Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **53**, 1 (2002), 55–81

Differential equations driven by fractional Brownian motion

David Nualart*

Universitat de Barcelona, Facultat de Matemàtiques,

Gran Via, 585, 08007, Barcelona, Spain

E-mail: nualart@mat.ub.es

Aurel Răşcanu

Universitatea "Alexandru Ioan Cuza", Facultatea de matematică,

B-dul Carol I, nr. 11, 6600, Iaşi, Romania

E-mail: rascanu@uaic.ro

Received September 13, 2001. Revised November 20, 2001

Abstract

A global existence and uniqueness result of the solution for multidimensional, time dependent, stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$ is proved. It is shown, also, that the solution has finite moments. The result is based on a deterministic existence and uniqueness theorem whose proof uses a contraction principle and a priori estimates.

1. Introduction

Let $B = \{B_t, t \geq 0\}$ be a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$. That is, B is a centered Gaussian process with the covariance function (see [16])

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

Notice that if $H = \frac{1}{2}$, the process B is a standard Brownian motion, but if $H \neq \frac{1}{2}$, it does not have independent increments. From (1.1) it follows that $\mathbb{E}|B_t - B_s|^2 = |t-s|^{2H}$. As a consequence, the process B has $\alpha-$Hölder continuous paths for all $\alpha \in (0,H)$.

Keywords: Stochastic differential equation, Brownian motion.

MSC2000: 60H05, 60H07.

* Supported by the MCyT Grant number BFM2000-0598 and the INTAS project 99-0016.

The definition of stochastic integrals with respect to the fractional Brownian motion has been investigated by several authors (see, for instance, [1], [2], [6], [7], [8], [13]).

In the case $H > 1/2$ one can use a pathwise approach to define integrals with respect to the fractional Brownian motion. In fact, if $\{u_t, t \geq 0\}$ is a stochastic process whose trajectories are λ-Hölder continuous with $\lambda > 1-H$, then the Riemann-Stieltjes integral $\int_0^T u_s dB_s$ exists for each trajectory, due to the results by Young [21]. More refined results have been obtained in [3] by Ciesielski, Kerkyacharian and Roynette (see also [18]) for processes u with trajectories in the Besov space $\mathcal{B}_{p,1}^{1-H}$, where $\frac{1}{p} < H < 1 - \frac{1}{p}$. The fractional Brownian motion has trajectories in the Besov space $\mathcal{B}_{p,\infty}^{H}$, and the following inequality holds:

$$\left\| \int_0^{\cdot} u_t dB_t \right\|_{\mathcal{B}_{p,\infty}^{H}} \leq C \parallel u \parallel_{\mathcal{B}_{p,1}^{1-H}} \|B\|_{\mathcal{B}_{p,\infty}^{H}} .$$

In [22] and [23] Zähle has defined pathwise integrals $\int_0^T u_t dB_t$ for processes with paths in the fractional Sobolev type space $I_{0+}^{1-\beta}(L^2(0,T))$. The indefinite integral $\int_0^t u_s dB_s$ is a continuous process provided that, in addition, u is bounded.

In this paper we are interested in multidimensional stochastic differential equations of the form

$$X_t = X_0 + \int_0^t \sigma\left(s, X_s\right) dB_s + \int_0^t b(s, X_s) ds, \tag{1.2}$$

where B is a fBm with Hurst parameter $H > 1/2$, and the integral with respect to B is a pathwise Riemann-Stieltjes integral. This kind of equation has been studied by several authors ([10], [12], [13], [14], [19]).

In [14], Lyons considered integral equations

$$x_t = x_0 + \sum_{j=1}^{m} \int_0^t \sigma^j(x_s) dg_s^j,$$

$0 \leq t \leq T$, where the g^j are continuous functions with bounded p-variation on $[0, T]$ for some $p \in [1, 2)$. This equation has a unique solution in the space of continuous functions of bounded p-variation if each g^j has a Hölder continuous derivative of order $\alpha > p-1$. Taking into account that the fBm of Hurst parameter H has locally bounded p-variation for $p > 1/H$, the result proved in [14] can be applied to equations driven by a fBm provided the nonlinear coefficient has a Hölder continuous derivative of order $\alpha > 1/H - 1$. Using this approach based on the notion of p-variation and the general limit theorem proved by Lyons in [15] for differential equations driven by geometric rough paths, Coutin and Qian [5], [4] have established the existence of strong solutions and a Wong-Zakai type approximation limit for stochastic differential equations driven by a fractional Brownian motion with parameter $H > 1/4$.

In [19] Ruzmaikina establishes an existence-uniqueness theorem for ordinary differential equations with Hölder continuous forcing. The global solution is constructed, first, in small time intervals where the contraction principle can be applied, provided the Hölder constant is small enough. The integral $\int_0^T f dg$ is defined in the sense of Young [21], assuming that the functions f and g are Hölder continuous of orders β and γ, respectively, with $\beta + \gamma > 1$. This result is applied to stochastic differential equations driven by a fractional Brownian motion with parameter $H > 1/2$.

In [23] the existence and uniqueness of solutions is proved for differential equations driven by a fractional Brownian motion with parameter $H > 1/2$, in a small random interval, provided the diffusion coefficient is a contraction in the space $W_{2,\infty}^{\beta}$, where $1/2 < \beta < H$. Here $W_{2,\infty}^{\beta}$ denotes the Besov-type space of bounded measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\int_0^T \int_0^T \frac{(f(t) - f(s))^2}{|t-s|^{2\beta+1}} dsdt < \infty.$$

In this paper we follow the approach of Zähle, and we present a general result on the existence and uniqueness of solution for multidimensional, time dependent, stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. This result is based on a deterministic existence and uniqueness theorem whose proof uses a contraction principle. Our result is global and based on a priori estimate.

The organization of the paper is as follows. In Section 2 we state our main result. Section 3 contains some basic facts about fractional integrals and derivatives. The extended Stieltjes integral that we use is defined in Section 4, which also provides the basic estimates in Lemma 4.1. Section 5 contains the deterministic result, and, finally, in Section 6 we derive the existence and uniqueness result for stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$.

2. Main result

Let $\frac{1}{2} < H < 1$, $1 - H < \alpha < \frac{1}{2}$ and $d \in \mathbb{N}^*$. Denote by $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ the space of measurable functions $f : [0,T] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

For any $0 < \lambda \leq 1$, denote by $C^{\lambda}\left(0,T;\mathbb{R}^d\right)$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_{\lambda} := \|f\|_{\infty} + \sup_{0\leq s<t\leq T} \frac{|f(t) - f(s)|}{(t-s)^{\lambda}} < \infty,$$

where $\|f\|_{\infty} := \sup_{t\in[0,T]} |f(t)|$. We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}(0,T;\mathbb{R}^d) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \subset C^{\alpha-\varepsilon}(0,T;\mathbb{R}^d).$$

Consider the equation on $\mathbb{R}^d$

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(s, X_s)\, dB_s^j + \int_0^t b^i(s, X_s) ds, \ \ t \in [0,T], \tag{2.3}$$

$i = 1, ..., d$, where the processes B^j, $j = 1, ..., m$ are independent fractional Brownian motions with Hurst parameter H defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, X_0 is a d-dimensional random variable, and the coefficients $\sigma^{i,j}, b^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are measurable functions. Set $\sigma = \left(\sigma^{i,j}\right)_{d\times m}$, $b = \left(b^i\right)_{d\times 1}$ and for a matrix $A = \left(a^{i,j}\right)_{d\times m}$ and a vector $y = \left(y^i\right)_{d\times 1}$ denote $|A|^2 = \sum_{i,j} \left|a^{i,j}\right|^2$ and $|y|^2 = \sum_i \left|y^i\right|^2$.

Let us consider the following assumptions on the coefficients, which are supposed to hold for $\mathbb{P}$-almost all $\omega \in \Omega$. The constants M_N, L_N, K_0 and the function b_0 may depend on ω.

($\mathbf{H}_1$) $\sigma(t,x)$ is differentiable in x, and there exist some constants $0 < \beta, \delta \le 1$ and for every $N \ge 0$ there exists $M_N > 0$ such that the following properties hold:

$$(H_\sigma): \begin{cases} i) \quad \text{Lipschitz continuity} \\ |\sigma(t,x) - \sigma(t,y)| \le M_0 |x-y|, \quad \forall\, x \in \mathbb{R}^d, \forall\, t \in [0,T] \\ ii) \quad \text{Local Hölder continuity} \\ |\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)| \le M_N |x-y|^\delta, \\ \qquad\qquad \forall\, |x|, |y| \le N, \forall t \in [0,T], \\ iii) \quad \text{Hölder continuity in time} \\ |\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \le M_0 |t-s|^\beta \\ \qquad\qquad \forall\, x \in \mathbb{R}^d, \forall\, t, s \in [0,T] \end{cases}$$

for each $i = 1, ..., d$.

($\mathbf{H}_2$) There exists $b_0 \in L^\rho\left(0,T;\mathbb{R}^d\right)$, where $\rho \ge 2$, and for every $N \ge 0$ there exists $L_N > 0$ such that the following properties hold:

$$(H_b): \begin{cases} i) \quad \text{Local Lipschitz continuity} \\ |b(t,x) - b(t,y)| \le L_N |x-y|, \ \forall\, |x|, |y| \le N, \forall t \in [0,T], \\ ii) \quad \text{Boundedness} \\ |b(t,x)| \le L_0 |x| + b_0(t), \ \forall x \in \mathbb{R}^d, \forall t \in [0,T]. \end{cases}$$

($\mathbf{H}_3$) There exist $\gamma \in [0,1]$ and $K_0 > 0$ such that

$$|\sigma(t,x)| \le K_0 \left(1 + |x|^\gamma\right), \ \forall x \in \mathbb{R}^d, \forall t \in [0,T]. \tag{2.4}$$

Let

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}.$$

The main result of our paper is the following theorem on the existence and uniqueness of a solution for the stochastic differential equation (2.3).

