Some examples of Gorenstein liaison in codimension three

ROBIN HARTSHORNE

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail: robin@math.berkeley.edu

ABSTRACT

Gorenstein liaison seems to be the natural notion to generalize to higher codimension the well-known results about liaison of varieties of codimension 2 in projective space. In this paper we study points in $\mathbb{P}^3$ and curves in $\mathbb{P}^4$ in an attempt to see how far typical codimension 2 results will extend. While the results are satisfactory for small degree, we find in each case examples where we cannot decide the outcome. These examples are candidates for counterexamples to the hoped-for extensions of codimension 2 theorems.

For curves in projective three-space $\mathbb{P}^3_k$, the theory of liaison, or linkage, is classical, and is now a well-understood framework for the classification of algebraic space curves [21], [12]. This theory has been successfully extended to schemes of codimension 2 in any projective space $\mathbb{P}^n$ [2], [30], [26], [22, Ch. 6]. Recently a number of efforts have been made to find a suitable extension of these results in codimension ≥ 3 [22]. Traditional liaison uses complete intersections to link one scheme to another. In codimension 2, the property of being a complete intersection is equivalent to being arithmetically Gorenstein [33]. Thus there are two natural ways to generalize. It appears that complete intersection liaison is too fine a relation to give analogous results in higher codimension. Thus attention has been focussed on Gorenstein liaison, and a number of recent results have created an optimistic attitude that much of the codimension 2 case will carry over naturally to higher codimension [22], [16], [4], [25], ... The purpose of this paper is to give some examples of Gorenstein liaison for points in $\mathbb{P}^3$ and for curves in $\mathbb{P}^4$, which suggest that the situation in codimension ≥ 3 may be more complicated than was initially suspected.

For points in $\mathbb{P}^3$, we show first that any set of n points in general position in a plane or on a nonsingular quadric surface can be obtained from a single point by a sequence of ascending Gorenstein biliaisons (see §1 below for definitions of these terms). Thus any set of n points in a plane or on a quadric surface is glicci (in the Gorenstein liaison class of a complete intersection). On a nonsingular cubic surface, we can still show that any set of n points in general position is glicci, but we have to use ascending and descending biliaisons and simple liaisons to prove this. For a set of n points in general position in $\mathbb{P}^3$, we show that for $n \leq 19$ it is glicci, but we are unable to prove this for $n = 20$. Thus a set of 20 points in general position in $\mathbb{P}^3$ becomes a candidate for a possible counterexample to the hope that all ACM schemes are glicci.

In $\mathbb{P}^4$, various classes of ACM curves have been shown to be glicci, in particular, determinantal curves and ACM curves lying on general smooth ACM rational surfaces in $\mathbb{P}^4$ [16]. We show that any general ACM curve of degree ≤ 9 or degree 10 and genus 6 in $\mathbb{P}^4$ is glicci. Then we study ACM curves of degree 20 and genus 26. There are determinantal curves of this degree and genus, but we show that a general curve in the irreducible component of the Hilbert scheme containing the determinantal curves cannot be obtained by ascending Gorenstein biliaison from a line. We do not know if this curve is glicci, so we propose it as a candidate for an example of an ACM curve that is not glicci.

For curves in $\mathbb{P}^4$, we consider the set of curves with Rao module k, i.e., of dimension 1 in 1 degree only. We call such a curve *minimal* if its Rao module occurs in degree 0. We show that there are minimal curves of every degree ≥ 2. Then we give examples that suggest that there are curves in the liaison class of two skew lines that cannot be reached by ascending Gorenstein biliaison from a minimal curve; and that there are other curves with Rao module k that are not in the liaison class of two skew lines. We will describe the examples and the evidence for these statements below, but in most cases we cannot prove anything.

I hope that further study of these examples and others will establish whether these guesses are correct or not, and help clarify some of the major questions concerning Gorenstein liaison in codimension ≥ 3.

I would like to thank Juan Migliore for his book [22], which clearly sets out the case for Gorenstein liaison, and which stimulated this research. I would also like to thank him and Rosa Miró-Roig and Uwe Nagel for sharing their unpublished papers with me. Lastly, I would like to thank the referee for many helpful suggestions, and in particular for an idea that led to a great improvement of Example 4.6.

1. Basic results and questions

Let V_1 and V_2 be two equidimensional closed subschemes without embedded components, of the same dimension r in $\mathbb{P}^n_k$, the n-dimensional projective space over an algebraically closed field k. We say V_1 and V_2 are *linked by a complete intersection*

scheme X, if X is a complete intersection scheme of dimension r containing V_1 and V_2, and if

$$\mathcal{I}_{V_2,X} \cong \mathrm{Hom}(\mathcal{O}_{V_1}, \mathcal{O}_X), \text{ and}$$
$$\mathcal{I}_{V_1,X} \cong \mathrm{Hom}(\mathcal{O}_{V_2}, \mathcal{O}_X).$$

Using the language of generalized divisors on Gorenstein schemes [13, 4.1] we can say equivalently V_1 and V_2 are linked by a complete intersection X if and only if there is a complete intersection scheme S of dimension $r+1$ containing V_1 and V_2, such that X is an effective divisor in the linear system $|mH|$ on S for some $m > 0$, where H is the hyperplane section of S, and $V_2 = X - V_1$ as generalized divisors on S.

The equivalence relation generated by complete intersection linkage is called *CI-liaison*. If the equivalence can be accomplished by an even number of links, we speak of *even CI-liaison*.

A scheme X in $\mathbb{P}^n$ is called *arithmetically Gorenstein* (AG) if its homogeneous coordinate ring R/I_X is a Gorenstein ring, where $R = k[x_0, \ldots, x_n]$ is the homogeneous coordinate ring of $\mathbb{P}^n$, and I_X is the (saturated) homogeneous ideal of X. If, in the first definition above, we require that X be an arithmetically Gorenstein scheme, instead of a complete intersection, then we say that V_1 and V_2 are *linked by an AG scheme*. The equivalence relation generated by this kind of linkage is called *Gorenstein liaison* (or *G-liaison* for short); if the equivalence can be accomplished by an even number of G-links, then we speak of *even Gorenstein liaison*.

One way of obtaining AG schemes is as follows. Let S be an *arithmetically Cohen-Macaulay* (ACM) scheme in $\mathbb{P}^n$ (this means that the homogeneous coordinate ring R/I_S is a Cohen-Macaulay ring). Assume also that S satisfies the property G_1 (Gorenstein in codimension 1 [13, p. 291]), so that we can use the language of generalized divisors. Then any effective divisor X in the linear system $|mH - K|$ on S, where $m \in \mathbb{Z}$, H is the hyperplane section, and K is the canonical divisor, is an arithmetically Gorenstein scheme [22, 4.2.8].

Suppose now that V_1 and V_2 are divisors on an ACM scheme S of dimension $r+1$ satisfying G_1, let X be an effective divisor in the linear system $|mH - K|$ for some m, and suppose that $V_2 = X - V_1$ as generalized divisors on S. Then it is easy to see that V_1 and V_2 are linked by the AG scheme X (cf. proof of [13, 4.1] and note that since S satisfies G_1, X is an almost Cartier divisor on S [13, p. 301]). In this case we will say that V_1 and V_2 are *strictly G-linked*. We do not know whether the equivalence relation generated by strict G-linkages is equivalent to the G-liaison defined above, so we will call it *strict G-liaison*, and if it is accomplished in an even number of steps, *strict even G-liaison*.

Combining two strict G-linkages gives the following result.

Proposition 1.1 [16, 5.14], [22, 5.2.27].

Let V_1 and V_2 be effective divisors on an ACM scheme S satisfying G_1. Suppose that $V_2 \in |V_1 + hH|$ for some $h \in \mathbb{Z}$. Then V_1 and V_2 can be strictly G-linked in two steps.

Note that even though the statement in [22, 5.2.27] requires S smooth, the proof given there works for S ACM satisfying G_1 if one takes into account that any divisor $X \in |mH - K|$ is almost Cartier [13, 2.5].

The proposition above motivates the following definition. In the situation of (1.1), we say that V_2 is obtained by an *elementary Gorenstein biliaison of height h* from V_1 [22, 5.4.7]. Because of the proposition, an elementary G-biliaison is a strict even G-liaison. If $h \geq 0$, we call the biliaison *ascending*.

Now we can state some of the main questions raised by trying to generalize codimension 2 results to higher codimension.

Question 1.2. a) *Does strict Gorenstein liaison generate the same equivalence relation (G-liaison) as Gorenstein liaison?*

b) *Do the elementary Gorenstein biliaisons generate the same equivalence relation as even G-liaison?*

In codimension 2, both questions reduce to CI-liaison, for which the answers to parts a) and b) are both yes [13, 4.1, 4.4].

Question 1.3. a) *Is every ACM subscheme of $\mathbb{P}^n$ in the G-liaison class of a complete intersection (in which case we say it is glicci)?*

b) *Can every glicci scheme be obtained by a finite sequence of ascending elementary G-biliaisons from a scheme (of the same dimension) of degree 1?*

In codimension 2, part a) is the classical theorem of Gaeta [11], [29], ... Part b) seems likely to be true, though I do not know a reference. Note in b) it would be equivalent to ask for ascending elementary G-biliaisons starting with any complete intersection scheme. In higher codimension, many special cases of a) have been shown to be true [22], [16], [25], [4]. Closely related is the theorem of Migliore and Nagel [24] that every ACM subscheme X of $\mathbb{P}^n$ has a flat deformation to a glicci scheme, and there is also a glicci scheme with the same Hilbert function as X.

For the following questions we limit the discussion to curves (locally Cohen–Macaulay schemes of dimension 1) for simplicity. For a curve $C \subseteq \mathbb{P}^n$ we define its *Rao module* to be the finite length graded R-module $M = \oplus_{l \in \mathbb{Z}} H^1(\mathcal{I}_C(l))$, where $\mathcal{I}_C$ is the ideal sheaf of C. It is easy to see that even G-liaison preserves the Rao module, up to shift of degrees [22, 5.3.3].

Question 1.4. *Does the Rao module characterize the even G-liaison class of a curve? In other words, if C and C' are two curves with $M_{C'} \cong M_C(h)$ for some $h \in \mathbb{Z}$, are C and C' in the same even G-liaison class? (In codimension 2, this is the well-known theorem of Rao [30].)*

Now we come to the problem of the structure of an even G-liaison class. Let $C \subseteq \mathbb{P}^n$ be a curve, and let $\mathcal{L}$ be the class of all curves C' in the even G-liaison class of C. The Rao modules of curves in $\mathcal{L}$ are all isomorphic up to shift. As long as the Rao module is not zero (which is equivalent to saying the curves are not ACM), one knows that there is a minimal leftward shift of M that can occur [22, 1.2.8]. We denote by

$\mathcal{L}_0$ the subset of $\mathcal{L}$ consisting of those curves with the leftmost possible shift of the Rao module, and we call these *minimal curves*. Let $\mathcal{L}_h$ denote the set of curves with Rao module shifted h places to the right from $\mathcal{L}_0$, for each $h \geq 0$. Then $\mathcal{L} = \cup \mathcal{L}_h$ for $h \geq 0$, and each one of these $\mathcal{L}_h$ for $h \geq 0$ is nonempty [22, 1.2.8].

In codimension 2 a biliaison class $\mathcal{L}$ satisfies the *Lazarsfeld–Rao property* [22, 5.4.2]. It says that a) $\mathcal{L}_0$ is a single irreducible family of curves, and b) any curve $C \in \mathcal{L}_h$ can be obtained by a finite sequence of ascending biliaisons from a minimal curve, plus if necessary a deformation with constant cohomology within the class $\mathcal{L}_h$. (But even for curves in $\mathbb{P}^3$ it is not known if these deformations are necessary [21, IV, 5.4, p. 93].) Easy examples show that in codimension 3, $\mathcal{L}_0$ need not consist of a single irreducible family of curves [22, 5.4.8]. So we rephrase the question somewhat.

