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Harmonic Bergman spaces with small exponents in the unit ball

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Abstract

The dual of the weighted harmonic Bergman space $h^p_\alpha(\mathbb{B})$ is shown to be the harmonic Bloch space under certain volume integral paring for $0 < p < 1$ and $-1 < \alpha < \infty$ on the unit ball of $\mathbb{R}^n$.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball in $\mathbb{R}^n$, $n \geq 2$. The *weighted harmonic Bergman space* $h^p_\alpha = h^p_\alpha(\mathbb{B})$ for $p > 0$ and $\alpha > -1$ is the space of harmonic functions in $L^p(\mathbb{B}, d\nu_\alpha)$, where $d\nu_\alpha(x) = (1 - |x|^2)^\alpha dx$ and dx is the normalized Lebesgue measure in $\mathbb{B}$. For any $f \in L^p(\mathbb{B}, d\nu_\alpha)$, write

$$||f||_{p,\alpha} = \left[\int_{\mathbb{B}} |f(x)|^p d\nu_\alpha(x)\right]^{1/p}.$$

If $p = \infty$, we denote h^∞ the set of all bounded harmonic functions in $\mathbb{B}$. It is a subset of *harmonic Bloch space* $\mathcal{B}$, which consists of all harmonic function f on $\mathbb{B}$ such that

$$||f||_{\mathcal{B}} = \sup_{x\in\mathbb{B}} \left(1 - |x|^2\right)\left|\nabla f(x)\right| < \infty.$$

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To study the weighted harmonic Bergman spaces in the unit ball, Coifman and Rochberg [4] constructed the continuous projection P_α of $L^p(\mathbb{B}, d\nu_\alpha)$ onto h^p_α for $p \in (1, \infty)$ and $\alpha \in \mathbb{N}$. (For the case of $\alpha \in (-1, \infty)$ or more general weights, we refer to see [5] and [3].) As a consequence, the following duality can be obtained (see [3] for example):

$$h^{p\,*}_\alpha = h^q_\alpha$$

for $p \in (1, \infty)$ and $q = p/(p-1)$, under the paring

$$< f, g >= \int_{\mathbb{B}} f(x)\overline{g(x)} d\nu_\alpha(x), \qquad f \in h^p_\alpha, \quad g \in h^q_\alpha.$$

By extending the result of Coifman and Rochberg to the case of $p = 1$, Djrbashian and Shamoian [5] proved that $h^{1\,*}_\alpha = \mathcal{B}$; see also [13].

The purpose of this paper is to consider the remaining cases, namely identifying the dual spaces of h^p_α when $p \in (0, 1)$ and $\alpha \in (-1, \infty)$.

Let $0 < p < 1$ and $-1 < \alpha < \infty$. The space $L^p(\mathbb{B}, d\nu_\alpha)$ is a quasi-Banach space; i.e. it is a complete metric space with the metric $d(f, g) = ||f - g||^p_{p,\alpha}$ satisfying the properties $d(f, g) = d(f-g, 0)$ and $d(\lambda f, 0) = |\lambda|^p d(f, 0)$ for $\lambda \in \mathbb{C}$. The Bergman space h^p_α is closed in $L^p(\mathbb{B}, d\nu_\alpha)$. Let $h^{p\,*}_\alpha$ denote the space of all bounded linear functionals on h^p_α. Then $h^{p\,*}_\alpha$ is a Banach space with the norm

$$||F|| = \sup\{|F(f)| : ||f||_{p,\alpha} \leq 1\}.$$

Our main result is the following theorem.

Theorem A

Suppose $0 < p < 1$ *and* $-1 < \alpha < \infty$. *Then*

$$h^{p\,*}_\alpha = \mathcal{B}$$

under the integral paring

$$< f, g >= \lim_{r \to 1^-} \int_{\mathbb{B}} f(rx)\overline{g(x)}(1 - |x|^2)^{-n+(n+\alpha)/p} dx$$

where $f \in h^p_\alpha$ *and* $g \in \mathcal{B}$.

We refer to [14] and [4] for the version of holomorphic Bergman spaces.

To prove the main result, we need to establish some basic results in the theory of weighted harmonic Bergman spaces and harmonic Bloch space in the unit ball. These will be included in Sections 2-5.

2. Bergman projections

In this section, we show that Bloch space appears as the image of the bounded functions under the weighted Bergman projections, which extends the result of Stroethoff [13] in the unweighted case.

We shall be using the following notation: for $x, y \in \mathbb{B}$, we will write in polar coordinates by $x = |x|x'$ and $y = |y|y'$.

