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On the refinements of a polyhedral subdivision

Francisco Santos*

Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria, E–39005 Santander, Spain

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Abstract

Let $\pi{:}P{\to}Q$ be an affine projection map between two polytopes P and Q. Billera and Sturmfels introduced in 1992 the concept of *polyhedral subdivisions of* Q *induced by* π (or π-induced) and the *fiber polytope* of the projection: a polytope $\Sigma(P,\pi)$ of dimension $\dim(P){-}\dim(Q)$ whose faces are in correspondence with the *coherent* π-induced subdivisions (or π-coherent subdivisions).

In this paper we investigate the structure of the poset of π-induced refinements of a π-induced subdivision. In particular, we define the *refinement polytope* associated to any π-induced subdivision S, which is a generalization of the fiber polytope and shares most of its properties.

As applications of the theory we prove that if a point configuration has non-regular subdivisions then it has non-regular triangulations and we provide simple proofs of the existence of non-regular subdivisions for many particular point configurations.

Introduction

In 1992, Billera and Sturmfels [6] introduced the concept of polyhedral subdivisions of Q induced by an affine projection map $\pi : P \to Q$ between two polytopes (or π-*induced*) and the *fiber polytope* of the projection: a polytope $\Sigma(P, \pi)$ of dimension $\dim(P) - \dim(Q)$ whose faces are in correspondence with the *coherent* π-induced subdivisions (or π-coherent subdivisions). See also [26, Chapter 9]. This was a natural generalization and a clarification of the theory of *secondary polytopes* developed by Gelfand, Kapranov and Zelevinsky [8, 14].

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There is a certain amount of recent literature concerning this theory (see [2, 3, 4, 5, 6, 7, 10, 12, 13, 15, 19, 20, 21] and the survey article [22]), mainly in connection with the so-called Baues problem, stated by Billera, Kapranov and Sturmfels [7], which asked whether the refinement poset of all proper π-induced subdivisions of a polytope projection is homotopy equivalent to a sphere. Although a negative example was found by Rambau and Ziegler [21] the particular and important cases of P being a simplex or a cube remain open.

Here we show that the theory of fiber polytopes generalizes nicely to the study of lower ideals of the poset of π-induced subdivisions, i.e., to the study of π-induced refinements of a given π-induced subdivision S. In particular, for any π-induced subdivision S of Q one can define a certain *π-refinement polytope* $\Sigma(S,\pi)$ (Theorem 1.3) with the following properties:

(a) If S is the trivial subdivision, then $\Sigma(S,\pi)$ equals the fiber polytope $\Sigma(P,\pi)$ (Theorem 1.3, part 3).
(b) The faces of $\Sigma(S,\pi)$ correspond to certain π-induced refinements of S which we call π-coherent refinements of S (Theorem 2.4, parts 2 and 4) and which include all the π-coherent subdivisions which refine S.
(c) If S is a π-coherent subdivision then $\Sigma(S,\pi)$ equals the face of the fiber polytope corresponding to S (Theorem 1.3, parts 2 and 4).
(d) If S' refines S then $\Sigma(S',\pi)$ is strictly contained in $\Sigma(S,\pi)$ (Theorem 2.4, parts 1 and 3 and Corollary 2.5).

It is important to stress that the refinement polytope considered here is not the same as the *generalized secondary polytope* of the subdivision S, considered in [1, Section 2.12] and [24, Section 4.2]. That polytope has properties (a) and (b) above, but neither (c) nor (d). See more details in Remark 2.7.

With the aid of this theory we are able to prove several nice properties of the poset $\Omega(S,\pi)$ of π-induced refinements of a π-induced subdivision S, such as the following ones:

- The poset is "atomic", meaning by this that any subdivision S is the least common upper bound of its *tight* refinements (Proposition 2.3; a π-induced subdivision is called tight if it has no proper π-induced refinements, i.e., if it is an atom in the poset).
- A subdivision S is π-coherent if and only if all of its tight refinements are π-coherent (Theorem 2.8). In particular, if a polytope projection produces non-coherent subdivisions, then it produces non-coherent tight subdivisions (Corollary 2.9). This allows us to give simple proofs of the existence of non-regular subdivisions for several particular point configurations (Section 3, Examples 3.2).
- If the refinement polytope $\Sigma(S,\pi)$ has dimension 1, then the poset of proper π-induced refinements of S is isomorphic to the poset of proper faces of a cube of a certain dimension (Theorem 4.3).
- In particular, the elements of height 1 in the poset have exactly two proper refinements, which are both tight (Corollary 4.5). This suggests the definition of a π-flip between two tight π-induced subdivisions (Definition 4.7). If P is a simplex, if P is a cube or if $\dim(Q) = 1$ this definition coincides respectively (at least in

non-degenerate cases) with the standard notions of bistellar flip between triangulations of a point configuration, cube-flip between cubical tilings of a zonotope and polygon move between monotone paths in a polytope (Section 5).

The structure of the paper is as follows: Section 1 is a review of the concepts and previous results in the theory of fiber polytopes and π-induced subdivisions, and ends with the definition of the π-refinement polytope. Section 2 contains the main results on π-refinement polytopes and π-coherent refinements. Section 3 applies the theory to existence of non-regular triangulations. Section 4 analyzes the case where the π-refinement polytope is 1-dimensional and Section 5 is devoted to the concept of π-flip and its relation to bistellar flips, cube flips and polygon moves in the cases mentioned above. Several examples and open questions are included throughout the paper.

1. Refinement polytopes

Subdivisions of a point configuration

By a point configuration $\mathcal{A}$ in $\mathbb{R}^d$ we mean a finite labelled subset of $\mathbb{R}^d$. We admit $\mathcal{A}$ to have repeated points, which are distinguished by their labels. The following formalization of *polyhedral subdivisions* of $\mathcal{A}$ comes from [14, Section 7.2]. Equivalent ones can be found in [3, 6, 8, 15, 22, 26]. We will call *faces* of $\mathcal{A}$ the subsets where affine functionals take their maximum. $\mathcal{A}$ and $\emptyset$ are considered faces. Given two subsets B_1 and B_2 of $\mathcal{A}$ we say that they *intersect properly* if the following two conditions hold:

- $\mathrm{conv}(B_1) \cap \mathrm{conv}(B_2)$ is a face F of both polytopes $\mathrm{conv}(B_1)$ and $\mathrm{conv}(B_2)$ (possibly empty).
- $F \cap B_1 = F \cap B_2$.

A subset of $\mathcal{A}$ is said to be *full-dimensional* if it affinely spans $\mathcal{A}$ and *simplicial* if it is affinely independent. Following [8] and [14] we define:

DEFINITION 1.1. A *(polyhedral) subdivision* of $\mathcal{A}$ is a collection S of full-dimensional subsets of $\mathcal{A}$ which intersect pairwise properly and cover $\mathrm{conv}(\mathcal{A})$ in the sense that $\cup_{B\in S}\mathrm{conv}(B) = \mathrm{conv}(\mathcal{A})$. The elements of S are called *cells* of the subdivision.

The set of subdivisions of $\mathcal{A}$ is partially ordered by the *refinement relation*

$$S_1 \leq S_2 \qquad :\Longleftrightarrow \qquad \forall B_1 \in S_1,\ \exists B_2 \in S_2,\ B_1 \subset B_2.$$

The poset of subdivisions of $\mathcal{A}$ has a unique maximal element which is the *trivial* or *improper* subdivision $\{\mathcal{A}\}$. The maximal proper elements are called *coarse subdivisions* and the minimal elements are the subdivisions all of whose faces are simplicial, which are called *triangulations* of $\mathcal{A}$.

Baues posets and π-induced subdivisions

Let $P \subset \mathbb{R}^p$ be a polytope and $\pi : \mathbb{R}^p \to \mathbb{R}^q$ be an affine projection map. We denote by $\mathrm{vert}(P)$ the set of vertices of P. Let $\mathcal{A} = \pi(\mathrm{vert}(P))$, with each point in $\mathcal{A}$ labelled by the vertex of P of which it is considered to be the image ($\mathcal{A}$ may have repeated points). A subdivision S of $\mathcal{A}$ is said to be it π-induced if each of its cells equals $\pi(B)$ for some face B of the point configuration $\mathrm{vert}(P)$.

Observe that, since π is a bijection between $\mathrm{vert}(P)$ and $\mathcal{A}$, the subdivision S completely describes which faces of $\mathrm{vert}(P)$ project to cells of S, even if different geometric faces of the polytope P have the same image under π. We will use the following notation: if S is a π-induced subdivision of $\mathcal{A}$ and B is a cell of S, P^B will denote the face of the polytope P for which $\pi(\mathrm{vert}(P^B)) = B$ and P^S will denote the union of all such faces, for the different cells in S.

We will call *Baues poset* of the polytope projection $\pi : P \to \pi(P)$ the poset of all π-induced subdivisions of $\mathcal{A}$ (excluding the trivial one $\{\mathcal{A}\}$) partially ordered by refinement. We will denote it $\Omega(P, \pi)$. Its minimal elements are called *tight* π-induced subdivisions. A π-induced subdivision S is tight if and only if P^S has pure dimension equal to the dimension of $\pi(P)$. Equivalently, if for every face F of P contained in P^S one has $\dim(F) = \dim(\pi(F))$. Faces with this property will be called *tight* faces of P. Observe that the refinement ordering in $\Omega(P, \pi)$ coincides with the inclusion ordering in the set $\{P^S : S \text{ is a } \pi\text{-induced subdivision}\}$.