Theorem 2.1

Suppose that X_0 is an $\mathbb{R}^d$-valued random variable, the coefficients $\sigma(t,x)$ and $b(t,x)$ satisfy assumptions (H_1) and (H_2) with $\beta > 1-H$, $\delta > \frac{1}{H} - 1$. Then

I. If $\alpha \in (1-H, \alpha_0)$ and $\rho \geq 1/\alpha$, then there exists a unique stochastic process $X \in L^0\left(\Omega, \mathcal{F}, \mathbb{P}; W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)\right)$ solution of the stochastic equation (2.3) and, moreover, for $\mathbb{P}$-almost all $\omega \in \Omega$

$$X(\omega,\cdot) = \left(X^i(\omega,\cdot)\right)_{d\times 1} \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right).$$

II. Moreover, if $\alpha \in (1-H, \alpha_0 \wedge (2-\gamma)/4)$, $\rho \geq 1/\alpha$, $X_0 \in L^\infty\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d\right)$, the assumption (H_3) is satisfied and the constants M_N, L_N, K_0 and the function b_0 are independent of ω, then the solution X satisfies

$$E\left\|X\right\|_{\alpha,\infty}^p < \infty\ ,\ \forall p \geq 1\,.$$

3. Fractional integrals and derivatives

Let $a,b \in \mathbb{R}$, $a<b$. Denote by $L^p(a,b)$, $p \geq 1$, the usual space of Lebesgue measurable functions $f:[a,b] \to \mathbb{R}$ for which $\|f\|_{L^p(a,b)} < \infty$, where

$$\|f\|_{L^p(a,b)} = \begin{cases} \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty \\ \text{ess sup}\left\{|f(x)| : x \in [a,b]\right\}, & \text{if} \quad p = \infty. \end{cases}$$

Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,dy$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\,dy\,,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler function. Let $I_{a+}^\alpha(L^p)$ (resp. $I_{b-}^\alpha(L^p)$) the image of $L^p(a,b)$ by the operator I_{a+}^α (resp. I_{b-}^α). If $f \in I_{a+}^\alpha(L^p)$ (resp. $f \in I_{b-}^\alpha(L^p)$) and $0<\alpha<1$ then the Weyl derivative

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy\right) 1_{(a,b)}(x) \tag{3.5}$$

$$\left(\text{resp.}\quad D^{\alpha}_{b-}f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \times \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}dy\right)1_{(a,b)}(x)\right) \tag{3.6}$$

is defined for almost all $x \in (a,b)$ (the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all $x \in (a,b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

Recall from [20] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then

$$I^{\alpha}_{a+}(L^p) = I^{\alpha}_{b-}(L^p) \subset L^q(a,b).$$

- If $\alpha > \frac{1}{p}$ then

$$I^{\alpha}_{a+}(L^p) \cup I^{\alpha}_{b-}(L^p) \subset C^{\alpha-1/p}(a,b).$$

The following inversion formulas hold:

$$I^{\alpha}_{a+}\left(D^{\alpha}_{a+}f\right) = f, \quad \forall f \in I^{\alpha}_{a+}(L^p)$$

and

$$D^{\alpha}_{a+}\left(I^{\alpha}_{a+}f\right) = f, \quad \forall f \in L^1(a,b),$$

and the same statements are true for $(I^{\alpha}_{b-}, D^{\alpha}_{b-})$.

4. Generalized Stieltjes integrals

4.1. Definition of generalized Stieltjes integrals

Following [22] we can give the definition of the generalized Stieltjes integral (fractional integral) of f with respect to g. Let $f(a+) = \lim_{\varepsilon\searrow 0} f(a+\varepsilon)$, $g(b-) = \lim_{\varepsilon\searrow 0} g(b-\varepsilon)$ (supposing that the limits exist and are finite) and define

$$f_{a+}(x) = (f(x) - f(a+))\,1_{(a,b)}(x),$$
$$g_{b-}(x) = (g(x) - g(b-))\,1_{(a,b)}(x).$$

DEFINITION 4.1 (Generalized Stieltjes Integral). Suppose that f and g are functions such that $f(a+)$, $g(a+)$ and $g(b-)$ exist, $f_{a+} \in I^{\alpha}_{a+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$. Then the integral of f with respect to g is defined by

$$\int_a^b f dg = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f_{a+}(x)\,D^{1-\alpha}_{b-}g_{b-}(x)\,dx + f(a+)\left(g(b-) - g(a+)\right). \tag{4.7}$$

Remark 4.1. If $\alpha p < 1$, under the assumptions of the preceding definition we have $f \in I_{a+}^{\alpha}(L^p)$ and (4.7) can be rewritten as

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) \, D_{b-}^{1-\alpha} g_{b-}(x) \, dx, \tag{4.8}$$

which is determined for general functions $f \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$.

Remark 4.2. If $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$, it is proved in [22] that the conditions of the above definition and remark are fulfilled and we may choose $p = q = \infty$ and $\alpha < \lambda$, $1 - \alpha < \mu$. Moreover the integral $\int_a^b f dg$ coincides with the Riemann-Stieltjes integral.

The linear spaces $I_{a+}^{\alpha}(L^p)$ are Banach spaces with respect to the norms

$$\|f\|_{I_{a+}^{\alpha}(L^p)} = \|f\|_{L^p} + \left\|D_{a+}^{\alpha} f\right\|_{L^p} \sim \left\|D_{a+}^{\alpha} f\right\|_{L^p},$$

and the same is true for $I_{b-}^{\alpha}(L^p)$. If $0 < \alpha < 1/p$ then the norms of the spaces $I_{a+}^{\alpha}(L^p)$ and $I_{b-}^{\alpha}(L^p)$ are equivalent, and for $a \le c < d \le b$ the restriction of $f \in I_{a+}^{\alpha}(L^p(a,b))$ to (c,d) belongs to $I_{c+}^{\alpha}(L^p(c,d))$ and the continuation of $f \in I_{c+}^{\alpha}(L^p(c,d))$ by zero beyond (c,d) belongs to $I_{a+}^{\alpha}(L^p(a,b))$. As a consequence, if $f \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ then the integral $\int_a^b 1_{(c,d)} f dg$ in the sense of (4.8) exists for any $a \le c < d \le b$ and we have

$$\int_c^d f dg = \int_a^b 1_{(c,d)} f dg, \tag{4.9}$$

whenever the left-hand side is determined in the sense of (4.8).

Fix a parameter $0 < \alpha < 1/2$. Denote by $W_T^{1-\alpha,\infty}(0,T)$ the space of measurable functions $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly,

$$C^{1-\alpha+\varepsilon}(0,T) \subset W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T), \quad \forall \varepsilon > 0.$$

Moreover, if g belongs to $W_T^{1-\alpha,\infty}(0,T)$, its restriction to $(0,t)$ belongs to $I_{t-}^{1-\alpha}(L^{\infty}(0,t))$ for all t and

$$\begin{aligned} \Lambda_{\alpha}(g) &:= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| \left(D_{t-}^{1-\alpha} g_{t-} \right)(s) \right| \\ &\le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty. \end{aligned}$$

We also denote by $W_0^{\alpha,1}(0,T)$ the space of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^{\alpha}} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy \, ds < \infty.$$

The restriction of $f \in W_0^{\alpha,1}(0,T)$ to $(0,t)$ belongs to $I_{0+}^{\alpha}(L^1(0,t))$ for all t.

Notice that if f is a function in the space $W_0^{\alpha,1}(0,T)$, and g belongs to $W_T^{1-\alpha,\infty}(0,T)$, then the integral $\int_0^t fdg$ exists for all $t \in [0,T]$ and by (4.9) we have

$$\int_0^t fdg = \int_0^T f\mathbf{1}_{(0,t)}dg,$$

which implies the additivity of the indefinite integral. Furthermore the following estimate holds

$$\left|\int_0^t fdg\right| \le \Lambda_\alpha(g) \|f\|_{\alpha,1}.$$

Indeed, by the definition (4.8) we can write

$$\int_0^t fdg = (-1)^\alpha \int_0^t \left(D_{0+}^\alpha f\right)(s) \left(D_{t-}^{1-\alpha} g_{t-}\right)(s)\, ds.$$

Hence,

$$\begin{aligned}\left|\int_0^t fdg\right| &\le \sup_{0<s<t} \left|\left(D_{t-}^{1-\alpha} g_{t-}\right)(s)\right| \int_0^t \left|\left(D_{0+}^\alpha f\right)(s)\right|\, ds \\ &\le \Lambda_\alpha(g) \|f\|_{\alpha,1}. \end{aligned} \tag{4.10}$$

In the next section we will derive more precise estimates for this indefinite integral.

4.2 A priori estimates

Fix $0 < \alpha < \frac{1}{2}$. Given two functions $g \in W_T^{1-\alpha,\infty}(0,T)$ and $f \in W_0^{\alpha,1}(0,T)$ we denote

$$G_t(f) := \int_0^t fdg.$$

The following proposition provides the basic estimate for iterative calculus in the Banach fixed point theorem applied to the differential equations considered in this paper.

Proposition 4.1

I. Let $g \in W_T^{1-\alpha,\infty}(0,T)$ and $f \in W_0^{\alpha,1}(0,T)$. Then for all $s < t$, the following estimates

$$|G_t(f) - G_s(f)| \le \Lambda_\alpha(g) \int_s^t \left(\frac{|f(r)|}{(r-s)^\alpha} + \alpha \int_s^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\right) dr \tag{4.11}$$

and

$$\begin{aligned}|G_t(f)| + \int_0^t \frac{|G_t(f) - G_s(f)|}{(t-s)^{\alpha+1}}\, ds \;&\le\; \Lambda_\alpha(g)\, c_{\alpha,T}^{(1)} \\ &\times \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha}\right) \left(|f(r)| + \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\right) dr, \end{aligned} \tag{4.12}$$

hold.

II. If $f \in W_0^{\alpha,\infty}(0,T)$ then $G_\cdot(f) \in C^{1-\alpha}(0,T)$ and

$$\|G(f)\|_{1-\alpha} \le \Lambda_\alpha(g) c_{\alpha,T}^{(2)} \|f\|_{\alpha,\infty}. \tag{4.13}$$

The constants $c_{\alpha,T}^{(i)}$, $i \in \{1,2\}$ depend, only, on α and T.

