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Gorenstein liaison of some curves in $\mathbb{P}^4$

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Abstract

Despite the recent advances made in Gorenstein liaison, there are still many open questions for the theory in codimension ≥ 3. In particular we consider the following question: given two curves in $\mathbb{P}^n$ with isomorphic deficiency modules (up to shift), can they be evenly Gorenstein linked? The answer to this is yes for curves in $\mathbb{P}^3$, due to Rao, but for higher codimension the answer is not known. This paper will look at large classes of curves in $\mathbb{P}^4$ with isomorphic deficiency modules and show that they can be Gorenstein linked. However, we are not able to prove (or disprove) the general case.

1. Introduction

Liaison is, roughly speaking, the study of properties shared by two schemes whose union is either a complete intersection or an arithmetically Gorenstein scheme. It is well known that a complete intersection is arithmetically Gorenstein, and in codimension 2 these two notions coincide. Much of the work in liaison has focused on using complete intersections to link, but Gorenstein liaison appears to be a more natural approach for codimension ≥ 3. The difficulty with this arises when we try to find good arithmetically Gorenstein schemes (i.e. arithmetically Gorenstein schemes containing a given scheme of the same codimension). Fortunately, the authors of [5] have developed many useful tools for doing so, some of which shall be used in this paper.

Liaison theory in codimension 2, in particular for curves in $\mathbb{P}^3$, has become a well-understood and useful field of study in algebraic geometry. However, there are still many open problems for liaison theory in higher codimension. As the title suggests, this paper will take a look at how to G-link various curves in $\mathbb{P}^4$ with isomorphic

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deficiency modules. The original motivation for these examples comes from the hope for a nice generalization of the following theorem to curves of higher codimension.

Theorem 1.1 (Rao, [10]).

Let C and C' be curves in $\mathbb{P}^3$ with deficiency modules (Rao modules) $M(C)$ and $M(C')$. Then C is evenly linked to C' if and only if $M(C)$ is isomorphic to some shift of $M(C')$.

By no means does this paper prove the existence of such a generalization; thus it is apparent that more work needs to be done. There is, however, a more general result in one direction, which follows from the Hartshorne-Schenzel Theorem.

Corollary 1.2 ([8]).

Let C, $C' \subset \mathbb{P}^n$ be evenly linked curves of the same codimension. Then there is an integer p such that $M(C)(p) \cong M(C')$.

Thus, we are really looking for some insight on the following question.

Question 1.3 *Let C and C' be two curves in $\mathbb{P}^n$, $n \geq 4$. If $M(C) \cong M(C')(p)$ for some integer p, can we (evenly) G-link C to C'?*

In an attempt to enlighten the situation a bit, we will look at some curves in $\mathbb{P}^4$. We define a curve of type (P, d, t) to be a disjoint union of two plane curves (in $\mathbb{P}^4$), one of degree d and one of degree t ($d \leq t$), such that the two planes meet in the point P (but neither plane curve contains the point P). We should notice that when $d = 1$ the point P is not uniquely determined, since there are many planes that contain a line. (We will make sure to mention the differences that may arise in this case.) In this paper we will prove the following:

Lemma 1.4

Let C, $C' \subset \mathbb{P}^4$ be curves of type (P, d, t) and (P', d', t') respectively.
(i) For $d = 1$, $M(C) \cong M(C')$ if and only if $d = d' = 1$.
(ii) For $d \geq 2$, $M(C) \cong M(C')$ if and only if $P = P'$ and $d = d'$.

Theorem 1.5

Let C, C' be as in Lemma 1.4. Then we can (evenly) G-link C to C' if and only if $M(C) \cong M(C')$.

If we wish to classify all curves with deficiency module isomorphic (up to shift) to that of a curve of type (P, d, t) the natural place to start is with the minimal curves. We know from [8], Proposition 1.2.8 that there is a leftmost possible shift for any deficiency module. We shall call any curve with this leftmost shift *minimal*. As it turns out, a curve C of type (P, d, t) is minimal, but these are not the only minimal curves for deficiency module $M \cong M(C)$. We shall show that there are minimal curves with deficiency module M of every degree $\geq d + 1$, and we shall investigate whether or not all minimal curves can be G-linked. In codimension 2, the even liaison class (which is determined by the deficiency module) has the Lazarsfeld-Rao property. In particular, this says that if V_1 and V_2 are two minimal curves in the even liaison class, then there exists an irreducible flat family of curves (all in that same even liaison class) to which both V_1 and V_2 belong (see [8] Section 6.3). We already know that this property does not hold for G-liaison in codimension 3 or higher, since we have minimal curves in the same even liaison class of different degrees. Even still, we might hope that the minimal curves in an even liaison class of fixed degree form an irreducible family. Hartshorne, [4] Proposition 4.1, showed that this is the case for curves with deficiency module k. However, in Section 4 we will show that there are two curves that can be G-linked but are not in the same irreducible family (see Remark 4.6).

