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Fixed points of the Hardy–Littlewood maximal operator

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Abstract

Let $1 \leq p \leq +\infty$ be a real number and $\mathcal{M}$ be the centered Hardy–Littlewood maximal operator. Then, there exists a non-constant function $f \in L^p(\mathbb{R}^n)$ such that $\mathcal{M}(f) = f$, if and only if $n \geq 3$ and $n/(n-2) < p \leq +\infty$. We also prove that for every $n \geq 1$ and every $1 \leq p \leq +\infty$, there is no non-constant fixed point of the so-called centered "strong" maximal operator $\mathcal{N}$ in $L^p(\mathbb{R}^n)$.

1. Introduction

The classical centered Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on the Lebesgue space $L^1_{\text{loc}}(\mathbb{R}^n)$ by setting

$$\forall f \in L^1_{\text{loc}}(\mathbb{R}^n), \quad \mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| \, dy,$$

for every $x \in \mathbb{R}^n$. Here $|B_r|$ denotes the volume of the Euclidean ball B_r centered at the origin of $\mathbb{R}^n$ and with radius r. The maximal function is a classical tool in harmonic analysis but recently it has been successfully used in studying Sobolev functions and partial differential equations, see [1] and [2]. The celebrated theorem of Hardy, Littlewood and Wiener asserts that the maximal operator is bounded in $L^p(\mathbb{R}^n)$ for all $1 < p \leq \infty$ (see, Stein [3]):

$$\|\mathcal{M}(f)\|_p \leq C_p \, \|f\|_p. \tag{1}$$

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In this paper our goal is to study the existence of non-constant fixed points of the maximal operator $\mathcal{M}$ in the framework of Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 \le p \le +\infty$, (a locally integrable function f is a fixed point of $\mathcal{M}$, if and only if $\mathcal{M}(f) = f$).

It is well-known (and simple to check) that for every non-zero locally integrable function f on $\mathbb{R}^n$, there exists a constant $C > 0$ such that

$$\forall x \in \mathbb{R}^n, \quad \mathcal{M}(f)(x) \ge C \ (1 + |x|)^{-n}. \tag{2}$$

Consequently, the zero function is the only fixed point of $\mathcal{M}$ in $L^1(\mathbb{R}^n)$. Obviously, each positive constant is a fixed point of $\mathcal{M}$ in $L^\infty(\mathbb{R}^n)$. However, the question can be raised whether $\mathcal{M}$ has non-constant fixed points in $L^p(\mathbb{R}^n)$ when $p \in (1, +\infty]$. Our following result yields an affirmative answer.

Theorem 1

Let $1 \le p \le +\infty$ be a real number. Then, there exists a non-constant fixed point $f \in L^p(\mathbb{R}^n)$ of $\mathcal{M}$, if and only if $n \ge 3$ and $n/(n-2) < p \le +\infty$.

Before we come back to the proof of Theorem 1, we shall make some clarifying remarks.

Remark 1. A locally integrable function f on $\mathbb{R}^n$ is a fixed point of $\mathcal{M}$, if and only if f is a super-harmonic function (i.e., $\Delta(f) \le 0$; here Δ is the Laplace operator) and is positive. Indeed, based on Lebesgue's differentiation theorem, the relation

$$f(x) = \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r} f(x-y) \ dy \tag{3}$$

holds for almost every $x \in \mathbb{R}^n$, whenever $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore, one obtains

$$|f(x)| \le \mathcal{M}(f)(x) \quad \text{a.e. } \ x \in \mathbb{R}^n. \tag{4}$$

