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Gonality and Clifford index for real algebraic curves

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Abstract

Let X be a smooth connected projective curve of genus g defined over $\mathbb{R}$. Here we give bounds for the real gonality of X in terms of the complex gonality of X.

1. Introduction

Let X be a smooth connected projective curve of genus g defined over $\mathbb{R}$. The real gonality, $\mathrm{gon}(X,\mathbb{R})$, of X is the minimal integer k such that there is $L \in \mathrm{Pic}(X)$ with $\deg(L) = k, h^0(X,L) = 2$ and L defined over $\mathbb{R}$. Notice that any real line bundle computing $\mathrm{gon}(X,\mathbb{R})$ is spanned because the base locus of a real line bundle is a real divisor. We will give an example (see Example 2.2) of smooth real curve X with $X(\mathbb{R}) = \emptyset$ and such that there is a real line bundle L on X with $h^0(X,L) > 2$ and $\deg(L) < \mathrm{gon}(X,\mathbb{R})$. If $X(\mathbb{R}) \neq \emptyset$, then this phenomenon cannot occur (see Lemma 2.1). For any $R \in \mathrm{Pic}(X)$, set $\mathrm{Cliff}(R) = \deg(R) - 2(h^0(X,R)) + 2$; the integer $\mathrm{Cliff}(R)$ is called the Clifford index of R. The real Clifford index, $\mathrm{Cliff}(X,\mathbb{R})$, of X is the minimal integer $\mathrm{Cliff}(L)$, where L is a real line bundle on X with $h^0(X,L) \geq 2$ and $h^1(X,L) \geq 2$. The real Clifford dimension of X is the minimal integer $n \geq 1$ such that there is a real line bundle L on X with $h^0(X,L) \geq 2$, $\deg(L) \leq 2g-2$, $\mathrm{Cliff}(L) = \mathrm{Cliff}(X,\mathbb{R})$ and $h^0(X,L) = n+1$. If we drop the word "real" in the previous definitions we obtain respectively the usual notion of gonality, of Clifford index and of Clifford dimension. See [8] and [7], Theorem 2.1, for more on the Clifford index of complex projective curves. We will write $\mathrm{gon}(X)$ (resp. $\mathrm{Cliff}(X)$) for the usual complex gonality (resp. complex Clifford index) of X. By [7], Theorem 2.1, we have $\mathrm{Cliff}(X)+2 \leq \mathrm{gon}(X) \leq \mathrm{Cliff}(X)+3$. In this paper we will see that relation between $\mathrm{gon}(X,\mathbb{R})$ and $\mathrm{Cliff}(X,\mathbb{R})$ is quite different. Here we prove the following results.

Keywords: Real algebraic curves, real line bundles, gonality, real gonality.

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Theorem 1.1

Let X be a smooth projective curve of genus $g \geq 3$ defined over $\mathbb{R}$. Set $k := \mathrm{gon}(X)$ and assume $\mathrm{gon}(X, \mathbb{R}) > k$. If $X(\mathbb{R}) = \emptyset$ assume $g - k$ even. Then Cliff$(X, \mathbb{R}) \leq 2k - 4$. If $X(\mathbb{R}) \neq \emptyset$ we have $\mathrm{gon}(X, \mathbb{R}) \leq 2k - 2$.

Theorem 1.2

Let X be a smooth projective curve of genus $g \geq 3$ defined over $\mathbb{R}$ and with $X(\mathbb{R}) \neq \emptyset$. Set $k := \mathrm{gon}(X)$ and assume $\mathrm{gon}(X, \mathbb{R}) > k$ and $(k-1)(2k-3) - g$ odd. Then $\mathrm{gon}(X, \mathbb{R}) \leq 2k - 3$.

2. Proofs and examples

For any real algebraic scheme Y, σ will denote the complex conjugation on the set $Y(\mathbb{C})$. Thus $Y(\mathbb{R}) = \{P \in Y(\mathbb{C}) : \sigma(P) = P\}$.

Lemma 2.1

Let X be a smooth projective curve defined over $\mathbb{R}$ and with $X(\mathbb{R}) \neq \emptyset$. Assume the existence of a real line bundle L on X with $h^0(X, L) \geq 3$. Then there exists a real line bundle M on X with $h^0(X, M) = 2$, deg$(M) \leq$ deg$(L) - h^0(X, L) + 2$ and M spanned by its global sections.

