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Exchange rings satisfying the related comparability

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Abstract

In this paper, we investigate the related comparability over exchange rings. It is shown that an exchange ring R satisfies the related comparability if and only if for any regular $x \in R$, there exist a related unit $w \in R$ and a group G in R such that $wx \in G$.

Let R be an associative ring with identity. R is said to be an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M\oplus(\bigoplus_{i\in I} A_i')$ (see [2–5], [10–12], [15–16] and [20]). We know that regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero ([3, Theorem 7.2]) are all exchange rings. Very recently, Varadarajan showed that the ring of real valued continuous functions on a zero-dimensional completely regular space is an exchange ring ([19, Proposition 3.2]). We refer the reader to [2] for a survey of exchange rings.

Many authors have studied the comparability of modules over exchange rings (see [4], [6–12] and [17]). Following the first author (see [6] and [9–11]), an associative ring R

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is said to satisfy the related comparability, provided that for any idempotents $e, f \in R$ with $e = 1+ab$ and $f = 1+ba$ for some $a \in (1-e)R(1-f)$ and $b \in (1-f)R(1-e)$, there exists a central idempotent $u \in R$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. The class of exchange rings satisfying the related comparability is very large. It includes all exchange rings satisfying the general comparability [17], all exchange rings satisfying the comparability axiom, all exchange rings with stable range one, all one-sided unit-regular rings [7–8], all right self-injective regular rings, all right or left continuous regular rings, etc. In this paper, we give some necessary and sufficient conditions on an exchange ring R under which it satisfies the related comparability.

Throughout this paper, rings are associative with identity and modules are right unital modules. Let $\mathrm{B}(R)$ denote the Boolean algebra of all central idempotents in R, $\mathrm{J}(R)$ the Jacobson radical of R, and $\mathrm{U}(R)$ the set of all units of R. If A is an R-module, the notation $B \lesssim^{\oplus} A$ means that B is isomorphic to a direct summand of A. An element $w \in R$ is called a related unit if there exists some $e \in \mathrm{B}(R)$ such that ew is right invertible in eR and $(1-e)w$ is left invertible in $(1-e)R$ (see [9]).

Theorem 1

Let R be an exchange ring. Then the following are equivalent:

(1) R satisfies the related comparability.

(2) For every regular $x \in R$, there exists a related unit $w \in R$ such that wx is an idempotent of R.

(3) For every regular $x \in R$, there exist a related unit $w \in R$ and a group G in R such that $wx \in G$.

Proof. (1)⇒(2) It is trivial by [10, Theorem 2].

(2)⇒(1) Given any regular $x \in R$, there exists a related unit $w \in R$ such that $wx = e$ is an idempotent of R. Assume $x = xyx$ for a $y \in R$. From $xy + (1-xy) = 1$, we have $ey + w(1-xy) = w$. Since $w \in R$ is a related unit, we may assume $fws = f$ and $(1-f)tw = 1-f$ for $f \in \mathrm{B}(R)$ and $s, t \in R$. So $f\big(eys + w(1-xy)s\big) = f$, hence $f\big(eys + w(1-xy)s\big)(1-e) = f(1-e)$. This implies that $f\big(e + w(1-xy)s(1-e)\big) = f\big(1 - eys(1-e)\big)$. Clearly, $\big(1 + eys(1-e)\big)\big(1 - eys(1-e)\big) = 1$. Thus we have $f\big(1 + eys(1-e)\big)w\big(x + (1-xy)s(1-e)\big) = f$. That is, $f\big(x + (1-xy)s(1-e)\big)$ is left invertible in fR. Since $f\big(xy + (1-xy)\big) = f$ in fR, by [8, Lemma 1], there exists a $z_1 \in R$ such that $f\big(y + z_1(1-xy)\big) = u$ is right invertible in fR. Therefore, $fx = xf\big(y + z_1(1-xy)\big)x = xux$.

Since $(1-f)tw = 1-f$, one easily checks that

$$\begin{aligned}
&(1-f)(1-e)w(1-xy)t(1-e)w(1-xy)\\
&= (1-f)(1-e)w(1-xy)tw(1-xy) - (1-f)(1-e)w(1-xy)twxw(1-xy)\\
&= (1-f)(1-e)w(1-xy)(1-xy) - (1-f)(1-e)w(1-xy)xw(1-xy)\\
&= (1-f)(1-e)w(1-xy).
\end{aligned}$$

Set $g = (1-f)(1-e)w(1-xy)t(1-e)$. Obviously, we have $e\big(y + w(1-xy)\big) + (1-e)w(1-xy) = ey + w(1-xy) = w$. So $(1-f)\big(e(y+w(1-xy)) + gw(1-xy)\big) = (1-f)w$,