Proof. I. Using the additivity property of the indefinite integral and Definition (4.8) we obtain

$$
\begin{aligned}
|G_t(f) - G_s(f)| &= \left| \int_s^t f dg \right| \\
&= \left| \int_s^t D_{s+}^{\alpha}(f)(r) \left(D_{t-}^{1-\alpha} g_{t-} \right)(r)\, dr \right| \\
&= \Lambda_\alpha(g) \left(\int_s^t \frac{|f(r)|}{(r-s)^\alpha} dr + \alpha \int_s^t \int_s^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy dr \right),
\end{aligned}
$$

which implies (4.11).

Multiplying (4.11) by $(t-s)^{-\alpha-1}$ and integrating in s yields

$$
\begin{aligned}
\int_0^t \frac{|G_t(f) - G_s(f)|}{(t-s)^{\alpha+1}}\, ds \le \Lambda_\alpha(g) \int_0^t (t-s)^{-\alpha-1} \\
\times \left(\int_s^t \frac{|f(r)|}{(r-s)^\alpha} dr + \alpha \int_s^t \int_s^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy dr \right) ds. \quad (4.14)
\end{aligned}
$$

By the substitution $s = r - (t-r)y$ we have

$$
\int_0^r (t-s)^{-\alpha-1}(r-s)^{-\alpha} ds = (t-r)^{-2\alpha} \int_0^{r/(t-r)} (1+y)^{-\alpha-1} y^{-\alpha} dy\,, \qquad (4.15)
$$

and, on the other hand,

$$
\int_0^y (t-s)^{-\alpha-1} ds = \alpha^{-1} \left[(t-y)^{-\alpha} - t^{-\alpha} \right] \le \alpha^{-1}(t-y)^{-\alpha}. \qquad (4.16)
$$

Substituting (4.15) and (4.16) into (4.14) yields

$$
\begin{aligned}
\int_0^t \frac{|G_t(f) - G_s(f)|}{(t-s)^{\alpha+1}}\, ds \le \Lambda_\alpha(g) \Bigg[c_\alpha^{(1)} \int_0^t \frac{|f(r)|}{(t-r)^{2\alpha}} dr \\
+ \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} (t-y)^{-\alpha} dy dr \Bigg], \qquad (4.17)
\end{aligned}
$$

where

$$
c_\alpha^{(1)} = \int_0^\infty (1+y)^{-\alpha-1} y^{-\alpha} dy \; = B(2\alpha, 1-\alpha)
$$

and

$$
B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1} dt = \int_0^\infty \frac{t^{q-1}}{(1+t)^{p+q}} dt
$$

is the Beta function.

From (4.11) we can derive the following estimate

$$
|G_t(f)| \le \Lambda_\alpha(g) \left(\int_0^t \frac{|f(r)|}{r^\alpha} dr + \alpha \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy dr \right). \qquad (4.18)
$$

From (4.17) and (4.18) we obtain (4.12) with $c_{\alpha,T}^{(1)} = \max(c_{\alpha}^{(1)}, 1) + T^{\alpha}$, since

$$\alpha + (t-y)^{-\alpha} \le T^{\alpha}\left(r^{-\alpha} + (t-y)^{-2\alpha}\right).$$

Observe that

$$c_{\alpha,T}^{(1)} \le \frac{1}{\alpha(1-\alpha)} + T^{\alpha}.$$

II. From (4.18) we obtain

$$\|G_t(f)\|_{\infty} \le \Lambda_{\alpha}(g)\left(\frac{T^{1-\alpha}}{1-\alpha} + \alpha T\right)\|f\|_{\alpha,\infty}.$$

The relation (4.11) implies, for each $s < t$,

$$\left|\int_s^t f dg\right| \le \Lambda_{\alpha}(g)\frac{1}{1-\alpha}\max\{1, \alpha T^{\alpha}\}(t-s)^{1-\alpha}\|f\|_{\alpha,\infty},$$

and the indefinite integral $\int_0^t f dg$ is Hölder continuous of order $1-\alpha$. Hence, the inequality (4.13) holds with

$$c_{\alpha,T}^{(2)} = \frac{T^{1-\alpha}}{1-\alpha} + \alpha T + \frac{1}{1-\alpha}\max\{1, \alpha T^{\alpha}\}. \ \square$$

Recall that the space $W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ is a Banach space with respect to the norm

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]}\left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}ds\right).$$

and for $\lambda \ge 0$ a equivalent norm is defined by

$$\|f\|_{\alpha,\lambda} = \sup_{t\in[0,T]} e^{-\lambda t}\left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}\ ds\right).$$

Let $0 < \alpha < \frac{1}{2}$, $f \in W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ and $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$. Define

$$G_t^{(\sigma)}(f) = \int_0^t \sigma(s, f(s))\, dg_s,$$

where $\sigma = \left(\sigma^{i,j}\right)_{d\times m} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times m} = \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^d\right)$ satisfy the assumptions (H_1) with constant $\beta > \alpha$.

Proposition 4.2

If $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ then

$$G^{(\sigma)}(f) \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$$

Moreover for all $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$:

$$j) \ \left\| G^{(\sigma)}(f) \right\|_{1-\alpha} \leq \Lambda_\alpha(g) C^{(2)} \left(1 + \|f\|_{\alpha,\infty}\right),$$

$$jj) \ \left\| G^{(\sigma)}(f) \right\|_{\alpha,\lambda} \leq \frac{\Lambda_\alpha(g) C^{(3)}}{\lambda^{1-2\alpha}} \left(1 + \|f\|_{\alpha,\lambda}\right), \tag{4.19}$$

for all $\lambda \geq 1$, *where the constants* $C^{(2)}$ *and* $C^{(3)}$ *are independent of* λ, f, g *(they depend on* T, *the dimensions* d, m, *and the constants* $|\sigma(0,0)|$, M_0, α, β *from* (H_1)*).*

If $f, h \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ *are such that* $\|f\|_\infty \leq N$, $\|h\|_\infty \leq N$, *then*

$$\left\| G^{(\sigma)}(f) - G^{(\sigma)}(h) \right\|_{\alpha,\lambda} \leq \frac{\Lambda_\alpha(g) C_N^{(4)}}{\lambda^{1-2\alpha}} \left(1 + \Delta(f) + \Delta(h)\right) \|f-h\|_{\alpha,\lambda} \tag{4.20}$$

for all $\lambda \geq 1$, *where*

$$\Delta(f) = \sup_{r\in[0,T]} \int_0^r \frac{|f_r - f_s|^\delta}{(r-s)^{\alpha+1}}\, ds, \tag{4.21}$$

and the constants $C_N^{(4)}$ *are independent of* λ, f, h, g. *(*$C_N^{(4)}$ *depends on* T, *the dimensions* d, m, *and the constants from* (H_1)*).*

Proof. In order to simplify the presentation of the proof we assume $d = m = 1$. First we remark that if $f \in W_0^{\alpha,\infty}(0,T)$, then $\sigma(\cdot, f) \in W_0^{\alpha,\infty}(0,T)$ since

$$\begin{aligned} &|\sigma(r, f(r))| + \int_0^r \frac{|\sigma(r, f(r)) - \sigma(s, f(s))|}{(r-s)^{\alpha+1}}\, ds \\ &\qquad \leq C + M_0 |f(r)| + M_0 \int_0^r \frac{|f(r) - f(s)|}{(r-s)^{\alpha+1}}\, ds, \end{aligned}$$

and then

$$\|\sigma(\cdot, f)\|_{\alpha,\infty} \leq C + M_0 \|f\|_{\alpha,\infty}$$

with $C = M_0 \left(T^\beta + \frac{T^{\beta-\alpha}}{\beta-\alpha}\right) + |\sigma(0,0)|$. Hence $G^{(\sigma)}(f) \in C^{1-\alpha}(0,T)$. Clearly, by (4.13) the inequality (4.19-j) holds with $C^{(2)} = c_{\alpha,T}^{(2)}(C + M_0)$. From (4.12) we obtain

$$\begin{aligned} \left\| G^{(\sigma)}(f) \right\|_{\alpha,\lambda} \leq\ & \Lambda_\alpha(g) c_{\alpha,T}^{(1)} \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha}\right) \\ & \times \left(|\sigma(r, f(r))| + \int_0^r \frac{|\sigma(r, f(r)) - \sigma(s, f(s))|}{(r-s)^{\alpha+1}}\, ds \right) dr \\ \leq\ & \Lambda_\alpha(g) c_{\alpha,T}^{(1)} \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha}\right) \\ & \times \left(C + M_0 |f(r)| + M_0 \int_0^r \frac{|f(r) - f(s)|}{(r-s)^{\alpha+1}}\, ds \right) dr \\ \leq\ & \Lambda_\alpha(g) c_{\alpha,T}^{(1)} \lambda^{2\alpha-1} c_\alpha \sup_{r\in[0,T]} e^{-\lambda r} \\ & \times \left(C + M_0 |f(r)| + M_0 \int_0^r \frac{|f(r) - f(s)|}{(r-s)^{\alpha+1}}\, ds \right) \end{aligned}$$

because

$$\begin{aligned}\int_0^t e^{-\lambda(t-r)}\left((t-r)^{-2\alpha}+r^{-\alpha}\right)dr &= \int_0^t e^{-\lambda x}\left(x^{-2\alpha}+(t-x)^{-\alpha}\ \right)dx\\ &= \frac{1}{\lambda}\int_0^{\lambda t}\ e^{-y}\left(\lambda^{2\alpha}y^{-2\alpha}+\lambda^{\alpha}(\lambda t-y)^{-\alpha}\right)dy\\ &\le \lambda^{2\alpha-1}\left(\int_0^{\infty}e^{-y}y^{-2\alpha}dy+\sup_{z>0}\int_0^z e^{-y}(z-y)^{-\alpha}dy\right)\\ &:= \lambda^{2\alpha-1}c_\alpha .\end{aligned}$$

Hence (4.19-jj) holds. Remark that $c_\alpha \le \frac{1}{1-2\alpha}+4$.