Hartshorne has recently taken a look at similar examples in [4]. He even gives possible counterexamples to the proposed generalization of Theorem 1.1. Nonetheless, until an answer is found we press forward.

I would like to thank my advisor Juan Migliore for introducing me to this beautiful theory and for his guidance in writing this paper. I would also like to thank Robin Hartshorne and Scott Nollet for their correspondence about this work and the referee for suggestions on how to expand this paper.

2. Background and definitions

Throughout this paper, $\mathbb{P}^n$ will be the n-dimensional projective space over an algebraically closed field k, and $R = k[x_0, \ldots, x_n]$. For a closed subscheme X of $\mathbb{P}^n$, we denote by $\mathcal{I}_X$ its ideal sheaf and by $I_X = H^0_*(\mathbb{P}^n, \mathcal{I}_X)$ its saturated homogeneous ideal. For a curve $C \subset \mathbb{P}^n$, we will denote by $M(C) = H^1_*(\mathbb{P}^n, \mathcal{I}_C) = \oplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^n, \mathcal{I}_C(t))$ its deficiency module (or Rao module).

Definition 2.1. Let V_1 and V_2 be two non-empty, equidimensional schemes without embedded components. We say that V_1 and V_2 are *directly Gorenstein-linked*, or *directly G-linked*, by an arithmetically Gorenstein scheme (*aG* scheme for short) $X \subset \mathbb{P}^n$ if $I_X \subset I_{V_1} \cap I_{V_2}$ and we have $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$. In this case we say that V_2 is *residual* to V_1 in the scheme X (and vice versa).

Remark 2.2. When the two schemes V_1 and V_2 have no common component, as well as being non-empty and equidimensional without embedded components, the above definition is equivalent to the following: V_1 and V_2 are *directly G-linked* by the *aG* scheme X if $V_1 \cup V_2 = X$. Also, we say that V_1 and V_2 are *G-linked* if we can use a finite number of direct G-links to get from V_1 to V_2.

Gorenstein liaison, or G-liaison, is the study of the equivalence relation generated by G-linkage. Complete intersection linkage is defined similarly, replacing arithmetically Gorenstein with complete intersection in the above definition, and in that case we speak of CI-liaison. It has been shown that CI-liaison can be viewed as a theory about generalized divisors on complete intersection schemes [3]. The authors of [5] took a similar approach for G-liaison, viewing it as a theory of divisors on arithmetically

Cohen-Macaulay (aCM) schemes satisfying G_1 (Gorenstein in codimension 1, see [3]). We will use the following results to produce some G-links for our examples.

Theorem 2.3 (KMMNP, [5])**.**

Let $S \subset \mathbb{P}^n$ be an aCM scheme satisfying property G_1, and let K be a twisted canonical divisor on S (i.e. a subscheme of S defined by the vanishing of a regular section of $\omega_S(t)$ for some $t \in \mathbb{Z}$). Let $F \in I_K$ be a homogeneous polynomial of degree d such that F does not vanish on any component of S. Let H_F be the divisor cut out on S by F. Then any (effective) divisor on S in the linear system $|H_F - K|$ is arithmetically Gorenstein.

Corollary 2.4 (KMMNP, [5])**.**

Let K' be any divisor in the linear system $|H_F - K|$ as in Theorem 2.3. Let G be a homogeneous polynomial not vanishing on any component of S. Then $K' + H_G$ is also aG.

Remark 2.5. If K is a twisted canonical divisor on S, then we shall call any divisor K' of the form $H_G - K$ (as in Theorem 2.3) a *twisted anticanonical divisor*.