Question 1.5. a) *Describe the set $\mathcal{L}_0$ of minimal curves in an even G-liaison class.*

b) *Can every curve in $\mathcal{L}_h$ for $h > 0$ be obtained from a minimal curve by a finite sequence of ascending elementary G-biliaisons, followed possibly by a flat deformation within the family $\mathcal{L}_h$?*

An optimist might hope for positive answers to all these questions. However, the examples we give below suggest that many of the answers may be no.

2. Points in $\mathbb{P}^3$

Any scheme of dimension zero is ACM, so in this section we will address Question 1.3. The study of arbitrary zero-schemes, even in $\mathbb{P}^3$, becomes quite complicated, so we will direct our attention to sets of reduced points in general position. In this section the phrase "a general X has property Y" will mean that there is a nonempty Zariski open subset of the family of all X's having the property Y.

We begin with points in $\mathbb{P}^2$. In this case it is known from the theorem of Gaeta [22, 6.1.4] that any zero scheme in $\mathbb{P}^2$ is *licci* (in the liaison class of a complete intersection), but we give a slightly more precise statement for a general set of points and at the same time we illustrate in a simple case the technique we will use in the later propositions.

Proposition 2.1

A set of n general points in $\mathbb{P}^2$ can be obtained from a single point by a sequence of ascending elementary biliaisons.

Proof. By induction on n. For $n = 1$ there is nothing to prove. For $n = 2$, any 2 points lie on a line L. A single biliaison of height 1 on L reduces 2 points to 1 point. For $n = 3, 4, 5$, a set of n reduced points, no three on a line, lies on a nonsingular conic. These points can be obtained by an elementary biliaison of height 1 or 2 from a set of 1 or 2 points, and we are done by induction.

In general, let $n \geq 3$. Then there is an integer $d \geq 2$ such that $\frac{1}{2}(d-1)(d+2) < n \leq \frac{1}{2}d(d+3)$. Since curves of degree d in $\mathbb{P}^2$ form a linear system of dimension $\frac{1}{2}d(d+3)$, any set of n points will lie on a curve of degree d. Since the nonsingular

curves form a Zariski open subset of the family of all curves, a set of n *general* points in $\mathbb{P}^2$ (in the sense mentioned above) will lie on a nonsingular curve C of degree d, and will form a set of n general points on C. The genus of C is $g = \frac{1}{2}(d-1)(d-2)$. We use the fact that on a nonsingular curve of genus g, any divisor of degree $\geq g$ is effective. Let D be the divisor of n general points on C, and let H be the hyperplane class on C. We define a divisor D' as follows.

$$D' = \begin{cases} D - H & \text{if } n = \frac{1}{2}(d-1)(d+2) + 1 \\ D - 2H & \text{if } \frac{1}{2}(d-1)(d+2) + 2 \leq n \leq \frac{1}{2}d(d+3). \end{cases}$$

Then we verify that in either case, the degree of D' is $\geq g$, so that the divisor D' is effective, and secondly that $n' = \deg D' \leq \frac{1}{2}(d-1)(d+2)$.

Now, by induction on d, a general set of n' points can be obtained from a single point by ascending biliaisons. Since any D as above bilinks down to a D', it follows that by bilinking up n' general points on C, we obtain n general points on C, as required. $\square$

Proposition 2.2

A set of n general points on a (fixed) nonsingular quadric surface $Q \subseteq \mathbb{P}^3$ can be obtained from a single point by a finite number of ascending elementary G-biliaisons on Q.

Proof. The method is analogous to the proof of (2.1), except that now we use both types of ACM curves on Q. For $n = 1$, there is nothing to prove. For $n = 2$, we put 2 points on a twisted cubic curve, and then move them by linear equivalence (a biliaison of height 0) until they lie on a line on Q. On that line, we obtain 2 points by a biliaison of height 1 from 1 point. For $n = 3$, the points lie on a conic, and come from 1 point by biliaison. For $n = 4, 5$, the points lie on a twisted cubic curve, and reduce by biliaison to 1 or 2 points.

Now suppose $n \geq 6$. Then there is an integer $a \geq 2$ such that either

i) $a^2 + a \leq n \leq a^2 + 2a$, or

ii) $a^2 + 2a + 1 \leq n \leq a^2 + 3a + 1$.

In case i) we consider the complete intersection curve C of bidegree (a, a) on Q. It has degree $2a$ and genus $g = (a-1)^2$ and moves in a linear system of dimension $a^2 + 2a$. Hence n general points lie on a smooth such curve C, forming a divisor D. The divisor $D' = D - H$ on C has degree $n' = n - 2a$. Since $n' \geq a^2 - a > g$, the divisor D' is effective. On the other hand $n' \leq a^2$, so n' falls in the range i) for $a - 1$ unless $n' = a^2$, in which case it falls in range ii) for $a - 1$. By induction on a, a set of n' general points can be obtained by ascending elementary G-biliaisons from a point, so also can n points.

In case ii), we consider the ACM curve C of bidegree $(a, a+1)$ on Q. It has degree $d = 2a + 1$, genus $g = a(a-1)$, and moves in a linear system of dimension $a^2 + 3a + 1$ on Q. So n general points form a divisor D on a nonsingular such curve C. The divisor

$D' = D - H$ has degree $n' = n - 2a - 1 \geq a^2 > g$, so D' is effective. On the other hand $a^2 \leq n' \leq a^2 + a + 1$, which is range ii) for $a - 1$. So by induction on a again, we can obtain D' by biliaisons from a point, and D by a single elementary G-biliaison for D' on C. $\square$

Corollary 2.3

A set of n general points on a nonsingular quadric surface Q in $\mathbb{P}^3$ is in the strict even G-liaison class of a point. In particular, it is glicci.

Proposition 2.4

A set of n general points on a (fixed) nonsingular cubic surface S in $\mathbb{P}^3$ is in the same strict Gorenstein liaison class as a point on S.

Proof. A curve C of degree d and genus g on S moves in a linear system of dimension $d + g - 1$. If a set of n general points is to form a divisor D on C, we need $n \leq d + g - 1$. In that case the linear system $D - H$ on C has degree $\leq g - 1$, and hence may not be effective. Thus we cannot use Gorenstein biliaisons on the cubic surface. Instead, we will use strict Gorenstein liaison by AG divisors in the linear systems $|mH - K|$ on ACM curves C on S.

There are four types of smooth ACM curves on S, obtained by biliaison (of curves) on S from the line, the conic, the twisted cubic, and the hyperplane class H, which is a plane cubic curve. For $a \geq 1$ the four types are

  i) $d = 3a - 2$, $g = 1/2(3a^2 - 7a + 4)$
 ii) $d = 3a - 1$, $g = 1/2(3a^2 - 5a + 2)$
iii) $d = 3a$, $g = 1/2(3a^2 - 3a)$
iv) $d = 3a$, $g = 1/2(3a^2 - 3a + 2)$.

In each case, one of these curves C with degree d and genus g (we say *type* (d, g)) moves in a linear system of dimension $d + g - 1$. On an ACM curve C of type (d, g), we will consider only divisors of degree n, where $g \leq n \leq d + g - 1$. If n and n' are both in this range, and if $n + n' = \deg(mH - K)$ for some m, then a strict Gorenstein liaison by AG divisors in the linear system $|mH - K|$ will transform general divisors of degree n to general divisors of degree n' and vice versa. To explain this in more detail, let Z be a set of n general points on S. If $n \leq d + g - 1$, then Z is contained in a curve C as above. If $n' = \deg(mH - K) - n \geq g$, then there is an effective divisor Z' of degree n' such that $Z + Z' \in |mH - K|$. Thus Z and Z' are linked. The same arguments work in reverse, starting with Z', assuming $n' \leq d + g - 1$ and $n \geq g$. Hence there are Zariski open subsets U (resp. U') of the set of all subsets of n (resp. n') points of S such that each $Z \in U$ is linked to a $Z' \in U'$ and vice versa. If one of these is already known to be in the strict G-liaison class of a point, we conclude so is the other. We will write $n \leftrightarrow n'$ by $mH - K$ on (d, g).

For $n \leq 8$, we use the following liaisons.

  1) $1 \leftrightarrow 3$ by $H - K$ on $(4, 1)$
  2) $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$ by $2H - K$ on $(5, 2)$

3) $6 \leftrightarrow 8$ by $3H - K$ on $(6, 3)$

4) $4 \leftrightarrow 8$ and $5 \leftrightarrow 7$ by $3H - K$ on $(6, 4)$

5) $6 \leftrightarrow 7$ by $3H - K$ on $(7, 5)$.

These liaisons show that any set of $n \leq 8$ general points on S is in the strict G-liaison class of a point. Note that we must use ascending *and* descending liaisons to accomplish this. For example, if $n = 2$, the links go $2 \to 6 \to 7 \to 5 \to 3 \to 1$.

For $9 \leq n \leq 17$, we use the following links.

1) $9 \leftrightarrow 11$ by $4H - K$ on $(7, 5)$

2) $7 \leftrightarrow 13$ and $8 \leftrightarrow 12$ by $4H - K$ on $(8, 7)$

3) $12 \leftrightarrow 17$ and $13 \leftrightarrow 16$ by $5H - K$ on $(9, 9)$

4) $10 \leftrightarrow 17$; $11 \leftrightarrow 16$; $12 \leftrightarrow 15$; and $13 \leftrightarrow 14$ by $5H - K$ on $(9, 10)$.

Using these links a general set of $9 \leq n \leq 17$ points is linked down to a set of 7 or 8 points treated above.

For $n \geq 18$, we find an integer a such that $n_0 = \frac{3}{2}a(a - 1) \leq n < n_1 = \frac{3}{2}(a + 1)a$. We divide these n's into six ranges.

A) $n_0 \leq n \leq n_0 + 2$

B) $n_0 + 3 \leq n \leq n_0 + a - 1$

C) $n_0 + a, n_0 + a + 1$

D) $n_0 + a + 2 \leq n \leq n_0 + 2a - 2$

E) $n_0 + 2a - 1, n_0 + 2a$

F) $n_0 + 2a + 1 \leq n \leq n_0 + 3a - 1$.

In range A, we do

1) $n_0 \leftrightarrow n_0 + 2$ by $(2a - 2)H - K$ on type (i) curve.

2) $n_0 + 1 \leftrightarrow n_0 + 3a - 1$ and $n_0 + 2 \leftrightarrow n_0 + 3a - 2$ by $(2a - 1)H - K$ on type (iv).

In range B, we do

3) $n_0 + t \leftrightarrow n' < n_0$ by $(2a - 2)H - K$ on (i).

In range C, we do

4) $n_0 + t \leftrightarrow n' < n_0$ by $(2a - 2)H - K$ on (ii).

In range D, we do links by $(2a - 1)H - K$ on types (ii) and (iv). If a is odd, $a = 2k + 1$, start with $m = n_0 + 3k + 2$, link on type (iv). Then alternate linkages on type (ii) and (iv). This covers all values of n in range D, starting in the middle and spiraling outward, until finally we land in range E. If a is even, $a = 2k$, start with $m = n_0 + 3k$, and do a link on type (ii) first, then alternate (iv) and (ii).

In range E, link by $(2a - 1)H - K$ on type (iv), to land in range C.

In range F, link by $(2a - 1)H - K$ on type (iii) to land in range B or C.

In summary, ranges B and C link down to $n' < n_0$ so are ok by induction. Ranges E and F link down to ranges B and C. Range D spirals up and down until it lands in range E; and finally range A links up to range F. So for example, if $n = 18$ (range A), the links go $18 \to 20 \to 28 \to 22 \to 16 \to 13 \to 7$, which we did earlier. If $n = 54$ (range D), the links go $54 \to 55 \to 53 \to 56 \to 52 \to 40 \to 35 \to 27 \to 23 \to 15 \to 12 \to 8 \to 6$, treated above. $\square$

Corollary 2.5

A set of n general points on a smooth cubic surface in $\mathbb{P}^3$ is glicci.