The *generalized Baues problem* posed by Billera, Kapranov and Sturmfels [7] asked whether $\Omega(P, \pi)$ is always homotopy equivalent to a sphere of dimension $\dim(P) - \dim(\pi(P))$. In general the answer is negative, as an example of Rambau and Ziegler [21] showed. The parameters in this example are $\dim(P) = 5$, $\dim(\pi(P)) = 2$ and $\#\mathrm{vert}(P) = 10$. However, the cases of P being a simplex or a hypercube I^d are specially interesting and still open. In the simplex case $\Omega(P, \pi)$ is the poset of *all* the subdivisions of $\mathcal{A} = \pi(\mathrm{vert}(P))$. In the cube case it is the poset of all *zonotopal tilings* of the zonotope $\pi(I^d)$ and it is isomorphic to the *extension space* of the oriented matroid dual to the one realized by the generators of the zonotope (this isomorphism is the Bohne-Dress theorem on zonotopal tilings, see [26]). Positive answers are known for the following cases:

- $\dim(\pi(P)) = 1$ [7] or $\dim(P) - \dim(\pi(P)) \leq 2$ [21].
- P is a simplex and either $\dim(\pi(P)) = 2$ [13] or $\dim(P) - \dim(\pi(P)) = 3$ [4].
- P is a cube and either $\dim(\pi(P)) = 2$ or $\dim(P) - \dim(\pi(P)) = 3$ [21].
- P is a cyclic polytope and π the projection which forgets some of the coordinates (in particular, $\mathcal{A}$ is the vertex set of another cyclic polytope) [20, 3].

The cube case is actually equivalent to a special case of the simplex case. Indeed, the poset of zonotopal tilings of a zonotope of dimension d with n generators equals the poset of all subdivisions of a certain *Lawrence polytope* of dimension $n + d - 1$ with $2n$ vertices (see [15] and [23, Section 4]). It is also equivalent to the *extension space conjecture* of oriented matroid theory. More generally, the case of P being a product of simplices (in which $\Omega(\mathcal{A}, \pi)$ is the poset of mixed subdivisions of a Minkowski sum of point configurations) would follow from the case of P being a simplex, via the use of the Cayley trick [15]. We will come back to this in Section 5.

Fiber polytopes and π-coherent subdivisions

Every non-zero linear functional $w \in (\mathbb{R}^p)^*$ defines a π-induced subdivision as follows: the map π factors into a map $(\pi, w) : \mathbb{R}^p \to \mathbb{R}^q \times \mathbb{R}$ and the map $\rho : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$ which forgets the last coordinate. Let $\overline{\mathcal{A}}$ be the point configuration $(\pi, w)(\mathrm{vert}(P))$ in $\mathbb{R}^q \times \mathbb{R}$. A face of $\overline{\mathcal{A}}$ is called *upper* if its outer normal cone contains a vector with last coordinate strictly positive. The collection of upper facets of $\overline{\mathcal{A}}$ projects onto a subdivision S_w of $\mathcal{A}$. The subdivision is π-induced, since every face of $\overline{\mathcal{A}}$ is the projection of the vertex set of a face of P. We call S_w the *π-coherent subdivision of $\mathcal{A}$ for the functional w*. A subdivision of $\mathcal{A}$ is called π-coherent if it is the π-coherent subdivision for some functional.

The following is a different description of the π-induced subdivision S_w associated to a linear functional w: for any generic point $x \in \mathrm{conv}(\mathcal{A}) = \pi(P)$ let $(\pi^{-1}(x))^w$ be the face of the fiber $\pi^{-1}(x) \subset P$ on which w takes its maximum and let $F_{x,w}$ be the smallest face of P which contains $(\pi^{-1}(x))^w$. Then, $S_w = \{\pi(\mathrm{vert}(F_{x,w})) : x \in \mathrm{conv}(\mathcal{A})$ and x is generic$\}$.

Given a π-coherent subdivision S of $\mathcal{A}$, the collection of all functionals w for which $S = S_w$ is a relatively open polyhedral convex cone. The collection of all these cones for varying S is a polyhedral fan which covers $(\mathbb{R}^p)^*$. In fact, it is the normal fan of a certain polytope of dimension $\dim(P) - \dim(\pi(P))$ called the *fiber polytope* of the projection π, as proved in [6]. We will denote this polytope $\Sigma(P, \pi)$ and its precise definition is as follows: Let $\Gamma(P, \pi)$ denote the set of all piecewise linear sections $s : \pi(P) \to P$ for the projection π. For each such section, the average $\frac{1}{\mathrm{vol}(\pi(P))} \int_{\pi(P)} s(x) \mathrm{d}x$ is a point in the fiber $\pi^{-1}(O) \subset \mathbb{R}^p$ of π over the centroid O of $\pi(P)$. Let

$$\Sigma(P, \pi) := \left\{ \frac{1}{\mathrm{vol}(\pi(P))} \int_{\pi(P)} s(x) \mathrm{d}x : s \in \Gamma(P, \pi) \right\}.$$

Theorem 1.2 ([6])

$\Sigma(P, \pi)$ *is a polytope of dimension* $\dim(P) - \dim(\pi(P))$. *Its faces are in one-to-one correspondence with the π-coherent subdivisions of* $\mathcal{A} = \pi(\mathrm{vert}(P))$.

If the polytope P is a simplex, the π-coherent subdivisions of $\mathcal{A}$ are simply called *coherent* [14] or *regular* [8, 17]. The fiber polytope is called the *secondary polytope* of the point configuration $\mathcal{A}$.

Refinement polytopes

The following statement makes more explicit the bijection between π-coherent subdivisions of $\mathcal{A}$ and faces of $\Sigma(P, \pi)$:

Theorem 1.3

Let S be a π-induced subdivision for a certain polytope projection $\pi : P \to \pi(P)$. Let $\Gamma(S,\pi)$ be the subset of $\Gamma(P,\pi)$ consisting of sections with image in P^S. Let

$$\Sigma(S,\pi) := \left\{ \frac{1}{\mathrm{vol}(\pi(P))} \int_{\pi(P)} s(x)\mathrm{d}x : s \in \Gamma(S,\pi) \right\}.$$

Then,

1. *$\Sigma(S,\pi)$ is the Minkowski average of the fiber polytopes of the different cells in S. More precisely*
$$\Sigma(S,\pi) = \frac{1}{\mathrm{vol}(\pi(P))} \sum_{B\in S} \mathrm{vol}(\mathrm{conv}(B))\Sigma(P^B,\pi)$$
2. *If S is the π-coherent subdivision of $\mathcal{A}$ for a functional w, then $\Sigma(S,\pi)$ is the face of $\Sigma(P,\pi)$ which maximizes w.*
3. *$\Sigma(S,\pi) = \Sigma(P,\pi)$ if and only if S is the trivial subdivision $\{\mathcal{A}\}$.*
4. *$\Sigma(S,\pi)$ is a face of $\Sigma(\mathcal{A},\pi)$ if and only if S is a π-coherent subdivision.*

Proof. Decomposing the integral $\int_{\pi(P)} s(x)\mathrm{d}x$ via the subdivision S, for each section $s \in \Gamma(S,\pi)$, gives the formula in part 1.

For part 2, if S is π-coherent for a functional $w \in (\mathbb{R}^p)^*$, then for each cell $B \in S$, the maximum value of $w(\int_{\mathrm{conv}(B)} s(x)\mathrm{d}x)$ is taken on and only on the sections $s(x)$ with image contained in P^B. This proves the statement, and also the following claim which will be used for part 4: if S' is a subdivision with $\Sigma(S',\pi) \subset \Sigma(S,\pi)$ and S is π-coherent, then $P^{S'} \subset P^S$ and, hence, S' refines S.

Part 1 trivially shows that $\Sigma(\{\mathcal{A}\},\pi) = \Sigma(P,\pi)$. In order to prove part 3, suppose that S is not the trivial subdivision. Let B be any cell in S and x be a point in the relative interior of B. Let P^B be the face of P corresponding to B. The fact that S is not trivial implies that $P^B \cap \pi^{-1}(x)$ is a proper face of the fiber $\pi^{-1}(x)$. Let w be a functional whose maximum over $\pi^{-1}(x)$ is not taken in any point of $P^B \cap \pi^{-1}(x)$. Let S_w be the π-coherent subdivision for w. By the proof of part 2, the value of the functional w over $S(P,\pi)$ cannot be maximized in any point of $\Sigma(S,\pi)$. In particular, $\Sigma(S,\pi) \neq \Sigma(P,\pi)$.

The “if” in part 4 is implied by part 2 of the statement. For the “only-if”, suppose that $\Sigma(S,\pi)$ is a face of $\Sigma(P,\pi)$. Let w be a functional whose maximum over $\Sigma(P,\pi)$ is taken precisely in the face $\Sigma(S,\pi)$. By part 2, $\Sigma(S,\pi) = \Sigma(S_w,\pi)$, where S_w is the π-coherent subdivision of $\mathcal{A}$ for w. The last claim in the proof of part 2 implies that S refines S_w. Since S is assumed not to be π-coherent, it is a proper refinement of S_w.

$\Sigma(S_w,\pi)$ is the Minkowski average of the fiber polytopes $\Sigma(P^B,\pi)$ for the cells $B \in S_w$. $\Sigma(S,\pi)$ is, by part 3 applied to the different cells of S_w, a Minkowski sum of polytopes strictly contained in them. Thus, $\Sigma(S,\pi)$ is strictly contained in $\Sigma(S_w,\pi)$, which is a contradiction. □

DEFINITION 1.4. Let S be a π-induced subdivision for a polytope projection $P \to \pi(P)$. The polytope $\Sigma(S,\pi)$ of Theorem 1.3 will be called the *π-refinement polytope of the subdivision S.*

Remark 1.5. The projection $\pi : P^S \to \pi(P)$ induces a map $\mathcal{B} : \pi(P) \to 2^{\mathbb{R}^p}$ which associates to every point $x \in \pi(P)$ the restricted fiber $\pi^{-1}(x) \cap P^S$. This map is an example of what Billera and Sturmfels [6] call a *polytope bundle* and it is piecewise linear. The π-refinement polytope $\Sigma(P, \pi)$ is the Minkowski integral of the polytope bundle. In particular, Theorem 1.3 and Proposition 1.2 in [6] imply, respectively, parts 1 and 2 of the previous theorem.

