = (L^1(\mu), L^\infty(\mu))$ and $\bar{B} = (L^1(\nu), L^\infty(\nu))$ in Theorem 3.1.

Theorem 4.1

Let T be a sublinear operator satisfying that

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

is bounded with constant $1/(p-1)$. Then,

$$\sup_{t>0} \frac{\int_0^t (Tf)^*_\nu(s)\, ds}{(1+\log^+ t)} \preceq \int_0^\infty f^*_\mu(s) \left(1 + \log^+ \frac{1}{s}\right) ds\,.$$

Now, if the norm of the operator T blows up when p tends to p_0^+ instead of 1, we also have the following generalization of Yano's theorem:

Theorem 4.2

Let T be a sublinear operator satisfying that

$$T : L^{p}(\mu) \longrightarrow L^{p}(\nu),$$

is bounded with constant $1/(p-p_0)$, $(p_0 < p \leq p_1)$. Then,

$$\sup_{t>0} \left(\frac{\left(\int_0^t (Tf)^*_\nu(s)^{p_0}\, ds\right)^{1/p_0}}{1+\log^+ t} \right) \preceq \|f\|_{L^{p_0}(\mu)} + \int_0^1 \frac{\left(\int_0^t f^*_\mu(s)^{p_0}\, ds\right)^{1/p_0}}{t}\, dt\,.$$

Proof. If we take $\bar{A} = (L^{p_0}(\mu), L^\infty(\mu))$ and $\bar{B} = (L^{p_0}(\nu), L^\infty(\nu))$, we have, by Lemma 4.1, that the hypothesis of our Theorem 3.1 is satisfied and the result follows immediately from it, (5) and Proposition 3.1. □

2) Zygmund's Operators

If we now take $\bar{A} = (L^1(\mu), L^\infty(\mu))$ and $\bar{B} = (L^1(\nu), L^\infty(\nu))$ in Theorem 3.2, we obtain the following improvement of Zygmund's result (see [5]).

Theorem 4.3

Let T be a sublinear operator satisfying that, for every $p > p_0$,

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

with constant p. Then,

$$\sup_{t>0} \frac{\int_0^t (Tf)_\nu^*(s)\, ds}{(1+\log^+ \frac{1}{t})} \preceq \int_1^\infty f_\mu^{**}(t) \frac{dt}{t} + \|f\|_\infty \cdot$$

And, if the norm of the operator T blows up when p tends to p_1^- instead of ∞, we also have the following result:

Theorem 4.4

Let T be a sublinear operator satisfying that

$$T : L^p(\mu) \longrightarrow L^p(\nu),$$

is bounded with constant $1/(p_1 - p)$, $(p_0 \leq p < p_1)$. *Then,*

$$\sup_{t>0} \left(\frac{\left(\int_t^\infty (Tf)_\nu^*(s)^{p_1}\, ds\right)^{1/p_1}}{1+\log^+ \frac{1}{t}} \right) \preceq \|f\|_{p_1} + \int_1^\infty \frac{\left(\int_t^\infty f_\mu^*(s)^{p_1}\, ds\right)^{1/p_1}}{t}\, dt \cdot$$

Proof. Let us take $\bar{A} = (L^1(\mu), L^{p_1}(\mu))$ and $\bar{B} = (L^1(\nu), L^{p_1}(\nu))$. Then, by Lemma 4.2 we have that the hypothesis of our Theorem 3.2 is satisfied and the result follows immediately from it, (7) and Proposition 3.2. □

3) Riesz potentials

Let us consider

$$I_\alpha = \int_{\mathbb{R}^n} \frac{f(y)}{\|x-y\|^{n-\alpha}}\, dy \cdot$$

It is known (see [12], p. 117 or [1], p. 228) that

$$I_\alpha : L^p \longrightarrow L^q,$$

where $1/q = 1/p - \alpha/n$, $1 < p$ and $q < \infty$. Moreover, the constant of the above operator behaves like $1/(p-1)$ when p is near 1, and like q when q is near ∞.