Corollary 2.6

A set of $n \leq 19$ general points in $\mathbb{P}^3$ is glicci.

Proof. Indeed, since the cubic surfaces in $\mathbb{P}^3$ form a linear system of dimension 19, a set of $n \leq 19$ general points lie on a smooth cubic surface, and we can apply (2.4). Using (2.2) we can also see that a set of $n \leq 9$ general points can be obtained from a single point by ascending elementary G-biliaisons. However, we can get a stronger result, and another proof of (2.6) by another method. $\square$

Proposition 2.7

A set of $n \leq 19$ general points in $\mathbb{P}^3$ is in the strict Gorenstein liaison class of a point. Furthermore, if $n \neq 17, 19$, it can be obtained by a sequence of ascending elementary G-biliaisons from a point.

Proof. Again we use ACM curves (d, g) in $\mathbb{P}^3$, but now we need to know how many general points of $\mathbb{P}^3$ lie on such a curve. Call this number $m(d, g)$. This is no longer an elementary question, because the families of these curves form a Hilbert scheme, not a linear system. The question was studied in Perrin's thesis [28], and depends on semi-stability of the normal bundle. Here are his results, for the ACM curves we need:

(d, g)	m	reference in [28]
(1,0)	2	
(2,0)	3	
(3,0)	6	
(4,1)	8	
(5,2)	9	
(6,3)	12	p. 66
(7,5)	14	p. 87
(8,7)	16	p. 10; p. 116
(9,9)	18	p. 87
(10,11)	20	p. 66

In this table m has the naive value $2d$, except for $(2, 0)$, a conic, which lies in a plane, so can pass through at most 3 general points, and $(5, 2)$, which lies on a quadric surface, so can pass through at most 9 general points.

To prove our result, we use the following biliaisons and liaisons.

    1) $1 \leftrightarrow 2$ biliaison on $(1, 0)$
    2) $1 \leftrightarrow 3$ biliaison on $(2, 0)$
    3) $1, 2, 3 \leftrightarrow 4, 5, 6$ biliaison on $(3, 0)$
    4) $3, 4 \leftrightarrow 7, 8$ biliaison on $(4, 1)$
    5) $4 \leftrightarrow 9$ biliaison on $(5, 2)$
    6) $4, 5, 6 \leftrightarrow 10, 11, 12$ biliaison on $(6, 3)$

   7) $6, 7 \leftrightarrow 13, 14$ biliaison on $(7, 5)$

   8) $8, 9 \leftrightarrow 15, 16$ biliaison on $(8, 7)$

   9) $9 \leftrightarrow 18$ biliaison on $(9, 9)$

  10) $12 \leftrightarrow 17$ liaison by $5H - K$ on $(9, 9)$

  11) $11 \leftrightarrow 19$ liaison by $5H - K$ on $(10, 11)$. $\square$

Remark 2.8. If we consider a set of 20 general points in $\mathbb{P}^3$, none of the above methods works. They do not lie on a cubic surface, so we cannot apply (2.4). They do form a divisor D on an ACM curve $(10, 11)$, but $D - H$ has degree 10, less than the genus, so it may not be effective. Liaison by $5H - K$ would give a divisor of degree 10, which may not be effective. Liaison by $6H - K$ gives another general divisor of degree 20, so we get nowhere.

    It is conceivable that some upward liaison may eventually lead to a zero-scheme that can then be linked back down to a point. Or perhaps there are other AG schemes in $\mathbb{P}^3$ besides the ones of the form $mH - K$ on ACM curves that we have been using.

    On the other hand, it may simply be that 20 general points in $\mathbb{P}^3$ are not in the G-liaison class of a point, so we propose this as a potential counterexample to Question 1.3. If this is so, the cone over these 20 points would be an ACM curve in $\mathbb{P}^4$ that is not glicci.

Remark 2.9. In [16, 3.1], the authors prove that any standard determinantal scheme is glicci. Taking the $t \times t$ minors of a $t \times (t + 2)$ matrix of linear forms in $\mathbb{P}^3$ gives a zero-dimensional determinantal scheme of degree $\frac{1}{6}(t+2)(t+1)t$, which is glicci by the above result. For $t = 1, 2$, any set of 1 or 4 points in $\mathbb{P}^3$ is determinantal. However, for $t = 3, 4$, the dimension calculation in [16, 10.3] show that determinantal sets of 10 points have codimension 3 in zero-schemes of degree 10, and determinantal sets of 20 points have codimension 15 among zero-schemes of degree 20.

3. ACM curves in $\mathbb{P}^4$

In the literature, a number of special cases of ACM curves have been shown to be glicci [4], [16]. In this section we begin a systematic study of ACM curves of small degree in $\mathbb{P}^4$. We show that any general ACM curve of degree ≤ 9, or a general ACM curve of degree 10 and genus 6 can be obtained by ascending Gorenstein biliaisons from a line. On the other hand, we show that there is an irreducible component of the Hilbert scheme of curves of degree 20 and genus 26 whose general member is a smooth ACM curve that cannot be obtained by ascending Gorenstein biliaisons from a line. We propose this curve as a candidate for a possible counterexample to the question whether every ACM curve is glicci.

    We start by finding a lower bound on the genus of an ACM curve in $\mathbb{P}^4$.

Proposition 3.1

Let C be a nondegenerate (i.e., not contained in a hyperplane) ACM curve in $\mathbb{P}^4$, of degree d and arithmetic genus g. Then $d \geq 4$ and $g \geq G_{\min}(d)$, where

$$G_{\min}(d) = (s-1)d - \binom{s+2}{3} - \binom{s+2}{4} + 1,$$

and $s \geq 2$ is the unique integer for which $\binom{s+2}{3} \leq d < \binom{s+3}{3}$. Furthermore, if $g = G_{\min}(d)$, then $s = s_0(c)$, the least degree of a hypersurface containing C.

Proof. The simplest way to see this is to consider the h-vector of the curve [22, §1.4]. This is a sequence of positive integers $c_0 = 1, c_1, c_2, \ldots, c_r$, which determine the degree and genus of the curve according to the formulae

$$d = \sum_{i=0}^{r} c_i, \qquad g = \sum_{i=2}^{r}(i-1)c_i.$$

The c_i measure the Hilbert function of a graded ring $R = k[x_0, x_1, x_2]/J$ of finite length, since C has codimension 3. Since R is a quotient of a polynomial ring in three variables, we have $c_i \leq \binom{i+2}{2}$ for $i \geq 1$. The hypothesis C nondegenerate implies $c_1 = 3$. Thus $d \geq 4$. For a given value of d,, the genus will be minimized by making each c_i as large as possible for $i = 2, 3, \ldots$. Thus for $4 \leq d < 10$ the minimum g is attained by the h-vector $1, 3, d - 4$, with genus $g = d - 4$. For $10 \leq d < 20$, the minimum genus comes from the h-vector $1, 3, 6, d - 10$, with $g = 2d - 14$. For the general case, a short calculation with binomial coefficients gives the formula above.

The least degree $s_0(c)$ of a hypersurface containing C can be read from the h-vector as the least i for which $c_i < \binom{i+2}{2}$. For the h-vectors giving the minimum genus, this is just the number s defined above. $\square$

Remark 3.2. It seems reasonable to expect that for each $d \geq 4$ and $g = G_{\min}(d)$, the set of ACM curves of degree d and genus g in $\mathbb{P}^4$ should form an open subset of an irreducible component of the Hilbert scheme of curves in $\mathbb{P}^4$, and that a general such curve should be nonsingular, but we do not know how to prove this.

Notation 3.3. We will be dealing with curves on certain rational ACM surfaces in $\mathbb{P}^4$, so here we fix some terminology and notation.

The *smooth cubic scroll S* is obtained by blowing up one point $P \in \mathbb{P}^2$, and embedding in $\mathbb{P}^4$ by the complete linear system $H = 2l - e$, where l is the total transform of a line in $\mathbb{P}^2$, and e is the class of the exceptional divisor E. One knows that Pic $S = \mathbb{Z} \oplus \mathbb{Z}$, generated by l, e. We denote the divisor class $al - be$ by $(a; b)$.

The *Del Pezzo surface S* is obtained by blowing up five points $P_1, \ldots, P_5$, no three collinear in $\mathbb{P}^2$, and embedding in $\mathbb{P}^4$ by $H = 3l - \Sigma e_i$. In this case Pic $S = \mathbb{Z}^6$, and we denote the divisor class $al - \Sigma b_i e_i$ by $(a; b_1, \ldots, b_5)$. If some b's are repeated, we denote that with an exponent. Thus in the discussion of $(8,4)$ curves below, the divisor class $(5; 2^2, 1^3)$ means $(5; 2, 2, 1, 1, 1)$.

A *Castelnuovo surface* S in $\mathbb{P}^4$ is a smooth surface of degree 5 and sectional genus 2. It can be obtained by blowing up 8 points $P_0, P_1, \ldots, P_7$ in $\mathbb{P}^2$ and embedding by the linear system $H = (4; 2, 1^7)$ (see [27]). If the points P_i are no three collinear and no 6 on a conic, we call it a *general* Castelnuovo surface. Here Pic $S = \mathbb{Z}^9$, and we denote the divisor class $al - \Sigma b_i e_i$ by $(a; b_i)$.

A *Bordiga surface* is a smooth surface of degree 6 and sectional genus 3. It can be obtained by blowing up ten points $P_1, \ldots, P_{10}$ in $\mathbb{P}^2$ and embedding by the linear system $H = (4; 1^{10})$ [27]. If the points P_i are such that no three are collinear, no 6 on a conic, and no 10 on a cubic curve, we call it a *general* Bordiga surface.

Proposition 3.4

a) *If C is an integral nondegenerate ACM curve of degree $d \le 9$ in $\mathbb{P}^4$, the degree-genus pair (d, g) must be one of the following:* $(4, 0), (5, 1), (6, 2), (7, 3), (8, 4), (8, 5),$ $(9, 5), (9, 6), (9, 7)$.

b) *For each (d, g) pair in a), the set of nonsingular nondegenerate curves in $\mathbb{P}^4$ forms an open subset of an irreducible component of the Hilbert scheme, and*

c) *For each (d, g) pair as above, the general such curve is ACM and can be obtained by ascending Gorenstein biliaisons from a line.*

Proof. a) A lower bound on g is given by (3.1); an upper bound is given by the Castelnuovo bound for the genus of an integral curve (see, e.g., Rathmann [31]). This list gives all possible values of g between the lower and upper bounds.

b) For $g = d - 4$, the irreducibility is given by a theorem of Ein [7]. For $(d, g) = (8, 5)$, C is the canonical embedding of a non-hyperelliptic curve of genus 5, so the family is irreducible. For $(d, g) = (9, 6)$ the curve is a non-trigonal curve of genus 6, embedded by a linear system $D = K - P$, so the family is irreducible (I am indebted to E. Drozd for this observation). For $(d, g) = (9, 7)$, the family is irreducible by a theorem of Harris [8].

c) We do a case-by-case analysis.

For $(d, g) = (4, 0)$, up to automorphisms of $\mathbb{P}^4$, there is just one rational normal curve C of degree 4. It lies on a smooth cubic scroll S, having divisor class $(2; 0)$. If H denotes the hyperplane class $(2; 1)$, then $C - H = (0; -1)$, which is a line. Thus C is obtained by an ascending Gorenstein biliaison from a line on the surface S.