Remark 1.6. If S is a tight π-induced subdivision then $\Gamma(S, \pi)$ has only one element and, in particular, the refinement polytope $\Sigma(S, \pi)$ is a single point. This point equals

$$v_S := \frac{\sum_{B \in S} \mathrm{vol}(\mathrm{conv}(B)) O^B}{\mathrm{vol}(\pi(P))},$$

where O^B denotes the centroid of the face P^B of P.

Suppose, moreover, that P is a simplex with vertex set $\{e_1, \ldots, e_{p+1}\}$ and let $a_i = \pi(e_i)$, so that π-induced subdivisions coincide with the polyhedral subdivisions of the point configuration $\mathcal{A} := \{a_1, \ldots, a_{p+1}\}$. Then, the centroid O^B of each face can be rewritten as the average of its $q + 1$ vertices and the formula above takes the following form

$$v_S = \frac{\sum_{i=1}^{p+1} \left(\sum_{a_i \in B \in S} \mathrm{vol}(\mathrm{conv}(B)) \right) e_i}{(d + 1)\mathrm{vol}(\pi(P))}.$$

In other words, the i-th affine coordinate of the vertex $v_S \in \mathbb{R}^p$ equals, up to a normalization constant, the volume of the star of a_i in S. This is the standard way to express the vertex of the secondary polytope associated to a regular triangulation of $\mathcal{A}$ [8, 14].

2. π-coherent refinements of a π-induced subdivision

We are interested in the poset of π-induced subdivisions of $\mathcal{A}$ which refine a given one S. We will see that this poset behaves in many respects as $\Omega(P, \pi)$ and in particular that the faces of the above defined *π-refinement polytope* $\Sigma(S, \pi)$ are in correspondence with some π-induced refinements of S. As it happened with Theorem 1.3 some of our results can be proved from more general results concerning polytope bundles as in Section 1 of [6]. We will not discuss this in detail.

Throughout this section we fix $\pi : \mathbb{R}^p \to \mathbb{R}^q$ to be a linear projection map, P a polytope in $\mathbb{R}^p$ and $\mathcal{A} = \pi(\mathrm{vert}(P))$.

For each π-induced subdivision S of $\mathcal{A}$ we will call the poset of all refinements of S which are π-induced the *π-refinement poset* of S. We denote it $\Omega(S, \pi)$. In other words, $\Omega(S, \pi)$ is the lower ideal of S in the poset $\Omega(P, \pi)$.

For any linear functional $w \in (\mathbb{R}^p)^*$ and any cell B of a subdivision S it makes sense to consider the π-coherent subdivision B_w of B for the functional w.

Definition 2.1. Let S be a π-induced subdivision of $\mathcal{A}$. We call the subdivision $\mathrm{Ref}(S, \pi, w) := \cup_{B \in S} B_w$ the *π-refinement of S for the functional w*. (That $\mathrm{Ref}(S, \pi, w)$ is indeed a subdivision of $\mathcal{A}$ is proved in the next theorem).

A subdivision of $\mathcal{A}$ is called a *π-coherent refinement of S* if it can be obtained from S in this way.

Theorem 2.2

1. $\mathrm{Ref}(S, \pi, w)$ *is a π-induced subdivision of $\mathcal{A}$ and refines S.*
2. *If the π-coherent subdivision of $\mathcal{A}$ for a certain functional w refines S, then it equals* $\mathrm{Ref}(S, \pi, w)$.
3. *If S is itself π-coherent for a functional w_0, then for any $w \in \mathbb{R}^{p*}$ there is a sufficiently small positive $\epsilon \in \mathbb{R}$ such that* $\mathrm{Ref}(S, \pi, w)$ *is the π-coherent subdivision of $\mathcal{A}$ for the functional $w_0 + \epsilon w$.*

Proof. 1. Since each B_w is a refinement of the cell B of S, the cells in S_w cover $\mathcal{A}$. For the same reason, two cells in the same B_w intersect properly. Let $\tau \in B_w$ and $\tau' \in B'_w$ be cells in the refinements of B and B', for different cells B and B' in S. Then $F = B \cap B'$ is a common face of B and B' and both $\tau \cap F$ and $\tau' \cap F$ are cells in the π-coherent subdivision of F induced by w. Thus, $\tau \cap \tau'$ is a common face of τ and τ', and S_w is a subdivision. It is obvious that S_w refines S. Also, S_w is π-induced since the cells in each B_w are projections of faces of the corresponding P^B, which is itself a face of P.

2. Let S' be the π-coherent subdivision of $\mathcal{A}$ for w. Suppose that S' refines S. This implies that for every cell $B \in S$ the subset S'_B of S' consisting of cells contained in B is a subdivision of B. On the other hand, since S' is π-coherent for w, the subset S'_B in question must be the π-coherent subdivision of B for w. Hence, S' contains all the cells of $\mathrm{Ref}(S, \pi, w)$. Since two different subdivisions of $\mathcal{A}$ cannot be contained in one another, we conclude that $S' = \mathrm{Ref}(S, \pi, w)$.

3. Since the normal fan of $\Sigma(P, \pi)$ decomposes the line $\{w_0 + \epsilon w : \epsilon \in \mathbb{R}\}$ into a finite collection of segments, there exists a small positive real ϵ such that the polytope $(\pi, w_0 + \lambda w)(P)$ has the same combinatorial type and the same upper envelope for every $\lambda \in (0, \epsilon]$. We assume ϵ to have this property.

It is clear that in these conditions the face lattice of $(\pi, w_0 + \epsilon w)(P)$ is a refinement of the face lattice of $(\pi, w_0)(P)$. We want to see now that the upper envelope of $(\pi, w_0 + \epsilon w)(P)$ is a refinement of the upper envelope of $(\pi, w_0)(P)$. Let F be an upper facet of $(\pi, w_0 + \epsilon w)(P)$ and let F_0 be a facet of $(\pi, w_0)(P)$ containing F. Since the exterior normal to F has positive last coordinate, the exterior normal to F_0 has non-negative last coordinate. The exterior normal to F_0 cannot have zero last coordinate, because this would imply that $\pi(F_0)$ (and hence $\pi(F)$) is not full-dimensional. Thus, F_0 is an upper facet.

The above proves that the π-coherent subdivision of $\mathcal{A}$ produced by $w_0 + \epsilon w$ is a refinement of S. Part 2 of the statement gives the rest. □

The following observations are straightforward:

- The π-coherent refinements of a π-coherent subdivision S are exactly the π-coherent subdivisions which refine S (this is a consequence of part 2 of the previous result). In particular, the π-coherent refinements of the trivial subdivision $\{\mathcal{A}\}$ are exactly the π-coherent subdivisions of $\mathcal{A}$.
- The π-coherent refinements of a subdivision which is not π-coherent may or may not be π-coherent. A trivial example of this is that a non-coherent subdivision is

a π-coherent refinement of itself, for the functional $w = 0$. For a non-trivial one, see Example 2.6(b).

- Reciprocally, not all the π-induced subdivisions which refine a π-coherent one are π-coherent. The trivial subdivision gives a trivial example. For a non-trivial one, see Example 2.6(a).

The second part of the following result can be rephrased as "the Baues poset $\Omega(P, \pi)$ is *atomic*", although the word atomic is usually reserved to lattices.

Proposition 2.3

Let $\pi : P \to \pi(P)$ be a polytope projection. Let S be a π-induced subdivision. Let d be the dimension of $\pi(P)$.

1. *Let F be a face of P of dimension greater than d. If every tight d-face of F is contained in P^S then F itself is contained in P^S.*
2. *Let S' be a π-induced subdivision. If every tight refinement of S' refines S then S' refines S.*

Proof. 1. Let x be a generic point in $\pi(F)$. Being generic implies that every vertex of $\pi^{-1}(x) \cap F$ is contained in the relative interior of a d-face of F and that this d-face is tight. Hence, $\pi^{-1}(x) \cap F$ is contained in P^S. Since this holds for any generic point x, F is contained in P^S.

2. Let F be a face of P contained in $P^{S'}$. Let F' be a tight d-face of F. We will prove that $F' \subset P^S$, which implies by part 1 that $F \subset P^S$. Hence ${S'}^P \subset S^P$ and this implies that S' refines S.

Let w be a vector in the (relatively open) normal cone of the face F' of F. Then F' is a maximal face in $P^{\mathrm{Ref}(S',\pi,w)}$ and it is in $P^{S''}$ for any tight refinement S'' of $\mathrm{Ref}(S', \pi, w)$. By hypothesis, $P^{S''} \subset P^S$. □

Theorem 2.4

Let S be a π-induced subdivision for a certain polytope projection $\pi : P \to \pi(P)$. Let S' be a π-induced refinement of S. Then,

1. *$\Sigma(S', \pi) \subset \Sigma(S, \pi)$.*
2. *If S' is the π-coherent refinement of S for a functional w, then $\Sigma(S', \pi)$ is the face of $\Sigma(S, \pi)$ which maximizes w.*
3. *$\Sigma(S, \pi) = \Sigma(S', \pi)$ if and only if $S = S'$.*
4. *$\Sigma(S', \pi)$ is a face of $\Sigma(S, \pi)$ if and only if S' is a π-coherent refinement of S.*

Proof. Part 1 is trivial since $S' \leq S$ implies that $\Gamma(S', \pi) \subset \Gamma(S, \pi)$. In other words, if a polytope bundle is contained in a second one, the Minkowski integral of the first one is contained in that of the second.