Let $f,h\in W_0^{\alpha,\infty}(0,T)$ be such that $\|f\|_\infty \le N$, $\|h\|_\infty \le N$. We can write using (4.12)

$$\begin{aligned}&\left\|G^{(\sigma)}(f)-G^{(\sigma)}(h)\right\|_{\alpha,\lambda}\\ &\le\ \Lambda_\alpha(g)c_{\alpha,T}^{(1)}\sup_{t\in[0,T]}e^{-\lambda t}\int_0^t\Big\{\ \left((t-r)^{-2\alpha}+r^{-\alpha}\right)\Big(\left|\sigma\left(r,f(r)\right)-\sigma\left(r,h(r)\right)\right|\\ &\quad+\int_0^r\frac{\left|\sigma\left(r,f(r)\right)-\sigma\left(s,f(s)\right)-\sigma\left(r,h(r)\right)+\sigma\left(s,h(s)\right)\right|}{(r-s)^{\alpha+1}}\ ds\Big)\Big\}dr\\ &\le\ \Lambda_\alpha(g)c_{\alpha,T}^{(1)}\lambda^{2\alpha-1}c_\alpha\sup_{r\in[0,T]}e^{-\lambda r}\Big\{M_0\left|f(r)-h(r)\right|\\ &\quad+\int_0^r\frac{\left|\sigma\left(r,f(r)\right)-\sigma\left(s,f(s)\right)-\sigma\left(r,h(r)\right)+\sigma\left(s,h(s)\right)\right|}{(r-s)^{\alpha+1}}\ ds\Big\}.\end{aligned}$$

By Lemma 7.1 we obtain

$$\begin{aligned}&\left|\sigma\left(r,f(r)\right)-\sigma\left(s,f(s)\right)-\sigma\left(r,h(r)\right)+\sigma\left(s,h(s)\right)\right|\\ &\le M_0|f(r)-f(s)-h(r)+h(s)|+M_0|f(r)-h(r)|(r-s)^\beta\\ &\quad+M_N|f(r)-h(r)|\left(|f(r)-f(s)|^\delta+|h(r)-h(s)|^\delta\right).\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^r\frac{\left|\sigma\left(r,f(r)\right)-\sigma\left(s,f(s)\right)-\sigma\left(r,h(r)\right)+\sigma\left(s,h(s)\right)\right|}{(r-s)^{\alpha+1}}\ ds\\ &\le M_0\int_0^r\frac{|f(r)-f(s)-h(r)+h(s)|}{(r-s)^{\alpha+1}}\ ds\\ &\quad+\frac{M_0}{\beta-\alpha}|f(r)-h(r)|r^{\beta-\alpha}\\ &\quad+M_N|f(r)-h(r)|\left(\int_0^r\frac{|f(r)-f(s)|^\delta}{(r-s)^{\alpha+1}}\ ds+\int_0^r\frac{|h(r)-h(s)|^\delta}{(r-s)^{\alpha+1}}\ ds\right).\end{aligned}$$

Making use of the notation (4.21) we obtain

$$\left\|G^{(\sigma)}(f)-G^{(\sigma)}(h)\right\|_{\alpha,\lambda}\ \le\ \Lambda_\alpha(g)C_N^{(4)}\lambda^{2\alpha-1}\left(1+\Delta(f)+\Delta(h)\right)\|f-h\|_{\alpha,\lambda}$$

with $C_N^{(4)}=c_{\alpha,T}^{(1)}c_\alpha\left(1+\frac{T^{\beta-\alpha}}{\beta-\alpha}\right)(M_0+M_N)$. □

Finally, we shall give similar estimates on the ordinary Lebesgue integrals

$$F_t(f) = \int_0^t f(s)ds \quad \text{and} \quad F_t^{(b)}(f) = \int_0^t b(s, f(s))\, ds\,.$$

Proposition 4.3

Let $0 < \alpha < \frac{1}{2}$ and $f : [0,T] \to \mathbb{R}^d$ be a measurable function.

If $\sup_{t\in[0,T]} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds < \infty$ (in particular if $\|f\|_\infty < \infty$) then $F_\cdot(f) \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and

$$|F_t(f)| + \int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}}\, ds \le C_{\alpha,T} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds. \tag{4.22}$$

If $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ then $F_\cdot(f) \in C^1(0,T;\mathbb{R}^d)$,

$$|F_t(f) - F_s(f)| = \left| \int_s^t f dr \right| \le (t-s) \|f\|_\infty\,, \tag{4.23}$$

and

$$\|F(f)\|_{\alpha,\infty} = \left\| \int_0^\cdot f ds \right\|_{\alpha,\infty} \le C'_{\alpha,T} \|f\|_\infty\,. \tag{4.24}$$

Proof. I. We have

$$\begin{aligned}
|F_t(f)| + \int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}}\, ds &\le \int_0^t |f(s)|\, ds + \int_0^t (t-s)^{-\alpha-1} \left(\int_s^t |f(r)|\, dr \right) ds \\
&\le \int_0^t |f(s)|\, ds + \frac{1}{\alpha} \int_0^t (t-r)^{-\alpha} \quad |f(r)|\, dr \\
&\le C_{\alpha,T} \int_0^t (t-r)^{-\alpha} \quad |f_r|\, dr,
\end{aligned}$$

and hence (4.22), (4.23), (4.24) hold with $C_{\alpha,T} = T^\alpha + \frac{1}{\alpha}$ and $C'_{\alpha,T} = \frac{T^{1-\alpha}}{1-\alpha} C_{\alpha,T}$. □

Proposition 4.4

Assume that $b = \left(b^i\right)_{d\times 1} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the assumptions (H_2) with constant $\rho = \frac{1}{\alpha}$. If $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ then $F_\cdot^{(b)}(f) = \int_0^\cdot b(s, f(s))\, ds \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ and

$$\begin{aligned}
&j)\ \left\| F^{(b)}(f) \right\|_{1-\alpha} \le d^{(1)} \left(1 + \|f\|_\infty\right), \\
&jj)\ \left\| F^{(b)}(f) \right\|_{\alpha,\lambda} \le \frac{d^{(2)}}{\lambda^{1-2\alpha}} \left(1 + \|f\|_{\alpha,\lambda}\right),
\end{aligned} \tag{4.25}$$

for all $\lambda \ge 1$, where $d^{(i)}$ $i \in \{1,2\}$ are positive constants depending only on α, T, L_0, and $B_{0,\alpha} = \|b_0\|_{L^{1/\alpha}}$.

If $f, h \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ are such that $\|f\|_\infty \le N$, $\|h\|_\infty \le N$, then

$$\left\| F^{(b)}(f) - F^{(b)}(h) \right\|_{\alpha,\lambda} \le \frac{d_N}{\lambda^{1-\alpha}} \|f - h\|_{\alpha,\lambda} \tag{4.26}$$

for all $\lambda \ge 1$, where $d_N = C_{\alpha,T}\, L_N \Gamma(1-\alpha)$ depends on α, T and L_N from (H_2).

Proof. In order to simplify the presentation of the proof we assume $d = 1$. Let $f \in W_0^{\alpha,\infty}(0,T)$. Then for $0 \leq s < t \leq T$:

$$\begin{aligned}|F_t^{(b)}(f) - F_s^{(b)}(f)| &\leq \int_s^t |b(r, f(r))|\, dr \leq \int_s^t (L_0\, |f(r)| + b_0(r))\, dr \\ &\leq L_0\,(t-s)\, \|f\|_\infty + (t-s)^{1-\alpha} \left(\int_s^t (b_0(r))^{1/\alpha}\, dr \right)^\alpha \\ &\leq (L_0\, T^\alpha\, \|f\|_\infty + B_{0,\alpha})\,(t-s)^{1-\alpha},\end{aligned}$$

where $B_{0,\alpha} := \|b_0\|_{L^{1/\alpha}}$. As a consequence $F^{(b)}(f) \in C^{1-\alpha}(0,T)$ and the inequality (4.25-j) holds with $d^{(1)} = (L_0\, T^\alpha + B_{0,\alpha})\left(1 + T^{1-\alpha}\right)$.

By (4.22) we have:

$$\begin{aligned}&\left|F_t^{(b)}(f)\right| + \int_0^t \frac{|F_t^{(b)}(f) - F_s^{(b)}(f)|}{(t-s)^{\alpha+1}}\, ds \\ &\quad \leq C_{\alpha,T} \int_0^t \frac{|b(s, f(s))|}{(t-s)^\alpha} ds \\ &\quad \leq C_{\alpha,T} \int_0^t \frac{(L_0\, |f(s)| + b_0(s))}{(t-s)^\alpha} ds \\ &\quad \leq C_{\alpha,T} \left[L_0 \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds + \left(\int_0^t (t-s)^{-\alpha/(1-\alpha)}\, ds \right)^{1-\alpha} B_{0,\alpha} \right]. \end{aligned} \tag{4.27}$$

Hence

$$\begin{aligned}\left\| F^{(b)}(f) \right\|_{\alpha,\lambda} &\leq C_{\alpha,T}\, L_0 \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds \\ &\quad + C_{\alpha,T} \left(\frac{1-\alpha}{1-2\alpha} \right)^{1-\alpha} B_{0,\alpha} \sup_{t\in[0,T]} e^{-\lambda t} t^{1-2\alpha} \\ &\leq C_{\alpha,T}\, L_0 \lambda^{\alpha-1} \Gamma(1-\alpha) \sup_{s\in[0,T]} e^{-\lambda s}\, |f(s)| \\ &\quad + C_{\alpha,T}\, (1-2\alpha)^{-\alpha} (1-\alpha)^{1-\alpha} e^{2\alpha-1} B_{0,\alpha} \lambda^{2\alpha-1} \\ &\leq \lambda^{2\alpha-1} d^{(2)} \left(1 + \|f\|_{\alpha,\lambda} \right)\end{aligned}$$

for all $\lambda \geq 1$, that is (4.25-jv) with

$$d^{(2)} = C_{\alpha,T} \left[L_0\, \Gamma(1-\alpha) + (1-2\alpha)^{-\alpha} (1-\alpha)^{1-\alpha} e^{2\alpha-1} B_{0,\alpha} \right].$$