To apply our results, observe that if $p = 1$, $1/q = 1-\alpha/n$ and if $q = \infty$, $1/p = \alpha/n$. Therefore, we consider the pairs

$$\bar{A} = (L^1, L^{n/\alpha}) \quad \text{and} \quad \bar{B} = (L^{n/(n-\alpha)}, L^\infty).$$

Now, if $\theta = (1-1/p)/(1-\alpha/n)$, we have by Lemma 4.2, that $\Lambda(\varphi_\theta; \bar{A}) \subset L^p$, and, by Lemma 4.1 that $L^{n/n-\alpha} \subset M(\varphi_\theta; \bar{B})$. Moreover, the constants of the above embeddings do not blow up when p tends to either 1 or n/α.

Consequently,

$$I_\alpha : \bar{A}_{\theta,1} \longrightarrow \bar{B}_{\theta,\infty},$$

with constant $C/\theta(1-\theta)$, for every θ, and, hence, as an immediate consequence of Theorems 3.1 and 3.2, we obtain the following end-point estimates for the Riesz potential:

Theorem 4.5

The Riesz potential I_α satisfies the following estimates:

$$\sup_{t>0} \frac{\left(\int_0^t (I_\alpha f)^*(s)^{n/(n-\alpha)}\, ds\right)^{(n-\alpha)/n}}{1+\log^+ t} \preceq \|f\|_1 + \int_0^1 \left(\int_{t^{n/(n-\alpha)}}^\infty f^*(s)^{n/\alpha}\, ds\right)^{\alpha/n} dt$$

and,

$$\sup_{t>0} \frac{\left(\int_0^{t^{n/(n-\alpha)}} (I_\alpha f)^*(s)^{n/(n-\alpha)}\, ds\right)^{(n-\alpha)/n}}{t\left(1+\log^+ \frac{1}{t}\right)} \preceq \|f\|_{n/\alpha} + \int_1^\infty \frac{\left(\int_t^\infty f^*(s)^{n/\alpha}\, ds\right)^{\alpha/n}}{t} dt\cdot$$

4) Sobolev Embeddings

Let Ω be any domain in $\mathbb{R}^n$ and let $W^{1,p}(\Omega)$ be the classical Sobolev space

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_p + \|\nabla f\|_p,$$

where ∇f is the gradient of f, and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Then, it is known (see for example [8], p. 149) that, for every $1 \le p < n$

$$W_0^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega),$$

with constant $C/(n-p)^{(n-1)/n}$. Then, using our results we have the following end-point estimates:

Theorem 4.6

$$\left(\sup_{t>0} \frac{\left(\int_0^t f^{**}(s)^{n/(n-1)}\, ds\right)}{t(1+\log^+ \frac{1}{t})}\right)^{(n-1)/n} \preceq \|f\|_{W^{1,n}(\Omega)} + \int_1^\infty \frac{\left(\int_t^\infty \left(f^{**}(s) + |\nabla f|^{**}(s)\right)^n ds\right)^{1/n}}{t(1+\log^+ t)^{1/n}\, dt\cdot}$$

Proof. Let $\bar{A} = (W_0^{1,1}(\Omega), W_0^{1,n}(\Omega))$ and $\bar{B} = (L^{n/(n-1)}, L^\infty)$. Then, it is known that

$$K(t, f; \bar{B}) \approx \left(\int_0^{t^{n/(n-1)}} f^{**}(s)^{n/(n-1)}\, ds \right)^{(n-1)/n}$$

and, using Holmstedt's formula, we have that if $\bar{W} = (W_0^{1,1}(\Omega), W_0^{1,\infty}(\Omega))$

$$K(t, f; \bar{A}) \approx t \left(\int_{t^{n/(n-1)}}^{\infty} \Big(\frac{K(f, s; \bar{W})}{s} \Big)^n\, ds \right)^{1/n},$$

where, (see [7]),

$$K(t, f; \bar{W}) \approx t \Big(f^{**}(t) + |\nabla f|^{**}(t) \Big).$$

As in Lemmas 4.1 and 4.2, one can easily see that, if $\theta = n'/p'$,

$$L^{np/(n-p)} \subset \bar{B}_{\theta,\infty}$$

and

$$\bar{A}_{\theta,1} \subset W_0^{1,p}(\Omega),$$

both embeddings with constant independent of θ. From this and Proposition 3.2, the result follows immediately from Theorem 3.2. □