For parts 2, 3 and 4 the proofs of the analogue statements in Theorem 1.3 are equally valid here, with minor changes. □

Corollary 2.5

Let S be a π-induced subdivision for a certain polytope projection $\pi : P \to \pi(P)$. Then:

1. *The map $S' \mapsto \Sigma(S', \pi)$ is an isomorphism between the poset of π-coherent refinements of S and the poset of non-empty faces of the polytope $\Sigma(S, \pi)$. Moreover, the normal cone of the face $\Sigma(S', \pi)$ of $\Sigma(S, \pi)$ equals the collection of vectors w such that $S' = \mathrm{Ref}(S, \pi, w)$.*
2. *In particular, the vertices of $\Sigma(S, \pi)$ are the points $\{v_T : T$ is a tight refinement of $S\}$.*

EXAMPLE 2.6: Let π be the projection from a 5-simplex to the point configuration $\mathcal{A} = \{(4,0,0),(0,4,0),(0,0,4),(2,1+\epsilon,1-\epsilon),(1-\epsilon,2,1+\epsilon),(1+\epsilon,1-\epsilon,2)\}$, where ϵ is a sufficiently small real number, possibly zero. This is the smallest example of a configuration with non-coherent subdivisions [8, 14, 22, 26]. If $\epsilon = 0$, then $\mathcal{A}$ consists of the vertices of two homothetic triangles one inside another. If $\epsilon \neq 0$, then the interior triangle is slightly rotated, but the Baues poset is independent of this rotation.

Let the points in $\mathcal{A}$ be labelled $1, \dots, 6$ in the order we have written them. Consider the subdivison $S = \{456, 1245, 2356, 1346\}$, consisting of the central triangle surrounded by three quadrilaterals. The refinements of S are obtained by independently introducing one of the two diagonals in some or all of the quadrilaterals. Hence, $\Omega(S, \pi)$ is isomorphic to the face poset of a 3-cube.

(a) If $\epsilon = 0$, then S is coherent. Its corresponding face in the secondary polytope $\Sigma(\{A\}, \pi)$ is a hexagon. The two non-regular triangulations (and some other refinements of S) are not π-coherent refinements.

(b) If $\epsilon \neq 0$, then S is not coherent. In the secondary polytope, the former hexagonal facet is now "inflated" to three quadrilateral facets, corresponding to three refinements of S. The refinement polytope of S must contain these three facets and, hence, it is three dimensional. On the other hand, its face poset is a subposet of $\Omega(S, \pi)$, which is already the face poset of a 3-dimensional polytope. Hence, all the refinements of S are π-coherent, although some of them are not coherent.

Remark 2.7. Suppose that P is a simplex and let S be a subdivision of a point configuration $\mathcal{A} = \pi(P)$. A refinement S' of S is called *regular decomposition of S* in [1, Section 2.12] and *coherent refinement* of S in [24, Section 4.2] if it satisfies the following conditions:

(i) For each cell $B \in S$ there is a lifting function w_B defined on B such that S' restricted to $\mathrm{conv}(B)$ equals B_{w_B} and

(i) The lifting functions can be chosen in such a way that for every $B, B' \in S$, the function $w_B - w_{B'}$ defined on $B \cap B'$ is an affine function.

Our definition of π-coherent refinement is stronger than this, since we require $w_B = w_{B'}$ on $B \cap B'$. (This is called *strongly coherent refinement* in [24]).

This weaker notion of coherent refinement gives rise to different "refinement polytopes", called *generalized secondary polytopes* in [1] and [24]. As an example, in the subdivision S of Example 2.6 all the refinements of S are coherent in this wider sense and hence the generalized secondary polytope is combinatorially a 3-cube regardless of the value of ϵ. The generalized secondary polytopes are specially interesting in connection to the toric schemes associated to subdivisions of a point configuration.

Theorem 2.8

Let S be a π-induced subdivision. Then, the following conditions are equivalent:

1. *S is π-coherent.*
2. *All π-coherent refinements of S are π-coherent subdivisions.*
3. *All π-coherent refinements of S which are tight are π-coherent subdivisions.*

Proof. For the implication $1 \Rightarrow 2$, suppose that S is π-coherent, so that $\Sigma(S,\pi)$ is a face of $\Sigma(P,\pi)$. If S' is a π-coherent refinement of S then $\Sigma(S',\pi)$ is a face of $\Sigma(S,\pi)$ and, thus, of $\Sigma(P,\pi)$ (we have used parts 2 of Theorem 1.3 and of Theorem 2.4). By part 4 of Theorem 1.3, S' is π-coherent.

The implication $2 \Rightarrow 3$ is trivial. Let us prove $3 \Rightarrow 1$. We will use induction on the number of proper refinements of S. Thus, we can assume that the implication $3 \Rightarrow 1$ holds for every proper refinement of S.

Let $S_1,\dots,S_k$ be the maximal proper π-coherent refinements of S, which are in bijection with the facets of the π-refinement polytope $\Sigma(S,\pi)$. By inductive hypothesis, $S_1,\dots,S_k$ are π-coherent subdivisions.

Let $w_1,\dots,w_k \in \mathbb{R}^{p*}$ be linear functionals so that S_i is the π-coherent refinement of S for w_i $(i=1,\dots,k)$. In particular, w_i restricted to the affine span of $\Sigma(S,\pi)$ represents the exterior normal of the i-th facet of $\Sigma(S,\pi)$. Scaling the w_i with positive constants we can assume that the functional $w := \sum_{i=1}^k w_i$ is constant on $\Sigma(S,\pi)$ and, hence, that the π-coherent refinement of S for the functional w is S itself.

By part 2 of Theorem 1.3, S_i is the π-coherent subdivision of the projection $\pi : P \to \pi(P)$ for the functional w_i $(i=1,\dots,k)$. We claim that this implies that S is the π-coherent subdivision for the functional w. In fact, let us call S_w this latter π-coherent subdivision. Since the π-coherent subdivision for w_i refines S for every i, S_w refines S too (here we are just using that on each fiber $\pi^{-1}(x) \cap P$ the normal cone to the face which projects to a cell of S is convex).

Hence, by part 2 of Theorem 2.2, S_w is the π-coherent refinement of S for the functional w. Since w is constant on $\Sigma(S,\pi)$, $\Sigma(S_w,\pi) = \Sigma(S,\pi)$ and, by part 3 of Theorem 2.4, $S = S_w$. □

Corollary 2.9

Let $P \to \pi(P)$ be a polytope projection. The following statements are equivalent:

1. *Every π-induced subdivision is π-coherent.*
2. *Every tight π-induced subdivision is π-coherent.*
3. *(The order complex of) $\Omega(P,\pi)$ is homeomorphic to a sphere of dimension $\dim(P) - \dim(\pi(P))$.*

Proof. That condition 1 implies both 2 and 3 is trivial. The implication from 2 to 1 is a direct consequence of Theorem 2.8.

For the implication from 3 to 1, observe that the subposet of $\Omega(P,\pi)$ consisting of π-coherent subdivisions is the poset of proper faces of the polytope $\Sigma(P,\pi)$, which is homeomorphic to a sphere of dimension $\dim(P) - \dim(\pi(P))$. The implication follows from the fact that a sphere cannot be a proper subset of a sphere of the same dimension (see e.g. [18, p.217, Exercise 6.9]). $\square$

DEFINITION 2.10. We will call *height* of a π-induced subdivision S the maximum of the lengths of all the refinement chains of π-induced subdivisions having S as maximal element (so that tight π-induced subdivisions have height zero and the height of every other π-induced subdivision equals one plus the maximum height of its proper π-induced refinements).

Corollary 2.11

If S is a non-trivial π-induced subdivision with height greater or equal than the dimension of the fiber polytope, then there exists a tight π-induced but not π-coherent subdivision which refines S. $\square$

Question 2.12 Is the converse of Corollary 2.11 also true? In other words, does there exist a polytope projection π which has non-coherent subdivisions but in which all proper π-induced subdivisions have height strictly lower than the dimension of the fiber polytope? We do not know any such example.

EXAMPLE 2.13: Suppose that T is a π-induced subdivision which refines another π-induced subdivision S and that $\mathrm{height}(S) - \mathrm{height}(T) > 1$. Does this imply that there is another π-induced subdivision S' in between S and T? The answer is no, as the following example shows.

Let $\mathcal{A}$ be the point configuration consisting of the twelve vertices of a regular icosahedron together with its centroid. Let π be the natural projection from a 12-simplex onto $\mathcal{A}$, so that every subdivision of $\mathcal{A}$ is π-induced. Let S be the trivial subdivision, which has height at least $13 - 3 - 1 = 9$ (in fact, at least 10 as we will see in Example 3.2.2).

The twenty facets of the icosahedron can be divided into six adjacent pairs and eight single triangles in such a way that each pair is adjacent to four single triangles and each single triangle to three pairs. (Once a pair is formed there is a unique way to form the other ones). Let T be the subdivision of $\mathcal{A}$ obtained coning the centroid to each single triangle and to each pair, so that the cells of T are eight tetrahedra and six triangular bipyramids. T has height equal to six, since each bipyramid can be refined independently and has height 1. However, it is easy to check that T is a coarse subdivision of $\mathcal{A}$.

3. Non-regular triangulations

In this section we will assume that P is a simplex, and let $\mathcal{A} = \pi(\mathrm{vert}(P))$. Every polyhedral subdivision of $\mathcal{A}$ is π-induced and $\Omega(P, \pi)$ is simply denoted $\Omega(\mathcal{A})$. The π-coherent subdivisions in this case are usually called *regular*. The fiber polytope $\Sigma(\mathcal{A})$ associated with the projection from a simplex is the *secondary polytope* of $\mathcal{A}$ [8, 14] and has dimension $\#\mathcal{A} - \dim(\mathcal{A}) - 1$.