We denote by $P(x, y')$ the Poisson kernel in $\mathbb{B}$. It is known that

$$\begin{aligned} P(x,y) &= \frac{1-|x|^2|y|^2}{||y|x-y'|^n} \\ &= \sum_{m,j} |x|^m|y|^m Y_j^m(x')Y_j^m(y'), \qquad x \in \mathbb{B},\ y \in \overline{\mathbb{B}}, \end{aligned}$$

where Y_j^m is real function and $\{Y_j^k\}_j$ is the real orthogonal basis on $\partial\mathbb{B}$ of spherical harmonics of degree k (see [2]). Any harmonic function on $\mathbb{B}$ can be represented as

$$f(x) = \sum_{m,j} f_{m,j}|x|^m Y_j^m(x'), \tag{2.1}$$

where $f_{m,j} \in \mathbb{C}$ and the convergence is uniform on compacts in $\mathbb{B}$.

For $-2 < \beta < \infty$ and $x, y \in \mathbb{B}$, we define the weighted Bergman kernels

$$Q_\beta(x,y) = \sum_{m,j} \frac{\Gamma(\beta+1+m+n/2)}{\Gamma(m+n/2)}|x|^m|y|^m Y_j^m(x')Y_j^m(y'). \tag{2.2}$$

Note that

$$Q_\beta(x,y) = Q_\beta(y,x), \quad Q_\beta(x,y) = \overline{Q_\beta(y,x)}.$$

When $\beta = -1$, we know that $Q_{-1}(x,y) = P(x,y)$. When $\beta = 0$, we denote $Q(x,y) = Q_0(x,y)$ and it is known in [2] that

$$Q(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x\cdot y - 2n - 4)|x|^2|y|^2 + n}{2||y|x-y'|^{n+2}}. \tag{2.3}$$

But in general $Q_\beta(x,y)$ has no explicit formula in closed form.

For $-1 < \beta < \infty$, we consider the corresponding integral operators, so called the weighted Bergman projections,

$$P_\beta f(x) = C_\beta \int_{\mathbb{B}} Q_\beta(x,y)f(y)d\nu_\beta(y),$$

where $C_\beta = \frac{2}{n\Gamma(\beta+1)}$.

Theorem 2.1 ([4], [5])

Suppose $1 < p < \infty$, $-1 < \alpha < \infty$ and $\beta > (1+\alpha)/p - 1$. Then P_β is a continuous projection from $L^p(\mathbb{B}, d\nu_\alpha)$ onto h^p_α.

Theorem 2.2

For any $\beta \in (-1,\infty)$, P_β is bounded from $L^\infty(\mathbb{B})$ onto $\mathcal{B}$.

Proof. The boundedness of P_β from $L^\infty(\mathbb{B})$ to $\mathcal{B}$ is given by Theorems 7.6 and 7.5 in [5]. In particular $P_\beta L^\infty(\mathbb{B}) \subset \mathcal{B}$. When $\beta = 0$, Stroethoff [13] proved that $\mathcal{B} \subset P_0 L^\infty(\mathbb{B})$. We shall show that $\mathcal{B} \subset P_\beta L^\infty(\mathbb{B})$ for any $\beta \in (-1,\infty)$.

To prove this we use Green's formula in the form

$$\int_{\mathbb{B}} (u\triangle v - v\triangle u)\, dV = \int_{\partial\mathbb{B}} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} \right) dS,$$

where dV denotes the Lebesgue volume measure on $\mathbb{B}$, dS denotes the surface-area measure on $\partial\mathbb{B}$, and $\partial/\partial\mathbf{n}$ stands for differentiation with respect to the outward unit normal.

Assume $u, v \in h^\infty$. Since $\beta + 2 > 1$, by Green's formula we have

$$\int_{\mathbb{B}} v(y) \triangle\big((1-|y|^2)^{\beta+2} u(y)\big)\, dy = 0.$$

By the direct computation

$$\triangle\big((1-|y|^2)^{\beta+2}u(y)\big) = -4(\beta+2)(1-|y|^2)^\beta \big[(1-|y|^2)R_s u(y) - (\beta+1)u(y)\big],$$

where $R_s = R + sI$, $s = \beta + 1 + n/2$, I is the identity operator, and $Ru(y) = \nabla u(y)\cdot y$ is the radial derivative. Consequently,

$$\int_{\mathbb{B}} v(y)\big[(1-|y|^2)R_s u(y) - (\beta+1)u(y)\big]\, d\nu_\beta(y) = 0.$$

Now taking $v(y) = Q_\beta(x,y)$ for fixed $x \in \mathbb{B}$ we get

$$P_\beta\big[(1-|x|^2)R_s u(x) - (\beta+1)u(x)\big] = 0.$$

Namely,

$$u(x) = P_\beta\left[\frac{1}{\beta+1}\big((1-|x|^2)R_s u(x)\big)\right], \qquad x \in \mathbb{B}, \quad u \in h^\infty. \tag{2.4}$$