Since we shall be speaking of divisors on schemes, a definition is in order. A *divisor* on a subscheme S of $\mathbb{P}^n$ will be an equidimensional, locally Cohen-Macaulay, codimension one subscheme of S. Taking a hypersurface section of our scheme S is the most basic, and most useful, type of divisor. If $F \in R_t$ is a homogeneous polynomial not vanishing on any component of S (i.e. $[I_S : F] = I_S$), then H_F is the divisor cut out on S by F. If our scheme S is aCM then we denote the set of all divisors cut out by hypersurfaces of degree t by $|tH|$.

This paper focuses on how to use the above theorem to produce G-links between different curves in $\mathbb{P}^4$, and not why we can do so. If the reader wishes a more thorough background on these techniques, and liaison theory in general, see [5], [8] and [9].

3. Main results

Let X_1, $X_2 \subset \mathbb{P}^4$ be two (disjoint) plane curves such that neither curve intersects the plane of the other. That is, $\Lambda_1 \cap \Lambda_2 = P$ (a point), where Λ_1 and Λ_2 are the planes of X_1 and X_2 respectively. (Notice that this implies neither curve contains the point P.) Also, assume that $deg\ X_1 = d$ and $deg\ X_2 = t \geq d$. Finally, let $C = X_1 \cup X_2$. For the rest of this paper we shall call C a curve of type (P, d, t), and all the curves that we speak of shall be thought of as being inside $\mathbb{P}^4$.

We need to be careful when $d = 1$. In this case X_1 is a line that doesn't intersect Λ_2. For any point $P \in \Lambda_2$ we can find a plane Λ_1 containing X_1 such that $\Lambda_1 \cap \Lambda_2 = P$. In other words, the point P is not uniquely determined by the curve $C = X_1 \cup X_2$, which will allow us to have more freedom in the constructions we will use later in this paper. Thus, the results will still hold when $d = 1$ (in fact they are a bit more general, since the deficiency module will not depend on our choice of P), and we will make sure to note the differences when necessary. In this case we shall call C a curve of type $(-, 1, t)$.

One of the goals of this paper is to show that any curve of type (P, d, t) can be G-linked to any curve of type (P, d, s). Part of the motivation for trying to do this came from the following observation:

Lemma 3.1

Let Λ_1, Λ_2 be linear subspaces of $\mathbb{P}^n$ meeting in a single point P, where $I_P = (x_0, x_1, \ldots, x_{n-1})$. Let C_1, C_2 be aCM curves in Λ_1, Λ_2 respectively, not both containing P, and let $C = C_1 \cup C_2$. Let d be the smallest number among the degrees of hypersurfaces of Λ_1 containing C_1 and not P, or hypersurfaces of Λ_2, containing C_2 and not P. Then

(i) $M(C) \cong M_d = R/(x_0, x_1, \ldots, x_{n-1}, x_n^d)$, and

(ii) For $d \geq 2$ the module structure of $M(C)$ determines and is determined by the point P.

Proof. First notice that we have $I_C = I_{X_1} \cap I_{X_2}$. Thus we have the following exact sequence

$$0 \to I_C \to I_{X_1} \oplus I_{X_2} \to I_{X_1} + I_{X_2} \to 0.$$

Then we can sheafify and take cohomology to get

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I_C \arrow[r] & I_{X_1} \oplus I_{X_2} \arrow[rr] \arrow[dr] & & R \arrow[r] & M(C) \arrow[r] & 0 \\
& & & I_{X_1} + I_{X_2} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Notice that $H^0_*(\widetilde{I_{X_1} + I_{X_2}}) = R$ since X_1 and X_2 are disjoint. Also, $H^1_*(\mathcal{I}_{X_1} \oplus \mathcal{I}_{X_2}) = 0$ since both curves are aCM. Now we see that $M(C) \cong R/(I_{X_1} + I_{X_2})$, and the ideal $I_{X_1} + I_{X_2}$ has n independent generators in degree 1 (the n linear forms that define the point P). Also, by hypothesis $I_{X_1} + I_{X_2}$ contains x_n^d. Hence, $M(C) \cong M_d$. For $d = 1$ we get $M(C) \cong k$ (in degree 0). To see that the module structure determines and is determined by P (for $d \geq 2$) all we need to do is notice that the linear forms that annihilate $M(C)$ are exactly the ones in I_P. $\square$

Remark 3.2. In particular this lemma shows that a curve C of type (P, d, t) has deficiency module $M(C) \cong M_d$. This is essentially the same proof used in Example 1.5.4 from [8].