Corollary 2.9 says that $\mathcal{A}$ has non-regular subdivisions if and only if it has non-regular triangulations. This is interesting since the triangulations of $\mathcal{A}$ are easier to enumerate than the subdivisions. For example, in [2] the authors, after computing all the triangulations of the cyclic polytopes $C(7,3)$, $C(8,3)$ and $C(8,4)$ and checking that they are regular prove that all the subdivisions are regular too by somewhat sophisticated arguments (see Lemma 4.6 in [2]). Our result saves this part of the work.

In the following we will produce simple proofs of existence of non-regular triangulations for some particular point configurations.

Lemma 3.1

Let S be a subdivision of a point configuration $\mathcal{A}$. Let $B_1, \ldots, B_k$ be the list of cells of S which are not simplices. Let h_i be the dimension of the secondary polytope of B_i. Suppose further that the facets of each B_i are simplices, except perhaps for those contained in the boundary of conv$(\mathcal{A})$. Then, S has height at least $h_1 + \cdots + h_k$.

Proof. The conditions on the facets of the B_i's imply that the common face of any pair of them is a simplex. Thus, the refinements of S are obtained refining the B_i's independently. In particular, $\Omega(S, \pi)$ equals the direct product of the Baues posets of each of the B_i's, and each of these has height at least h_i. □

Example 3.2: The following point configurations have non-regular triangulations:

1. *The six vertices of two parallel triangles in the plane, one inside another.*

   Let T_1 denote the outer triangle and T_2 the inner one. Let a_i, b_i and c_i denote the vertices of T_i. Then, the subdivision $S = \{\{a_1a_2b_1b_2\}, \{a_1a_2c_1c_2\}, \{b_1b_2c_1c_2\}, \{a_2b_2c_2\}\}$ satisfies the conditions of Lemma 3.1. and has height 3, the dimension of the secondary polytope. This is the same configuration and subdivision as in Example 2.6.

2. *The vertices of any 3-polytope with more vertices than facets, together with an interior point of it.*

   Let Q be any 3-polytope with more vertices than facets and let a be a point in its interior. Consider the subdivision S obtained coning a to the facets of Q, which satisfies the conditions of Lemma 3.1. Calling V, E and F the numbers of vertices, edges and facets of Q, the height of S is easily seen to be at least $2E - 3F$, which equals $2V - F - 4$ by Euler's formula. By our hypothesis, this number is at least $V - 3$, the dimension of the secondary polytope.

   Observe that every simple 3-polytope other than the tetrahedron is a valid Q for this example. Also, that essentially the same proof applies if Q is any polytope obtained by a small perturbation of a 3-polytope with more vertices than facets or if Q is an icosahedron. For the icosahedron, divide its boundary into ten pairs of two adjacent triangles and cone these pairs to the interior point. This produces a subdivision of height 10 while the secondary polytope has dimension $13-4=9$. Example 2.13 also implies that this point configuration has non-regular triangulations.

3. *The configuration consisting of the centroids of the 15 non-empty faces of a 3-dimensional simplex.*

   Consider the 3-simplex subdivided into four combinatorial 3-cubes, each of them being the star of a vertex in the first barycentric subdivision of the 3-simplex. In each of the four 3-cubes so obtained we cut the inner corner (incident to the centroid of the 3-simplex). This produces a subdivision S of the 3-simplex into four 3-simplices and four 3-polytopes with seven vertices and all but the three external facets simplicial. This subdivision satisfies the conditions of Lemma 3.1 and has height at least $4 \times (7-3-1) = 12$, which is greater than the dimension $15-3-1=11$ of the secondary polytope.

4. *The vertices of a 4-cube.*

   Let a be a particular vertex of the 4-cube. The vertex figure of the 4-cube at a is precisely a 3-simplex divided into four combinatorial 3-cubes as in the previous example. Thus, the 4-cube can be subdivided into four cones over 3-cubes with apex at a. Cutting vertices in these four 3-cubes as we did in the previous configuration produces a subdivision of the 4-cube with eight cells, four of which are 4-simplices and the other four have eight vertices. This subdivision satisfies the conditions of Lemma 3.1. Again, this subdivision has height at least $4 \times (8-4-1) = 12$, which is bigger than $16-4-1=11$.

5. *The 3-dimensional configuration in $\mathbb{R}^4$ consisting of the 12 points $e_i - e_j$ $(i,j = 1,2,3,4; i \neq j)$ together with the origin.*

   A different (affinely equivalent) description of the point configuration in question is that it consists of the centroid and the 12 vertices of a cuboctahedron, where a cuboctahedron is the convex hull of the mid-points of the edges of a regular 3-cube. After removing two square pyramids with base at two opposite square facets and apex at the centroid of the cuboctahedron, the rest of the cuboctahedron can be subdivided into four (non-regular) octahedra. This gives a subdivision satisfying the conditions of Lemma 3.1 and of height $2+2+2+2+1+1$ (2 for each of the four octahedra and 1 for each of the 2 square pyramids), which is bigger than $12-3-1$.

6. *The vertices of the product $\Delta_3 \times \Delta_3$ of two 3-dimensional simplices.*

   Let us embed $\Delta_3 \times \Delta_3$ in $\mathbb{R}^4 \times \mathbb{R}^4$ having as vertices the 16 points (e_i, e_j), $i,j = 1,\dots,4$. Let $\mathcal{A}$ be this set of vertices. The projection $\Pi : \mathbb{R}^4 \times \mathbb{R}^4 \mapsto \mathbb{R}^4$ defined by $(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) \mapsto (x_1-y_1, x_2-y_2, x_3-y_3, x_4-y_4)$ sends $\mathcal{A}$ to the point configuration $\mathcal{A}_0$ of the previous example, identifying the four vertices (e_i, e_i) at the origin O of $\mathbb{R}^4$. The fact that $\dim(\mathcal{A}) - \dim(\mathcal{A}_0) = 3 = \dim(\{(e_i,e_i) : i = 1,\dots,4\}) - \dim(\{O\})$ implies that if $B \cup \{O\}$ is a subset of $\mathcal{A}_0$ with k elements and dimension l, then $B \cup \{(e_i, e_i) : i = 1, \ldots, 4\}$ is a subset of $\mathcal{A}$ with $k+3$ elements and dimension $l+3$ (we are slightly abusing notation, identifying $\mathcal{A} \setminus \{(e_i, e_i) : i = 1, \ldots, 4\}$ with $\mathcal{A}_0 \setminus \{O\}$ by the projection Π). Hence, the lifted cell in $\mathcal{A}$ is full-dimensional or simplicial if and only if the cell in $\mathcal{A}_0$ had those properties. Moreover, if two such cells in $\mathcal{A}_0$ intersect properly then the corresponding lifted cells intersect properly too.

In particular, the subdivision S_0 of $\mathcal{A}_0$ described in the previous example, consisting of 4 octahedra and 2 square pyramids, lifts to a family S of 6 full-dimensional cells in $\mathcal{A}$ which intersect properly. We want to show that S is a subdivision satisfying the conditions of Lemma 3.1. If this is so, then it is clear that it has height $2+2+2+2+1+1 = 10$, which is more than the dimension (9) of the secondary polytope of $\Delta_3 \times \Delta_3$.

- S_0 can be refined to a triangulation T_0 with 20 simplices, all of them incident to O. For this, refine the square pyramids arbitrarily and refine the octahedra using the diagonal containing O. This triangulation T_0 lifts to a collection T of 20 full-dimensional simplices which intersect properly in $\mathcal{A}$. Since $\Delta_3 \times \Delta_3$ is a lattice polytope of normalized volume 20, T is a triangulation of $\mathcal{A}$. Since each simplex of T is contained in a cell of S, S is a subdivision of $\mathcal{A}$.
- The interior common facets between cells of S are obtained lifting the interior common facets between cells of S_0, all of which are incident to O and are simplices. This implies that they are also simplices in S.

Remark 3.3. For most of the point configurations in the above list non-regular triangulations were previously known (see [10] for the 4-cube and the product of two tetrahedra and [12] for the cuboctahedron). However, the proof presented here is probably the simplest existing one. In particular, our proof relies only on the combinatorics and not the geometry of the point configuration, where by “combinatorics” we mean the oriented matroid $\mathcal{M}(\mathcal{A})$ of affine dependencies between the points of $\mathcal{A}$. This is interesting since the Baues poset of $\mathcal{A}$ (and in particular whether or not $\mathcal{A}$ has any non-regular triangulations) depends only on the oriented matroid $\mathcal{M}(\mathcal{A})$, while the regularity of a specific triangulation depends also on the geometry.

In particular, observe that if the example 1 is slightly perturbed so that the two triangles become non-parallel, our proof still implies that the configuration has non-regular triangulations, while any “geometric” proof would have to be adapted to the perturbed case; the configuration moves from having two different non-regular triangulations to having only one.

Remark 3.4. Since the property of having only regular triangulations for a point configuration $\mathcal{A}$ depends only on its oriented matroid $\mathcal{M}(\mathcal{A})$, a natural question is whether this property is minor closed, i.e., closed under the oriented matroid operations of deletion and contraction.

It is easy to check that the property is closed under deletion: if T is a non-regular triangulation of $\mathcal{A} \setminus \{p\}$ then the triangulation T' of $\mathcal{A}$ obtained joining to p the facets of T which are visible from T is non-regular.