So, a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a fixed point of $\mathcal{M}$, if and only if f is positive and satisfies

$$\forall r > 0, \quad \frac{1}{|B_r|} \int_{B_r} f(x-y) \ dy \le f(x) \ \text{ a.e. } \ x \in \mathbb{R}^n,$$

which concludes our claim. □

Remark 2. Let $f \in L^p(\mathbb{R}^n)$ be a non-constant fixed point of $\mathcal{M}$; let φ be a positive function belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x) \ dx = 1$. Then, there exists some $t > 0$ such that the function $f_t = f * \varphi_t(x)$, where $\varphi_t(x) = t^{-n} \ \varphi(x/t)$, is also a non-constant fixed point of $\mathcal{M}$ belonging to $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Indeed, obviously, for every $t > 0$, $f_t \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. To see that f_t is a fixed point of $\mathcal{M}$, according to the estimate (4), it suffices to establish that

$$f_t \ge \mathcal{M}(f_t).$$

This last estimate is deduced by applying Fubini theorem and the fact that $\mathcal{M}$ commutes with translations (i.e., $\tau_\alpha \mathcal{M} = \mathcal{M} \tau_\alpha$ with $\tau_\alpha f(x) = f(x-\alpha)$): for every $r>0$, let $\chi_r = \frac{1}{|B_r|}\chi_{B_r}$. Then, we have $\mathcal{M}(f_t) = \sup_{r>0} \chi_r * f_t$ and

$$\begin{aligned}\chi_r * (\varphi_t * f)(x) &= \int_{\mathbb{R}^n} \varphi_t(y)\ (\chi_r * \tau_y f)(x)\ dy \\ &\le \int_{\mathbb{R}^n} \varphi_t(y)\ \mathcal{M}(\tau_y f)(x)\ dy = \varphi_t * \mathcal{M}(f)(x) = f_t.\end{aligned}$$

Finally, using the Lebesgue differentiation theorem, there exists some $t>0$ such that f_t is non-constant, since f is non-constant. This proves our claim. □

2. Proof of Theorem 1

2.1. Proof of Theorem 1, Part 1

Let $n \ge 3$ and $n/(n-2) < p \le +\infty$. Before we prove that the maximal operator has a non-constant fixed point $f \in L^p(\mathbb{R}^n)$, we shall first recall some facts about the Riesz potentials (for more details, see for instance Stein [3], page 116). Classically, the Riesz potentials $(I_\alpha)_{\alpha\in[0,n)}$ are defined in $\mathcal{S}(\mathbb{R}^n)$ by setting

$$I_\alpha(g) = \mathcal{F}^{-1}\big[(2\pi|\xi|)^{-\alpha}\ \mathcal{F}g\big]$$

where $\mathcal{F}g(\xi) = \int_{\mathbb{R}^n} e^{2\pi i(x|\xi)}\ g(x)\ dx$ with $(x|\xi) = x_1\xi_1 + \cdots + x_n\xi_n$, and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

The fundamental result of Hardy–Littlewood–Sobolev asserts that (see, for example, Stein [3], page 119), for every $1 < q < p < +\infty$ and $0 < \alpha < n/q$ such that $1/p = 1/q - \alpha/n$, we have

$$\|I_\alpha(g)\|_p \le C_{p,q}\ \|g\|_q \tag{5}$$

and the following representation integral

$$I_\alpha(g)(x) = c_\alpha \int_{\mathbb{R}^n} |x-y|^{-n+\alpha}\ g(y)\ dy, \tag{6}$$

where Γ is the gamma function and

$$c_\alpha = \pi^{-n/2}\ 2^{-\alpha}\frac{\Gamma\big(n/2-\alpha/2\big)}{\Gamma(\alpha/2)}.$$

Now, we are ready to establish our claim.

Step 1. Here we deal with the case $n \ge 3$ and $n/(n-2) < p < +\infty$. On the one hand, we notice that for every $1 < q < \min(p, n/2)$ and every $2 < \alpha < n/q$ such that $1/p = 1/q - \alpha/n$, we have the following identity –which is valid for the class $\mathcal{S}(\mathbb{R}^n)$

by the means of Fourier transform, and extends to $L^q(\mathbb{R}^n)$ via the Hardy–Littlewood–Sobolev result recalled above–

$$\forall g \in L^q(\mathbb{R}^n), \quad \Delta(I_\alpha(g)) = -I_{\alpha-2}(g). \tag{7}$$

On the other hand, for every non zero positive function $g \in L^q(\mathbb{R}^n)$, the equations (6) (applied to $\alpha - 2$), (7) and Remark 1 imply that $I_\alpha(g)$ is a fixed point of $\mathcal{M}$. Thanks to Hardy–Littlewood–Sobolev result, $I_\alpha(g)$ is a non-constant fixed point of $\mathcal{M}$ and belongs to $L^p(\mathbb{R}^n)$. This concludes Step 1.