For $(d, g) = (5, 1)$, suppose given a smooth nondegenerate $(5, 1)$ curve C in $\mathbb{P}^4$. Let C_0 be the abstract elliptic curve, and let D_0 be the divisor corresponding to $\mathcal{O}_C(1)$. Then C is obtained by embedding C_0 with the complete linear system $|D_0|$. Choose F a divisor of degree 3 on C_0, and use $|F|$ to embed C_0 as a nonsingular cubic curve C_1 in $\mathbb{P}^2$. Choose a point $P \in C_1$. Blow up P in $\mathbb{P}^2$ and embed by the linear system $H = 2l - e$ to get a nonsingular cubic scroll S in $\mathbb{P}^4$. The image of C_1 will be a $(5, 1)$ curve $C_2 \subseteq \mathbb{P}^4$, obtained by embedding C_0 with the linear system $|2F - P|$. By adjusting the choice of P, we may arrange that $D_0 \sim 2F - P$. Then C and C_2 will differ by an automorphism of $\mathbb{P}^4$. We conclude that C lies on a smooth cubic scroll S', and has divisor class $(3; 1)$ on S'. Then $C - H = (1; 0)$ is a conic. The conic in turn can be obtained ascending Gorenstein biliaison from a line on a plane. Thus C is obtained by ascending Gorenstein biliaisons from a line.

A similar argument shows that every smooth nondegenerate $(6,2)$ curve C in $\mathbb{P}^4$ appears as a divisor of type $(4;2)$ on a smooth cubic scroll. Then $C - H = (2;1)$ is a twisted cubic curve, which can in turn be obtained by an ascending Gorenstein biliaison on a quadric surface in $\mathbb{P}^3$.

The case $(d,g) = (7,3)$ is a little more complicated. We will consider only a smooth nondegenerate non-hyperelliptic $(7,3)$ curve C in $\mathbb{P}^4$. Let C_0 be the abstract curve of genus 3, and D_0 the divisor giving the embedding C. Let C_1 be the canonical embedding of the non-hyperelliptic curve C_0 as a smooth plane quartic curve in $\mathbb{P}^2$. Choose five points $P_1, \ldots, P_5$ on C_1 with no 3 collinear. Blow up $P_1, \ldots, P_5$ and embed by $3l - \Sigma e_i$ to get a Del Pezzo surface S in $\mathbb{P}^4$. The image of C_1 is then a smooth $(7,3)$ curve $C_2 \subseteq S$ with divisor class $(4;1^5)$, which is an embedding of C_0 by the divisor $3K - \Sigma P_i$. We would like to choose the P_i so that $3K - \Sigma P_i \sim D_0$, i.e., $\Sigma P_i \sim 3K - D_0$.

Case 1. The divisor $3K - D_0$ can be represented by 5 points, no three collinear. In this case we find that C is contained in a Del Pezzo surface S', with divisor class $(4;1^5)$. Then $C - H = (1;0^5)$, which is a twisted cubic curve, so C can be obtained from a line by ascending Gorenstein biliaisons.

Case 2. If Case 1 does not occur, one sees easily that $3K - D_0 \sim K + P$ for some point P. In this case we take S to be the smooth cubic scroll obtained by blowing up P. Then C_2 is a divisor of type $(4;1)$ on S, and is an embedding of C_0 by $2K - P \sim D_0$. So C lies on a cubic scroll, and $C - 2H = (0;-1)$ is a line.

Note the two types of non-hyperelliptic $(7,3)$ curves can be distinguished by the property that the Case 1 curves have only finitely many trisecants, while the Case 2 curves have infinitely many trisecants.

For a smooth $(8,4)$ curve C we use a different technique. For this curve, $h^0(\mathcal{I}_C(2)) = 2$, so C is contained in a unique complete intersection surface $S = F_2 \cap F_2'$. Since the family of $(8,4)$ curves is irreducible, and since a general complete intersection surface $S = F_2 \cap F_2'$ is a smooth Del Pezzo surface containing a smooth $(8,4)$ curve C in the divisor class $(5;2^2,1^3)$, we conclude that a general such C lies on a smooth Del Pezzo surface S, with divisor class $(5;2^2,1^3)$. Then $C - H = (2;1^2,0^3)$, which is a nondegenerate smooth $(4,0)$ curve in $\mathbb{P}^4$. Thus using the case of $(4,0)$ above, we conclude that C can be obtained by ascending Gorenstein biliaisons from a line.

The case $(9,6)$ is similar to $(8,4)$, because again $h^0(\mathcal{I}_C(2)) = 2$, we conclude that a general smooth $(9,6)$ curve C lies on a Del Pezzo surface S with divisor class $(6;2^4,1)$. In this case $C - 2H = (0;0^4,-1)$ which is a line.

A smooth nondegenerate $(8,5)$ curve C in $\mathbb{P}^4$ is the canonical embedding of a non-hyperelliptic genus 5 curve. According to the theorem of Petri [32] if the curve is not trigonal, then C is the complete intersection of three quadric hypersurfaces $C = F_2 \cap F_2' \cap F_2''$. Let $S = F_2 \cap F_2'$. Then C is the divisor $2H$ on S, and $C - H$ is an elliptic quartic curve in $\mathbb{P}^3$, which can be obtained from a conic by biliaison on a quadric surface. Thus C is obtained by ascending Gorenstein biliaisons from a line.

If the curve is trigonal, then C lies on a smooth cubic scroll $S = F_2 \cap F_2' \cap F_2''$. It has divisor class $(5;2)$, so $C - 2H$ is $(1;0)$, a conic, and we conclude again. Note in this case $C = H - K$ is arithmetically Gorenstein, even though it is not a complete intersection.

For $(d, g) = (9, 5)$, arguments like the ones above show that we can embed the general genus 5 curve as a plane quintic with a double point, and thus obtain a general $(9, 5)$ curve C on a Castelnuovo surface with divisor class $(5; 2, 1^7)$. Then $C - H = (1, 0^8)$ is a $(4, 0)$ curve and we use our earlier result.

Finally, every smooth $(9, 7)$ curve C, as a curve of maximal genus, lies on a cubic surface S as the divisor $3H$, by a theorem of Harris [8]. Then $C - 2H$ is a twisted cubic curve, and we are done. □

Corollary 3.5

A general smooth ACM curve of degree $d \leq 9$ in $\mathbb{P}^4$ is glicci.

Remark 3.6. The glicciness of ACM curves lying on general ACM surfaces in $\mathbb{P}^4$ and of integral ACM curves of degree ≤ 7 was already proven in [16, §8]. Our contribution is to show that a general smooth ACM curve of degree ≤ 9 actually does lie on a smooth rational ACM surface, and to check the possibly stronger property that they can be obtained from a line by ascending Gorenstein biliaisons (cf. Question 1.3b).

Note that our proof actually shows *every* smooth curve with $(d, g) = (4, 0), (5, 1),$ $(6, 2)$ is ACM and can be obtained by ascending Gorenstein biliaisons from a line. The same can be said for $(7, 3)$ curves, by extending the analysis above: one can show that a hyperelliptic $(7, 3)$ curve lies on a cubic scroll or the cone over a twisted cubic curve, and in both cases is obtained from a smooth $(4, 0)$ curve by biliaison.

For the next case of $(8, 4)$ curves, the situation is more complicated. There are smooth hyperelliptic $(8, 4)$ curves on a cubic scroll, but they are not ACM. Since an ACM $(8, 4)$ curve lies on a unique complete intersection surface $S = F_2 \cap F_2'$, to study *all* smooth $(8, 4)$ curves, one would presumably have to study the possible singular surfaces S. One approach is to use Riemann–Roch on the surface to show that the divisor $C - H$ is effective, but then one has to deal with not necessarily irreducible $(4, 0)$ curves.

The analysis becomes increasingly complex for the remaining cases, so we do not know if c) holds for all smooth ACM curves of the given degree and genus.

Remark 3.7. For $d \geq 10$, the family of smooth non-degenerate ACM curves of given (d, g) in $\mathbb{P}^4$ may not be irreducible. The first example is $(d, g) = (10, 9)$, for which there are two different families of such curves lying on smooth cubic scrolls.

EXAMPLE 3.8: We consider smooth $(10, 6)$ curves in $\mathbb{P}^4$. Note that all the curves in Proposition 3.4, being ACM of degree ≤ 9, are contained in quadric hypersurfaces, since their hyperplane section is ≤ 9 points and is contained in a quadric surface of $\mathbb{P}^3$. The case $(d, g) = (10, 6)$ is the first case where there are smooth ACM curves not contained in a quadric hypersurface.

By the theorem of Ein [7], the family of smooth $(10, 6)$ curves in $\mathbb{P}^4$ is irreducible. To show that a general $(10, 6)$ curve is ACM, it suffices, by semicontinuity, to exhibit one. A divisor of type $(5; 1^{10})$ on a general Bordiga surface S is the transform of a plane quintic curve, which can be taken to be smooth, so we get a smooth $(10, 6)$ curve C on S. For this curve $C - H = (1; 0^{10})$ is a smooth $(4, 0)$ curve, which is ACM, so C

is also ACM. Note however that the curve just described is not general in the variety of moduli of curves of genus 6, because it has a g_5^2: a representation as a plane quintic curve.

Next, let C_0 be an abstract curve of genus 6, with general moduli. Then C_0 admits a birational representation as a plane curve $C_1 \subseteq \mathbb{P}^2$ with four nodes P_1, P_2, P_3, P_4, no three collinear [1]. Choose six additional points $P_5, \ldots, P_{10}$ on C_1 in general position. Blow up $P_1, \ldots, P_{10}$ to obtain a Bordiga surface S, containing the proper transform $C_2 \subseteq S$ of C_1. Then C_2 is a smooth $(10, 6)$ curve in $\mathbb{P}^4$ with general moduli. Since the curve has genus 6, by varying the choice of the six points $P_5, \ldots, P_{10}$, we can obtain any general divisor class on C_2 as its hyperplane section. We conclude that the general $(10, 6)$ curve C in $\mathbb{P}^4$ is contained in a general Bordiga surface S with divisor class $(6; 2^4, 1^6)$. Then $C - H = (2; 1^4, 0^6)$, which is a smooth $(4, 0)$ curve, so C can be obtained by ascending Gorenstein biliaisons from a line.

To study the $(10, 6)$ curves in more detail, we note that on a general Bordiga surface S, there are eight divisor classes (up to permutation of the P_i) containing $(10, 6)$ curves. They are

$$D_1 = (5; 1^{10})$$
$$D_2 = (6; 2^4, 1^6)$$
$$D_3 = (7; 2^9, 0)$$
$$D_4 = (7; 3, 2^6, 1^3)$$
$$D_5 = (8; 3^3, 2^6, 1)$$
$$D_6 = (8; 4, 2^9)$$
$$D_7 = (9; 3^6, 2^4)$$
$$D_8 = (10; 3^{10}).$$

Of these D_3 and D_6 have Rao module k. They will be discussed in the next section. The remaining 6 cases are ACM. The first three of these, D_1, D_2, and D_4, can be obtained by Gorenstein biliaison from $(4, 0)$ curves on S. However, $D_5 - H$, $D_7 - H$, and $D_8 - H$ are not effective divisors so these curves cannot be obtained by Gorenstein biliaison on this surface S.

Using the arithmetically Gorenstein divisor $3H - K$ on S, of degree 20, the divisor class D_i is Gorenstein-linked to D_{8-i}. It follows that D_5, D_7, D_8 are glicci (as observed in [16, §8]). However, we do not know whether or not these curves may be obtained by ascending Gorenstein biliaison on some other surface.

EXAMPLE 3.9: For our last example, we will study ACM $(20, 26)$ curves in $\mathbb{P}^4$. Note that all the curves in the earlier part of this section, plus all the ACM curves lying on rational ACM surfaces in $\mathbb{P}^4$, which were proved to be glicci in [16, §8], lie on cubic hypersurfaces in $\mathbb{P}^4$. So by analogy with our findings for points in $\mathbb{P}^3$ in §2 above, we might expect that all ACM curves contained in cubic hypersurfaces in $\mathbb{P}^4$ would be glicci. This also suggests that in looking for counterexamples to ACM $\Rightarrow$ glicci (Question 1.3), we should look at curves not contained in a cubic hypersurface.