However, the property is not closed under contraction: let $\mathcal{A} \subset \mathbb{R}^3$ be the point configuration $a_1 = (2,0,0)$, $a_2 = (0,2,0)$, $a_3 = (0,0,2)$, $a_4 = (1,0,0)$, $a_5 = (0,1,0)$,

$a_6 = (0, 0, 1)$, and $a_7 = (-1, -1, -1)$. The contraction $\mathcal{A}/a_7$ is (affinely equivalent to) the planar point configuration that we have discussed in Example 3.2.1.In particular, $\mathcal{A}/a_7$ has non-regular triangulations. On the other hand, $\mathcal{A}$ has only regular triangulations. Indeed, the following two assertions are easy to check. Observe that $\mathcal{A}/a_7$ has five symmetry classes of triangulations, four of them regular:

- Each regular triangulation of $\mathcal{A}/a_7$ is the link of the point a_7 in a unique triangulation T' of $\mathcal{A}$. This triangulation is regular, by Lemma 2.2 in [11] where it is proved that every regular triangulation of $\mathcal{A}/a$ is the link of the point a in at least one regular triangulation of $\mathcal{A}$.
- The two non-regular triangulations of $\mathcal{A}/a_7$ are not links of a_7 in triangulations of $\mathcal{A}$. In other words, the truncated triangular pyramid $\mathrm{conv}(a_1, a_2, a_3, a_4, a_5, a_6)$ cannot be triangulated so that the triangulation of its boundary agrees with the non-regular triangulations of $\mathcal{A}/a_7$.

This shows that although the arguments in Examples 3.2.4 and 3.2.6. are based in a contraction technique, the contraction alone is not enough.

Question 3.5 We can further ask whether the property of *not having* non-regular triangulations can be characterized by a finite list of excluded minors. Since the property is not closed under contraction, we ask this for each fixed dimension.

The answer is yes if $d \geq 3$ for point configurations $\mathcal{A}$ *in general position*, meaning by this that any $\dim(\mathcal{A}) + 1$ points are independent, as a combination of the following two results:

1. The following higher dimensional generalization of Erdös-Szekeres Theorem [9, Proposition 9.4.7]: for any fixed dimension d and any integer n there is an integer N such that any point configuration in $\mathbb{R}^d$ containing at least N points in general position contains as a minor the oriented matroid of a cyclic polytope $C(n, d)$.
2. The existence of non-regular triangulations of any cyclic polytope $C(n, d)$ with $d \geq 3$ and $n \geq d + 6$ [2].

These two results imply that any point configuration in $d \geq 3$ with enough points in general position has non-regular triangulations. This is clearly not true in $d = 2$, since the vertex set of any n-gon has only regular triangulations. For $d = 2$ we conjecture that any 2-dimensional point configuration which has non-regular triangulations contains either an 8-element or a 6-element subconfiguration which has non-regular triangulations.

4. The Baues poset for almost-fine subdivisions. Flips

DEFINITION 4.1. Let S be a π-induced subdivision for a polytope projection $\pi : P \to \pi(P)$. We will call *rank* of S the dimension of the π-refinement polytope $\Sigma(S, \pi)$.

It is clear from Theorem 2.2 that the height of any π-induced subdivision is greater or equal than its rank. Also, that a π-induced subdivision has rank 0 if and only if it has height 0 (and if and only if it is tight). In this section we will be interested in the π-refinement posets of subdivisions of rank 1. Let us first see how to compute the rank of a subdivision:

Proposition 4.2

Let S be a π-induced subdivision of a polytope projection $\pi : P \to \pi(P)$. For each $B \in S$, let L_B be the linear subspace parallel to a fiber $\pi^{-1}(x) \cap P^B$ of the projection $\pi : P^B \to \mathrm{conv}(B)$ for any x in the relative interior of $\mathrm{conv}(B)$ and let

$$L_S := \sum_{B \in S} L_B.$$

Then, L_S is the linear subspace parallel to $\Sigma(S, \pi)$. In particular, the rank of S equals $\dim(L_S)$.

Proof. If S is the trivial subdivision, this is a well known fact (the affine span of the fiber polytope equals the fiber over the centroid of $\pi(P)$). For a non-trivial S, the statement follows from the decomposition of the π-refinement polytope $\Sigma(S, \pi)$ as a Minkowski sum of the fiber polytopes of the cells $B \in S$ (Theorem 1.3.1). □

Theorem 4.3

Let S be a π-induced subdivision of rank 1. Then the poset $\Omega(S, \pi)$ of π-induced refinements of S is isomorphic to the poset of proper non-empty faces of a cube of dimension $\mathrm{height}(S)$*. In particular, it is homeomorphic to a sphere of dimension* $\mathrm{height}(S) - 1$.

Proof. Throughout this proof let $|S|$ represent the polyhedral complex induced by a polyhedral subdivision S. $|S|$ is a collection of polytopes which covers $\pi(P)$ and which is closed under taking faces. Its maximal elements are the convex hulls of the cells in S.

By Proposition 4.2 the fiber $\pi^{-1}(x) \cap P^S$ of every point $x \in \pi(P)$ is either a point or a segment parallel to $\Sigma(S, \pi)$. Let U denote the subset of $\pi(P)$ consisting of points whose fiber is a segment.

- **Claim 1:** *U is open in $\pi(P)$. Proof:* Let C be the union of the (closed) cells of $|S|$ which do not intersect U. C is clearly closed and disjoint from U. Moreover, the relative interior of every cell F of $|S|$ is contained in either C or U, depending on whether F is the projection of a face of P of the same dimension or of one dimension more. Hence, U and C are complements of each other and U is open.

Let us globally choose a positive and negative direction in the fibers of the points in U. Every refinement S' of S is characterized by the map $\phi_{S'} : U \to \{-, 0, +\}$ which to a point $x \in U$ associates the sign $-$ or $+$ if $\pi^{-1}(x) \cap P^{S'}$ is the negative or positive end of the segment $\pi^{-1}(x) \cap P^S$ (respectively) or 0 if $\pi^{-1}(x) \cap P^{S'} = \pi^{-1}(x) \cap P^S$.

- **Claim 2:** *$\phi_{S'}$ is continuous in U for every π-induced refinement S' of S. Proof:* Let us call U_0, U_+ and U_- the inverse images by $\phi_{S'}$ of 0, + and $-$. U_0 is open in $\pi(P)$ (and hence in U) by Claim 1 applied to the subdivision S'. That U_+ and

U_- are open in $\pi(P)$ (and hence in U) can be proved with the same argument: if x is a point in U_+ then the relative interior of any face of $|S|$ containing x is contained in U_+ (and the same for U_-).

Saying that $\phi_{S'} : U \to \{-,0,+\}$ is continuous is equivalent to saying that it is constant on each connected component of U. Moreover, the following converse of Claim 2 is trivial: any locally constant map $\phi : U \to \{-,0,+\}$ represents a π-induced refinement of S. Thus, the set of π-induced refinements of S is in bijection with the set of maps from $\{U_1,\dots,U_k\}$ to $\{+,0,-\}$, where $U_1,\dots,U_k$ are the connected components of U (which are clearly a finite number). This set of maps is in natural bijection with the faces of a cube of dimension k, and this bijection induces a poset isomorphism between $\Omega(S,\pi)$ and the poset of proper non-empty faces of the k-dimensional cube. The rest of the statement is trivial. □

It is interesting to observe that the proof above is valid also if S has *local rank* equal to 1, meaning by this that for any $B \in S$, L_B has dimension 0 or 1 (or, equivalently, $\dim(P^B) \geq \dim(B)+1$). This occurs in Example 2.13. The only change needed in the proof is that the choice of a positive and negative direction for each fiber is local, i.e., made independently in each connected component of U.

Question 4.4 In what other cases is it possible to prove that the poset $\Omega(S,\pi)$ is homeomorphic or at least homotopy equivalent to a sphere? It would be interesting to prove it for the cases $\dim(\mathcal{A}) = 1$ or $\operatorname{rank}(S) = 2$. It might be that the existing proofs when S is the trivial subdivision [7, 21] can be adapted here.

Corollary 4.5

Let S be a π-induced subdivision. Then, S has height 1 if and only if it has exactly two proper refinements. In this case the two refinements are tight.

Proof. If S has height 0 then it has no proper refinements. If S has height at least 2, then it has rank at least 1 and at least three proper refinements: at least two tight ones (vertices of $\Sigma(S,\pi)$) and at least one non-tight one, in a chain of length at least two.

Finally, if S has height 1, then it has rank 1 because $\operatorname{height}(S) \geq \operatorname{rank}(S)$ and rank 0 would imply height 0. In this case, the previous result says that $\Omega(S,\pi)$ is the poset of faces of a segment. □

EXAMPLE 4.6: We will see in Section 5 that if P is a simplex or a cube (more generally, any product of simplices) then rank 1 implies height 1. This is not true in general. For example, the natural projection between the cyclic polytopes $C(6,4)$ and $C(6,2)$ has π-induced subdivisions of rank 1 and height 2 (in a certain coordinatization), as shown in [2, Section 6].

It is even easy to construct subdivisions of rank 1 and arbitrarily large height: Let P_0 be the regular prism over an n-gon for an even n, i.e., the 3-polytope with the following $2n$ vertices:

$$a_k = \left(\cos\left(\frac{2\pi k}{n}\right),\ \sin\left(\frac{2\pi k}{n}\right), 1\right) \quad \text{and} \quad b_k = \left(\cos\left(\frac{2\pi k}{n}\right),\ \sin\left(\frac{2\pi k}{n}\right), -1\right),$$

for $k = 0, \ldots, n-1$. Let P be the slightly non-regular antiprism obtained truncating P_0, whose vertices are the a_i's and the mid-points of consecutive b_i's. Let $c_i = (b_i + b_{i+1})/2$ be such a mid-point for each $i = 0, \ldots, n-1$, where it is understood that $b_n = b_0$. Let π be the projection $(x, y, z) \mapsto x$ which maps P to the segment $[-1, 1]$. Let S be the subdivision consisting of the cells $\{\pi(a_i), \pi(c_i), \pi(a_{i+1})\}$, for $i = 0, \ldots, \frac{n}{2} - 1$. Then, S has rank 1 (L_S is a vertical segment) and height $n/2$.

Definition 4.7. Let S_1 and S_2 be two tight π-induced subdivisions. We will say that they *differ by a π-flip* if they are the two proper refinements of a certain π-induced subdivision of height 1. We will call it π-flips the π-induced subdivisions of height 1.