Step 2. We assume that $n \geq 3$. To prove that $\mathcal{M}$ has a non-constant fixed point $f \in L^\infty(\mathbb{R}^n)$, we use the known result (cf. Step 1) for a finite p: we choose a non-constant fixed point $f \in L^p(\mathbb{R}^n)$ of $\mathcal{M}$. Then, we consider (cf. Remark 2) the non-constant fixed point f_t of $\mathcal{M}$ which belongs to $L^\infty(\mathbb{R}^n)$ (since $\varphi_t \in L^{p'}(\mathbb{R}^n)$ with $1/p + 1/p' = 1$). This completes the proof of Theorem 1, Part 1. □

2.2. Proof of Theorem 1, Part 2

We assume that there exists a non-constant fixed point of $\mathcal{M}$ in $L^p(\mathbb{R}^n)$ and we shall prove that $n \geq 3$ and $n/(n-2) < p \leq +\infty$. We split the proof into two steps.

Step 1. Here we shall prove that necessarily $n \geq 3$. We proceed by contradiction. We prove, if $n = 1, 2$, there exists no non-constant fixed point of $\mathcal{M}$ in $L^p(\mathbb{R}^n)$ whenever $1 \leq p \leq +\infty$.

• Let $f \in L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, be a non-constant fixed point of $\mathcal{M}$. According to Remark 2, we may assume that $f \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$. Therefore, f is a positive concave function:

$$\forall x \in \mathbb{R}, \quad f(x) \leq f'(x_0)\ (x - x_0) + f(x_0) \tag{8}$$

where we may choose some $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$. But, the estimate (8) is not possible, since f is a non-constant positive function belonging to $L^p(\mathbb{R})$, $1 \leq p \leq +\infty$.

• Again, there is no non-constant fixed point $f \in L^p(\mathbb{R}^2)$ whenever $1 \leq p \leq +\infty$. If this would be false, as we have said, we may assume that $f \in C^\infty(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Let $x_0 \in \mathbb{R}^2$ and $\varepsilon \in (0,1)$, set $t_\varepsilon = f(x_0) - \varepsilon$. We find $\delta > 0$ such that $f(x) > t_\varepsilon$ if $|x - x_\varepsilon| \leq \delta$. Now, for $\delta < |x - x_\varepsilon| < R$, we consider the function

$$G(x) = f(x) + \frac{t_\varepsilon}{\log(R) - \log(t_\varepsilon)}\big(\log(|x - x_0|) - \log(t_\varepsilon)\big) - t_\varepsilon.$$

Since $x \to \log(|x - x_0|)$ is harmonic on $\mathcal{C}_R = \{x \in \mathbb{R}^2 :\ \delta < |x - x_0| < R\}$ and f is super-harmonic on $\mathcal{C}_R$, then G is super-harmonic on $\mathcal{C}_R$. Furthermore, G is positive on the boundary of $\mathcal{C}_R$. Hence, the maximum principle yields that G is positive on the closure of $\mathcal{C}_R$. So, by passing to limits ($R \to +\infty$) for fixed x, this yields $f(x) \geq t_\varepsilon$ everywhere. Since $\varepsilon > 0$ is arbitrary, f is constant, which is a contradiction. Henceforward, we suppose $n \geq 3$.