Now we will establish some facts which will be useful for producing our links. When $C = X_1 \cup X_2 \subset \mathbb{P}^4$ is a disjoint union of two lines, let P be any point not contained in the hyperplane containing C. Then if we let Λ_i be the plane spanned by

X_i and P, we can think of $C \subset \Lambda_1 \cup \Lambda_2$ as a curve of type $(P, 1, 1)$. This construction helps make sense of the following lemmas, when $d = 1$.

Lemma 3.3

Let $C = X_1 \cup X_2$ be a curve of type (P, d, d). Then we can find a form $F \in R_d$ such that F cuts out X_1 on Λ_1 and X_2 on Λ_2.

Proof. Without loss of generality, let $I_P = (x_0, x_1, x_2, x_3), I_{\Lambda_1} = (x_0, x_1)$ and $I_{\Lambda_2} = (x_2, x_3)$. Now let $F \in R_d$ such that F meets Λ_1(and *not* Λ_2) properly, i.e. $[I_{\Lambda_1} : F] = I_{\Lambda_1}$ and $[I_{\Lambda_2} : F] \neq I_{\Lambda_2}$. Then we can find a homogeneous polynomial $G \notin I_{\Lambda_2}$ such that $GF \in I_{\Lambda_2}$. However, I_{Λ_2} is a prime ideal which forces $F \in I_{\Lambda_2} \subset I_P$. Now we notice that since X_1 and X_2 are plane curves not containing P, any form of degree d that cuts out either curve cannot be in I_P, thus it must cut out a curve on both planes Λ_1 and Λ_2.

This allows us to see that we have $I_{X_1} = (x_0, x_1, x_4^d + G_2)$ and $I_{X_2} = (x_2, x_3, x_4^d + G_1)$ where $G_1 \in (I_{\Lambda_1})_d$ and $G_2 \in (I_{\Lambda_2})_d$. Then the form $F = x_4^d + G_2 + G_1 \in (I_{X_1} \cap I_{X_2})_d$ as desired. (Note that $F \notin I_P$.) $\square$

Lemma 3.4

Let $C = X_1 \cup X_2$, $C' = X_1' \cup X_2'$ be two curves of type (P, d, d) such that there exists $F \in R_d$ which cuts out all four plane curves X_1, X_2, X_1', and X_2'. Then C can be linked to C'.

Proof. (Note: the links used here will be CI-links.)

Let $\Lambda_1, \Lambda_2, \Lambda_1{}'$ and $\Lambda_2{}'$ be the planes of X_1, X_2, X_1' and X_2' respectively. Recall that $\Lambda_1 \cap \Lambda_2 = P = \Lambda_1{}' \cap \Lambda_2{}'$. If it turns out that Λ_i meets $\Lambda_j{}'$ in a line for all $1 \leq i, j \leq 2$, then the surface $S = \Lambda_1 \cup \Lambda_2 \cup \Lambda_1{}' \cup \Lambda_2{}'$ is a complete intersection. We see this since S is a cone with vertex P over a complete intersection stick figure in $\mathbb{P}^3$. The form F cuts out $C \cup C'$ on S, making $C \cup C'$ a complete intersection as well. Hence C is linked to C'.

We cannot rely on this to be the situation however, so we consider the case where $\Lambda_i \cap \Lambda_j{}' = P$ for all $1 \leq i, j \leq 2$. Notice now that we can find two planes Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = P$ and Γ_i meets each of the planes Λ_1, Λ_2 and $\Lambda_1{}'$ in a line, for $i = 1, 2$ (see remark below). Let $S = \Lambda_1 \cup \Lambda_2 \cup \Gamma_1 \cup \Gamma_2$ and $T = \Lambda_2 \cup \Lambda_1{}' \cup \Gamma_1 \cup \Gamma_2$. Then we see from above that, using the same form F, C is linked to a curve of type (P, d, d) on $\Gamma_1 \cup \Gamma_2$ which is in turn linked to a curve of type (P, d, d), C'', on $\Lambda_2 \cup \Lambda_1{}'$. Repeating this process we can link C'' to C', which gets us what we want.

All other possible configurations of the planes Λ_1, Λ_2, $\Lambda_1{}'$ and $\Lambda_2{}'$ can be handled in a similar fashion. $\square$

Remark 3.5. Something more should be said about why we can find Γ_1 and Γ_2 used in the proof above. If H_1 is a general hyperplane not containing the point P, then it cuts out three skew lines on the planes Λ_1, Λ_2 and $\Lambda_1{}'$, one on each. Then we can find a trisecant γ_1 connecting those lines and let Γ_1 be the cone over γ_1 through the point

P. We can do the same with another general hyperplane H_2 not containing P to get Γ_2.

With these two lemmas in place we can now handle the more general situation.