$e = e^2$, $g = g^2$, and $eg = ge = 0$. Consequently, $(1-f)e\big(y+w(1-xy)\big) = (1-f)e\big(e(y+w(1-xy)) + gw(1-xy)\big) = (1-f)ew$ and $(1-f)gw(1-xy) = (1-f)g\big(e(y+w(1-xy)) + gw(1-xy)\big) = (1-f)gw$. Therefore $(1-f)(e+g)w = (1-f)w$, and then $(1-f)\big(e+(1-f)(1-e)w(1-xy)t(1-e)\big)w = (1-f)\big(e+(1-e)w(1-xy)t(1-e)\big)w = (1-f)\big(e(1-ew(1-xy)t(1-e))+w(1-xy)t(1-e)\big)w = (1-f)\big(e+w(1-xy)t(1-e)\big(1+ew(1-xy)t(1-e)\big)\big)\big(1-ew(1-xy)t(1-e)\big)w = (1-f)w\big(x+(1-xy)t(1-e)\big(1+ew(1-xy)t(1-e)\big)\big)\big(1-ew(1-xy)t(1-e)\big)w = (1-f)w$. From $(1-f)tw = 1-f$, we deduce that $(1-f)(x+(1-xy)t(1-e)\big(1+ew(1-xy)t(1-e)\big)\big)\big(1-ew(1-xy)t(1-e)\big)w = (1-f)$. This yields that $x+(1-xy)t(1-e)\big(1+ew(1-xy)t(1-e)\big)$ is right invertible in $(1-f)R$. In view of [8, Lemma 1], we can find $z_2 \in R$ such that $(1-f)\big(y+z_2(1-xy)\big) = v$ is left invertible in $(1-f)R$. So $(1-f)x = x(1-f)\big(y+z_2(1-xy)\big)x = xvx$. Thus $x = fx + (1-f)x = x(u+v)x$, where $u+v$ is a related unit, as required.

(2)⇒(3) It is clear because every idempotent in R is a group member.

(3)⇒(2) For any regular $x \in R$, there exist a related unit $w \in R$ and a group G in R such that $wx \in G$. Thus, the element $wx \in R$ has a group inverse $(wx)^{\#} \in R$. One easily checks that $wx\big((wx)^{\#} + 1 - (wx)^{\#}wx\big)wx = wx$ and $(wx)^{\#} + 1 - (wx)^{\#}wx = \big(wx + 1 - (wx)^{\#}wx\big)^{-1} \in \mathrm{U}(R)$. Clearly, $\big((wx)^{\#} + 1 - (wx)^{\#}wx\big)w$ is a related unit and $\big((wx)^{\#} + 1 - (wx)^{\#}wx\big)wx$ is an idempotent of R, as asserted. □

Corollary 2

Let R be an exchange ring. Then the following are equivalent:

(1) R satisfies the related comparability.

(2) For every regular $x \in R$, there exists a related unit $w \in R$ such that $x \in wG$ for some group G in R.

Proof. (1)⇒(2) Given any regular $x \in R$, there exists a $y \in R$ such that $x = xyx$. Since $yx+(1-yx) = 1$, by [10, Theorem 4] and [7, Proposition 1], we have a $z \in R$ such that $x + z(1-yx) = w \in R$ is a related unit. Hence $x = xyx = \big(x + z(1-yx)\big)yx = w(yx) \in wG$ for a group G in R.

(2)⇒(1) For any regular $x \in R$, we have a $y \in R$ such that $x = xyx$ and $y = yxy$. So there is a related unit $w \in R$ such that $y \in wG$ for a group G in R. Hence $y = wg$ for some group member $g \in R$. From $xy+(1-xy) = 1$, we see that $xwg+(1-xy) = 1$. Clearly, $g \in R$ is unit-regular. Assume that $g = ue$ with $u \in \mathrm{U}(R)$ and $e = e^2 \in R$. We have $xwue + (1-xy) = 1$, so $e + (1-e)(1-xy) = 1 - (1-e)xwue$. Set $z = wu(1-e)$. Then $wg + z(1-xy) = wu(1+(1-e)xwue)^{-1} \in R$ is a related unit. Therefore $wu(1+(1-e)xwue)^{-1}x = \big(wg + z(1-xy)\big)x = yx$ is an idempotent of R. By virtue of Theorem 1, we conclude that R satisfies the related comparability. □

Corollary 3

Let R be an exchange ring. Then the following are equivalent:

(1) R satisfies the related comparability.