It is important to mention that, when the domain Ω is bounded, the right hand side of Theorem 4.6 reduces to $\|f\|_{W^{1,n}(\Omega)}$ and, that, in this case our result recovers the well known Trudinger's Sobolev embedding theorem.

Moreover, as far as we know, our estimates are new, for unbounded domains.

5. Relationship with the method of Jawerth and Milman

In the theory of Jawerth and Milman (see [9]), the authors work with operators T such that

$$T : \bar{A}_{\mu_\theta,1;J} \longrightarrow \bar{B}_{\sigma_\theta,\infty;K}$$

is bounded with constant $\|T\|_\theta$.

First of all, the space $\bar{B}_{\sigma_\theta,\infty;K} = M(\psi_\theta; \bar{B})$, where $\psi_\theta(t) = t/\sigma_\theta(t)$. However, the first difference of their method with ours is that for the domain spaces they work with the J-method of interpolation, while we consider the spaces $\Lambda(\varphi_\theta; \bar{A})$ which are easily seen to be K-spaces.

In the classical real case when $\mu_\theta(t) = \sigma_\theta(t) = t^\theta$, we get that

$$\bar{A}_{\mu_\theta,1;J} = \bar{A}_{\theta,1;K} = \Lambda(\varphi_\theta; \bar{A}),$$

with $\varphi_\theta(t) = t^{1-\theta}$ and, hence, both theories coincide, although the techniques developed are completely different.

Now, following the notation introduced in Remark 2.1, let

$$D_\Sigma(s) = h(1,s), \qquad R_\Sigma(t) = R_{D_\Sigma}(t)$$

and

$$R_\Delta(t) = \frac{t}{h(t,1)}, \qquad D_\Delta(s) = D_{R_\Delta}(s).$$

Then, the following results hold.

Theorem 5.1

$$\sum_{\theta\in\Theta} \{\|T\|_\theta \Lambda(\varphi_\theta;\overline{A})\} = \Lambda(D_\Sigma;\overline{A})$$

Proof. To show the first embedding, it is enough to see that

$$\Lambda(\varphi_\theta;\overline{A}) \subset \Lambda(D_\Sigma;\overline{A}),\ \theta\in\Theta,$$

with constant $\|T\|_\theta$. Now, by definition of D_Σ, we have that

$$D_\Sigma(s) \le \|T\|_\theta \varphi_\theta(s)$$

or equivalently

$$\int_0^t dD_\Sigma(s) \le \|T\|_\theta \int_0^t d\varphi_\theta(s)$$

and by Hardy's lemma

$$\begin{aligned}\|a\|_{\Lambda(D_\Sigma;\overline{A})} &\le \int_0^\infty k(a,s;\overline{A})dD_\Sigma(s) \le \|T\|_\theta \int_0^\infty k(a,s;\overline{A})d\varphi_\theta(s)\\ &= \|T\|_\theta \|a\|_{\Lambda(\varphi_\theta;\overline{A})},\end{aligned}$$

from which the result follows.