Theorem 3.6

Let C be a curve of type (P, d, t) and let C' be a curve of type (P, d, s), $d \geq 2$. Then C can be G-linked to C'. Also, a curve C of type $(-, 1, t)$ can be G-linked to ANY curve C' of type $(-, 1, s)$.

Proof. First we deal with the case $d = 1$. Let $C = X_1 \cup X_2$ and $C' = X_1' \cup X_2'$, where $X_2 \subset \Lambda_2$ and $X_2' \subset {\Lambda_2}'$ are plane curves of degree t and s respectively. (When $t = 1$ we let Λ_2 be any plane in $\mathbb{P}^4$ containing X_2 that is disjoint from X_1. Likewise when $s = 1$.) If $\Lambda_2 \cap {\Lambda_2}' = P$ (a point that lies on neither X_2 nor X_2'), then we let $\Lambda_1({\Lambda_1}'$ resp.) be the plane spanned by P and the line X_1 (X_1' resp.). At this point we can think of C as a curve of type $(P, 1, t)$ and C' as a curve of type $(P, 1, s)$, and we can proceed with the rest of this proof. (If not, we can find an intermediate curve D of type $(-, 1, r)$ that satisfies these conditions with respect to both C and C'.)

Let $C = X_1 \cup X_2$, $X_1 \subset \Lambda_1$ and $X_2 \subset \Lambda_2$. Now let Γ_1, Γ_2 be two planes such that $\Gamma_1 \cap \Gamma_2 = P$, and Λ_i meets Γ_j in a line for all $1 \leq i, j \leq 2$. Then $S = \Lambda_1 \cup \Lambda_2 \cup \Gamma_1 \cup \Gamma_2$ is a complete intersection surface containing C. By Lemma 3.3 we know we can find a homogeneous polynomial $F \in R_t$ that cuts out a complete intersection curve, H_F, on S containing C. The residual to C in H_F is a curve $D \subset T = \Lambda_1 \cup \Gamma_1 \cup \Gamma_2$, and the surface T is aCM satisfying G_1.

Now we need to figure out what a twisted canonical divisor on T looks like. We can see that $I_{\Gamma_1} = (v, w)$, $I_{\Lambda_1} = (w, x)$, $I_{\Gamma_2} = (x, y)$ and $I_T = (vx, wx, wy)$ where the linear forms v, w, x and y define the point P. Then the presentation matrix for the canonical module (shifted by 5) of T is $A = \left(\begin{smallmatrix} x & 0 & -y \\ 0 & w & -v \end{smallmatrix}\right)$. Let A' be the concatenation of A with the matrix $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, which is homogeneous. Let $M_{A'}$ be the cokernel of the map represented by A', and let I be the ideal of the annihilator of the module $M_{A'}$. We can see that $I = (v - y, w, x)$, which defines a line containing P on the middle plane Λ_1. By step II of the proof of Proposition 5.12 in [5] we know that I defines a twisted canonical divisor, K, on T. Let L be any general linear form cutting out K on T, and let $K' = H_L - K$ (notice that K' is two lines through P, one on Γ_1 and one on Γ_2). We know by Corollary 2.4 that any divisor of the form $rH + K'$ is arithmetically Gorenstein. In particular, the divisor $E = (D \cup X_1) + K'$ is arithmetically Gorenstein, and the residual to D in E is $X_1 + K'$.

Now let $G \in (I_{X_1})_d$ be a form that meets T properly (i.e. $[I_T : G] = I_T$). Then the divisor $H_G + K'$ on T is aG and contains $X_1 + K'$. The residual to $X_1 + K'$ in $H_G + K'$ is a curve of type (P, d, d). Note that we used a single form $G \in (I_{X_1})_d$ to define this curve of type (P, d, d) which lies on $\Gamma_1 \cup \Gamma_2$.