(2) For every regular $x \in R$, there exists a related unit $w \in R$ such that $x \in xwxR \cap Rxwx$.

Proof. (1)⇒(2) is trivial by [10, Theorem 2].

(2)⇒(1) For any regular $x \in R$, we have $x \in xwxR \bigcap Rxwx$ for a related unit $w \in R$. Thus, there exist $s, t \in R$ such that $x = xwxs$ and $x = txwx$. It is easy to verify that $wx = (wx)^2 s \in (wx)^2 R$. Furthermore, we see that $wx = wtxwx = wt(txwx)wx = wt^2x(wx)^2 \in R(wx)^2$. Hence $wx \in (wx)^2R \cap R(wx)^2$. It follows by [15, Theorem 1] that wx is a group member in R. Using Theorem 1, we complete the proof. □

Analogously to the consideration in [11, Corollary 9], we claim that an exchange ring R satisfies the related comparability if and only if whenever $eR \cong fR$ with idempotents $e, f \in R$, there exists a related unit $w \in R$ such that $ew = wf$.

Let $e \in \mathrm{B}(R)$ and $f : M \to N$ be an R-module homomorphism. We denote by f_e the homomorphism $Me \to Ne$ given by $f_e(me) = f(m)e$ for $m \in M$.

Lemma 4

Let R be an exchange ring. Suppose for any right R-module decompositions $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong R^n \cong A_2$, there exist $C, D, E \leq M$ such that $M = C \oplus D \oplus B_1 = C \oplus E \oplus B_2$ with $De = 0$ and $E(1-e) = 0$ for some $e \in \mathrm{B}(R)$. Then the matrix ring $M_n(R)$ is an exchange ring satisfying the related comparability.

Proof. Clearly, $M_n(R)$ is an exchange ring. Take $M = R^n \oplus R^n$. With respect to this decomposition, let $p_i : M \to R^n$ be the projections and $q_i : R^n \to M$ the injections for $i = 1, 2$. Set $A_1 = q_1(R^n)$ and $B_1 = \mathrm{Ker} p_1$. Then $M = A_1 \oplus B_1$ with $A_1 \cong R^n$. Suppose $ax + b = 1$ in $\mathrm{End}_R R^n$. Set $f = ap_1 + bp_2$ and $g = q_1x + q_2$. We easily check that $fg = ax + b = 1$. Thus, $M = g(R^n) \oplus \mathrm{Ker} f$. Let $A_2 = g(R^n)$ and $B_2 = \mathrm{Ker} f$. Then $M = A_2 \oplus B_2$ with $A_2 \cong R^n$. By hypothesis, we can find $C, D, E \leq M$ such that $M = C \oplus D \oplus B_1 = C \oplus E \oplus B_2$ with $De = 0$ and $E(1-e) = 0$ for some $e \in \mathrm{B}(R)$.

Since $De = 0$, we have $Me = Ce \oplus B_1e = Ce \oplus Ee \oplus B_2e$. Let $k : Ce \to Me$ be the injection from Ce to Me. Then $Ce = k(Ce)$, and so $Me = k(Ce) \oplus (\mathrm{Ker} p_1)e = k(Ce) \oplus \mathrm{Ker}(p_1)_e$. Therefore, $(p_1)_e k$ is an isomorphism. On the other hand, we have $Me = k(Ce) \oplus Ee \oplus \mathrm{Ker} f_e$. Let $h : R^n e \cong A_2 e \cong Ce \oplus Ee \to Ce \cong A_1 e \cong R^n e$ be the projection. It can be verified that $Ee \oplus \mathrm{Ker} f_e = \mathrm{Ker}(hf_e)$. So $Me = k(Ce) \oplus \mathrm{Ker}(hf_e)$. Hence, $hf_e k$ is an isomorphism, and then $h\big(a_e(p_1)_e k + b_e(p_2)_e k\big)$ is an isomorphism. Consequently, $a_e + b_e(p_2)_e k\big((p_1)_e k\big)^{-1}$ is left invertible in $\mathrm{End}_R R^n e$. So $ae^* + bp_2k'e^*$ is left invertible in $(\mathrm{End}_R R^n)e^*$, where $e^* : R^n \to R^n$ is given by $e^*(x_1, \cdots, x_n) = (ex_1, \cdots, ex_n)$. Clearly, $e^* \in \mathrm{B}(\mathrm{End}_R R^n)$ because $e \in \mathrm{B}(R)$. Likewise, $a(1-e^*) + bp_2 l(1-e^*)$ is right invertible in $(\mathrm{End}_R R^n)(1-e^*)$. Thus, we show that $a + bp_2(k'e^* + l(1-e^*)) \in \mathrm{End}_R R^n$ is a related unit. By [10, Theorem 4], $M_n(R) \cong \mathrm{End}_R R^n$ satisfies the related comparability. □

In [11, Theorem 4], the author claimed that related comparability over exchange rings is Morita invariant. Unfortunately, there is a gap in that proof. Now we prove this fact by a new route as follows.