The first example of an ACM curve in $\mathbb{P}^4$ not contained in a cubic hypersurface will have h-vector $1, 3, 6, 10$ (cf. proof of 3.1). It has degree 20 and genus 26. For existence

of such curves, we let C be the determinantal curve defined by the 4×4 minors of a 4×6 matrix of general linear forms. A general such curve will be smooth, ACM, of degree 20 and genus 26. The family of such determinantal curves has dimension ≤ 69, by [16, 10.3].

The method of [17, 3.7] shows that C is linearly equivalent to $H + K$ on an ACM surface S in $\mathbb{P}^4$, of degree 10 and sectional genus 11, defined by the 4×4 minors of a 4×5 matrix of general linear forms, where H denotes the hyperplane section of S, and K denotes the canonical class of S. Furthermore, a similar argument using [17, 3.1] shows that the curve C_0 defined by the 3×3 minors of a 3×5 matrix of linear forms will be linearly equivalent to K on S. This latter curve C_0 also appears in the divisor class $2H_0 + K_0$ on the surface S_0 defined by 3×3 minors of a 3×4 matrix of linear forms. (I am grateful to J. Migliore for pointing out the paper [17] and explaining to me how to obtain these linear equivalences.)

Now S_0 is just the Bordiga surface, and C_0 is an ACM $(10, 6)$ curve, discussed earlier. By the linear equivalence $C_0 \sim 2H_0 + K_0$ on S_0 we recognize that C_0 is in the class $(5; 1^{10})$, which we called D_1 in (3.8) above. These are isomorphic to plane curves of degree 5 and thus are not general in the moduli of curves of genus 6.

Since C_0 can be obtained by ascending Gorenstein biliaison from a line, and since $C \sim C_0 + H = K + H$ as the ACM surface S, we conclude that the determinantal $(20, 26)$ curve C can also be obtained by ascending Gorenstein biliaison from a line.

Next, I claim that the only way to obtain an ACM $(20, 26)$ curve D in $\mathbb{P}^4$ by ascending Gorenstein biliaison is from an ACM $(10, 6)$ curve C_1 on an ACM surface S_1 of degree 10 and sectional genus 11, as $D \sim C_1 + H$ on S_1. Indeed, suppose that $D \sim C_1 + H$ on some ACM surface S_1. Then C_1 is an ACM curve of type (d_1, g_1) in $\mathbb{P}^4$, while H is an ACM curve of type (d_2, g_2) in $\mathbb{P}^3$. From this we get $(20, 26) = (d_1 + d_2, g_1 + g_2 + d_1 - 1)$. For each d_1 (resp. d_2) we know the minimum possible genus of an ACM curve in $\mathbb{P}^4$ (resp. $\mathbb{P}^3$)—cf. 3.1. Looking at these, we find that $g_1 + g_2 + d_1 - 1 > 26$ in all cases except $(d_1, g_1) = (10, 6)$ and $(d_2, g_2) = (10, 11)$. Thus any $(20, 26)$ curve D that can be obtained by ascending Gorenstein biliaison must lie on a surface S_1 of degree 10 and sectional genus 11.

Now we look at the dimensions of some families of $(20, 26)$ curves. By [16, 10.3], the family of determinantal curves C as above has dimension ≤ 69. On the other hand, each component of the Hilbert scheme of $(20, 26)$ curves in $\mathbb{P}^4$ has dimension $\geq 5d + 1 - g = 75$. So we see immediately that a general element of an irreducible component of $H_{20,26}$ cannot be determinantal. However, there may be other $(20, 26)$ curves C' on S, not determinantal themselves, but linearly equivalent to C, obtainable by ascending Gorenstein biliaison on S.

So let us find the dimension of the complete linear system $|C|$ on S. From the exact sequence
$$\mathcal{O} \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$
we see that $\dim_S |C| = h^0(\mathcal{O}_C(C)) = C^2 + 1 - g + h^1(\mathcal{O}_C(C))$. We also have a resolution of $\mathcal{O}_S$
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5)^4 \to \mathcal{O}_{\mathbb{P}^4}(-4)^5 \to \mathcal{O}_S \to 0$$

coming from its matrix representation. From this we find $h^2(\mathcal{O}_S) = 4$ and $p_a(S) = 4$. On the surface S we have $H^2 = \deg S = 10$. From the adjunction formula for H, which is a $(10, 11)$ curve, we find $H \cdot K = 10$. And from the formula [14, p. 434] for surfaces in $\mathbb{P}^4$, we find $K^2 = 5$. Now $C = H + K$, so we get $C^2 = 35$. Also, since $C = H + K$, from the Kodaira Vanishing Theorem we have $H^1(\mathcal{O}_S(C)) = H^2(\mathcal{O}_S(C)) = 0$. Thus $h^1(\mathcal{O}_C(C)) \cong h^2(\mathcal{O}_S) = 4$. So we find

$$\dim_S |C| = 35 + 1 - 26 + 4 = 14.$$

The family of ACM surface S has dimension 60 (for example by the formula of Ellingsrud [10]), so we find that the family of $(20, 26)$ curves in $\mathbb{P}^4$ that can be linearly equivalent to C on such a surface S has dimension ≤ 74. In particular, a general curve in an irreducible component of $H_{20,26}$ does not arise in this way.

There may also be other linear equivalence classes of $(20, 26)$ curves on S of larger dimension. Indeed, this is what does happen with the $(10, 6)$ curves studied in (3.8) above: the curves linearly equivalent to the determinantal curves were all of type D_1 on the Bordiga surface, and these curves are not general in the moduli of genus 6 curves, while a general genus 6 curve appears as an ACM curve in a different linear system D_2 on the Bordiga surface. So we must see if something analogous happens with the $(20, 26)$ curves.

First, we look on a general ACM surface S of degree 10 and sectional genus 11. According to a theorem of Lopez [19, III.4.2], Pic $S = \mathbb{Z} \oplus \mathbb{Z}$ generated by H and K. We look for divisors $mH + nK$ with degree 20 and genus 26. There are only two possibilities: $C = H + K$ or $C' = 4H - 2K$. In the latter case we compute $C'^2 = 20$. Therefore, by Clifford's theorem, $h^0(\mathcal{O}_{C'}(C')) - 1 \leq 10$, so $\dim_S |C'| \leq 11$. Thus the family of such curves C' in $\mathbb{P}^4$ has dimension ≤ 71. So we see that a general ACM $(20, 26)$ curve in $\mathbb{P}^4$ cannot lie on a *general* ACM surface S of degree 10 and sectional genus 11.

Now let us estimate the dimension of a family of smooth $(20, 26)$ curves D, general in an irreducible component of $H_{20,26}$ containing the determinantal curves C above, and lying on a non-general ACM surface X of degree 10 and sectional genus 11. We will make use of the Clifford index of a curve.

Recall that the *gonality* of a curve C is the least d for which there exists a linear system g^1_d on the curve. The *Clifford index* of the curve is the minimum of $d - 2r$, taken over all linear systems g^r_d with $r \geq 1$ and $0 < d \leq g - 1$. For most curves, the Clifford index is equal to $\mathrm{gon}(C) - 2$, computed by a g^1_d. Curves for which this is not so are *Clifford exceptional* curves, and have been studied by Martens [20] and Eisenbud et al. [9].

If C is the determinantal $(20, 26)$ curve studied above, then $C \sim H + K$ on the surface S. The hyperplane section H is a $(10, 11)$ curve in $\mathbb{P}^3$, obtained as $C_0 + H_0$ on a nonsingular quartic surface in $\mathbb{P}^3$. Here C_0 is a nonhyperelliptic $(6, 3)$ curve having gonality 3; H_0 is a plane quartic curve, also having gonality 3. Hence, by [15], H has gonality ≥ 6. (In fact the gonality is equal to 6 because H must have a 4-secant.) The curve K on S is the determinantal $(10, 6)$ curve discussed above, isomorphic to a plane quintic curve, with gonality 4. So applying [15] again, we find the gonality

of C is ≥ 10. It follows from the study of Clifford exceptional curves in [20] and [9] that C is not exceptional, so we conclude Cliff $C \geq 8$. (On the other hand, the linear system $|K|$ on S cuts out a g_{15}^3 on C, so Cliff $C \leq 9$. I suspect Cliff $C = 9$, but don't know how to prove that.) Since we are considering a curve D that is general in an irreducible component of $H_{20,26}$ containing C, we may assume also that Cliff $D \geq 8$.

Now we consider a smooth $(20, 26)$ curve D with Cliff $D \geq 8$, contained in a smooth ACM surface of degree 10 and sectional genus 11 in $\mathbb{P}^4$, and we want to estimate the dimension of the linear system $|D|$ on S. As above, we find

$$\dim_S |D| = D^2 + 1 - g + h^1(\mathcal{O}_D(D)).$$

Since D is a $(20, 26)$ curve, the adjunction formula gives

$$D^2 + D \cdot K = 50.$$

Let us denote $D \cdot K$ by b. Then $D^2 = 50 - b$. On the other hand, let us consider the linear system $|D \cdot K|$ on D. It has dimension $a - 1$, where $a = h^0(\mathcal{O}_D(K))$. Since $K_D = (D + K) \cdot D$, we also have $h^1(\mathcal{O}_D(D)) = a$. The linear system $|D \cdot K|$ thus has dimension $a - 1$ and degree b. Our hypothesis Cliff $D \geq 8$ thus implies $b - 2a + 2 \geq 8$, or $b \geq 2a + 6$.

Now we can compute

$$\begin{aligned}
\dim_S |D| &= D^2 + 1 - g + h^1(\mathcal{O}_D(D)) \\
&= 50 - b + 1 - 26 + a \\
&= 25 + a - b.
\end{aligned}$$

Since $b \geq 2a + 6$, we find

$$\dim_S |D| \leq 19 - a.$$

Now from the exact sequence

$$0 \to \mathcal{O}_X(K - D) \to \mathcal{O}_X(K) \to \mathcal{O}_D(K) \to 0,$$

we find $a = h^0(\mathcal{O}_D(K)) \geq h^0(\mathcal{O}_X(K)) = h^2(\mathcal{O}_X) = 4$. Thus

$$\dim_S |D| \leq 15.$$

On the other hand, our surface S is not general, so it moves in a family of dimension at most 59, so the dimension of the family of curves that arise in this way is at most 74.

In conclusion, we see that there exists an irreducible component of the Hilbert scheme $H_{20,26}$ of smooth $(20, 26)$ curves in $\mathbb{P}^4$ (namely one containing the determinantal curves) whose general member is an ACM curve that does not lie on an ACM surface S of degree 10 and sectional genus 11, and so cannot be obtained by ascending Gorenstein biliaison from a line. We propose this curve as a possible candidate for a counterexample to ACM $\Rightarrow$ glicci (Question 1.3).

4. Curves in $\mathbb{P}^4$ with Rao module k

Let $\mathcal{M}$ be the set of all locally CM curves in $\mathbb{P}^4$ with Rao module k (i.e., of dimension one in one degree only). One knows that the Rao module must occur in a nonnegative degree [22, 1.3.11(b)], and that there are curves with Rao module k in degree 0 (e.g., two skew lines). So we denote by $\mathcal{M}_h$ the set of curves with Rao module k in degree h, and note that $\mathcal{M} = \cup_{h \geq 0} \mathcal{M}_h$.

Let $\mathcal{L} \subseteq \mathcal{M}$ be the subset of those curves in the G-liaison class of two skew lines, and let $\mathcal{L}_h = \mathcal{L} \cap \mathcal{M}_h$. Then $\mathcal{L} = \cup_{h \geq 0} \mathcal{L}_h$, and the curves in $\mathcal{L}_0$ are the *minimal* curves defined in §1 above.

In this section we will study the curves in $\mathcal{M}$, with a view to elucidating Questions 1.4 and 1.5 above.