Since $|Ru(x)| \le |\nabla u(x)|$, we find that $\frac{1}{\beta+1}(1-|x|^2)R_s u(x)$ is bounded in $\mathbb{B}$ for any $u \in h^\infty$. This proves $h^\infty \subset P_\beta L^\infty(\mathbb{B})$, which yields the result $\mathcal{B} \subset P_\beta L^\infty(\mathbb{B})$ by applying the same approach as in the case of $\beta = 0$ ([13, p. 59]). More precisely, let $u \in \mathcal{B}$ and apply (2.4) to its dilates u_r, defined by $u_r(x) = u(rx)$. Note that $R_s u = su + Ru$ and $(R_s u)_r = R_s u_r$, as pointed in [13],

$$(1-|x|^2)R_s u_r(x) \longrightarrow (1-|x|^2)R_s u(x)$$

in $L^2(\mathbb{B}, d\nu_\alpha)$ as $r \to 1^-$. In fact, we consider the integral

$$\int_{\mathbb{B}} \left|(1-|x|^2)R_s u(rx) - (1-|x|^2)R_s(u)\right|^2 d\nu_\alpha(x)$$

and split the integral into two parts: $\delta\mathbb{B}$ and $\mathbb{B}\backslash\delta\mathbb{B}$. Since $(1-|x|^2)R_s u(x) \in L^2(\mathbb{B}, d\nu_\alpha)$, we can make the integral over $\mathbb{B} \setminus \delta\mathbb{B}$ arbitrary small by choosing δ close enough to 1. Once δ is fixed, the integral over $\delta\mathbb{B}$ clearly approaches 0 as $r \to 1^-$. Take $\alpha \in (-1, -1 + p(\beta + 1))$. By the boundedness of P_β on $L^2(\mathbb{B}, d\nu_\alpha)$ and the fact that $u_r \to u$, we see that (2.4) holds for any $u \in \mathcal{B}$. This completes the proof. □

3. Fractional derivatives

In order to identify the dual space of h^p_α when $0 < p < 1$, we need to introduce a certain type of fractional differentiation and integration.

Let $h(\mathbb{B})$ denote the space of all harmonic functions in $\mathbb{B}$ and equip $h(\mathbb{B})$ with the topology of uniform convergence on compact subsets. Thus, a linear operator T on $h(\mathbb{B})$ is continuous if and only if $Tf_n \to Tf$ uniformly on compact subsets whenever $f_n \to f$ uniformly on compact subsets.

Lemma 3.1

For every $\beta \in (-1, \infty)$, there exists a unique continuous linear operator D^β on $h(\mathbb{B})$ such that

$$D^\beta_x[Q(x, y)] = Q_\beta(x, y), \qquad x,\ y \in \mathbb{B}.$$

Proof. Recall that

$$Q_\beta(x, y) = \sum_{m,j} \frac{\Gamma(\beta + 1 + m + n/2)}{\Gamma(m + n/2)} |x|^m |y|^m Y^m_j(x')Y^m_j(y')$$

and

$$Q(x, y) = \sum_{m,j} \frac{\Gamma(1 + m + n/2)}{\Gamma(m + n/2)} |x|^m |y|^m Y^m_j(x')Y^m_j(y').$$

If we define on monomials by

$$D^\beta\big(|x|^m Y^m_j(x')\big) = \frac{\Gamma(\beta + 1 + m + n/2)}{\Gamma(1 + m + n/2)} |x|^m Y^m_j(x') \tag{3.1}$$

for all m and j, and extend D^β linearly to the whole space $h(\mathbb{B})$, then the resulting operator D^β has the desired properties. The uniqueness follows from the series expansion of harmonic functions. This completes the proof. □

It is easy to see that the operator D^β can also be represented by

$$D^\beta f(x) = \lim_{r\to 1^-} \frac{2}{n} \int_{\mathbb{B}} Q_\beta(x, y) f(ry) dy, \qquad x \in \mathbb{B}.$$

In fact, this can be verified on monomials and so that on any harmonic functions by applying the polar coordinates formula in integration:

$$\int_{\mathbb{B}} f(x)dx = n \int_0^1 r^{n-1} \int_{\partial\mathbb{B}} f(r\eta)d\sigma(\eta)dr,$$

where $d\sigma$ is the normalized surface measure on $\partial\mathbb{B}$. In particular, the limit above always exists. If $f \in h^\infty$, then

$$D^\beta f(x) = \frac{2}{n}\int_{\mathbb{B}} Q_\beta(x,y)f(y)dy.$$

Lemma 3.2

For every $\beta \in (-1,\infty)$, the operator D^β is invertible on $h(\mathbb{B})$.

Proof. We define an operator D_β on monomials by

$$D_\beta\big(|x|^m Y_j^m(x')\big) = \frac{\Gamma(1+m+n/2)}{\Gamma(\beta+m+n/2)}|x|^m Y_j^m(x')$$

and extend D_β linearly to the whole space $h(\mathbb{B})$. Then the operator D_β is continuous linear operator on $h(\mathbb{B})$, and it is the inverse of D^β. □

When $\beta > 0$, the operator D^β and D_β can be considered as a fractional derivative and integral of order β respectively.