Lemma 5

If R satisfies the related comparability, then so does $M_n(R)$.

Proof. Suppose $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong R^n \cong A_2$. Then we have decompositions $A_1 = A_{11} \oplus \cdots \oplus A_{1n}$ and $A_2 = A_{21} \oplus \cdots \oplus A_{2n}$ with all A_{ij} isomorphic to R. By virtue of [10, Theorem 4], we can find $C_1, D_1, E_1 \subseteq M$ such that $M = C_1 \oplus D_1 \oplus (A_{12} \oplus \cdots \oplus A_{1n} \oplus B_1) = C_1 \oplus E_1 \oplus (A_{22} \oplus \cdots \oplus A_{2n} \oplus B_2)$ with $D_1 e_1 = 0$ and $E_1(1 - e_1) = 0$ for some $e_1 \in \mathrm{B}(R)$. We easily check that $D_1 \oplus A_{12} \cong R \cong E_1 \oplus A_{22}$. Using [10, Theorem 4] again, we have $C_2, \cdots, C_n, D_n, E_n \leq M$ such that $M = (C_1 \oplus \cdots \oplus C_n) \oplus D_n \oplus B_1 = (C_1 \oplus \cdots \oplus C_n) \oplus E_n \oplus B_2$ with $D_n e_n = 0$ and $E(1 - e_n) = 0$ for some $e_n \in \mathrm{B}(R)$. In view of Lemma 4, we complete the proof. □

As an immediate consequence of the lemma above, we show that if R is an exchange ring satisfying the related comparability, then so is $\mathrm{End}_R A$ for any finitely generated projective right R-module A because the related comparability over exchange rings can be inherited by their corner rings.

Theorem 6

Let R be an exchange ring and n a positive integer. Then the following are equivalent:

(1) R satisfies the related comparability.

(2) For every regular $A \in M_n(R)$, there exist an idempotent matrix $E \in M_n(R)$ and a related unit $W \in M_n(R)$ such that $A = EW$.

(3) For every regular $A \in M_n(R)$, there exists a related unit $W \in M_n(R)$ such that WA is an idempotent matrix.

Proof. (1)⇔(2) By Lemma 5 and [10, Corollary 3], we know that R satisfies the related comparability if and only if so does $\mathrm{M}_n(R)$. Thus, the result follows by an argument of the first author [11, p. 4213].

(1)⇔(3) In view of Theorem 1, Lemma 5 and [10, Corollary 3], we immediately obtain the result. □

Corollary 7

Let R be a regular ring and n a positive integer. Then the following are equivalent:

(1) R satisfies the related comparability.

(2) For every $A \in M_n(R)$, there exist an idempotent matrix $E \in M_n(R)$ and a related unit $W \in M_n(R)$ such that $A = EW$.

(3) For every $A \in M_n(R)$, there exists a related unit $W \in M_n(R)$ such that WA is an idempotent matrix.

Proof. Since R is regular, so is $\mathrm{M}_n(R)$. Hence the result follows by Theorem 6. □

Now we extend [7, Theorem 10] and [15, Theorem 4] as follows.

Theorem 8

Let R be an exchange ring. Then the following are equivalent:

(1) R is either a local ring or a Boolean ring.

(2) For every nonzero regular $x \in R$, there exists a unique related unit $w \in R$ such that wx is an idempotent of R.

Proof. (1)⇒(2) If R is a local ring, then each idempotent of R is trivial and each related unit of R is a unit. If R is a Boolean ring, then each related unit is trivial. Thus, we claim that for every nonzero regular $x \in R$, there exists a unique related unit $w \in R$ such that wx is an idempotent.