Step 2. Here we assume that $n \geq 3$, and we shall prove that $n/(n-2) < p \leq +\infty$. Let $f \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be a non-constant fixed point of $\mathcal{M}$. Since $f(0) = \mathcal{M}(f)(0) > 0$, there exists some $t > 0$ such that

$$\forall x \in \mathbb{R}^n, \ \ |x| \leq t \Longrightarrow f(x) \geq t.$$

Since the function $|\cdot|^{2-n}$ is harmonic on $\mathcal{D}_R = \{x \in \mathbb{R}^n : \ t < |x| < R\}$ and f is super-harmonic on $\mathcal{D}_R$, then the function

$$H(x) = f(x) - t^{n-1} \ |x|^{2-n} + t^{n-1} \ R^{2-n}$$

is super-harmonic on $\mathcal{D}_R$. Furthermore, H is positive on the boundary of $\mathcal{D}_R$. Again, the maximum principle yields that H is positive on the closure of $\mathcal{D}_R$. So, by passing to limits ($R \to +\infty$), this yields $f(x) \geq t^{n-1} \ |x|^{2-n}$ everywhere. Using the assumption $f \in L^p(\mathbb{R}^n)$ and polar coordinates, we obtain $n/(n-2) < p \leq +\infty$. This concludes the proof of Theorem 1. □

Before we conclude this paper, we would like to make some remarks. For the so-called centered "strong" maximal operator $\mathcal{N}$ (see below for its definition), the picture is different. Denote by $x = (x_1, \ldots, x_n)$ points in $\mathbb{R}^n$. For a locally integrable function f on $\mathbb{R}^n$, define

$$\mathcal{N}(f)(x) = \sup_{r_1>0} \cdots \sup_{r_n>0} \frac{2^{-n}}{r_1 \cdots r_n} \int_{x_1-r_1}^{x_1+r_1} \cdots \int_{x_n-r_n}^{x_n+r_n} |f(y_1, \cdots, y_n)| \ dy_n \cdots dy_1.$$

Corollary 1

For every $1 \leq p \leq +\infty$ and every $n \geq 1$, there is no non-constant fixed point $f \in L^p(\mathbb{R}^n)$ of the operator $\mathcal{N}$.

Proof. We proceed by contradiction by assuming that there exists a non-constant fixed point $f \in L^p(\mathbb{R}^n)$ of $\mathcal{N}$ (with $1 \leq p \leq +\infty$ and $n \geq 1$). Similar arguments like in Remark 2 lead us to assume that f is continuous in $\mathbb{R}^n$.

First, assume $p < +\infty$; via Fubini's theorem, there exists $u \in \mathbb{R}^{n-1}$ such that the function $f_u(x_1) = f(x_1, u)$ is a non-constant function belonging to $L^p(\mathbb{R})$. The equation $\mathcal{N}(f) = f$ leads, for every $x_1 \in \mathbb{R}$ and every $r > 0$, to

$$f_u(x_1) \geq \frac{1}{2r} \int_{x_1-r}^{x_1+r} \left(\frac{1}{|Q_\rho|} \int_{Q_\rho} f(y_1, \cdots, y_n) \ dy_n \cdots dy_2 \right) \ dy_1 \tag{9}$$

where $u = (u_1, \cdots, u_{n-1})$ and $Q_\rho = \{x \in \mathbb{R}^{n-1} : \max_{1 \leq i \leq n-1} |x_i - u_i| \leq \rho\}$. By passing to limits ($\rho \to 0$) in the estimate (9), continuity of f and Fatou lemma yield

$$\forall x_1 \in \mathbb{R}, \quad f_u(x_1) \geq \frac{1}{2r} \int_{x_1-r}^{x_1+r} f_u(y_1) \ dy_1 \tag{10}$$

for every $r > 0$. Therefore, f_u is a non-constant fixed point of $\mathcal{M}$ in $L^p(\mathbb{R})$. But this is impossible (see Theorem 1).

Finally, if $p = \infty$, there exists $i = 1, \cdots, n$ and $(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n) \in \mathbb{R}^{n-1}$ such that the function ψ defined by

$$\psi(y) = f(u_1, \cdots, u_{i-1}, y, u_{i+1}, \cdots, u_n)$$

is non-constant and belongs to $L^\infty(\mathbb{R})$. The same arguments given before yield that ψ is a non-constant fixed point of $\mathcal{M}$ in $L^\infty(\mathbb{R})$, which is a contradiction (see Theorem 1). This concludes the proof of Corollary 1. □

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