Proof. Set $x := h^0(X, L) - 2$. For a general $P \in X(\mathbb{C})$ (i.e. for all points of $X(\mathbb{C})$ except at most finitely many ones) we have $h^0(X, L(-P)) = h^0(X, L) - 1$. Since $X(\mathbb{R})$ is infinite, the same is true if we take as P a sufficiently general point of any connected component of $X(\mathbb{R})$. Iterating this trick we see that for any connected component T of $X(\mathbb{R})$ there are $P_1, \dots, P_x \in T$ such that $h^0(X, L(-P_1 - \dots - P_x)) = 2$. The line bundle $L(-P_1 - \dots - P_x)$ is real and $\deg(L(-P_1 - \dots - P_x)) = \deg(L) - h^0(X, L) + 2$. Let M be the subsheaf of spanned by $h^0(X, L(-P_1 - \dots - P_x))$. We have $\deg(M) \leq \deg(L(-P_1 - \dots - P_x))$ and $h^0(X, M) = h^0(X, L(-P_1 - \dots - P_x)) = 2$. Since the base locus of a real line bundle is defined over $\mathbb{R}, M$ is real. □

Proof of 1.1. Take $L \in \mathrm{Pic}(X)(\mathbb{C})$ computing $\mathrm{gon}(X)$. Hence $\deg(L) = k, h^0(X, L) = 2$ and L is spanned by its global sections. Since $\mathrm{gon}(X, \mathbb{R}) > k, L$ is not defined over $\mathbb{R}$. We have $\deg(\sigma^*(L)) = k, h^0(X, \sigma^*(L)) = 2$ and $\sigma^*(L)$ is spanned by its global sections. Set $M := L \otimes \sigma^*(L)$. The line bundle M has degree $2k$ and it is spanned by its global sections.

First claim: The line bundle M is defined over $\mathbb{R}$.

Proof. By its very definition M is σ-invariant. If $X(\mathbb{R}) \neq \emptyset$ this implies that M is defined over $\mathbb{R}$ (see e.g. [2], Proposition 4.1.2 (i)). In the general case we take a σ-invariant open covering $\{U_i\}_{i \in I}$ of $X(\mathbb{C})$ such that $L|U_i$ is trivial. For all indices $i, j \in I$ with $i \neq j$, let $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ a cocycle defining L. Since $\{U_i\}_{i \in I}$ is σ-invariant, $\sigma^*(L)$ has as $\{U_i\}_{i \in I}$ trivializing open covering of $X(\mathbb{C})$ with $\sigma(g_{ij})$ as associated cocycle. Hence M has $\{U_i\}_{i \in I}$ as trivializing open covering with $|g_{ij}|^2$ as cocycle. Since $|g_{ij}|^2$ is real, M is real.

Second claim: We have $h^0(X, M) \geq 4$.

Proof of the second claim: See $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric surface in $\mathbb{P}^3$. If L and $\sigma^*(L)$ are not isomorphic, then the morphism $\phi : X \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ has not as image a smooth conic. Thus $\phi(X)$ spans $\mathbb{P}^3$ and this implies $h^0(X, M) \geq 4$. Now assume L and $\sigma^*(L)$ isomorphic (over $\mathbb{C}$). Since L is not defined over $\mathbb{R}$, this is possible only if $X(\mathbb{R}) = \emptyset$ (see e.g. [2], Proposition 4.1.2 (i)). By [2], Proposition 4.1.2 (iii), we have $k \equiv g - 1 \bmod (2)$, contradiction.

By the two claims we have $\text{Cliff}(X, \mathbb{R}) \leq \text{Cliff}(M) \leq 2k - 6$. Since $\deg(M) = 2k \leq g - 3 + h^0(X, M)$, we have $h^1(X, M) \geq 2$ and hence M may be used to compute the Clifford index. Take any $P \in X(\mathbb{C})$. The line $\langle \phi(P), \phi(\sigma(P)) \rangle \subset \mathbb{P}^3$ spanned by $\phi(P)$ and $\phi(\sigma(P))$ is defined over $\mathbb{R}$. Hence the composition of ϕ with the projection from the line $\langle \phi(P), \phi(\sigma(P)) \rangle$ induces a morphism $f : X \to \mathbb{P}^1$ defined over $\mathbb{R}$ and with $\deg(f) \leq 2k - 2$. If $X(\mathbb{R}) \neq \emptyset$ Lemma 2.1 gives the existence of a real line bundle R on X with $\deg(R) \leq 2k - 2, R$ spanned by its global sections and $h^0(X, R) = 2$. □