Lemma 3.3

Suppose $-1 < \beta < \infty$ and f, g are bounded harmonic functions in $\mathbb{B}$. Then

$$\int_{\mathbb{B}} f(x)\overline{g(x)}dx = \frac{1}{\Gamma(\beta+1)}\int_{\mathbb{B}} D^\beta f(x)\overline{g(x)}(1-|x|^2)^\beta dx.$$

Proof. The desired identity follows from the integral form of D^β, the reproducing property of P_β, and Fubini's theorem. □

4. Estimate of Bergman kernels

In this section, we present the estimates on the gradient of fractional derivative of weighted Bergman kernels. These improve and extend the corresponding results in [5] and [10].

Theorem 4.1

Let $-1 < \alpha, \beta < \infty$. There exists a positive constant C such that for any $x, y \in \mathbb{B}$

(i) $|D_x^\beta Q_\alpha(x,y)| \le C\big||y|x - y'\big|^{-(n+\alpha+\beta)}$;

(ii) $|\nabla_x D_x^\beta Q_\alpha(x,y)| \le C\big||y|x - y'\big|^{-(n+\alpha+\beta+1)}$.

To prove this theorem, we need some lemmas.

Lemma 4.2

Let $0 < \delta < \lambda$. *Then, for any points* x *and* y *in* $\mathbb{B}$,

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}} dt \leq \frac{8^{\lambda}}{\delta(\lambda-\delta)} \frac{1}{||y|x-y'|^{\lambda-\delta}}.$$

Proof. Note that for any $t \in [0,1]$ and $x, y \in \mathbb{B}$

$$||y|x-y'| \leq 2|t|y|x-y'|. \tag{4.1}$$

Indeed, from the triangle inequality we have

$$\begin{aligned} &|t|y|x-y'| \geq 1-t, \\ &|t|y|x-y'| \geq ||y|x-y'| - (1-t), \end{aligned} \tag{4.2}$$

so that summing them up to yield (4.1).

If $||y|x-y'| \geq 1$, then from (4.1) we have $|t|y|x-y'| \geq 1/2$. Combining this with the inequality $||y|x-y'| \leq 2$, we have

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}} dt \leq \frac{2^{\lambda}}{\delta} \leq \frac{2^{2\lambda-\delta}}{\delta} \frac{1}{||y|x-y'|^{\lambda-\delta}}.$$

Now assume $||y|x-y'| < 1$ and denote $r = 1 - ||y|x-y'|$, then $0 < r < 1$ and $1 - r = ||y|x-y'|$. From (4.1) and (4.2) we have

$$1 - rt = 1 - t + t||y|x-y'| \leq 3|t|y|x-y'|.$$

As a result

$$\begin{aligned} \int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}} dt &\leq 3^{\lambda} \int_0^1 \frac{(1-t)^{\delta-1}}{(1-tr)^{\lambda}} dt \\ &\leq 3^{\lambda} \frac{\lambda}{\delta(\lambda-\delta)} \frac{1}{(1-r)^{\lambda-\delta}} \\ &\leq C \frac{1}{||y|x-y'|^{\lambda-\delta}}. \end{aligned}$$

This completes the proof. □

Let F be the hypergeometric function (see [6], [12]):

$$F(a, b; c; s) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} s^k$$

for $a, b, c \in \mathbb{R}$ and c neither zero nor a negative integer, where the Pochhammer symbol $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \in \mathbb{N}$. We need some known properties of hypergeometric functions:

(i) Bateman's integral formula

$$F(a,b;c+\mu;s)=\frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)}\int_0^1 t^{c-1}(1-t)^{\mu-1}F(a,b;c;ts)dt \tag{4.3}$$

with $c,\mu>0$ and $s\in(-1,1)$.

(ii) For any integer m ([12, p. 69])

$$\begin{aligned} F(-m,b;c;1)&=\frac{(c-b)_m}{(c)_m},\\ F(-m,a+m;c;1)&=\frac{(-1)^m(1+a-c)_m}{(c)_m}. \end{aligned} \tag{4.4}$$

The following identity furnishes the hypergeometric function with an integral representation; see [11, p. 40] for the special case.

Lemma 4.3

Let $t>1$, $\lambda\in\mathbb{R}$ and $r\in(-1,1)$, then

$$\int_{-1}^1 \frac{(1-u^2)^{(t-3)/2}}{(1-2ru+r^2)^\lambda}du=\frac{\Gamma(\frac{t-1}{2})\Gamma(\frac12)}{\Gamma(\frac t2)}F\left(\lambda,\lambda+1-\frac t2;\frac t2;r^2\right). \tag{4.5}$$

Proof. Let $C_m^\lambda(u)$ be the Gegenbauer polynomials. They can be defined by the generating function

$$(1-2ru+r^2)^{-\lambda}=\sum_{m=0}^\infty C_m^\lambda(u)r^m, \tag{4.6}$$

where

$$\begin{aligned} C_{2m}^\lambda(u)&=(-1)^m\frac{(\lambda)_m}{m!}F(-m,m+\lambda;1/2;u^2),\\ C_{2m+1}^\lambda(u)&=(-1)^m\frac{(\lambda)_m}{m!}2uF(-m,m+\lambda+1;3/2;u^2). \end{aligned} \tag{4.7}$$