Proposition 4.1

a) $\mathcal{M}_0$ contains curves of every degree $d \geq 2$.

b) For each $d \geq 2$, the set of curves in $\mathcal{M}_0$ of degree d forms an irreducible family, whose general member is the disjoint union $C = C' \cup L$ of a plane curve C' of degree $d - 1$ and a line L, not meeting the plane of C'.

c) Every curve in $\mathcal{M}_0$ is in the G-liaison class of two skew lines, i.e. $\mathcal{L}_0 = \mathcal{M}_0$.

Proof. a) The case of two skew lines in $\mathbb{P}^3$ is well-known [21, Example 6.2, p. 34]. For $d \geq 3$, let $C = C' \cup L$ as described in b). Clearly $h^0(\mathcal{O}_C) = 2$, so $h^1(\mathcal{I}_C) = 1$. On the other hand, since C' and L are contained in disjoint sublinear spaces of $\mathbb{P}^4$, it is clear that $H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \to H^0(\mathcal{O}_C(n))$ is surjective for $n \geq 1$, so $C \in \mathcal{M}_0$.

b) I claim any degree 2 curve C in $\mathbb{P}^4$ with $M = k$ lies in $\mathbb{P}^3$. If the curve is reduced, it is two lines, hence in a $\mathbb{P}^3$. If it is not reduced, then it is a double structure on a line L, and we have an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_C \to \mathcal{O}_L \to 0$$

where $\mathcal{L}$ is an invertible sheaf on L. Then there is a surjective map $u : \mathcal{I}_L/\mathcal{I}_L^2 \to \mathcal{L} \to 0$, and $\mathcal{L} \cong \mathcal{O}_L(a)$ for some a. Since $\mathcal{I}_L/\mathcal{I}_L^2 \cong \mathcal{O}_L(-1)^3$, the map u is given by three sections of $\mathcal{O}_L(a+1)$. If $a = -1$, we get a double line in a plane. If $a = 0$, there is a linear form x killed by u, so C lies in the $\mathbb{P}^3$ defined by $x = 0$. If $a > 0$, then the exact sequence

$$H^0(\mathcal{O}_L(-1))^3 \to H^0(\mathcal{O}_L(a)) \to H^1(\mathcal{I}_C) \to 0$$

shows the Rao module is bigger than k.

Thus a curve of degree 2 with $M = k$ lies in a $\mathbb{P}^3$, so these form an irreducible family whose general member is two skew lines.

So now let $d \geq 3$. Then C cannot be contained in $\mathbb{P}^3$, because of the Lazarsfeld–Rao property for curves in $\mathbb{P}^3$, so $h^0(\mathcal{O}_C) = 2$, because of the Rao module, and $h^0(\mathcal{O}_C(1)) = 5$. Let $A = H^0_*(\mathcal{O}_C)$. This is a graded S-algebra, where $S = k[x_0, x_1, x_2, x_3, x_4]$, and in particular A_0 is a 2-dimensional k-algebra. We consider two cases.

Case 1. A_0 is reduced, hence isomorphic to $k \times k$ as a k-algebra. Then A_0 contains two orthogonal idempotents e', e'', such that $e' + e'' = 1$, $e'^2 = e'$, $e''^2 = e''$, and $e'e'' = 0$. Hence C is the disjoint union of two curves C', C'', defined by the vanishing of e', e'', respectively. Let H', H'' be the linear spans of the curves C', C''. Then $h^0(\mathcal{O}_C(1)) = h^0(\mathcal{O}_{H'}(1)) + h^0(\mathcal{O}_{H''}(1)) = 5$. So one of these, say H', is a plane, and the other, H'' is a line L. Thus C' is a plane curve in H', and $C = C' \cup L$ as required. Note that H', H'' do not meet since $h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = h^0(\mathcal{O}_{H'}(1)) + h^0(\mathcal{O}_{H''}(1))$.

Case 2. A_0 is non-reduced, in which case it is isomorphic to the ring $k[\epsilon]/(\epsilon^2)$. Let $f \in A_0$ be a nonzero element with $f^2 = 0$. Now $A_1 \cong S_1$ is the k-vector space generated by x_0, x_1, x_2, x_3, x_4. Multiplication by f on A_1 is a nilpotent linear map with $f^2 = 0$. Furthermore, since C is locally CM, the kernel of f acting on A_1 must have dimension ≤ 3. Otherwise f would be supported at a point. So f has rank ≥ 2. Now from the structure of nilpotent transformations it follows (after a linear change of coordinates) that $fx_0 = x_2$, $fx_1 = x_3$, $fx_2 = fx_3 = fx_4 = 0$. Hence we can identify the S-algebra A as

$$A \cong S[f]/\big((f^2, fx_0 - x_2, fx_1 - x_3, fx_2, fx_3, fx_4) + I_C\big)$$

where $I_C \subseteq S$ is the homogeneous ideal of C.

Now let H' be the plane $x_2 = x_3 = 0$, and let C' be the curve obtained from $C \cap H$ by removing its embedded points, if any. Then there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0.$$

Since C' is a plane curve, $h^0(\mathcal{O}_{C'}) = 1$, and so $h^0(\mathcal{L}) = 1$. Furthermore note that the image of f in $\mathcal{O}_{C'}$ is annihilated by x_0, x_1, x_2, x_3, x_4, hence is 0. So f generates $h^0(\mathcal{L})$. Now f is annihilated by x_2, x_3, x_4, so it has support on the line $L : x_2 = x_3 = x_4 = 0$. Thus $\mathcal{L}$ is an $\mathcal{O}_L$-module, it is torsion-free since C is locally CM, and contains the submodule $\mathcal{O}_L$ generated by f. Hence $\mathcal{L} \cong \mathcal{O}_L$, generated by f.

Now it is clear that C consists of the plane curve C' of degree $d - 1$, containing the line L, plus a multiplicity two structure on L with $p_a = -1$. This is the limit of a flat deformation of the disjoint unions $C' \cup L$ described above, as the skew line L approaches a line in the curve C'.

So the curves in $\mathcal{M}_0$ of any degree $d \geq 2$ form an irreducible family.

c) Let $C \in \mathcal{M}_0$ have degree d. The case $d = 2$ in $\mathbb{P}^3$ is well-known, so we may assume $d \geq 3$. First consider the disjoint union $C = C' \cup L$ as in b). Take a hyperplane $\mathbb{P}^3$ containing L and meeting the plane H' of C' in a line L', skew to L, and not a component of C'. Let Q be a nonsingular quadric surface in that $\mathbb{P}^3$ containing L and L'. Then $S = H' \cup Q$ is an ACM surface of degree 3 in $\mathbb{P}^4$. Note that its negative canonical divisor $-K$ consists of a conic in H' plus a divisor of bidegree $(1, 2)$ on Q, meeting L' in the same two points as the conic, where $(1, 0)$ is the class of L. (Here we leave some details to the reader.) Now given C, there is an AG divisor X in the linear system $(d - 3)H - K$ on S containing C [22, 4.2.8]. The linked curve D is a divisor of bidegree $(d - 3, d - 1)$ on Q, which is in the biliaison class of two skew lines on Q. Thus C is in $\mathcal{L}_0$.

In the special case where C is a plane curve C' containing a line L, plus a double structure on L as above, we use exactly the same construction, except that now the hyperplane $\mathbb{P}^3$ meets H' in L, and the quadric surface Q contains the double structure on L. The same liaison works, using the theory of generalized divisors [13]. $\square$

Remark 4.2. The fact that $\mathcal{L}_0$ is not a single irreducible family was observed by Migliore [22, 5.4.8], who gave the example of a curve of degree 3 in $\mathcal{L}_0$. His student Lesperance [18] has independently proved 3.1a), c) in the case of reduced curves. Lesperance has also shown [18, 4.5] that for other Rao modules, the set of minimal curves of given degree need not be irreducible. Thus (4.1*b*) is special to the case of Rao module $M = k$.

EXAMPLE 4.3: Let C be a smooth curve of degree 5 and genus 0 in $\mathbb{P}^4$, not contained in any $\mathbb{P}^3$. It is the projection of the rational normal curve Γ in $\mathbb{P}^5$ from a point not lying on any secant line of Γ. A little elementary geometry shows that C has a unique trisecant E. If C meets E in three distinct points, then the three points of intersection of C and E determine a unique isomorphism (of abstract $\mathbb{P}^1$'s) from C to E fixing those three points. Let S be the surface formed as the closure of the set of lines joining corresponding points of C and E. Then S is a rational cubic scroll in $\mathbb{P}^4$.

On S, our rational quintic C has divisor class $(4; 3)$. The linear system $C - H = (2; 2)$ contains a disjoint union of two rulings of the surface S. Hence C is obtained by one elementary G-biliaison from two skew lines. In particular, $C \in \mathcal{L}_1$.

If E is a degenerate trisecant, i.e., a tangent line meeting the curve again, or an inflectional tangent, we can still show $C \in \mathcal{L}_1$ as follows. The smooth $(5, 0)$ curves in $\mathbb{P}^4$ form an irreducible family, so C is a specialization of the general type described above. Hence C must lie on a cubic surface in $\mathbb{P}^4$. It cannot lie on a reducible surface, since C is not in $\mathbb{P}^3$. The only other irreducible cubic surface is the cone over a twisted cubic curve, and that surface contains no smooth $(5, 0)$ curves. Hence C is on a smooth rational cubic scroll, and the previous argument applies.

EXAMPLE 4.4: We consider smooth curves of type $(6, 1)$ (degree 6 and genus 1) in $\mathbb{P}^4$, not contained in any $\mathbb{P}^3$. Then $h^1(\mathcal{I}_C(1)) = 1$ and $h^0(\mathcal{I}_C(2)) \geq 3$.

Case 1. If three quadric hypersurfaces containing C intersect in a surface, then that surface must be a cubic rational scroll S (reason: the degree of S must be ≤ 3; C is not contained in a plane or a quadric surface, and there is no $(6, 1)$ curve on the cone over a twisted cubic curve). In this case $C = (3; 0)$ on S and $C - H = (1; -1)$, which contains the disjoint union of a conic and a line. Thus C is in $\mathcal{L}_1$ and is obtained by a single elementary G-biliaison from a curve of degree 3 in $\mathcal{L}_0$. This curve C has infinitely many trisecants, formed by the rulings of S.

Case 2. Three quadric hypersurfaces containing C will intersect in a complete intersection curve X of degree 8 and genus 5. The residual intersection D will be a curve of degree 2. D cannot be a plane curve, because then it would meet C in 5 points, and projection from the plane of D would be a birational map of C to a line, which is impossible. Hence D is two skew lines or a nonplanar double structure on a line. By reason of the genus of X, D will be either two trisecants of C or a single

trisecant. Note also that these are all the trisecants of C, because any trisecant of C must be contained in each quadric hypersurface containing C, hence in X.

Case 2a. An example of a $(6, 1)$ curve with two trisecants can be obtained on a Del Pezzo surface, as a divisor of type $(3; 1^3, 0^2)$. In this case $C - H = (0; 0^3, -1^2)$, which is a disjoint union of two lines, so C is in $\mathcal{L}_1$ and is obtained by one elementary G-biliaison from the minimal curve of degree 2 in $\mathcal{L}_0$. This curve C has two trisecants, the lines $F_{45} = (1; 0^3, 1^2)$ and $G = (2; 1^5)$.

Case 2b. An example of a $(6, 1)$ curve with one trisecant can be obtained as follows. We project the Veronese surface V in $\mathbb{P}^5$ from a point in a plane containing a conic of V, so as to obtain a quartic surface S in $\mathbb{P}^4$ with a double line L. A general cubic curve in $\mathbb{P}^2$ gives a $(6, 1)$ curve in V meeting the conic in three points that project to distinct points of the line L in S. Thus the image $C \subseteq S$ of this curve will be a smooth $(6, 1)$ curve having L as a trisecant. Now the surface S is smooth except for a double line and two pinch points, hence locally CM. Its general hyperplane section is an integral curve in $\mathbb{P}^3$ of degree 4, arithmetic genus 1, with one node. This is a complete intersection in $\mathbb{P}^3$, hence S is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$ [22, 1.3.3], so it must contain every trisecant of C. But S contains no lines except L, so C has a unique trisecant. Since C is linked to a double structure on L, C is in the CI-liaison class of two skew lines, so C is in $\mathcal{L}_1$. Note that $C - H$ is not effective on S, so C cannot be obtained by an elementary Gorenstein biliaison on S. However, it seems likely that C also lies on a normal singular Del Pezzo surface on which it can be obtained by an elementary Gorenstein biliaison from two skew lines.