Proof of 1.2. We use every notation introduced in the proof of Theorem 1.1. First assume that ϕ is an embedding. For a general $P \in \phi(X(\mathbb{C}))$ (i.e. for all points of $\phi(X(\mathbb{C}))$ except finitely many ones), the projection of X from P is a plane curve, Y, of degree $2k - 1$, birational to X and with exactly $(k-1)(2k-3) - g$ ordinary nodes as only singularities. If $P \in \phi(X(\mathbb{R}))$, then Y is defined over $\mathbb{R}$. Since $(k-1)(2k-3) - g$ is odd and $\text{Sing}(Y)$ is defined over $\mathbb{R}$, there is at least one real singular point of Y. This is equivalent to the existence of a line D of $\mathbb{P}^3$ defined over $\mathbb{R}$, with $P \in D$ and with $\text{card}(D \cap (X(\mathbb{C}))) = 3$. The linear projection of $\mathbb{P}^3 \backslash D$ from D onto $\mathbb{P}^1$ induces a morphism $h : X \to \mathbb{P}^1$ of degree $2k - 3$ and defined over $\mathbb{R}$. Hence we conclude in this case. Now assume that ϕ is not an embedding. First assume the existence of $P_1 \in \text{Sing}(\phi(X)) \cap \mathbb{P}^3(\mathbb{R})$. The linear projection from P_1 induces a morphism $h_1 : X \to \mathbb{P}^2$ defined over $\mathbb{R}$ and with $\deg(h_1) \leq 2k - 2$. Hence we conclude in this case by Lemma 2.1. Now assume that there is no such point P_1 but that ϕ is birational. By assumption there is $P_2 \in \text{Sing}(\phi(X)) \cap (\mathbb{P}^3(\mathbb{C}) \backslash \mathbb{P}^3(\mathbb{R}))$. Since ϕ is real, $\sigma(P_2) \in \text{Sing}(\phi(X))$. We have $\sigma(P_2) \neq P_2$. The line D_2 spanned by P_2 and $\sigma(P_2)$ is defined over $\mathbb{R}$. The linear projection from D_2 induces a morphism $h_2 : X \to \mathbb{P}^1$ of degree $2k - 4$ and defined over $\mathbb{R}$. Hence we conclude in this case by Lemma 2.1. Now assume that ϕ is not birational. Let $s \geq 2$ be its degree. Take two sufficiently general points Q_1, Q_2 of $X(\mathbb{R})$. The linear projection from the line spanned by $\phi(Q_1)$ and $\phi(Q_2)$ induces a morphism $h_2 : X \to \mathbb{P}^1$ of degree $2k - 2s$ and defined over $\mathbb{R}$. Hence we conclude in this case by Lemma 2.1. □

Example 2.2: Fix an integer $d \geq 4$ and let $X \subset \mathbb{P}^2$ be a smooth plane curve defined over $\mathbb{R}$ (i.e. defined by a homogeneous polynomial with real coefficient and with no complex singular point). We have $\text{gon}(X) = d - 1$, $\text{Cliff}(X) = d - 4, O_X(1)$ is the only line bundle computing $\text{Cliff}(X)$ and for any $R \in \text{Pic}(X)(\mathbb{C})$ with $\deg(R) = d - 1$ and $h^0(X, R) \geq 2$ there is a unique $P \in X(\mathbb{C})$ with $R \cong O_X(1)(-P)$ (see [9] or [6], p. 6, or [1] or [5]). Since $O_X(1)$ is real, we have $\text{Cliff}(X, \mathbb{R}) = \text{Cliff}(X)$. The descriptions of all line bundles on X computing $\text{gon}(X)$ shows that $\text{gon}(X, \mathbb{R}) = \text{gon}(X)$ if and only if $X(\mathbb{R}) \neq \emptyset$. If d is odd, then $X(\mathbb{R}) \neq \emptyset$. If d is even there are curves X as above with $X(\mathbb{R}) \neq \emptyset$ and curves as above with $X(\mathbb{R}) = \emptyset$. Now assume d even, $d \geq 6$ and $X(\mathbb{R}) = \emptyset$. For any $Q \in \mathbb{P}^2(\mathbb{R})$ the linear projection $\mathbb{P}^2 \backslash \{Q\} \to \mathbb{P}^1$ induces a degree d