To calculate the integral in (4.5), we apply (4.6) and (4.7). Then we deduce that it is only needed to evaluate the integral

$$\int_{-1}^1 (1-u^2)^{(t-3)/2}F(-m,m+\lambda;1/2;u^2)du,$$

or rather, an integral over the interval $(0,1)$ by the simple change of variables $t=u^2$. For this integral, we first use Bateman's integral formula (4.3) with $s=1$ then apply (4.4), so that it can be represented by Pochhammer symbols. What comes out of the calculation of the integral in (4.5) then turns out to be a series which by the definition is a hypergeometric function as desired. □

Lemma 4.4

Let $\alpha>-1$ and $\beta\in\mathbb{R}$. Then for any $x\in\mathbb{B}$

$$\int_{\mathbb{B}}\frac{(1-|y|^2)^\alpha}{||x|y-x'|^{n+\alpha+\beta}}dy\approx\begin{cases}(1-|x|^2)^{-\beta}, & \beta>0,\\ \log\frac{1}{1-|x|^2}, & \beta=0,\\ 1, & \beta<0.\end{cases}$$

The notion $a(x)\approx b(x)$ means the ratio $a(x)/b(x)$ has a positive finite limit as $|x|\to1$.

Proof. Denote the above integral by $J_{\alpha,\beta}(x)$. From Stirling's formula we need only to show

$$J_{\alpha,\beta}(x) = \frac{\Gamma(\frac{n}{2}+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{n}{2}+1)} F\left(\frac{n+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \alpha+\frac{n}{2}+1; |x|^2\right).$$

For any continuous function f of one variable and any $\eta \in \partial\mathbb{B}$, we have the formula (see [2, p. 216])

$$\int_{\partial\mathbb{B}} f(<\zeta,\eta>) d\sigma(\zeta) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-u^2)^{(n-3)/2} f(u) du,$$

where the symbol $<\zeta,\eta>$ stands for the inner product in $\mathbb{R}^n$. Taking $f(u) = (1 - 2ru + r^2)^{-(n+\alpha+\beta)/2}$ for fixed $r \in (0,1)$ and combining with Lemma 4.3 we have

$$\begin{aligned}\int_{\partial\mathbb{B}} (1-2r<\zeta,\eta>+r^2)^{-(n+\alpha+\beta)/2} d\sigma(\zeta) &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} \frac{(1-u^2)^{(n-3)/2}}{(1-2ru+r^2)^{(n+\alpha+\beta)/2}} du \\ &= F\left(\frac{n+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \frac{n}{2}; r^2\right).\end{aligned}$$

Consequently, from the polar coordinates formula we get

$$\begin{aligned}J_{\alpha,\beta}(x) &= n\int_0^1 r^{n-1}(1-r^2)^\alpha \int_{\partial\mathbb{B}} (1-2r|x|<x',\eta>+r^2|x|^2)^{-(n+\alpha+\beta)/2} d\sigma(\zeta) \\ &= C\int_0^1 r^{n-1}(1-r^2)^\alpha F\left(\frac{n+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \frac{n}{2}; r^2|x|^2\right) dr.\end{aligned}$$

The assertion now follows from Bateman's integral formula (4.3). □

Proof of Theorem 4.1 We first consider the case $\beta = 0$ and $\alpha = k \in \{0\} \cup \mathbb{N}$. Since $Q_\alpha(x,y) = Q_\alpha(|y|x, y')$, we may assume $y \in \partial\mathbb{B}$.

Fix $y' \in \partial\mathbb{B}$ and denote $r = |x|$. We have the induced formula

$$Q_{k+1}(x,y') = \left(r\frac{\partial}{\partial r} + \left(k+1+\frac{n}{2}\right)\right) Q_k(x,y')$$

and the formula

$$x \cdot \nabla_x Q_{k+1}(x,y') = r\frac{\partial}{\partial r} Q_{k+1}(x,y').$$

Recall that $Q_{-1}(x,y) = P(x,y)$ is the Poisson kernel. Starting from $P(x,y)$, by induction on $k \in \{-1,0\} \cup \mathbb{N}$ we find

$$\begin{aligned}|\frac{\partial}{\partial r} Q_k(x,y')| &\le C|x-y'|^{-(n+k+1)}, \\ |Q_k(x,y')| &\le C|x-y'|^{-(n+k)}, \\ |\nabla_x Q_{k+1}(x,y')| &\le C|x-y'|^{-(n+k+1)}.\end{aligned}$$

It means $|Q_k(x,y)| \le C||y|x-y'|^{-(n+k)}$ and $|\nabla_x Q_{k+1}(x,y)| \le C||y|x-y'|^{-(n+k+1)}$ for $k \in \{0\} \cup \mathbb{N}$.