We will call *graph of tight π-induced subdivisions* the graph whose vertices are the tight π-induced subdivisions and whose edges are the π-flips connecting them. We denote it $G(P, \pi)$. For any π-induced subdivision S, we will denote $G(S, \pi)$ the subgraph of $G(P, \pi)$ induced by the tight refinements of S.

If S_0 is a π-flip and S_1 and S_2 are its two tight refinements, then any π-induced subdivision coarser than S_1 and S_2 is coarser than S_0 as well, by part 2 of Proposition 2.3. This implies that $G(S, \pi)$ is homeomorphic to the subgraph of the 1-skeleton of $\Omega(S, \pi)$ induced by subdivisions of height at most 1. The following result is analogue to Lemma 8 in [22].

Proposition 4.8

Let $\pi : P \to \pi(P)$ be a polytope projection. Let S be a π-induced subdivision. The following conditions are equivalent:

1. *The graph $G(S', \pi)$ is connected for every π-induced refinement S' of S.*
2. *The refinement poset $\Omega(S', \pi)$ is connected for every π-induced refinement S' of S.*

Proof. (1) $\Rightarrow$ (2) For any particular subdivision S', if the graph $G(S', \pi)$ is connected then all the tight π-induced subdivisions are connected in $\Omega(S', \pi)$ by π-flips. Any non-tight subdivision can be refined to a tight one.

(2) $\Rightarrow$ (1) We want to show that if $S_0, \ldots, S_k$ is a path in $\Omega(S', \pi)$ connecting two tight refinements S_0 and S_k of S then there is a path connecting S_0 and S_k and using only subdivisions of height 0 or 1 (i.e., tight subdivisions or flips). Let h be the maximum height of a subdivision in the path $S_0, \ldots, S_k$. We will use induction on h.

Any subdivision S_i of height h in the path is between two subdivisions S_{i-1} and S_{i+1} of height lower than h which refine S_i. By part (2) applied to S_i, there is a path connecting S_{i-1} and S_{i+1} in $\Omega(S_i, \pi)$ and this path consists of subdivisions of height less than h. Replacing each subdivision of height h for such a path we obtain a path from S_0 to S_k with subdivisions of height less than h. $\square$

5. Special cases

Here we study flips in the particular cases of P being a simplex, P being a cube and $\dim(\pi(P)) = 1$. In these three cases π-flips are equal (at least in generic situations) to *geometric bistellar flips*, *cube-flips* and *polygon moves*, respectively.

Triangulations and geometric bistellar flips

We consider here the case where P is a simplex. An interesting feature of this case is that the bad behaviour exhibited in Example 4.6 cannot occur:

Proposition 5.1

Let $\pi : P \to \pi(P)$ be a polytope projection. If P is a simplex then any π-induced subdivision S of rank 1 has height 1.

Proof. Let C be the intersection of all the faces of the simplex P which contain a segment parallel to the 1-dimensional vector space L_S. Since P is a simplex, C is a face of P and contains a segment parallel to L_S. For each $B \in S$, L_B is either trivial or equals L_S, and the latter happens if and only if P^B contains C.

Observe that $\dim(C) = \dim(\pi(C)) + 1$ and, hence, the projection $C \to \pi(C)$ induces two non-trivial subdivisions of $\pi(C)$, which correspond to two refinements of S. Conversely, any refinement of a non-tight cell B of S induces a π-induced subdivision of the projection $C \to \pi(C)$. Clearly, in a refinement of S all the non-tight cells are refined inducing the same π-induced subdivision of the projection $\pi : C \to \pi(C)$.

Hence, the proper refinements of S are in bijection with the subdivisions induced by the projection $\pi : C \to \pi(C)$. This means that S has two proper refinements and, by Corollary 4.5, it has height 1. $\square$

The following is the standard definition of *geometric bistellar flip* in a triangulation, see [14, Chapter 7] or [8, 12, 22]. We intend to show that this notion coincides with our notion of π-flip.

Let $\mathcal{A}$ be a point configuration. Using the terminology of matroid theory, we call a minimal affinely dependent subset of $\mathcal{A}$ a *circuit* (see [9] or [26] for details). The unique (up to a scalar factor) dependence equation in a circuit divides its elements into two parts $Z = Z_+ \cup Z_-$ containing respectively the elements with positive and negative coefficient in the equation. These two parts are sometimes referred to as the *Radon partition* of Z and the pair (Z_+, Z_-) is called an *oriented circuit.* A circuit Z can be triangulated in exactly two ways:

$$T_+(Z) := \{\text{conv}(Z - \{p\}) : p \in Z_+\} \qquad T_-(Z) = \{\text{conv}(Z - \{p\}) : p \in Z_-\}.$$

Definition 5.2. Let T be a triangulation of $\mathcal{A}$ (i.e., a tight π-induced subdivision for the canonical projection π which sends the vertices of a simplex P to the elements of $\mathcal{A}$) and $(Z_+, Z_-) \subset \mathcal{A}$ an oriented circuit of $\mathcal{A}$. Suppose that the following conditions are satisfied:

1. The triangulation $T_+(Z)$ is a subcomplex of T.
2. All the maximum-rank simplices of $T_+(Z)$ have the same link L in T. In particular, $T_+(Z) * L$ is a subcomplex of T. Here and in what follows we denote by $A * B$ the *join* of two simplicial complexes A and B, i.e., the simplicial complex $\{a \cup b : a \in A, b \in B\}$.

In these conditions we can obtain a new triangulation T' of $\mathcal{A}$ by replacing the subcomplex $T_+(Z) * L$ of T with the complex $T_-(Z) * L$. This operation of changing the triangulation is called a *geometric bistellar flip* (or a *flip*, for short) supported on the circuit (Z_+, Z_-). We say that T and T' are *geometric bistellar neighbors*. We call the flip of type (k, l) if Z_+ and Z_- have k and l elements respectively.

Proposition 5.3

Let $\pi : P \to \pi(P)$ be a polytope projection where P is a simplex and let $\mathcal{A} = \pi(\mathrm{vert}(P))$. Then, two triangulations T and T' of $\mathcal{A}$ differ by a bistellar flip if and only if they differ by a π-flip.

Proof. Suppose first that T and T' differ by a bistellar flip. Using the notation of Definition 5.2, we have that $S := T \setminus (T_+(Z)*L) \cup (Z*L) = T' \setminus (T_-(Z)*L) \cup (Z*L)$ is a subdivision of $\mathcal{A}$ refined by both T and T'. Let us see that it has no other refinements. Any non-simplicial cell in S is of the form $Z * \sigma$ for an affinely independent set σ. Its only two refinements are $T_+(Z) * \sigma$ and $T_-(Z) * \sigma$. Moreover, if a non-simplicial cell of S is refined using $T_+(Z)$ then any other non-simplicial cell is refined in the same way (and the same happens for $T_-(Z)$). Hence, T and T' are the only two refinements of S and S has height 1 by Corollary 4.5.

Reciprocally, suppose that S is a height 1 subdivision and that T and T' are its proper refinements. We want to prove that T and T' satisfy the conditions of Definition 5.2. Let B any non-simplicial cell of S. Since L_B has dimension 1, P^B is a simplex of dimension $d+1$, hence B has $d+2$ elements and it contains a unique circuit Z. Moreover, this circuit Z is independent of the choice of B. In fact, let C be the minimal face of P containing a segment parallel to L_S, as in the proof of Proposition 5.1. We saw there that $\dim(C) - \dim(\pi(C)) = 1$ and that C is contained in P^B for any non-simplicial cell B of S. In particular, $\pi(\mathrm{vert}(C))$ contains the circuit Z contained in any non-simplicial cell B.

As a conclusion, the non-simplicial part of S has the form $Z * L$ where L is a simplicial subcomplex of S, T and T'. Hence, T and T' differ by a bistellar flip on the circuit Z. $\square$

Mixed subdivisions. The Cayley Trick

Let $P_1 \subset \mathbb{R}^{p_1}, \dots, P_r \subset \mathbb{R}^{p_r}$ be a finite family of polytopes. Let

$$\Pi_M : P_1 \times \cdots \times P_r \to \Pi_M(P_1 \times \cdots \times P_r)$$

be a projection of the product of these polytopes. If O_i denotes the origin in $\mathbb{R}^{p_i}$ we can decompose Π_M into the projections

$$\begin{aligned} \pi_i : P_i &\to \pi_i(P_i) \\ x &\mapsto \Pi_M\big(O_1, \dots, O_{i-1}, x, O_{i+1}, \dots, O_r\big)\,. \end{aligned}$$

We have that

$$\Pi_M(P_1 \times \cdots \times P_r) = \mathrm{M}\big(\pi_1(P_1), \dots, \pi_r(P_r)\big)$$

where M denotes the Minkowski sum of polytopes.

On the other hand, we call *Cayley embedding* of $\pi_1(P_1), \dots, \pi_r(P_r)$ the following point configuration in $\mathbb{R}^{r-1} \times \mathbb{R}^d$. Let $e_1, \dots, e_r$ be a fixed affine basis in $\mathbb{R}^{r-1}$ and $\mu_i : \mathbb{R}^d \to \mathbb{R}^{r-1} \times \mathbb{R}^d$ be the affine inclusion given by $\mu_i(x) = (e_i, x)$. Then we define

$$C\big(\pi_1(P_1), \dots, \pi_r(P_r)\big) := \mathrm{conv}\left(\cup_{i=1}^r \mu_i(\pi_i(P_i))\right).$$

The Cayley embedding of polytopes from complementary affine subspaces equals the *join product* of them. (For our purposes the join product $P_1 * \cdots * P_r$ of several polytopes with $P_i \subset \mathbb{R}^{p_i}$ can be defined to be their Cayley embedding $C(P_1, \dots, P_r) \subset \mathbb{R}^{r-1} \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_r}$.) We have the following natural projection.

$$\begin{aligned} \Pi_C : &P_1 * \ldots * P_r \to C(\pi_1(P_1), \dots, \pi_r(P_r)), \\ &(e_i, p_i) \mapsto \big(e_i, \pi_i(p_i)\big). \end{aligned}$$

The Cayley trick is a natural bijection between the subdivisions induced by the projections Π_M and Π_C. The bijection is easier to state and understand looking at the family P^S of faces of P associated to a subdivision induced by a projection $\pi : P \to \pi(P)$.