(2)⇒(1) For any nonzero regular $x \in R$, there exists a unique related unit $w \in R$ such that wx is an idempotent. Assume $x = xyx$, $ews = e$, and $(1-e)tw = 1-e$, where $y, s, t \in R$ and $e \in \mathrm{B}(R)$. As $wxwx = wx$, we have $x(1-e)wx = (1-e)x$. We also have $wxy + w(1-xy) = w$. Thus, $(ewx)ys + ew(1-xy)s = ews = e$, and then $ewxys(1-wx) + ew(1-xy)s(1-wx) = e - ewx$. Therefore, we see that $ewx + ew(1-xy)s(1-wx) = e - ewxys(1-wx)$. Consequently, $\big(1 + wxys(1-wx)\big)ew\big(x + (1-xy)s(1-wx)\big) = e$. By virtue of [8, Proposition 1], we can find $z \in R$ such that $ey + ez(1-xy) = ed$ is right invertible in eR. Hence, $ex = exyx = x\big(ey + ez(1-xy)\big)x = x(ed)x$, and then $x = ex + (1-e)x = x\big(ed + (1-e)w\big)x$. Since $ed + (1-e)w$ is a related unit and $\big(ed + (1-e)w\big)x$ is an idempotent of R, by the uniqueness, we know that $w = ed + (1-e)w$. Thus, $x = xwx$, and wx is an idempotent.

Since $1 - x(1-xw) = \big(1 + x(1-xw)\big)^{-1} \in \mathrm{U}(R)$, $w\big(1 - x(1-xw)\big)$ is a related unit. By the uniqueness, we know that $w = w\big(1 - x(1-xw)\big)$. Hence, $wx = wx^2w$. On the other hand, one can verify that $\big(1 - (1-wx)x\big)wx = wx - x + wx^2$ and $\big(wx - x + wx^2\big)^2 = wx - x + wx^2$. Clearly, $\big(1 - (1-wx)x\big)w$ is a related unit because $1 - (1-wx)x = \big(1 + (1-wx)x\big)^{-1} \in \mathrm{U}(R)$. Consequently, $w = \big(1 - (1-wx)x\big)w$. Therefore, $xw = wx^2w$.

Inasmuch as $wx = wx^2w$ and $(1-e)tw = 1-e$, it follows that $x(1-e) = x^2(1-e)w$. Likewise, $xe = ewx^2 = x^2ew$ because $wx = xw = wx^2w$. Consequently, $x = xe + x(1-e) = x^2w$ is strongly π-regular. Since R is an exchange ring, x is unit-regular. According to [20, Theorem 3], R has stable range one, hence it is directly finite. So $w \in \mathrm{U}(R)$. Assume $x \neq x^2$. Set $u = wxw$. It is easy to verify that $x(u + 1 - xu)x = x$ with $u + 1 - xu = (x + 1 - xu)^{-1} \in \mathrm{U}(R)$. By the uniqueness, we have $w = u + 1 - xu = wxw + 1 - xw$, hence $w(1-xw) = 1 - xw$ is an idempotent. By the uniqueness again, we know that $w = 1$ if $1 - xw \neq 0$. Clearly, $w = 1$ implies $x = x^2$, a contradiction. So $xw = 1$, and then $x \in \mathrm{U}(R)$. Therefore, every nonzero regular element of R is an idempotent or a unit.

Assume $a \in R$ is a regular element but not an idempotent, and $x, y \in R$ are nonzero regular elements such that $xy = 0$. By the consideration above, we see that $a \in \mathrm{U}(R)$, ax is regular, and x, y are idempotents. If $ax = (ax)^2$, then $x = xax$, which implies $a = 1$, a contradiction. If $ax \in \mathrm{U}(R)$, then $x \in \mathrm{U}(R)$. This implies $y = 0$, a contradiction. Therefore, we conclude that either all regular elements of R are idempotents, or for any regular $x, y \in R$, $xy = 0$ implies $x = 0$ or $y = 0$.

Suppose there exist regular elements which are not idempotents. Given any $a \in R$, there is $f = f^2 \in R$ such that $f = am$ and $1 - f = (1-a)n$ for some $m, n \in R$. As $f(1-f) = 0$, we see that $f = 0$ or $f = 1$. Thus, a or $1 - a$ is right invertible in R. In this case, R is a local ring. That is, R is a local ring or every regular element in R is an idempotent.

Assume that every regular element in R is an idempotent. Then the group of units of R is trivial, and so $\mathrm{J}(R) = 0$, and also R does not have nonzero nilpotent elements. If e is an idempotent in R, every element in $eR(1-e)$ is nilpotent and so $eR(1-e) = (1-e)Re = 0$ for every idempotent e in R. It follows that all the idempotents of R are central. Note that the property that every regular element is idempotent is inherited by any factor ring R/I. This is due to the fact that regular elements lifted modulo I, see [1, Lemma 2.1]. Now take any primitive ideal P of R. Then R/P is a primitive exchange ring with all idempotents central. Therefore, its unique idempotents are 0 and 1. It follows that R/P is a division ring and then R/P must be in fact the field with 2 elements. Since R is semiprimitive we conclude that R is a Boolean ring, as asserted. □

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