For non-integer α, denote $k = [\alpha]+1$. By definition we have

$$Q_\alpha(x,y) = \frac{2}{\Gamma(k-\alpha)} \int_0^1 t^{n+2\alpha+1}(1-t^2)^{k-\alpha-1} Q_k(tx,ty)dt$$

and

$$\nabla_x Q_\alpha(x,y) = \frac{2}{\Gamma(k-\alpha)} \int_0^1 t^{n+2\alpha+1}(1-t^2)^{k-\alpha-1} \nabla_x Q_k(tx,ty)dt.$$

The desired results follow from Lemma 4.2 and the estimates on Q_k and ∇Q_k.

In the above, we use induction and start from $P(x,y)$ to obtain the estimates of $Q_\alpha(x,y)$ and $\nabla_x Q_\alpha(x,y)$. In the similar way, we can use induction and start from $Q_\alpha(x,y)$ for given α to obtain the estimate on $R_\beta(x,y) \equiv D_x^\beta Q_\alpha(x,y)$ and $\nabla_x R_\beta(x,y)$. In fact, all the formulas above remain true if $Q_\alpha(x,y)$ is replaced by the corresponding $R_\beta(x,y)$. This completes the proof. $\Box$

5. An inequality

The following lemma will be needed in the proof of the main result. It is essentially due to Fefferman and Stein [7].

Lemma 5.1

For every $0 < p \le 1$ and $-1 < \alpha < \infty$, there exists a positive constant C such that

$$\int_{\mathbb{B}} |f(x)|(1-|x|^2)^{-n+(n+\alpha)/p} dx \le C||f||_{p,\alpha}$$

for all $f \in h_\alpha^p$.

To prove this result, we need to establish the subharmonic behavior of $|f|^p$ on Möbius invariant metric balls for any harmonic function f.

For any $y, w \in \mathbb{R}^n$, it is easy to verify

$$||y|w - (1-|w|^2)y'| = ||w|y - (1-|w|^2)w'|,$$

so that

$$||y|^2 w - (1-|w|^2)y| = |y|||w|y - (1-|w|^2)w'|. \tag{5.1}$$

For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in $\mathbb{B}$. It is an involution automorphism of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form (see [1])

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{||x|a - x'|^2}, \quad a, x \in \mathbb{B}. \tag{5.2}$$

From (5.1) we have

$$|\varphi_a(x)| = \frac{|x-a|}{||a|x-a'|} = \frac{|x-a|}{||x|a-x'|}. \tag{5.3}$$

By (5.2), a simple calculation yields

$$|\varphi_a(x)-a| = \frac{|x|}{||a|x-a'|}(1-|a|^2). \tag{5.4}$$

For any $a \in \mathbb{B}$ and $\delta \in (0,1)$, we denote

$$E(a,\delta) = \{x \in \mathbb{B} : |\varphi_a(x)| < \delta\},$$
$$B(a,\delta) = \{x \in \mathbb{B} : |x-a| < \delta\}.$$

Clearly, $E(a,\delta) = \varphi_a(B(0,\delta))$.

Lemma 5.2

Let $x, w \in \mathbb{B}$ and $y \in E(w,\delta)$. Then

$$\frac{1-\delta}{1+\delta}||x|w-x'| \le ||x|y-x'| \le \frac{1+\delta}{1-\delta}||x|w-x'|.$$

Proof. From (5.3) we have $|\varphi_y(w)| = |\varphi_w(y)|$, so that $y \in E(w,\delta)$ is equivalent to $w \in E(y,\delta)$. By the symmetricity, we need only to prove the right inequality. Since $||x|y-x'| \le ||x|(y-w)| + ||x|w-x'|$, it is enough to show

$$|y-w| \le \frac{2\delta}{1-\delta}||x|w-x'|$$

for any $y \in E(w,\delta)$.

Denote $\eta = \varphi_w(y)$, then $y = \varphi_w(\eta)$ and $|\eta| < \delta$. From (5.4) and the simple inequality $1-|w| \le ||x|w-x'|$, we get

$$|y-w| = |\varphi_w(\eta)-w| \le \frac{\delta}{1-\delta}(1-|w|^2) \le \frac{\delta}{1-\delta}2||x|w-x'|,$$

as desired. □

Lemma 5.3

Let $\delta \in (0,1)$ and $a \in \mathbb{B}$, then

$$B\left(a, \frac{\delta}{2}(1-|a|^2)\right) \subset E(a,\delta) \subset B\left(a, \frac{\delta}{1-\delta}(1-|a|^2)\right). \tag{5.5}$$

Proof. If $x \in B\big(a, \frac{\delta}{2}(1-|a|^2)\big)$, then from (5.3)

$$|\varphi_a(x)| = \frac{|x-a|}{||a|x-a'|} \le \frac{|x-a|}{1-|a|} < \delta.$$

If $x \in E(a,\delta)$, then from (5.4)

$$|x-a| = |\varphi_a(\varphi_a(x)) - a| = \frac{|\varphi_a(x)|}{||a|\varphi_a(x) - a'|}(1-|a|^2) < \frac{\delta}{1-\delta}(1-|a|^2).$$