Let $f, h \in W_0^{\alpha,\infty}(0,T)$ be such that $\|f\|_\infty \leq N$, $\|h\|_\infty \leq N$. Since

$$F_t^{(b)}(f) - F_t^{(b)}(h) = \int_0^t (b(s, f(s)) - b(s, h(s)))\, ds$$

then by (4.22), we have

$$\begin{aligned}\left\|F^{(b)}(f)-F^{(b)}(g)\right\|_{\alpha,\lambda} &\le C_{\alpha,T}\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \frac{|b(s,f(s))-b(s,h(s)|}{(t-s)^{\alpha}}ds\\ &\le C_{\alpha,T}\,L_N \sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \frac{|f(s)-h(s)|}{(t-s)^{\alpha}}ds\\ &\le C_{\alpha,T}\,L_N\,\|f-h\|_{\alpha,\lambda}\,\lambda^{\alpha-1}\Gamma(1-\alpha)\\ &=\frac{1}{\lambda^{1-\alpha}}d_N\,\|f-h\|_{\alpha,\lambda}\ ,\end{aligned}$$

where $d_N = C_{\alpha,T}\,L_N\Gamma(1-\alpha)$. □

Remark 4.3. If $L_N \equiv L_0$, $\forall N > 0$ then, in Proposition 4.4 the constant d_N is independent of N. Similarly, if $M_N \equiv M_0$, $\forall N > 0$ then, in Proposition 4.2 the constant $C_N^{(4)}$ is independent of N.

5. Deterministic differential equations

Let $0 < \alpha < 1/2$ be fixed. Let $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$. Consider the deterministic differential equation on $\mathbb{R}^d$

$$x_t^i = x_0^i + \int_0^t b^i(s,x_s)ds + \sum_{j=1}^m \int_0^t \sigma^{i,j}(s,x_s)\,dg_s^j,\ \ t\in[0,T]\,, \tag{5.28}$$

$i = 1,...,d$, where $x_0 \in \mathbb{R}^d$, and the coefficients $\sigma^{i,j}, b^i : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ are measurable functions satisfying the assumptions (H_1) and (H_2), respectively with $\rho = 1/\alpha$, $0 < \beta,\delta \le 1$ and

$$0<\alpha<\alpha_0=\min\left\{\frac{1}{2},\beta,\frac{\delta}{1+\delta}\right\}.$$

Then we can state the main result of this section.

Theorem 5.1

Equation (5.28) has a unique solution $x \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. Moreover the solution is $(1-\alpha)$-Hölder continuous.

Proof. First we remark that if $x \in W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ is a solution of Equation (5.28), then $x \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right)$. Indeed, for all $u \in W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ we have $G^{(\sigma)}(u) \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right)$ by Proposition 4.2 and $F^{(b)}(u) \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right)$ by Proposition 4.4. Hence,

$$x = x_0 + F^{(b)}(x) + G^{(\sigma)}(x) \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right).$$

Uniqueness. Let x and $\widetilde{x}$ be two solutions and choose N such that $\|x\|_{1-\alpha} \le N$, and $\|\widetilde{x}\|_{1-\alpha} \le N$. Then

$$x_t = x_0 + F_t^{(b)}(x) + G_t^{(\sigma)}(x)\ \text{ and }\ \widetilde{x}_t = x_0 + F_t^{(b)}(\widetilde{x}) + G_t^{(\sigma)}(\widetilde{x})$$

and by Propositions 4.2 and 4.4 we have

$$
\begin{aligned}
\|x-\widetilde{x}\|_{\alpha,\lambda} &\leq \left\|F^{(b)}(x)-F^{(b)}(\widetilde{x})\right\|_{\alpha,\lambda}+\left\|G^{(\sigma)}(x)-G^{(\sigma)}(\widetilde{x})\right\|_{\alpha,\lambda} \\
&\leq \frac{d_N}{\lambda^{1-\alpha}}\|x-\widetilde{x}\|_{\alpha,\lambda}+\frac{1}{\lambda^{1-2\alpha}}\Lambda_\alpha(g)C_N^{(4)}\left(1+\Delta(x)+\Delta(\widetilde{x})\right)\|x-\widetilde{x}\|_{\alpha,\lambda},
\end{aligned}
$$

for all $\lambda \geq 1$, where

$$
\begin{aligned}
\Delta(x)+\Delta(\widetilde{x}) &= \sup_{r\in[0,T]}\int_0^r \frac{|x_r-x_s|^\delta}{(r-s)^{\alpha+1}}\,ds+\sup_{r\in[0,T]}\int_0^r \frac{|\widetilde{x}_r-\widetilde{x}_s|^\delta}{(r-s)^{\alpha+1}}\,ds \\
&\leq 2N\sup_{r\in[0,T]}\int_0^r \frac{(r-s)^{(1-\alpha)\delta}}{(r-s)^{\alpha+1}}\,ds \\
&= 2N\frac{T^{\delta-\alpha(1+\delta)}}{\delta-\alpha(1+\delta)}=C_N.
\end{aligned}
$$

If we put λ sufficiently large such that

$$
\frac{d_N}{\lambda^{1-\alpha}}+\frac{1}{\lambda^{1-2\alpha}}\Lambda_\alpha(g)C^{(4)}C_N\leq\frac{1}{2}
$$

then we obtain

$$
\frac{1}{2}\|x-\widetilde{x}\|_{\alpha,\lambda}\leq 0,
$$

and as a consequence $x=\widetilde{x}$.

Existence. We shall prove the existence of the solution by a fixed point argument in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ using Lemma 7.2.

Consider the operator $\mathcal{L}: W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)\to C^{1-\alpha}\left(0,T;\mathbb{R}^d\right)\subset W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ defined by

$$
\begin{aligned}
x=\mathcal{L}(u) &= x_0+\int_0^\cdot b(s,u_s)ds+\int_0^\cdot \sigma(s,u_s)\,dg_s \\
&= x_0+F^{(b)}(u)+G^{(\sigma)}(u).
\end{aligned}
$$

By Propositions 4.2 and 4.4 we have for all $\lambda\geq 1$:

$$
\begin{aligned}
\|\mathcal{L}(u)\|_{\alpha,\lambda} &\leq |x_0|+\left\|F^{(b)}(u)\right\|_{\alpha,\lambda}+\left\|G^{(\sigma)}(u)\right\|_{\alpha,\lambda} \\
&\leq |x_0|+\frac{d^{(2)}}{\lambda^{1-2\alpha}}\left(1+\|u\|_{\alpha,\lambda}\right)+\frac{1}{\lambda^{1-2\alpha}}\Lambda_\alpha(g)C^{(3)}\left(1+\|u\|_{\alpha,\lambda}\right).
\end{aligned}
$$

Let $\lambda=\lambda_0$ be sufficiently large such that

$$
\frac{1}{\lambda_0^{1-2\alpha}}\left(d^{(4)}+\Lambda_\alpha(g)C^{(5)}\right)\leq\frac{1}{2}.
$$

If $\|u\|_{\alpha,\lambda_0}\leq 2(1+|x_0|)$, then $\|\mathcal{L}(u)\|_{\alpha,\lambda_0}\leq 2(1+|x_0|)$ and hence

$$
\mathcal{L}(B_0)\subset B_0
$$

where $B_0 = \left\{ u \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) : \|u\|_{\alpha,\lambda_0} \le 2(1+|x_0|) \right\}$. Hence condition i) of Lemma 7.2 is fulfilled with B_0 the ball centered at the origin of radius $r_0 = 2(1+|x_0|)$ with respect to the metric ρ_0 associated to the norm $\|\cdot\|_{\alpha,\lambda_0}$.

Remark that for all $u \in B_0$

$$\|u\|_{\alpha,\infty} \le 2(1+|x_0|)\, e^{\lambda_0 T}. \tag{5.29}$$

Once again, from Propositions 4.2 and 4.4 we have for all $u, v \in B_0 \supset \mathcal{L}(B_0)$ and for all $\lambda \ge 1$:

$$\begin{aligned}
&\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\alpha,\lambda} \\
&\quad \le \left\| F^{(b)}(u) - F^{(b)}(v) \right\|_{\alpha,\lambda} + \left\| G^{(\sigma)}(u) - G^{(\sigma)}(v) \right\|_{\alpha,\lambda} \\
&\quad \le \frac{d_{N_0}}{\lambda^{1-\alpha}} \|u - v\|_{\alpha,\lambda} + \frac{1}{\lambda^{1-2\alpha}} \Lambda_\alpha(g) C_{N_0}^{(4)} (1 + \Delta(u) + \Delta(v)) \|u - v\|_{\alpha,\lambda} \\
&\quad \le \frac{C_1}{\lambda^{1-2\alpha}} (1 + \Delta(u) + \Delta(v)) \|u - v\|_{\alpha,\lambda},
\end{aligned} \tag{5.30}$$

where $C_1 = d_{N_0} + \Lambda_\alpha(g) C_{N_0}^{(4)}$ and

$$\Delta(u) = \sup_{r \in [0,T]} \int_0^r \frac{|u_r - u_s|^\delta}{(r-s)^{\alpha+1}}\, ds.$$

Clearly $\Delta : W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \to [0,+\infty]$ is a lower semicontinuous function since the convergence in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ implies the uniform convergence.