Theorem 5.4 ([15])

Let

$$\Pi_M : P_1 \times \cdots \times P_r \to \mathrm{M}\big(\pi_1(P_1), \dots, \pi_r(P_r)\big)$$

and

$$\Pi_C : P_1 * \cdots * P_r \to C\big(\pi_1(P_1), \dots, \pi_r(P_r)\big)$$

be two polytope projections in the conditions above.

1. *If S is a Π_M-induced subdivision then every maximal face in $(P_1 \times \cdots \times P_r)^S$ is of the form $F_1 \times \cdots \times F_r$ for certain faces F_i of each P_i and moreover the family of faces*
$$\big\{F_1 * \cdots * F_r : F_1 \times \cdots \times F_r \in (P_1 \times \cdots \times P_r)^S\big\}$$
equals $(P_1 * \cdots * P_r)^{S'}$ for a certain Π_C-induced subdivision S'.
2. *Conversely, if S is a Π_C-induced subdivision then every maximal face in $(P_1 * \cdots * P_r)^S$ is of the form $F_1 * \cdots * F_r$ for certain faces F_i of each P_i and moreover the family of faces*
$$\big\{F_1 \times \cdots \times F_r : F_1 * \cdots * F_r \in (P_1 * \cdots * P_r)^S\big\}$$
equals $(P_1 \times \cdots \times P_r)^{S'}$ for a certain Π_M-induced subdivision S'.

Suppose now that each P_i is a simplex. Then the join product $P_1 * \cdots * P_r$ is also a simplex and, in particular, every Π_C-induced subdivision of rank 1 has height 1. Since the Baues posets of the projections Π_M and Π_C are isomorphic by Theorem 5.4 it is natural to expect that also every Π_M-induced subdivision of rank 1 has height 1. This follows from the following result, based on [25, Theorem 5.1].

Proposition 5.5

Suppose that $P_1, \dots, P_r$ are simplices. Let

$$\Pi_M : P_1 \times \cdots \times P_r \to \mathrm{M}\big(\pi_1(P_1), \dots, \pi_r(P_r)\big)$$

and

$$\Pi_C : P_1 * \cdots * P_r \to C\big(\pi_1(P_1), \dots, \pi_r(P_r)\big)$$

be two polytope projections in the conditions above. Let S be a Π_M-induced subdivision and S' a Π_C-induced subdivision which correspond to each other as in Theorem 5.4. Then the polytopes $\Sigma(S, \Pi_M)$ and $\Sigma(S', \Pi_C)$ are normally equivalent. In particular, they have the same dimension.

Before going into the proof, let us recall that two polytopes are said to be *normally equivalent* [6] or *strongly isomorphic* [25] if they lie in the same affine space and they have the same normal fan. The polytopes $\Sigma(S, \Pi_M)$ and $\Sigma(S', \Pi_C)$ of the previous statement can be considered to lie in the same affine space since the fibers of the projections Π_M and Π_C are both canonically isomorphic to the products of the fibers of the projections π_i.

Proof. If S and S' are the trivial subdivisions then the statement is just Theorem 5.1 in [25]. For arbitrary subdivisions, recall that $\Sigma(S, \Pi_M)$ equals the Minkowski sum of the fiber polytopes $\Sigma(B, \Pi_M)$ for the different cells $B \in S$ (and the same for S'). Since each $\Sigma(B, \Pi_M)$ is normally equivalent to the corresponding $\Sigma(B', \Pi_C)$ and since the normal fan of a Minkowski sum equals the common refinement of the normal fans of the summands, the result holds.

Remark: In the statement of [25, Theorem 5.1] the parameter r (number of polytopes P_i) equals the parameter d (dimension of the ambient space of the projections $\pi_i(P_i)$). However, this assumption is not used in the proof and it is posed because the case $d = r$ is interesting for the context of that paper. Even more, the same proof works also without the assumption that the polytopes P_i are simplices. $\square$

Zonotopal tilings and cubical flips

Here we assume that P is a cube, i.e., a product of segments. This is a particular case of the previous one so, in particular, it will be still true that rank 1 implies height 1, by Proposition 5.5.

If P is a cube of some dimension r, then its projection $\pi(P)$ is the Minkowski sum of r segments, i.e., a zonotope. The π-induced subdivisions coincide with the *zonotopal tilings* of $\pi(P)$. The tight ones are the *cubical tilings*, i.e., the subdivisions of $\pi(P)$

all of which cells are cubes. The natural notion of elementary change between cubical tilings is that of a cube-flip (see [22]) which is usually defined as follows: Let S be a cubical tiling of a zonotope $\pi(P)$ and let $d = \dim(\pi(P))$. Suppose that there is an interior vertex v of S which is incident to exactly $d+1$ cells. Then these cells form a convex zonotope of dimension d with $d+1$ generators, which has exactly two cubical tilings. One of them is contained in S. Switching to the other one produces a new cubical tiling S' of $\pi(P)$, and S and S' are said to differ by a cube-flip. For example, cube-flips in dimension 2 correspond to switching between the two decompositions of a hexagon into three parallelograms and in dimension 3 to switching between the two dissections of a rhombic dodecahedron into four combinatorial 3-cubes.

Let us say that a π-flip S for a polytope projection $\pi : P \to \pi(P)$ is *non-degenerate* if there is only one non-tight cell in S and all of its facets are tight.

Proposition 5.6

Let $P \to \pi(P)$ be a polytope projection where P is a cube. Then, two cubical tilings differ by a cube-flip if and only if they differ by a non-degenerate π-flip.

Proof. The 'only-if' is trivial: the $d+1$ cubes in which a cube-flip is made are a subdivision of a non-tight cell all of whose facets are tight. For the 'if', let S be the π-flip between T and T'. Let B be its unique non-tight cell. It has $\dim(L_B) = 1$, since S has rank 1, and hence, P^B is a $(d+1)$-cube. Because of non-degeneracy, the projection $\pi : P^B \to B$ has $(d+1)$ upper facets and $(d+1)$ lower facets, i.e., B has two cubical tilings both with $d+1$ cells, as in the definition of a cube-flip. □

The question arises of what "degenerate cube-flips" look like. Suppose that a cubical tiling T of $\pi(P)$ contains one of the two cubical tilings of a zonotope Z of dimension k with $k+1$ minimally dependent generators. What are the conditions necessary for the switch at the zonotope Z to be possible? As in the case of triangulations, the condition is related to the links, with the following definition:

DEFINITION 5.7. Let Z be a zonotope of dimension d generated by the segments $a_1, \dots, a_r$. For any subset $B \subset \{a_1, \dots, a_r\}$ we will denote Z_B the Minkowski sum of its elements. Let S be a zonotopal tiling of Z. Let Z_B be a Minkowski sum of a subset B of $\{a_1, \dots, a_r\}$. We call *zonotopal link* of B in S the set

$$\operatorname{link}_S(B) := \big\{W : Z_B + Z_W \text{ is a cell of } S\big\}.$$

Let $k \le d$ be an integer and let $B_1, \dots, B_{k+1}$ be different independent subsets of $\{a_1, \dots, a_r\}$ of cardinality k. If

1. $\cup_{i=1}^{k+1} B_i$ has $k+1$ elements (i.e., if $\sum_{i=1}^{k+1} B_i$ is a zonotope generated by $k+1$ elements of $\{a_1, \dots, a_r\}$) and
2. All the B_i have the same zonotopal link L in S,

then removing from S all the cells $B_i + W$, $i = 1, \dots, k+1$ and $W \in L$ and inserting the cells $B'_i + W$, where $B'_1, \dots, B'_{k+1}$ is the other cubical tiling of $\sum_{i=1}^{k+1} B_i$ one gets a new zonotopal tiling S'. We say that S and S' differ by a zonotopal flip.

With this definition it is easy to prove that:

Proposition 5.8

Let $\pi : P \to \pi(P)$ be the projection from a cube P to a zonotope $\pi(P)$. Two cubical tilings S_1 and S_2 of $\pi(P)$ differ by a π-flip if and only if they differ by a zonotopal flip. The π-flip is non-degenerate (i.e, the zonotopal flip is a cube flip) if and only if the parameter k of Definition 5.7 equals the dimension of $\pi(P)$.

Monotone paths and polygon flips

Here we suppose that $\dim(\pi(P)) = 1$. There is a unique (up to a constant) linear functional f on P which is constant on each fiber of the projection π. The π-induced subdivisions are the *cellular strings* on the polytope P with respect to f and the tight ones are the monotone paths in the direction of f (see [7]). The standard notion of elementary move between two monotone paths is that of a *polygon move* (see [22]): two monotone paths differ by a polygon move if they are different only in the boundary of a 2-face of P. As it happened in the case of zonotopal tilings, polygon moves correspond exactly to non-degenerate π-flips, but there are also some "degenerate polygon moves" which consist essentially in simultaneously moving through a family of 2-faces of P all of which have an edge parallel to a common direction. For example, let P be the octahedron $\{(x, y, z) \in \mathbb{R}^3 : |x|+|y|+|z| \leq 1\}$ and let $\pi : (x, y, z) \mapsto z$ be the projection to a vertical segment. There are four monotone paths, all of them π-coherent, but no non-degenerate polygon flip at all. Any π-flip involves two different 2-faces of P.

In this case π-induced subdivisions of rank 1 may have height greater than 1, as Example 4.6 shows.

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