We can repeat this whole process with C' (with planes ${\Lambda_1}'$, ${\Lambda_2}'$, ${\Gamma_1}'$ and ${\Gamma_2}'$). Now if $\Lambda_1 \cap {\Lambda_1}' = P$ then we can pick $G = G' \in R_d$ above such that G cuts out X_1 and X_1' (hence G also cuts out the corresponding curves of type (P, d, d)), by Lemma 3.3. Then using Lemma 3.4 we are done. If $\Lambda_1 \cap {\Lambda_1}' \neq P$ then we can find a third curve $C'' = X_1'' \cup X_2''$ of type (P, d, r) such that $\Lambda_1 \cap \Lambda_1'' = P = \Lambda_1' \cap \Lambda_1''$. Using the above proof we can link C to C'' and C'' to C'. $\square$

Remark 3.7. At first glance the divisor $X_1 + K'$ appears to be a curve type $(P, 2, d)$ when $d \geq 2$, which would be a problem. However, since K' contains the point P, $X_1 + K'$ is not a curve of type $(P, 2, d)$, and it can be easily shown that the deficiency module of $X_1 + K'$ is as desired.

There are other curves in $\mathbb{P}^4$ with deficiency module isomorphic to that of a curve C of type (P, d, t). For instance, let $D = \lambda \cup Y \subset \mathbb{P}^4$ be the disjoint union of a line λ and a plane curve Y of degree $d \geq 2$, such that λ meets the plane of Y in the point P. Notice that such a curve is degenerate and $M(D) \cong M(C)$, by Lemma 3.1. Trying to keep hope alive that there is a nice extension of Rao's theorem to curves of higher codimension, we link C to D as well.

Remark 3.8. When d = 1, we get a disjoint union of two lines, which is a curve of type $(-, 1, 1)$. This has already been taken care of in the last theorem.

Corollary 3.9

Let C be a curve of type (P, d, t), and let $D = \lambda \cup Y \subset \mathbb{P}^4$ be as above. Then D can be G-linked to C.

Proof. Let Λ_1 be the plane of Y, let Γ_1 be a plane containing λ and let Γ_2 be a third plane containing P such that Λ_1 meets both Γ_1 and Γ_2 in a line and $\Gamma_1 \cap \Gamma_2 = P$. Then $T = \Lambda_1 \cup \Gamma_1 \cup \Gamma_2$ is aCM satisfying G_1. Let $\gamma_2 \subset \Gamma_2$ be a line through P such that $\gamma_2 \cap \Lambda_1 = P$. Then as before we have $\lambda \cup \gamma_2 = K'$, a twisted anticanonical divisor on T. Thus if $F \in (I_Y)_d$ cuts out Y, then $H_F + K'$ is aG. The residual to D in $H_F + K'$ is a curve $D' \cup \gamma_2$, where D' is of type (P, d, d), $D' \subset \Gamma_1 \cup \Gamma_2$.

Now we can find a plane Λ_2 (not containing γ_2) that meets Λ_1 in the point P only while meeting each of the planes Γ_1 and Γ_2 in a line. Once again, the surface $S = \Lambda_1 \cup \Lambda_2 \cup \Gamma_1 \cup \Gamma_2$ is a complete intersection. Let L be a general linear form that cuts out γ_2 on Γ_2, and let $F \in (I_Y)_d$, as above, cutting out D'. Now if $G = FL$, then the divisor cut out on S by G, H_G, is a complete intersection containing $D' \cup \gamma_2$. The residual to $D' \cup \gamma_2$ in H_G is a curve D'' on $W = \Lambda_1 \cup \Gamma_1 \cup \Lambda_2$. With a slight abuse of notation we can see that $D'' = H_G - X$ where H_G is the hypersurface section cut out by G on W and X is a curve of degree d on Γ_1 cut out by F.

If K'' is a twisted anticanonical divisor on W (i.e. two lines through P, one on Λ_1 and one on Λ_2), the divisor $H_G + K''$ (on W) is aG, and the residual to D'' in $H_G + K''$ is $X + K''$. Finally, the divisor $H_F + K''$ is also aG and contains $X + K''$. The residual to $X + K''$ is a curve of type (P, d, d) on $\Lambda_1 \cup \Lambda_2$. $\square$

We say that two equidimensional subschemes without embedded components are *evenly G-linked* if they can be G-linked in an even number of steps. It is well known that even G-linkage also generates an equivalence relation, known as *even G-liaison* (note that we have an analogous definition for *even CI-liaison*). Theorem 1.1 (Rao) speaks of even-liaison, and as it turns out the examples in this paper are evenly G-linked. However, we should notice here that these deficiency modules are self-dual, so the distinction between liaison and even liaison becomes less significant. Indeed, notice that the curve C of type (P,d,t) is linked in 3 steps to a curve of type (P,d,d). This allows us, using a suitable number of intermediate curves (of type (P,d,d)), to G-link a curve of type (P,d,t) to a curve of type (P,d,s) in either an even or an odd number of steps.