If $u \in \mathcal{L}(B_0)$ then there exists $\overline{u} \in B_0$ such that $u = \mathcal{L}(\overline{u}) \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right)$. Then from Propositions 4.2 and 4.4 and using (5.29) we get

$$\begin{aligned}
\|u\|_{1-\alpha} = \|\mathcal{L}(\overline{u})\|_{1-\alpha} &\le |x_0| + \left\| F^{(b)}(u) \right\|_{1-\alpha} + \left\| G^{(\sigma)}(u) \right\|_{1-\alpha} \\
&\le |x_0| + d^{(1)} (1 + \|u\|_\infty) + \Lambda_\alpha(g) C^{(2)} \left(1 + \|u\|_{\alpha,\infty}\right) \\
&\le |x_0| + \left(d^{(1)} + \Lambda_\alpha(g) C^{(2)}\right) \left(1 + e^{\lambda_0 T} 2(1+|x_0|\right) =: C_2.
\end{aligned}$$

As a consequence

$$\begin{aligned}
\Delta(u) &= \sup_{r \in [0,T]} \int_0^r \frac{|\mathcal{L}(\overline{u})_r - \mathcal{L}(\overline{u})_s|^\delta}{(r-s)^{\alpha+1}}\, ds \\
&\le \sup_{r \in [0,T]} \int_0^r \frac{C_2 (t-s)^{(1-\alpha)\delta}}{(r-s)^{\alpha+1}}\, ds \\
&= \frac{C_2}{\delta - \alpha(1+\delta)} T^{\delta - \alpha(1+\delta)} =: C_4.
\end{aligned} \tag{5.31}$$

Therefore, from (5.30) and (5.31) condition ii) from Lemma 7.2 is satisfied for the metric associated with the norm $\|\cdot\|_{\alpha,1}$ and $\varphi(u) = C_1 (1/2 + \Delta(u))$.

From (5.30) and (5.31) we have

$$\|\mathcal{L}(u)-\mathcal{L}(v)\|_{\alpha,\lambda} \le \frac{C_1(1+2C_4)}{\lambda^{1-2\alpha}}\|u-v\|_{\alpha,\lambda},$$

for all $u,v\in\mathcal{L}(B_0)$. Let $\lambda=\lambda_2$ be sufficiently large such that

$$\frac{C_1(1+2C_4)}{\lambda^{1-2\alpha}} \le \frac{1}{2}.$$

Then

$$\|\mathcal{L}(u)-\mathcal{L}(v)\|_{\alpha,\lambda_2} \le \frac{1}{2}\|u-v\|_{\alpha,\lambda_2}, \quad \forall u,v\in\mathcal{L}(B_0),$$

that is the condition iii) from Lemma 7.2 is satisfied. Hence there exists $x\in\mathcal{L}(B_0)$ such that

$$x=\mathcal{L}(x)=x_0+\int_0^{\cdot} b(s,x_s)ds+\int_0^{\cdot}\sigma(s,x_s)\,dg_s.$$

This completes the proof of the theorem. □

Suppose now that the coefficient σ satisfies assumption (H_3) from Section 2. In that case we can derive a useful estimate on the solution as follows. Notice that (H_1) implies that the growth condition (2.4) holds for $\gamma=1$.

Proposition 5.1

Suppose that the coefficient σ satisfies the assumptions of Theorem 5.1, and in addition, assumption (H_3) holds. Then, the solution x of Equation (5.28) satisfies

$$\|x\|_{\alpha,\infty} \le C_1\exp\left(C_2\Lambda_\alpha(g)^\kappa\right),$$

where

$$\kappa=\begin{cases}\dfrac{1}{1-2\alpha} & \text{if } \gamma=1\\ >\dfrac{\gamma}{1-2\alpha} & \text{if } \dfrac{1-2\alpha}{1-\alpha}\le\gamma<1\\ \dfrac{1}{1-\alpha} & \text{if } 0\le\gamma<\dfrac{1-2\alpha}{1-\alpha}\end{cases}$$

and the constants C_1 and C_2 depend on T, α, γ, and the constants that appear in conditions (H_1), (H_2) and (H_3).

Proof. We denote by C a generic positive constant, depending on T, α, γ, and the constants that appear in assumptions (H_1), (H_2) and (H_3). Let $x_t=x_0+G_t^{(\sigma)}(x)+F_t^{(b)}(x)$. Using (4.18) we obtain

$$\begin{aligned}\left|G_t^{(\sigma)}(x)\right| &\le \Lambda_\alpha(g)\left(\int_0^t\frac{|\sigma(s,x_s)|}{s^\alpha}ds+\alpha\int_0^t\int_0^s\frac{|\sigma(s,x_s)-\sigma(r,x_r)|}{(s-r)^{\alpha+1}}drds\right)\\ &\le \Lambda_\alpha(g)\left(K_0\int_0^t\frac{1+|x_s|^\gamma}{s^\alpha}ds+\alpha M_0\int_0^t\int_0^s\frac{|x_s-x_r|}{(s-r)^{\alpha+1}}\,drds\right.\\ &\quad\left.+\frac{\alpha M_0 t^{\beta-\alpha+1}}{(\beta-\alpha)(\beta-\alpha+1)}\right)\\ &\le C\Lambda_\alpha(g)\left[1+\int_0^t\left(s^{-\alpha}|x_s|^\gamma ds+s^{-\alpha}\int_0^s\frac{|x_s-x_r}{(s-r)^{\alpha+1}}\,dr\right)ds\right].\end{aligned}$$

On the other hand, using (4.17) we can write

$$\int_0^t \frac{|G_t^{(\sigma)}(x) - G_s^{(\sigma)}(x)|}{(t-s)^{\alpha+1}} \, ds \leq \Lambda_\alpha(g) \left(c_\alpha^{(1)} \int_0^t \frac{|\sigma(s, x_s)|}{(t-s)^{2\alpha}} ds + \int_0^t \int_0^s \frac{|\sigma(s, x_s) - \sigma(r, x_r)|}{(s-r)^{\alpha+1}} (t-r)^{-\alpha} dr ds \right),$$

which leads to

$$\int_0^t \frac{|G_t^{(\sigma)}(x) - G_s^{(\sigma)}(x)|}{(t-s)^{\alpha+1}} \, ds \leq C\Lambda_\alpha(g) \left[1 + \int_0^t \left((t-s)^{-2\alpha} |x_s|^\gamma + (t-s)^{-\alpha} \int_0^s \frac{|x_s - x_r|}{(s-r)^{\alpha+1}} \, dr \right) ds \right].$$

On the other hand, from (4.27) we get

$$\left| F_t^{(b)}(x) \right| + \int_0^t \frac{|F_t^{(b)}(x) - F_s^{(b)}(x)|}{(t-s)^{\alpha+1}} \, ds \leq C \left[1 + \int_0^t (t-s)^{-\alpha} |x_s| ds \right].$$

Set

$$h_t = |x_t| + \int_0^t \frac{|x_t - x_s|}{(t-s)^{\alpha+1}} \, ds.$$

Then the above estimates lead to the following inequality

$$h_t \leq C(1 + \Lambda_\alpha(g)) \left[1 + \int_0^t \left[(t-s)^{-\varepsilon(\gamma)} + s^{-\alpha} \right] h_s ds \right] \tag{5.32}$$

where the exponent $\varepsilon(\gamma)$ depends on the values of γ, according to the following three different cases:

i) If $\gamma = 1$ we take $\varepsilon(\gamma) = 2\alpha$.

ii) If $\frac{1-2\alpha}{1-\alpha} \leq \gamma < 1$, take $\varepsilon(\gamma) > 1 + \frac{2\alpha-1}{\gamma}$. Applying Hölder inequality with $\delta' = \varepsilon(\gamma)\gamma > 2\alpha + \gamma - 1$ we obtain

$$\int_0^t |x_s|^\gamma (t-s)^{-2\alpha} ds \leq \left(\int_0^t |x_s| (t-s)^{-\delta'/\gamma} ds \right)^\gamma \left(\int_0^t (t-s)^{(-2\alpha+\delta')/(1-\gamma)} ds \right)^{1-\gamma} \leq C \left(1 + \int_0^t |x_s| (t-s)^{-\varepsilon(\gamma)} ds \right).$$

This allows to deduce (5.32) in this case.

iii) If $0 \leq \gamma < \frac{1-2\alpha}{1-\alpha}$, take $\varepsilon(\gamma) = \alpha$, and use the same argument as in the step ii).

The inequality (5.32) implies

$$h_t \leq C(1 + \Lambda_\alpha(g)) \left[1 + \left(1 + T^{\varepsilon(\gamma)-\alpha} \right) \int_0^t t^{\varepsilon(\gamma)} (t-s)^{-\varepsilon(\gamma)} s^{-\varepsilon(\gamma)} h_s ds \right].$$

As a consequence, the Gronwall-type Lemma 7.6 yields

$$\begin{aligned}\|x\|_{\alpha,\infty} &\le (1+\Lambda_\alpha(g))\, C d_\alpha \exp\left(c_\alpha T \left(C\left(1+\Lambda_\alpha(g)\right)\right)^{1/[1-\varepsilon(\gamma)]}\right) \\ &\le C_1 \exp\left(C_2 \Lambda_\alpha(g)^{1/[1-\varepsilon(\gamma)]}\right).\end{aligned}$$

which completes the proof. □

6. Stochastic integrals and equations with respect to the fractional Brownian motion

Fix a parameter $1/2 < H < 1$. Let $B = \{B_t, t \in [0,T]\}$ be a fractional Brownian motion with parameter H defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that (1.1) implies

$$\mathbb{E}(|B_t - B_s|^2) = |t-s|^{2H},$$

and, as a consequence, for any $p \ge 1$,

$$\|B_t - B_s\|_p = (\mathbb{E}(|B_t - B_s|^p))^{1/p} = c_p |t-s|^H. \tag{6.33}$$

By Lemma 7.5, proved in the Appendix, the random variable

$$G = \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left|\left(D_{t-}^{1-\alpha} B_{t-}\right)(s)\right| \tag{6.34}$$

has moments of all orders. As a consequence, if $u = \{u_t,\ t \in [0,T]\}$ is a stochastic process whose trajectories belong to the space $W_T^{\alpha,1}(0,T)$, with $1-H < \alpha < 1/2$, the pathwise integral $\int_0^T u_s dB_s$ exists in the sense of Definition 4.1 and we have the estimate

$$\left|\int_0^T u_s dB_s\right| \le G \|u\|_{\alpha,1}.$$

Moreover, if the trajectories of the process u belong to the space $W_0^{\alpha,\infty}(0,T)$, then the indefinite integral $U_t = \int_0^t u_s dB_s$ is Hölder continuous of order $1-\alpha$, and the estimates (4.13) hold.