To prove the opposite embedding, it is enough by Proposition 2.3, to prove it for a characteristic pair. Let (a,E) be a characteristic pair of $\bar{A}$. Then, for every $\theta\in\Theta$,

$$\|a\|_{\sum_{\theta\in\Theta}\{\|T\|_\theta\Lambda(\varphi_\theta;\overline{A})\}} \preceq \|T\|_\theta \|a\|_{\Lambda(\varphi_\theta;\overline{A})} \le \|T\|_\theta \varphi_\theta(|E|),$$

and taking infimum in θ, we get

$$\|a\|_{\sum_{\theta\in\Theta}\{\|T\|_\theta\Lambda(\varphi_\theta;\overline{A})\}} \le \inf_\theta \{\|T\|_\theta\varphi_\theta(|E|)\} = D_\Sigma(|E|). \ \square$$

Proposition 5.1

The space $\Lambda(D_\Sigma;\overline{A})$ is the least minimal Lorentz space so that

$$\Lambda(\varphi_\theta;\overline{A}) \subset \Lambda(D_\Sigma;\overline{A}),$$

with norm less than or equal to $\|T\|_\theta$.

Proof. Let ϕ be such that, for every $\theta \in \Theta$,

$$\Lambda\big(\varphi_\theta;\overline{A}\big) \subset \Lambda\big(\phi;\overline{A}\big),$$

with norm less than or equal to $\|T\|_\theta$. Then, if (a,E) is a characteristic pair for $\bar{A}$, we get that

$$\int_0^\infty k(a,s;\overline{A})d\phi(s) \le \|T\|_\theta \int_0^\infty k(a,s;\overline{A})d\varphi_\theta(s) \preceq \|T\|_\theta\varphi_\theta(|E|)$$

and hence,

$$\|a\|_{\Lambda(\phi;\overline{A})} = \int_0^\infty k(a,s;\overline{A})d\phi(s) \preceq \inf_\theta \{\|T\|_\theta\varphi_\theta(|E|)\} = D_\Sigma(|E|).$$

Therefore

$$\Lambda(D_\Sigma;\overline{A}) \subset \Lambda(\phi;\overline{A}). \ \square$$

Now, the Σ-method asserts that (under appropriate conditions)

$$T : \sum_{\theta\in\Theta} \|T\|_\theta\Lambda(\varphi_\theta;\bar{A}) \longrightarrow \sum_{\theta\in\Theta} M(\varphi_\theta;\bar{B}),$$

and hence it is interesting to know the relation between the range spaces in both theories. That is, between $\sum_{\theta\in\Theta} M(\varphi_\theta;\bar{B})$ and $M(R_\Sigma;\bar{B})$.

Proposition 5.2

If $M(\rho;\overline{B})$ satisfies that $M(\psi_\theta;\overline{B}){\subset}M(\rho;\overline{B})$, $\theta \in \Theta$, with constant 1, then

$$M(R_\Sigma;\overline{B}) \subset M(\rho;\overline{B})$$

Proof. We have that

$$\rho(t) \le \inf_\theta \psi_\theta(t) := \lambda(t)$$

and since

$$\lambda(t) \le \psi_\theta(t) = \frac{\|T\|_\theta\varphi_\theta(s)t}{\frac{\|T\|_\theta\varphi_\theta(s)t}{\psi_\theta(t)}} \le \frac{\|T\|_\theta\varphi_\theta(s)t}{\inf_\theta \frac{\|T\|_\theta\varphi_\theta(s)t}{\psi_\theta(t)}} = \frac{\|T\|_\theta\varphi_\theta(s)t}{h(t,s)},$$

we have that

$$\lambda(t) \le \inf_\theta \frac{\|T\|_\theta\varphi_\theta(s)t}{h(t,s)} = \frac{th(1,s)}{h(t,s)},$$

and hence

$$\lambda(t) \le t\inf_s \frac{h(1,s)}{h(t,s)} = \frac{t}{\sup_s \frac{h(t,s)}{h(1,s)}}.$$