4. Minimal curves

Something that is important to the structure of an even liaison class (at least in codimension 2) is the notion of minimal elements. A *minimal* element of an even liaison class is one with the leftmost possible shift of the corresponding deficiency modules. In [6] Migliore showed that for curves in $\mathbb{P}^3, \dim M(C)_i$ has to be strictly increasing for i non-positive. Fortunately, this proof easily extends to curves in $\mathbb{P}^4$ which tells us that the curves we have considered in this paper are in fact minimal.

In [4] Hartshorne considers (minimal) curves with deficiency module k in degree 0, among other things. In particular he shows the following:

Proposition 4.1
(a.) There are curves $C \subset \mathbb{P}^4$ of every degree ≥ 2 with $M(C) \cong k$ (concentrated in degree 0).
(b.) For each $d \geq 2$, the set of all such curves of degree d forms an irreducible family, whose general member is the disjoint union $C = C' \cup L$ of a plane curve C' of degree $d-1$ and a line L, not meeting the plane of C'.
(c.) Every such curve is in the G-liaison class of two skew lines.

Proof. See [4] Proposition 4.1. □

We should notice that the curves of type $(-,1,t)$ in this paper are exactly the curves of degree $t+1$ which are the general members of the irreducible families mentioned in Proposition 4.1(b.) above. Thus, it is natural to wonder if these results will extend to all minimal curves in $\mathbb{P}^4$ with deficiency module $M_d \cong R/(x_0,x_1,x_2,x_3,x_4^d)$ for $d \geq 2$. We ask the following:

Question 4.2 *(a.) What are the possible degrees for minimal curves C with $M(C) \cong M_d, d \geq 2$?*

(b.) Do the curves C, with deg $C = d'$ and $M(C) \cong M_d$, form an irreducible family for all possible d' from part (a.)?

(c.) Are all (minimal) curves with $M(C) \cong M_d$ in the same G-liaison class for each d?

Hartshorne has given us our answer for $d = 1$, so we shall restrict ourselves to $d \geq 2$. (Notice also that for $d \geq 2$, by our choice of M_d we have fixed the point P, where $I_P = (x_0,x_1,x_2,x_3)$). We have already seen that we have curves C with $M(C) \cong M_d$ and either *deg* $C = d' = d+1$ or *deg* $C = d' \geq 2d$. However, these are not the only minimal curves with these deficiency modules. Consider the following:

Proposition 4.3

Let $M_d \cong R/(x_0, x_1, x_2, x_3, x_4^d)$ for $d \geq 2$. Then

(i.) There are curves $C \subset \mathbb{P}^4$ with $M(C) \cong M_d$ of every degree $d' \geq d+1$.

(ii.) There are no reduced curves in $\mathbb{P}^4$ of degree $\leq d$ with deficiency module M_d.

Proof. Consider the curve $C = C_1 \cup C_2$ which again is a disjoint union of two plane curves whose corresponding planes meet only in the point P. This time we assume that C_1 does not contain P while C_2 does, and we assume that $deg\ C_1 = d$ and $deg\ C_2 \geq 1$. Lemma 3.1 tells us that $M(C) \cong M_d$. Thus there are minimal curves of every degree $d' \geq d+1$ with deficiency module isomorphic to M_d.

Suppose that $C \subset \mathbb{P}^4$ is a reduced curve with $M(C) \cong M_d$, $d \geq 2$. If C is degenerate, then we know that $deg\ C \geq d+1$ by the Lazarsfeld-Rao property. Thus we assume that C is non-degenerate. By Theorem 1.2.6(b) [8] we know that C has to have two connected components, $C = C_1 \cup C_2$. Now consider the exact sequence

$$0 \to H^0(\mathcal{I}_C(1)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathcal{O}_C(1)) \to M(C)_1 \to 0.$$