Proof of Theorem 2.1 The existence and uniqueness of a solution follows directly from the deterministic Theorem 5.1. Moreover the solution is Hölder continuous of order $1-\alpha$.

By condition (2.4) and Proposition 5.1 we obtain

$$\|X\|_{\alpha,\infty} \le C_1 \exp\left(C_2 G^{\,\kappa}\right),$$

where G is the random variable defined in (6.34). Hence, for all $p \ge 1$

$$E\|X\|_{\alpha,\infty}^p \le C_1 E \exp\left(p C_2 G^\kappa\right) < \infty \tag{6.35}$$

provided $\kappa < 2$. This implies that (6.35) holds if $\gamma/4 + 1/2 < 1 - \alpha < H$. If $\gamma = 1$ this means $H > 3/4$ and $\alpha < 1/4$. □

7. Appendix

In this section we will recall some technical tools used in the paper and we will show some technical lemmas that have been used in the proof of Theorem 5.1.

Lemma 7.1

Let $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying assumption (H_1). Then for all $N > 0$ and $|x_1|, |x_2|, |x_3|, |x_4| \leq N$:

$$\begin{aligned} &\left|\sigma(t_1,x_1) - \sigma(t_2,x_2) - \sigma(t_1,x_3) + \sigma(t_2,x_4)\right| \\ &\leq M_0 \left|x_1 - x_2 - x_3 + x_4\right| + M_0 \left|x_1 - x_3\right| \left|t_2 - t_1\right|^{\beta} \\ &\quad + M_N \left|x_1 - x_3\right| \left(\left|x_1 - x_2\right|^{\delta} + \left|x_3 - x_4\right|^{\delta}\right). \end{aligned}$$

Proof. By the mean value theorem we can write

$$\begin{aligned} \sigma(t_1,x_1) &- \sigma(t_2,x_2) - \sigma(t_1,x_3) + \sigma(t_2,x_4) \\ &= \int_0^1 (x_1 - x_3)\partial_x\sigma(t_1, \theta x_1 + (1-\theta)x_3)d\theta \\ &\quad - \int_0^1 (x_2 - x_4)\partial_x\sigma(t_2, \theta x_2 + (1-\theta)x_4)d\theta \\ &= \int_0^1 (x_1 - x_2 - x_3 + x_4)\partial_x\sigma(t_2, \theta x_2 + (1-\theta)x_4)d\theta \\ &\quad + \int_0^1 (x_1 - x_3)\left[\partial_x\sigma(t_1, \theta x_1 + (1-\theta)x_3) - \partial_x\sigma(t_2, \theta x_2 + (1-\theta)x_4)\right]d\theta. \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} &\left|\sigma(t_1,x_1) - \sigma(t_2,x_2) - \sigma(t_1,x_3) + \sigma(t_2,x_4)\right| \\ &\leq M_0 \left|x_1 - x_2 - x_3 + x_4\right| + M_0 \left|x_1 - x_3\right| \left|t_2 - t_1\right|^{\beta} \\ &\quad + M_N \left|x_1 - x_3\right| \left(\left|x_1 - x_2\right|^{\delta} + \left|x_3 - x_4\right|^{\delta}\right). \;\square \end{aligned}$$

We present now a fixed point result used in the paper.

Lemma 7.2

Let (X,ρ) be a complete metric space and ρ_0, ρ_1, ρ_2 some metrics on X equivalent to ρ. If $\mathcal{L} : X \to X$ satisfies:

i) there exists $r_0 > 0$, $x_0 \in X$ such that if $B_0 = \{x \in X : \rho_0(x_0, x) \leq r_0\}$ then

$$\mathcal{L}(B_0) \subset B_0,$$

ii) there exists $\varphi : (X, \rho) \to [0, +\infty]$ *lower semicontinuous function and some positive constants* C_0, K_0 *such that denoting* $N_\varphi(a) = \{x \in X : \varphi(x) \le a\}$

$$\begin{aligned} &a) \quad \mathcal{L}(B_0) \subset N_\varphi(C_0), \\ &b) \quad \rho_1(\mathcal{L}(x), \mathcal{L}(y)) \le K_0 \rho_1(x, y), \ \forall x, y \in N_\varphi(C_0) \cap B_0, \end{aligned}$$

iii) there exists $a \in (0, 1)$ *such that*

$$\rho_2(\mathcal{L}(x), \mathcal{L}(y)) \le a\, \rho_2(x, y), \ \forall x, y \in \mathcal{L}(B_0),$$

then there exists $x^* \in \mathcal{L}(B_0) \subset X$ *such that*

$$x^* = \mathcal{L}(x^*).$$

Proof. Let

$$x_{n+1} = \mathcal{L}(x_n), \quad n \in \mathbb{N}.$$

Then $x_n \in \mathcal{L}(B_0)$ and $\varphi(x_n) \le C_0$, for all $n \in \mathbb{N}^*$. Also

$$\rho_2(x_{n+1}, x_n) = \rho_2(\mathcal{L}(x_n), \mathcal{L}(x_{n-1})) \le a\rho_2(x_n, x_{n-1}) \le \ldots \le a^n \rho_2(x_1, x_0)$$

which yields

$$\rho_2(x_{n+p}, x_n) \le \frac{a^n}{1-a} \rho_2(x_1, x_0) \to 0, \text{ as } n \to \infty.$$

Since (X, ρ) is a complete metric space and B_0 is closed in X, then there exists $x^* \in B_0$ such that $x_n \to x^*$ and, moreover, by the lower semicontinuous property of φ, $\varphi(x^*) \le C_0$. From ii-b) we have

$$\rho_1(\mathcal{L}(x_n), \mathcal{L}(x^*)) \le K_0\, \rho_1(x_n, x^*) \to 0.$$

Hence $\mathcal{L}(x_n) \to \mathcal{L}(x^*)$ and $x^* = \mathcal{L}(x^*)$. □

The following lemma is the so-called Garsia-Rademich-Rumsey inequality (see [9]):

Lemma 7.3

Let $p \ge 1$, *and* $\alpha > p^{-1}$. *Then there exists a constant* $C_{\alpha,p} > 0$ *such that for any continuous function* f *on* $[0, T]$, *and for all* $t, s \in [0, T]$ *one has:*

$$|f(t) - f(s)|^p \le C_{\alpha,p} \ |t - s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} dx dy \tag{7.36}$$

(with the convention $0/0 = 0$*).*

The following lemmas provide basic inequalities for the fractional Brownian motion.

Lemma 7.4

Let $\{B_t : t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then for every $0 < \varepsilon < H$ and $T > 0$ there exists a positive random variable $\eta_{\varepsilon,T}$ such that $\mathbb{E}\left(|\eta_{\varepsilon,T}|^p\right) < \infty$ for all $p \in [1,\infty)$ and for all $s,t \in [0,T]$

$$|B(t) - B(s)| \leq \eta_{\varepsilon,T} |t-s|^{H-\varepsilon} \quad \text{a.s.} \tag{7.37}$$

Proof. Applying the inequality (7.36) with $\alpha = H - \varepsilon/2$ and $p = 2/\varepsilon$ we deduce for all $s,t \in [0,T]$

$$|B(t) - B(s)| \leq C_{H,\varepsilon} |t-s|^{H-\varepsilon} \xi,$$

where

$$\xi = \left(\int_0^T \int_0^T \frac{|B(r) - B(\theta)|^{2/\varepsilon}}{|r-\theta|^{(2H)/\varepsilon}} dr d\theta \right)^{\varepsilon/2}.$$

Let $q \geq 2/\varepsilon$. By Minkowski inequality and the estimate (6.33) we obtain

$$\begin{aligned} \|\xi\|_q^q &\leq \left(\int_0^T \int_0^T \frac{\left\| |B(r) - B(\theta)|^{2/\varepsilon} \right\|_{(q\varepsilon)/2}}{|r-\theta|^{(2H)/\varepsilon}} dr d\theta \right)^{(q\varepsilon)/2} \\ &\leq c_{\varepsilon,q} T^{q\varepsilon}. \end{aligned}$$

Hence, it suffices to take $\eta_{\varepsilon,T} = C_{H,\varepsilon}\xi$. □

Lemma 7.5

Let $\{B_t : t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$. If $1 - H < \alpha < \frac{1}{2}$ then

$$\mathbb{E} \sup_{0 \leq s \leq t \leq T} \left| D_{t-}^{1-\alpha} B_{t-}(s) \right|^p < \infty. \tag{7.38}$$

for all $T > 0$ and $p \in [1,\infty)$.

Proof. From (3.6) we have

$$\begin{aligned} \left| D_{t-}^{1-\alpha} B_{t-}(s) \right| \leq \frac{1}{\Gamma(\alpha)} \Bigg(&\frac{|B(t) - B(s)|}{(t-s)^{1-\alpha}} \\ &+ (1-\alpha) \int_s^t \frac{|B(s) - B(y)|}{(y-s)^{2-\alpha}} dy \Bigg). \end{aligned}$$

By (7.37) for $\varepsilon < \alpha - (1-H)$ there exists a random variable $\eta_{\varepsilon,T}$ with finite moments of all orders such that

$$\begin{aligned}\left|D_{t-}^{1-\alpha} B_{t-}(s)\right| &\le C_\alpha \eta_{\varepsilon,T}\left((t-s)^{H-\varepsilon-1+\alpha} + \int_s^t (s-y)^{H-\varepsilon-2+\alpha}\,dy\right)\\ &\le C_\alpha \eta_{\varepsilon,T} T^{H-\varepsilon-1+\alpha}\left(1+\frac{1}{H-\varepsilon-1+\alpha}\right),\end{aligned}$$

which yields clearly (7.38). □

Finally, let us give a version of the Gronwall lemma.