Therefore,

$$\sup_s \frac{h(t,s)}{h(1,s)} \le \frac{t}{\lambda(t)}$$

and since $\frac{t}{\lambda(t)}$ is quasi-concave, $E_\Sigma(t) \preceq \frac{t}{\lambda(t)}$, where $R_\Sigma(t) = t/E_\Sigma(t)$. Consequently

$$\|a\|_{M(\rho;\overline{B})} \le \|a\|_{M(\lambda;\overline{B})} \preceq \|a\|_{M(R_\Sigma;\overline{B})};$$

that is

$$M(R_\Sigma;\overline{B}) \subset M(\rho;\overline{B}). \ \square$$

Corollary 5.1

If $\sum_{\theta\in\Theta} M(\psi_\theta)$ is a maximal Lorentz space, then, for every pair $\bar{B}$,

$$M(R_\Sigma;\bar{B}) \subset \sum_{\theta\in\Theta} M(\psi_\theta;\bar{B}).$$

It is easy to see that, in the Yano's case, the hypothesis of the above corollary holds, since, if $1-\theta_0 = 1/p_0$ and $\varphi_\theta(t) = t^{1-\theta}$, then

$$\sum_{0<\theta\le\theta_0} M(\psi_\theta) = L^1 + L^{p_0,\infty}.$$

Moreover, by (4),

$$D_\Sigma(s) \approx s\Big(\log^+ \frac{1}{s}\Big)^\alpha + s^{1/p_0}\chi_{\{s>1\}}(s),$$

and

$$R_\Sigma(t) = \min\Big(t, t^{1/p_0}\Big),$$

and hence

$$M(R_\Sigma) = L^1 + L^{p_0,\infty} = \sum_{0<\theta\le\theta_0} M(\psi_\theta).$$

That is, the Σ-method coincide, in this particular case, with our method.

In particular, if $p_0 = \infty$, we get that

$$T : L(\log L)^\alpha \longrightarrow L^1 + L^\infty,$$

where

$$L(\log L)^\alpha = \left\{f;\ \int_0^1 f^*(t)\Big(\log\frac{1}{t}\Big)^\alpha dt < \infty\right\}.$$

Now, if we consider the space

$$L(1+\log L)^\alpha = \left\{f;\ \int_0^\infty f^*(t)\Big(1+\log^+\frac{1}{t}\Big)^\alpha dt < \infty\right\},$$

we have that $L(1+\log L)^\alpha \subset L(\log L)^\alpha$ and hence, the Σ-method asserts that, if T satisfies the hypotheses of Theorem 3.1 with $\theta_0 = 0$,

$$T : L(1+\log L)^\alpha \longrightarrow L^1 + L^\infty$$

is bounded, while with our method we can get a much smaller range space, namely

$$T : L(1 + \log L)^{\alpha} \longrightarrow M_0,$$

where, for every $p > 1$, (see [6])

$$M_0 \subset L^1 + L^{p,1} \subset L^1 + L^{p,\infty} \subset L^1 + L^{\infty},$$

and all the above embeddings are strict.

A similar argument can be done for the Δ-method. That is, from Theorem 3.1 in [9], we have that

$$M(R_{\Delta}; \bar{B}) = \bigcap_{\theta \in \Theta} \left(\frac{1}{\|T\|_{\theta}} M(\psi_{\theta}; \bar{B}) \right).$$

Also, it is easily seen that $M(R_{\Delta}; \bar{B})$ is the smallest maximal Lorentz space containing $M(\Psi_{\theta}; \bar{B})$ with norm $1/\|T\|_{\theta}$ and if $\bigcap_{\theta\in\Theta} \Lambda(\varphi_{\theta})$ is a minimal Lorentz space, then

$$\bigcap_{\theta \in \Theta} \Lambda(\varphi_{\theta}; \bar{A}) \subset \Lambda(D_{\Delta}; \bar{A}). \tag{8}$$

As before, in the Zygmund's case, if $1 - \theta_1 = 1/p_1$ and $\varphi_{\theta}(t) = t^{1-\theta}$, then

$$\bigcap_{\theta_1 \leq \theta < 1} \Lambda(\varphi_{\theta}) = L^{\infty} \cap L^{p_1,1},$$

and the embedding (8) is an equivalence. Therefore, we recover, in this particular case, the Δ-method.

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