Thus we see that any smooth nondegenerate $(6, 1)$ curve in $\mathbb{P}^4$ is in $\mathcal{L}_1$. The family of all such curves in $\mathbb{P}^4$ is irreducible [7]. The general type with two trisecants (Case 2a) is obtained by an elementary Gorenstein biliaison from a curve of degree 2 in $\mathcal{L}_0$, while the special type (Case 1) with infinitely many trisecants is obtained by Gorenstein biliaison from a curve of degree 3 in $\mathcal{L}_0$.

EXAMPLE 4.5: We consider nonsingular degree 7 genus 2 curves in $\mathbb{P}^4$, not contained in any $\mathbb{P}^3$. The family $H_{7,2}$ of all of these curves is irreducible, by Ein [7]. We see $h^1(\mathcal{I}_C(1)) = 1$, and there exist such curves with Rao module k on a Del Pezzo surface (see below), so by semicontinuity, the general such curve has Rao module k, i.e., it is in $\mathcal{M}_1$.

Next, note that $h^0(\mathcal{I}_C(2)) \geq 2$. If $h^0(\mathcal{I}_C(2)) > 2$, then the intersection of three quadric surfaces would either be a curve of degree 8, and then C would be linked to a line, hence ACM, which is impossible; or it would be a surface of degree 3, but there are no $(7, 2)$ curves on surfaces of degree 3 in $\mathbb{P}^4$. Hence $h^0(\mathcal{I}_C(2)) = 2$, and $h^1(\mathcal{I}_C(2)) = 0$, so by Castelnuovo–Mumford regularity, $C \in \mathcal{M}_1$. Thus all curves of $H_{7,2}$ are in $\mathcal{M}_1$.

Now look on a Del Pezzo surface S, and let $C = (4; 2, 1^3, 0)$. Then C is a smooth $(7, 2)$ curve, and $C - H = (1; 1, 0^3, -1)$ is a disjoint union of a conic and a line. Thus $C \in \mathcal{L}_1$, and C is obtained by an elementary Gorenstein biliaison from a curve of degree 3 in $\mathcal{L}_0$. Note that C has exactly four mutually skew trisecants, namely the lines F_{25}, F_{35}, F_{45}, and G on S.

If C is any smooth $(7,2)$ curve, we have seen that $h^0(\mathcal{I}_C(2)) = 2$. Let S be the complete intersection surface $F_2 \cdot F_2'$ of two quadric surfaces containing C. Then S is uniquely determined by C. It is a surface of degree 4, with sectional genus 1, but it may be singular. However, it must be irreducible, and hence has at most a line of singular points. Therefore C is an almost Cartier divisor on S, and we can apply the theory of generalized divisors. There is an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{L}(C) \to \omega_C(1) \to 0$$

[13, 2.1], making use of the fact that $\omega_S = \mathcal{O}_S(-1)$. Twisting by -1, and taking cohomology, we obtain

$$0 \to H^0(\mathcal{O}_S(-1)) \to H^0(\mathcal{L}(C-H)) \to H^0(\omega_C) \to H^1(\mathcal{O}_S(-1)),$$

where H denotes the hyperplane class on S. The two outside groups are 0, and $H^0(\omega_C)$ has dimension 2, so $H^0(\mathcal{L}(C-H)) \neq 0$. This shows that $C - H$ is effective on S. It is a divisor of degree 3, and must be in $\mathcal{L}_0$, so we see that any C in $\mathcal{M}_1$ is in $\mathcal{L}_1$, and is obtained by an elementary Gorenstein biliaison from a curve of degree 3 in $\mathcal{L}_0$.

For an example of a special $(7,2)$ curve, let $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let Γ be a line of bidegree $(1,0)$, and fix an involution σ on Γ. Take ϑ to be the linear system of those curves of bidegree $(1,2)$ on S_0 meeting Γ in a pair of the involution σ. Then ϑ maps S_0 to a surface S of degree 4 in $\mathbb{P}^4$ with a double line L_0 (the image of Γ). If C_0 is a general curve of bidegree $(2,3)$ on S_0, then the image C of C_0 in S is a smooth $(7,2)$ curve meeting L_0 in three points. It has four trisecants, namely the double line L_0 and the three rulings (images of $(0,1)$ curves in S_0) that meet L_0 at the points where C meets L_0. This curve is different from the general ones described above, because three of the trisecants meet the fourth one. Because of the general result above, C must arise by an elementary Gorenstein biliaison on S, but in this case the curve of degree 3 in $\mathcal{L}_0$ will be a nonreduced curve containing a double structure on the line L_0.

Next we look at a general Castelnuovo surface S'. On this surface, there are three different kinds of smooth $(7,2)$ curves, distinguished by their self-intersections, namely

$$C_1 = (4; 2, 1^5, 0^2) \qquad C_1^2 = 7$$
$$C_2 = (5; 2^4, 1^3, 0) \qquad C_2^2 = 6$$
$$C_3 = (5; 1^4, 2^4) \qquad C_3^2 = 5.$$

Of these C_1 is obtained by an elementary Gorenstein biliaison on S' from two skew lines, while $C_2 - H$ and $C_3 - H$ are not effective. Since we have seen above that every smooth $(7,2)$ curve arises by elementary Gorenstein biliaisons from a degree 3 curve in $\mathcal{L}_0$, this gives examples of curves that may be obtained by two different routes by elementary Gorenstein biliaisons from curves of two different degrees in $\mathcal{L}_0$.

In fact, I claim that every general $(7,2)$ curve arises also as a curve of type C_1 on a smooth Castelnuovo surface. To prove this in detail is rather long, so I will just give a sketch. Start with a smooth $(7,2)$ curve C on a smooth Del Pezzo surface S, say $C = (4; 2, 1^3, 0)$ as before. Choose a twisted cubic curve D and a conic Γ so that

$C + D + \Gamma = 3H$. (For example $D = (3; 1^4, 2)$ and $\Gamma = (2; 0, 1^4)$.) Let Π be the plane containing Γ. Then the two quadric hypersurfaces containing C meet Π in Γ, so a linear combination of them contains Π. So we may assume $S = F_2 \cdot F_2'$ where F_2 contains Π. By construction, there are cubic hypersurfaces F_3 containing $C + D + \Gamma$. Such an F_3 will meet Π in Γ plus a line. Adjusting F_3 by a linear form times F_2', we may assume that F_3 contains Π. Now $F_2 \cdot F_3 = \Pi \cup S'$, where S' is an ACM surface of degree 5, hence a Castelnuovo surface. Now one can verify that C on S' is a curve with self-intersection 7, like C_1 above, and that $C_1 - H$ is effective and represented by a curve of degree 2 in $\mathcal{L}_0$.

In conclusion, we see that every smooth $(7, 2)$ curve is in $\mathcal{L}_1$, and can be obtained by elementary Gorenstein biliaison from $\mathcal{L}_0$, in general by two different routes. This is in contrast to the $(6, 1)$ case above, where the curves are divided into two types, distinguished by which component of $\mathcal{L}_0$ they arise from.

EXAMPLE 4.6: We consider smooth $(10, 6)$ curves in $\mathbb{P}^4$ (cf. Example 3.8 above).

By Riemann–Roch applied to $\mathcal{O}_C(1)$ we see that an ACM $(10, 6)$ curve is nonspecial. Also we see that $\mathcal{O}_C(1)$ is special if and only if $\mathcal{O}_C(1)$ is a canonical divisor, and this is equivalent to $h^1(\mathcal{I}_C(1)) = 1$. The $(10, 6)$ curves in $\mathbb{P}^4$ with $\mathcal{O}_C(1)$ special are all projections of the canonical curves of genus 6 in $\mathbb{P}^5$. Since these form an irreducible family, we see that their projections, the canonical $(10, 6)$ curves in $\mathbb{P}^4$, form an irreducible family, and they all have $h^1(\mathcal{I}_C(1)) = 1$. A general such curve has Rao module k in degree 1, i.e., it is in $\mathcal{M}_1$. To see this, by semicontinuity, it is sufficient to exhibit one such. Let $C = (7; 2^9, 0)$ on a general Bordiga surface. This curve is the image of a plane septic curve with 9 double points, embedded in $\mathbb{P}^4$ by the linear system of quartics passing through the double points (and one further point). These are adjoint curves to C and so cut out the canonical linear series. Hence C is a canonical curve. Now $C - H = (3; 1^9, -1)$ is the disjoint union of a plane cubic curve and a line, which is a degree 4 curve in $\mathcal{L}_0$. Hence $C \in \mathcal{L}_1$ is obtained by an elementary Gorenstein biliaison from $\mathcal{L}_0$.

Now let us consider $(10, 6)$ curves in $\mathcal{M}_2$, i.e., with Rao module k in degree 2. Examples of such can be found on a general Castelnuovo surface S, for example $C_1 = (6; 3, 2, 1^6)$ and $C_2 = (6; 2^4, 1^4)$. Note that the curves C_1 are trigonal, while the curves C_2 can have general moduli. Both types can be obtained by Gorenstein biliaison on S, since $C_1 - H = (2; 1^2, 0^6)$ and $C_2 - H = (2; 0, 1^3, 0^4)$ are both $(5, 0)$ curves, hence in $\mathcal{L}_1$.

An examination of curves of minimal genus in $\mathcal{L}_1$ and ACM curves of minimal genus in $\mathbb{P}^3$ shows that the only way to obtain a $(10, 6)$ curve in $\mathcal{M}_2$ by Gorenstein biliaison is from a $(5, 0)$ curve in $\mathcal{L}_1$ on a surface of degree 5 and sectional genus 2 in $\mathbb{P}^4$, like the Castelnuovo surfaces.

Another example of a $(10, 6)$ curve in $\mathcal{M}_2$ is obtained by the curve C formed by the intersection of a smooth quintic elliptic scroll V with a hypersurface F of degree 2. If we take F to be a smooth quadric hypersurface, then by Klein's theorem [14, II.Ex. 6.5d] it contains no surfaces of odd degree, so C cannot be obtained by ascending Gorenstein biliaison from a curve in $\mathcal{L}_1$.

However one can show that C is in the G-liaison class of two skew lines by the following method, suggested by the referee. First note that two general cubic hypersurfaces F_3, F_3' containing V will link V to a Veronese surface W in $\mathbb{P}^4$. Thus C is linked by the complete intersection $F_2 \cap F_3 \cap F_3'$ to a curve $C' \subseteq W$, which is $W \cap F_2$. The curve C' is an $(8,3)$ curve, obtained from a plane curve of degree 4 by the 2-uple embedding of $\mathbb{P}^2$ and projection to $\mathbb{P}^4$.

Now W is not an ACM surface, but if we take a hyperplane section $\Gamma = W \cap \mathbb{P}^3$, then Γ is a $(4,0)$ curve in $\mathbb{P}^3$. It is contained in a unique nonsingular quadric surface $Q \subseteq \mathbb{P}^3$, and the union $W \cup Q$, meeting along Γ, will be an ACM surface of degree 6 in $\mathbb{P}^4$. We regard $C' \subseteq W$ as a curve on the surface $W \cup Q$. Now one can show (I leave some details to the reader) that $2H - K - C'$ on the surface $W \cup Q$ (where H, K denote the hyperplane section and canonical divisor) is a curve $D \cup \Gamma'$, where Γ' is a $(4,0)$ curve in $\mathbb{P}^3$, and D is a conic, not in $\mathbb{P}^3$, meeting Γ' in two points. Since $2H - K$ is an arithmetically Gorenstein curve on $W \cup Q$, we have thus linked C to C' and then to $D \cup \Gamma'$.