Lemma 7.6

Fix $0 \le \alpha < 1$, $a, b \ge 0$. Let $x : [0,\infty) \to [0,\infty)$ be a continuous function such that for each t

$$x_t \le a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds. \tag{7.39}$$

Then

$$x_t \le a\Gamma(1-\alpha)\sum_{n=0}^{\infty} \frac{\left(b\Gamma(1-\alpha)t^{1-\alpha}\right)^n}{\Gamma[(n+1)(1-\alpha)]} \le a d_\alpha \exp\left[c_\alpha t b^{1/(1-\alpha)}\right], \tag{7.40}$$

where c_a and d_α are positive constants depending only on α (as an example, one can set $c_\alpha = 2\left(\Gamma(1-\alpha)\right)^{1/(1-\alpha)}$ and $d_\alpha = 4e^2 \frac{\Gamma(1-\alpha)}{1-\alpha}$).

Proof. Define by an iteration procedure the sequences

$$\begin{aligned}&\beta_0(t) = 1 \quad \text{and} \quad R_0(t) = x_t\,,\\ &\beta_{n+1}(t) = bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} \beta_n(s)\,ds = bt^{1-\alpha}\int_0^1 (1-r)^{-\alpha} r^{-\alpha}\beta_n(rt)\,dr,\\ &R_{n+1}(t) = bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} R_n(s)\,ds\,.\end{aligned}$$

It is easy to verify by induction that for all $0 \le t \le T$

$$\begin{aligned}0 \le R_n(t) &\le \left(\sup_{s\in[0,T]} |x_s|\right)\beta_n(t)\,, \quad \text{and}\\ \beta_n(t) &= \Gamma(1-\alpha)\frac{\left(b\Gamma(1-\alpha)t^{1-\alpha}\right)^n}{\Gamma[(n+1)(1-\alpha)]}\,.\end{aligned}$$

Also by induction we have

$$x_t \le a\sum_{n=0}^{N}\beta_n(t) + R_{N+1}(t)\,. \tag{7.41}$$

Indeed for $N=0$ the relation (7.41) reduce to (7.39). Supposing that (7.41) holds for N, then

$$\begin{aligned} x_t &\leq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} x_s ds \\ &\leq a + bt^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha} \left[a \sum_{n=0}^{N} \beta_n(s) + R_{N+1}(s) \right] ds \\ &= a \sum_{n=0}^{N+1} \beta_n(t) + R_{N+2}(t) \,. \end{aligned}$$

The estimates (7.40) follow clearly now setting $x = b\Gamma(1-\alpha)t^{1-\alpha}$ and $\beta = 1-\alpha$ in the next lemma. $\square$

Lemma 7.7

For all $x \geq 0$ *and* $\beta > 0$,

$$S(x) := \sum_{n=0}^{\infty} \frac{x^n}{\Gamma((n+1)\beta)} \leq \frac{4e}{\beta} (1 \vee x)^{2/\beta - 1} \; e^{(1\vee x)^{1/\beta}} \leq \frac{4e^2}{\beta} e^{2x^{1/\beta}} \,. \tag{7.42}$$

Proof. Let $x \geq 1$. We split $S(x) = S_1(x) + S_2(x)$ where

$$\begin{aligned} S_1(x) &= \sum_{\substack{n\in\mathbb{N} \\ (n+1)\beta < 2}} \frac{x^n}{\Gamma((n+1)\beta)}, \\ S_2(x) &= \sum_{\substack{n\in\mathbb{N} \\ (n+1)\beta \geq 2}} \frac{x^n}{\Gamma((n+1)\beta)} \,. \end{aligned}$$

We have

$$S_1(x) \leq \frac{4e}{\beta} x^{2/\beta - 1}. \tag{7.43}$$

Indeed if $\beta \geq 2$ then $S_1(x) = 0$ and if $\beta < 2$ then

$$\begin{aligned} S_1(x) &\leq x^{2/\beta-1} \sum_{\substack{n\in\mathbb{N} \\ n < \frac{2}{\beta}-1}} \frac{1}{\Gamma((n+1)\beta)} \leq x^{2/\beta-1} \sum_{\substack{n\in\mathbb{N} \\ n < \frac{2}{\beta}-1}} e\,(n+1)\,\beta \\ &\leq \frac{4e}{\beta} x^{2/\beta-1} \end{aligned}$$

since

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt > e^{-1} \int_0^1 t^{a-1} dt = \frac{1}{ae}.$$

Denote

$$J_{\delta,\beta,k_0} = \{ n \in \mathbb{N} : [n\beta + \delta] = k_0 \} = \left\{ n \in \mathbb{N} : \frac{k_0 - \delta}{\beta} \leq n < \frac{k_0 - \delta + 1}{\beta} \right\}.$$

To estimate $S_2(x)$ remark that:

$\diamondsuit$ $\sum_{n\in J_{\delta,\beta,k_0}} 1 \le \frac{1}{\beta}, \quad \forall\,\delta \ge 0,\ \beta > 0,\ k_0 \in \mathbb{N},$

$\diamondsuit$ $x^n = \frac{1}{x}\left(x^{1/\beta}\right)^{(n+1)\beta} \le \frac{1}{x}\left(x^{1/\beta}\right)^{[(n+1)\beta]+1} = x^{2/\beta-1}\left(x^{1/\beta}\right)^{[(n+1)\beta]-1},$

$\diamondsuit$ $([a]-1)! \le \Gamma(a) \le [a]!$ for all $a \ge 2$.

Hence

$$\begin{aligned} S_2(x) &\le x^{2/\beta-1} \sum_{\substack{n\in\mathbb{N}\\(n+1)\beta\ge 2}} \frac{\left(x^{1/\beta}\right)^{[(n+1)\beta]-1}}{([(n+1)\beta]-1)!} \\ &\le x^{2/\beta-1}\sum_{k=1}^{\infty}\left(\sum_{n\in J_{\beta,\beta,k+1}} 1\right)\left(x^{1/\beta}\right)^k (k!)^{-1} \\ &\le x^{2/\beta-1}\,\frac{1}{\beta}\left(e^{x^{1/\beta}}-1\right) \end{aligned}$$

and then

$$S(x) = S_1(x) + S_2(x) \le \frac{4e}{\beta} x^{2/\beta-1}\, e^{x^{1/\beta}} \le 4e/\beta\, e^{2x^{1/\beta}}, \quad \forall\, x \ge 1$$

since $x^{2a-1} \le e^{x^a}$, for all $x \ge 1$, $a > 0$.

For $x \in [0,1]$ we have $S(x) \le S(1) \le \frac{4e^2}{\beta}$, which completes the proof. □

Remark. The right-hand side of equality (7.40) can be expressed in terms of the Mitag-Leffler function (see [17]):

$$a + a\sum_{n=1}^{\infty} b^n \frac{\Gamma(1-\alpha)^{n+1} t^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)]} = a + a\Gamma(1-\alpha)\left(E_{1-\alpha,1-\alpha}(b\Gamma(1-\alpha)t^{1-\alpha}) - 1\right),$$

and the asymptotic behavior of the Mittag-Leffler function leads also to the estimate (7.40).

Acknowledgement 1. The work was carried out during a stay of Aurel Răşcanu at the "Centre de Recerca Matemàtica" from Barcelona. He would like to thank the CRM for hospitality and generous support.

Acknowledgement 2. The authors would like to thank an anonymous referee for the careful reading of the manuscript and some helpful remarks.

References

1. E. Alòs, O. Mazet, and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* **29** (2001), 766–801.
2. P. Carmona and L. Coutin, Stochastic integration with respect to fractional Brownian motion, (preprint).
3. Z. Ciesielski, G. Kerkyacharian, and B. Roynette, Quelques espaces fonctionnels associés à des processus gaussiens, *Studia Math.* **107** (1993), 171–204.
4. L. Coutin and Z. Qian, Stochastic differential equations for fractional Brownian motions, *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), 75–80.
5. L. Coutin and Z. Qian, Stochastic analysis, rough paths analysis and fractional Brownian motions, (preprint).
6. W. Dai and C.C. Heyde, Itô's formula with respect to fractional Brownian motion and its application, *J. Appl. Math. Stochastic Anal.* **9** (1996), 439–448.
7. L. Decreusefond and A.S. Üstünel, Stochastic analysis of the fractional Brownian motion, *Potential Anal.* **10** (1998), 177–214.
8. D. Feyel and A. de la Pradelle, On fractional Brownian processes, *Potential Anal.* **10** (1999), 273–288.
9. A. Garsia, E. Rademich, and H. Rumsey, A real variable lemma and the continuity of paths of some Gaussian processes, *Indiana Univ. Math. J.* **20** (1970/1971), 565–578.
10. M.L. Kleptsyna, P.E. Kloeden, and V.V. Anh, Existence and uniqueness theorems for fBm stochastic differential equations, *Problems Inform. Transmission* **34** (1999), 332–341.
11. F. Klingenhöfer and M. Zähle, Ordinary differential equations with fractal noise, *Proc. Amer. Math. Soc.* **127** (1999), 1021–1028.
12. K. Kubilius, The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion, *Lith. Math. J.* **40** (2000), 104–110.
13. S.J. Lin, Stochastic analysis of fractional Brownian motions, *Stochastics Stochastics Rep.* **55** (1995), 121–140.
14. T. Lyons, Differential equations driven by rough signals, I, An extension of an inequality of L.C. Young, *Math. Res. Lett.* **1** (1994), 451–464.
15. T. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoamericana* **14** (1998), 215–310.
16. B.B. Mandelbrot and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10** (1968), 422–437.
17. I. Podlubny, *Fractional differential equation*, Academic Press, 1999.
18. R. Roynette, Mouvement brownien et espaces de Besov, *Stochastics Stochastics Rep.* **43** (1993), 221–260.
19. A.A. Ruzmaikina, Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion, *J. Statist. Phys.* **100** (2000), 1049–1069.
20. S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives*, *Theory and applications,* Gordon and Breach Science Publishers, Yvendon, 1993.
21. L.C. Young, An inequality of the Hölder type connected with Stieltjes integration, *Acta Math.* **67** (1936), 251–282.
22. M. Zähle, Integration with respect to fractal functions and stochastic calculus, I, *Probab. Theory Related Fields* **111** (1998), 333–374.
23. M. Zähle, On the link between fractional and stochastic calculus, in: *Stochastic Dynamics*, Bremen 1997, ed.: H. Crauel and M. Gundlach, pp. 305–325, Springer, 1999.