real morphism $f_Q : X \to \mathbb{P}^1$ not associated to a complete linear system. According to our conventions this is not sufficient to show that $\mathrm{gon}(X, \mathbb{R}) = d$. Indeed, we will see that $\mathrm{gon}(X, \mathbb{R}) = 2d - 4$. Fix points $P_1, P_2 \in X(\mathbb{C})$ with $\sigma(P_1) \neq P_2$ and such that the points $P_1, \sigma(P_1), P_2$ and $\sigma(P_2)$ are not collinear. It is easy to check that no 3 of the points $P_1, \sigma(P_1), P_2$ and $\sigma(P_2)$ are collinear. Set $D := P_1 + \sigma(P_1) + P_2 + \sigma(P_2)$. Thus D is an effective Cartier divisor with $\deg(D) = 4$ and D defined over $\mathbb{R}$. Thus $O_X(2)(-D)$ is a real line bundle of degree $2d - 4$. There is a pencil of conics passing through $P_1, \sigma(P_1), P_2$ and $\sigma(P_2)$. Since $d \neq 5$, we have $h^1(\mathbb{P}^2, O_{\mathbb{P}^2}(2 - d)) = 0$. Hence from the exact sequence

$$0 \to O_{\mathbb{P}^2}(2 - d) \to O_{\mathbb{P}^2}(2) \to O_X(2) \to 0 \tag{1}$$

we obtain $h^0(X, O_X(2)) = 6$. Thus $h^0(X, O_X(2)(-D)) = 2$. By [5], main result at p. 6, if $d - 1 < t \leq 2d - 5$ there is no complete base point free pencil of degree t on X. Thus $\mathrm{gon}(X, \mathbb{R}) = 2d - 4$.

EXAMPLE 2.3: Fix an integer $d \geq 4$ and $P \in \mathbb{P}^2(\mathbb{C}) \backslash \mathbb{P}^2(\mathbb{R})$. There is an irreducible real curve $Y \subset \mathbb{P}^2$ with $\deg(Y) = d$ and $\mathrm{Sing}(Y(\mathbb{C})) = \{P, \sigma(P)\}$. Let X be the normalization of Y. Thus X is a real curve of genus $(d - 1)(d - 2)/2 - 2$. Since $O_Y(1)$ is real, we have $\mathrm{Cliff}(X, \mathbb{R}) \leq d - 4$. We have $\mathrm{gon}(X) = d - 2$ and there are exactly two line bundles on X computing $\mathrm{gon}(X)$: the one induced by the linear projection $\mathbb{P}^2 \backslash \{P\} \to \mathbb{P}^1$ from P and the one induced by the linear projection $\mathbb{P}^2 \backslash \{\sigma(P)\} \to \mathbb{P}^1$ from $\sigma(P)$ (see the Main Lemma on p. 7 of [6]). Since none of them is real, we have $\mathrm{gon}(X) \geq d - 1$. Every spanned degree $d - 1$ line bundle on X is induced by the linear projection from a point $Q \in Y \backslash \{P, \sigma(P)\}$ (see the Main Lemma on p. 7 of [6]). Hence $\mathrm{gon}(X, \mathbb{R}) = d - 1$ if and only if $X(\mathbb{R}) \neq \emptyset$. We have $\mathrm{Cliff}(X) = d - 4$ (apply [7], Theorem 2.1, and the Main Lemma on p. 7 of [6]). Thus $\mathrm{Cliff}(X, \mathbb{R}) = d - 4$.

Remark 3.4. There is no hope that the examples described in 3.2 are the only ones with real Clifford dimension 2 and for which such phenomena do occur, as obvious applying the statements and/or the proofs of [1], [6], [3], [4] and [5] to other classes of plane integral curves defined over $\mathbb{R}$ and hence whose singularity scheme is σ-invariant, say having 4 nodes or having two triple points.

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