This completes the proof. □

Denote

$$d\tau(w) = (1-|w|^2)^{-n} dw,$$

which is a Möbius invariant measure on $\mathbb{B}$; see [1]. It is easy to see

$$\tau(E(a,\delta)) = \tau(B(0,\delta)) = n\int_0^\delta t^{n-1}(1-t^2)^{-n} dt.$$

Lemma 5.4

Let $p \in (0,\infty)$ and $\delta \in (0,1)$. Then there exists a positive constant C such that for any harmonic function f in $\mathbb{B}$

$$|f(x)|^p \le C\int_{E(x,\delta)} |f(w)|^p d\tau(w), \qquad x \in \mathbb{B}. \tag{5.6}$$

Proof. In Lemma 5.2, we take $x = y$ and $x = w$ respectively, then get

$$C^{-1}(1-|w|^2) \le 1-|y|^2 \le C(1-|w|^2), \qquad y \in E(w,\delta). \tag{5.7}$$

Assume f is harmonic in $\mathbb{B}$. Then $|f|^p$ has subharmonic behavior by the result of Fefferman and Stein [7]:

$$|f(x)|^p \le Cr^{-n}\int_{B(x,r)} |f|^p dx,$$

whenever $r < 1-|x|$, where C is a constant depending only on p and n. This yields (5.6) by taking $r = \frac{\delta}{2}(1-|x|^2)$ and then applying Lemma 5.3 and (5.7). □

Proof of Lemma 5.1 Assume $f \in h^p_\alpha$. From Lemma 5.4

$$|f(x)|^p \le C\int_{E(x,\delta)} |f(w)|^p d\tau(w).$$

Since $(1-|y|^2) \sim (1-|x|^2)$ for $y \in E(x,\delta)$, we can find a positive constant C such that

$$|f(x)| \le C(1-|x|^2)^{(\alpha+n)/p}||f||_{p,\alpha}$$

for all $f \in h^p_\alpha$. For $0 < p < 1$, we can write

$$|f(x)| = |f(x)|^p|f(x)|^{1-p}$$

and use the above inequality to estimate the second factor. Now integrate both sides with respect to $d\nu_\beta$ with $\beta = -n + (n+\alpha)/p$ and apply Fubini's theorem to yield the desired inequality. $\square$

6. Proof of main result

With all the preparations above, now we can give the proof of the main result. Our proof also suits to the case $p = 1$, with approach different from [5] and [13].

Theorem 6.1

Suppose $0 < p \leq 1$, $-1 < \alpha < \infty$ *and* $\beta = -n + (n+\alpha)/p$. *Then* $h_\alpha^{p\,*} = \mathcal{B}$ *under the integral paring*

$$< f, g >= \lim_{r \to 1^-} \int_{\mathbb{B}} f(rx)\overline{g(x)} d\nu_\beta(x), \qquad f \in h_\alpha^p,\ g \in \mathcal{B}.$$

Proof. Assume $F \in h_\alpha^{p\,*}$. Let $f \in h_\alpha^p$ and denote $f_r(x) = f(rx)$. It is easy to see

$$||f - f_r||_{p,\alpha} \to 0 \qquad \text{as} \quad r \to 1^-,$$

so that

$$F(f) = \lim_{r \to 1^-} F(f_r).$$

Fix $r \in (0,1)$. By Theorem 2.1 we can write

$$f_r(x) = P_0 f_r(x) = \frac{2}{n} \int_{\mathbb{B}} f_r(y) Q(x,y) dy, \qquad x \in \mathbb{B}.$$

We claim that

$$F(f_r) = \frac{2}{n} \int_{\mathbb{B}} f_r(y) F(Q(x,y)) dy,$$

where on the right hand side we think of F as acting with respect to the running variable x.

In fact, the interchange of the functional F and the integral can be justified as follows. Since f_r can be uniformly approximated by polynomials as in (2.1), it suffices to assume that f_r is a monomial in (2.1). Then we consider the integral on a slightly smaller ball $s\mathbb{B}$ with $s \in (0,1)$. Under these circumstances, the series expansion of the kernel function inside F given in (2.2) easily produces the desired equality. More precisely, let $f_r(x) = |x|^m Y_i^m(x')$, and fix $s \in (0,1)$. Let $y \in s\mathbb{B}$, then $Q(\cdot, y)$ is bounded harmonic in $\mathbb{B}$ by (2.3) or Theorem 4.1, such that

$$\begin{aligned}\int_{s\mathbb{B}} f_r(y) F(Q(x,y)) dy &= \int_{s\mathbb{B}} f_r(y) F\left(\sum_{k,j} (k + n/2) |x|^k |y|^k Y_j^k(x') Y_j^k(y') \right) dy \\ &= \int_{s\mathbb{B}} \sum_{k,j} (k + n/2) F\big(|x|^k Y_j^k(x')\big) Y_j^k(y') Y_i^m(y') |y|^{k+m} dy.\end{aligned}$$

By the dominated convergence theorem, we can interchange the sum and the integral above, since the series of the integrand above converges absolutely and uniformly in $\mathbb{B} \times s\mathbb{B}$, which follows directly from

$$\begin{aligned} |Y_j^k(y')| &\leq Ck^{(n-2)/2} \quad ([2]); \\ |F(|x|^k Y_j^k(x'))| &\leq C||F||k^{(n-2)/2}. \end{aligned} \tag{6.1}$$

Consequently, applying the polar coordinates formula in integration and the fact that $\{Y_j^k\}$ is orthogonal, then letting $s \to 1^-$ we get

$$\int_{\mathbb{B}} f_r(y)F(Q(x,y))dy = F(f_r)\int_{\mathbb{B}} (k+n/2)Y_i^m(y')Y_i^m(y')|y|^{2m}dy = \frac{n}{2}F(f_r),$$

which proves the claim.