For the last step, we take a quadric surface Q' containing D and meeting the quadric Q (which contains Γ') in a conic. Then $Q \cup Q'$ is a complete intersection quartic surface in $\mathbb{P}^4$, and on $Q \cup Q'$, $D \cup \Gamma' - H$ is 2 skew lines.

Thus C is an example of a curve with Rao module k, that cannot be obtained by ascending Gorenstein biliaison from a minimal curve, and yet is in the G-liaison class of 2 skew lines.

EXAMPLE 4.7: For our last example, we consider smooth $(11,7)$ curves in $\mathcal{M}_2$. To construct such curves on a general Bordiga surface S, take $C = (6; 2^3, 1^7)$. This is an $(11,7)$ curve, and $C - H = (2; 1^3, 0^7)$ is a smooth $(5,0)$ curve on S. Since $(5,0) \in \mathcal{L}_1$ by (4.3) above, we see that $C \in \mathcal{L}_2$, and is obtained from a minimal curve by two elementary Gorenstein biliaisons.

Next, I claim the only way to obtain an $(11,7)$ curve in $\mathcal{L}_2$ by two elementary G-biliaisons is the one just described. Indeed, the curves of minimal genus in $\mathcal{L}_1$ of degrees 4 to 7 are $(4,0)$, $(5,0)$, $(6,1)$, $(7,2)$. The minimal genus of ACM curves in $\mathbb{P}^3$ of complementary degree are $(7,5)$, $(6,3)$, $(5,2)$, $(4,1)$, which will give rise respectively to curves $(11,8)$, $(11,7)$, $(11,8)$, $(11,9)$ in $\mathcal{L}_2$. So an $(11,7)$ curve obtained by elementary G-biliaisons must be on the Bordiga surface or its specialization.

Since $h^0(\mathcal{O}_C(1)) = 5$, we see that $\mathcal{O}_C(1)$ is nonspecial, so we can compute the dimension of the Hilbert scheme of $(11,7)$ curves in $\mathbb{P}^4$ (which is irreducible by Ein [7]), by the usual formula $5d + 1 - g$. Thus the Hilbert scheme has dimension 49.

Now let us count the curves obtained by the construction above. The Bordiga surface moves in a family of dimension 36 (use, for example, the formula of Ellingsrud [10]). To find the dimension of the linear system $|C|$ on S, we use the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Thus $\dim_S |C| = h^0(\mathcal{O}_C(C))$. For the curve C of type $(6; 2^3, 1^7)$ mentioned above, we find $C^2 = 17$, so the divisor C^2 is nonspecial on C, and by Riemann–Roch, $h^0(\mathcal{O}_C(C)) = C^2 + 1 - g = 11$. Thus the dimension of the family of all curves of

this type on Bordiga surfaces is $\leq 11 + 36 = 47$. In particular, these curves are not general among all $(11, 7)$ curves.

But we wish to show more, namely that a general $(11, 7)$ curve does not lie on a Bordiga surface. So suppose now that C is any $(11, 7)$ curve on a Bordiga surface. I claim $C^2 \leq 17$. Indeed, we have $2g - 2 = C^2 + C.K$. On the Bordiga surface let $C = (a; b_1, \ldots, b_{10})$. The canonical divisor K can be written $K = (1; 0^{10}) - H$. So $2g - 2 = C^2 + a - d$, and $C^2 = 2g - 2 + d - a = 23 - a$. But in order to get a curve of genus 7, we must have $a \geq 6$. Thus $C^2 \leq 17$. Then the same argument as above shows that $h^0(\mathcal{O}_C(C)) \leq 11$, and we get the same dimension count, unless $\mathcal{O}_C(C)$ is a special divisor. But in that case $C^2 \leq 12$, and by Clifford's theorem $h^0(\mathcal{O}_C(C)) \leq 7$. Thus a general $(11, 7)$ curve does not lie on a Bordiga surface.

Next, observe that for any $(11, 7)$ curve in $\mathbb{P}^4$, $h^0(\mathcal{O}_C(2)) = 16$, so necessarily $h^1(\mathcal{I}_C(2)) \geq 1$. Since we have constructed curves C with $h^1(\mathcal{I}_C(2)) = 1$ and the other $h^1(\mathcal{I}_C(n)) = 0$ for $n \neq 2$, we conclude by semicontinuity that the general $(11, 7)$ curve in $\mathbb{P}^4$ has Rao module k in degree 2, i.e., it lies in $\mathcal{M}_2$. Since the general such curve does not lie on a Bordiga surface, by the above remarks, it cannot be obtained by ascending Gorenstein biliaisons from $\mathcal{L}_0$.

It is conceivable that the general $(11, 7)$ curve is linked by some ascending and descending G-liaisons to two skew lines, but this seems unlikely, so we propose the general $(11, 7)$ curve in $\mathbb{P}^4$ as a possible curve with Rao module k, not in the G-liaison class of two skew lines.

5. Conclusion

The examples presented in this paper would lead me to expect that for ACM schemes of codimension ≥ 3, some may be obtained by elementary Gorenstein biliaisons from a scheme of degree one; a broader class may be obtained by ascending and descending G-liaisons from a scheme of degree one; but that a general ACM scheme of high degree may not be in the G-liaison class of a complete intersection. Example 3.9 shows that at least one of the Questions 1.3a, 1.3b has no for an answer. Namely, for the general ACM $(20, 26)$ curve in $\mathbb{P}^4$, we must have either

a) it is ACM and not glicci, or

b) it is glicci, but cannot be obtained by ascending Gorenstein biliaisons from a curve.

For curves in $\mathbb{P}^n$, $n \geq 4$, with a given Rao module M, I would expect that the minimal curves form an infinite union of irreducible families; some curves in the family may be obtained by a sequence of ascending elementary Gorenstein biliaisons from a minimal curve; a larger class may be obtained by ascending and descending Gorenstein liaisons from a minimal curve; but that a general curve of high degree and genus with Rao module M is not in the G-liaison class of a minimal curve. Example 4.6 gives an example of a smooth $(10, 6)$ curve with Rao module k, that is in the Gorenstein liaison class of a minimal curve, but cannot be obtained by ascending Gorenstein biliaison from a minimal curve. Example 4.7 shows that either Question 1.4 has no for an

answer, or the deformation is necessary in Question 1.5b. Indeed, for the general $(11, 7)$ curve in $\mathbb{P}^4$ we must have either

a) it has Rao module k, but is not in the G-liaison class of two skew lines, or

b) it is in the liaison class of two skew lines, but cannot be obtained from a minimal curve by ascending Gorenstein biliaisons.

Based on this evidence, I would expect a no answer to Questions 1.3, 1.4, 1.5*b*. I have no idea about Question 1.2 since in this paper I used only strict Gorenstein liaisons and biliaisons. Also the question of even or odd liaison has not been addressed here, since it is irrelevant for ACM schemes and curves with Rao module k. This is a question that merits further study.

References

1. E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of algebraic curves, I*, Springer, New York, 1985.

2. E. Ballico, G. Bolondi, and J.C. Migliore, The Lazarsfeld–Rao problem for liaison classes of two-codimensional subschemes of $\mathbb{P}^n$, *Amer. J. Math.* **113** (1991), 117–128.

3. G. Bolondi and J.C. Migliore, The Lazarsfeld-Rao property on an arithmetically Gorenstein variety, *Manuscripta Math.* **78** (1993), 347–368.

4. M. Casanellas and R.M. Miró-Roig, Gorenstein liaison of curves in $\mathbb{P}^4$, *J. Algebra* **230** (2000), 656–664.

5. M. Casanellas and R.M. Miró-Roig, G-liaison of arithmetically Gorenstein schemes, (preprint).

6. M. Casanellas and R.M. Miró-Roig, Gorenstein liaison of divisors on standard determinantal schemes and on rational normal scrolls, *J. Pure Appl. Algebra,* to appear.

7. L. Ein, The irreducibility of the Hilbert scheme of smooth space curves, *Algebraic Geometry, Bowdoin 1985*, 83–87, Proc. Sympos. Pure Math. **46**, Amer. Math. Soc., Providence, R.I., 1987.

8. D. Eisenbud and J. Harris, *Curves in Projective Space*, Presses de l'Université de Montréal, Montreal, Que., 1982.

9. D. Eisenbud, H. Lange, G. Martens, and F.-O. Schreyer, The Clifford dimension of a projective curve, *Compositio Math.* **72** (1989), 173–204.

10. G. Ellingsrud, Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbb{P}^e$ à cône de Cohen-Macaulay, *Ann. Sci. École Norm. Sup. (4)* **8** (1975), 423–432.

11. F. Gaeta, Sulle curve sghembe algebriche di residuale finito, *Ann. Mat. Pura Appl. (4)* **27** (1948), 177–241.

12. R. Hartshorne, Classification of algebraic space curves, III, *Algebraic geometry and its applications*, 113–120, Springer, New York, 1994.

13. R. Hartshorne, Generalized divisors on Gorenstein schemes, *K-Theory* **8** (1994), 287–339.

14. R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

15. R. Hartshorne, Clifford index of ACM curves in $\mathbb{P}^3$, *Rend. Sem. Mat. Fis. Milano*, to appear.

16. J. Kleppe, J. Migliore, R.M. Miró-Roig, U. Nagel, and C. Peterson, Gorenstein liaison, complete intersection liaison invariants, and unobstructedness, *Mem. Amer. Math. Soc.* **154** (2001), n. 732.

17. M. Kreuzer, J. Migliore, U. Nagel, and C. Peterson, Determinantal schemes and Buchsbaum-Rim sheaves, *J. Pure Appl. Algebra* **150** (2000), 155–174.

18. J. Lesperance, Gorenstein liaison of some curves in $\mathbb{P}^4$, *Collect. Math.* **52** (2001), 219–230.

19. A. Lopez, Noether-Lefschetz theory and the Picard group of projective surfaces, *Mem. Amer. Math. Soc.* **89** (1991), n. 438.

20. G. Martens, Über den Clifford-Index algebraischer Kurven, *J. Reine Angew. Math.* **336** (1982), 83–90.

21. M. Martin-Deschamps and D. Perrin, Sur la classification des courbes gauches, *Astérisque* **184-185** (1990).

22. J.C. Migliore, Introduction to liaison theory and deficiency modules, *Progress in Mathematics* **165** (1998).

23. J.C. Migliore, Gorenstein liaison via divisors, *Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998),* 1–22, Sūrikaisekikenkyūsho Kōkyūroku **1078**, 1999.

24. J.C. Migliore and U. Nagel, Monomial ideals and the Gorenstein liaison class of a complete inter-section, *Compositio Math.,* to appear.

25. U. Nagel, Even liaison classes generated by Gorenstein linkage, *J. Algebra* **209** (1998), 543–584.

26. S. Nollet, Even linkage classes, *Trans. Amer. Math. Soc.* **348** (1996), 1137–1162.

27. C. Okonek, Moduli reflexiver Garben und Flächen von kleinem Grad in $\mathbb{P}^4$, *Math. Z.* **184** (1983), 549–572.

28. D. Perrin, Courbes passant par m points généraux de $\mathbb{P}^3$, *Mém. Soc. Math. France (N.S.)* **28/29** (1987), 138.

29. C. Peskine and L. Szpiro, Liaison des variétés algébriques, I, *Invent. Math.* **26** (1974), 271–302.

30. P. Rao, Liaison equivalence classes, *Math. Ann.* **258** (1981), 169–173.

31. J. Rathmann, The genus of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$, *Math. Z.* **202** (1989), 525–543.

32. B. Saint-Donat, On Petri's analysis of the linear system of quadrics through a canonical curve, *Math. Ann.* **206** (1973), 157–175.

33. J.-P. Serre, Sur les modules projectifs, Sém. Dubreil–Pisot (1960/61).