Fix $x \in \mathbb{B}$ and denote

$$h(y) = \overline{F(Q(x,y))} \qquad y \in \mathbb{B}.$$

By the series expansion of Q in (2.2), h is a bounded harmonic function in $\mathbb{B}$. From Lemma 3.3

$$F(f_r) = \frac{2}{n}\int_{\mathbb{B}} f_r(y)\overline{h(y)}dy = \frac{2}{n\Gamma(\beta+1)}\int_{\mathbb{B}} f_r(y)\overline{D^\beta h(y)}d\nu_\beta(y).$$

Let $g = \frac{2}{n\Gamma(\beta+1)}D^\beta h$ with $\beta = -n + (n+\alpha)/p$, then

$$F(f) = \lim_{r\to 1^-}\int_{\mathbb{B}} f_r(y)\overline{g(y)}d\nu_\beta(y)$$

for every $f \in h^p_\alpha$.

It remains to show that $g \in \mathcal{B}$. For simplicity we rewrite $g = D^\beta h$ by omitting the constant factor. Then

$$\overline{g(y)} = D_y^\beta F_x(Q(x,y)).$$

We claim that

$$\overline{g(y)} = F(D_y^\beta Q(x,y)),$$

and

$$\frac{\partial}{\partial y_j}\overline{g(y)} = F\left(\frac{\partial}{\partial y_j}D_y^\beta Q\right).$$

Indeed, the interchange of functional F and fractional or ordinary differentiation should be verified. This can be proved similarly as above, namely using the series expansions of kernel functions Q, $D^\beta Q$ and $\frac{\partial}{\partial y_j}D^\beta Q$. Notice that the action of D^β on monomials is given by (3.1), and the action of $\frac{\partial}{\partial y_j}$ on monomials Y_i^m can be represented as the linear combination of Y_l^{m-1}.

From the claim,

$$\big(1-|y|^2\big)\left|\frac{\partial}{\partial y_j}g(y)\right| = \left|F\Big[(1-|y|^2)\frac{\partial}{\partial y_j}D_y^\beta Q\Big]\right| = \big|F[u_y]\big|,$$

where

$$u_y(x) = \big(1-|y|^2\big)\frac{\partial}{\partial y_j}D_y^\beta Q(x,y).$$

Since $Q(x,y)=Q(y,x)$, by Theorem 4.1 we have

$$|u_y(x)| \le C\frac{1-|y|^2}{||y|x-y'|^{n+\beta+1}}.$$

It follows from Lemma 4.4 that

$$||u_y||_{p,\alpha} \le C.$$

Therefore

$$\big(1-|y|^2\big)\left|\frac{\partial}{\partial y_j}g(y)\right| = \big|F[u_y]\big| \le ||F||\ ||u_y||_{p,\alpha} \le C||F||.$$

This implies $(1-|y|^2)|\nabla g(y)| \le C||F||$.

Conversely, assume $g \in \mathcal{B}$. We now show that the formula

$$F(f) = \lim_{r\to 1^-}\int_{\mathbb{B}} f_r(x)\overline{g(x)}\big(1-|x|^2\big)^\beta dx, \qquad f \in h_\alpha^p,$$

defines a bounded linear functional in h_α^p, where $\beta = -n+(n+\alpha)/p$.

By Theorem 2.2, there exists a function $\varphi \in L^\infty(\mathbb{B})$, such that

$$g(x) = P_\beta\varphi(x) = C_\beta\int_{\mathbb{B}} Q_\beta(x,y)\varphi(y)d\nu_\beta(y), \qquad x \in \mathbb{B}.$$

Using Fubini's theorem and the reproducing property of P_β, we easily obtain

$$\int_{\mathbb{B}} f_r(x)\overline{g(x)}\big(1-|x|^2\big)^\beta dx = \int_{\mathbb{B}} f_r(y)\overline{\varphi(y)}\big(1-|y|^2\big)^\beta dy.$$

By Lemma 5.1, we have

$$F(f) = \int_{\mathbb{B}} f(x)\overline{\varphi(x)}\big(1-|x|^2\big)^\beta dx, \qquad f \in h_\alpha^p,$$

and

$$|F[f]| \le C||\varphi||_{L^\infty}||f||_{p,\alpha}. \ \square$$

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