From this we see that $h^0(\mathcal{O}_C(1)) = 6 = h^0(\mathcal{O}_{C_1}(1)) + h^0(\mathcal{O}_{C_2}(1))$. Now we have a few cases to consider. If C_1 is non-degenerate then $h^0(\mathcal{O}_{C_1}(1)) \geq 5$, which implies that $h^0(\mathcal{O}_{C_2}(1)) \leq 1$. This cannot happen if we wish C_2 to be a curve. If C_1 is a line then $h^0(\mathcal{O}_{C_1}(1)) = 2$ which implies that $h^0(\mathcal{O}_{C_2}(1)) = 4$ (i.e. C_2 is a non-degenerate curve in $\mathbb{P}^3$). Notice that C_2 also has to be aCM (otherwise we would get another contradiction using an exact sequence similar to the one above). Therefore, we can use the proof of Lemma 3.1 to compute $M(C)$. If indeed we have $M(C) \cong M_d$, then the initial degree of I_{C_2} has to be d. However, this forces $deg\ C_2 \geq d$. Finally, if we assume that C_1 is a plane curve (not a line), then we have $h^0(\mathcal{O}_{C_1}(1)) = 3$, which forces $h^0(\mathcal{O}_{C_2}(1)) = 3$ as well (that is, they are both plane curves). Once again we can use the proof of Lemma 3.1. Since $C = C_1 \cup C_2$ is a disjoint union, at most one of its two components can contain the point P (which has to be the point of intersection of the two planes involved here). What we know from the proof of Lemma 3.1 is that if C_1 doesn't contain P, then $deg\ C_1 \geq d$. Likewise for C_2. Thus we have $deg\ C \geq d+1$. $\square$

Remark 4.4. It is still unknown whether or not there are non-reduced curves of degree $\leq d$ with deficiency module M_d, although the existence of such a curve seems very unlikely. In any case, Proposition 4.3 gives us a partial answer to Question 4.2 (a.).

In correspondence with the author, Hartshorne pointed out a quick counterexample to Question 4.2(b). If we let C be a curve of type $(P, 2, 4)$ and let D be the (non-degenerate) union of a line with an aCM curve in $\mathbb{P}^3$ of degree 5 and genus 2, then we see that $deg\ C = deg\ D$, $M(C) \cong M(D) \cong M_d$ yet C and D have different genus. Thus, there is no chance for these two curves to be in the same irreducible family.

We can consider the following refined question: Do all the curves in $\mathbb{P}^4$ with fixed degree *and* genus, having deficiency module M_d, lie in the same irreducible family? We answer this with a counterexample. Consider $d = 2$. We have already seen that a curve C of type $(P, 2, 2)$ is a curve of degree 4 with $M(C) \cong M_2$. Now let us consider a curve $D = L \cup Y \subset \mathbb{P}^4$ which is a disjoint union of a line L and a twisted cubic Y. If we further assume that the hyperplane containing Y also contains the point P, while the line L, which also contains P, is not contained in the same hyperplane, then by Lemma 3.1 again we see that $M(D) \cong M_2$. Also we see that the genus of these two curves is -1, and they shall provide us with our counterexample:

Proposition 4.5

Let $C = X_1 \cup X_2$ be a curve of type $(P, 2, 2)$, and let $D = L \cup Y$ be the disjoint union of a line L and a twisted cubic Y. If we assume that the hyperplane of Y contains the point P and that L meets the hyperplane of Y in the point P only, then

(i.) $M(C) \cong M(D)$.

(ii.) The curves C and D are NOT contained in the same irreducible family.

Proof. With our assumptions, part (i.) follows directly from Lemma 3.1.

To prove part (ii.) we will look at the Hilbert scheme containing these two curves. Planes in $\mathbb{P}^4$ move in a 6 dimensional family, and conics in $\mathbb{P}^2$ move in a 5 dimensional family, thus the component of the Hilbert scheme to which C belongs has dimension (at least) 22. Likewise, lines in $\mathbb{P}^4$ move in a 6 dimensional family, and twisted cubics in $\mathbb{P}^4$ move in a 16 dimensional family. So, the component of the Hilbert scheme containing D also has dimension (at least) 22. Now, using the techniques from [11] Chapter 8, we can compute the dimension of the tangent space to the Hilbert scheme at these two points. Proposition 8.1 from [11] tells us that the Zariski tangent space of the Hilbert Scheme at C, T_C, is isomorphic to the global sections of the normal bundle, $\mathcal{N}_C$, of C in $\mathbb{P}^n$. In this case, since $C = X_1 \cup X_2$ is a disjoint union, we have $h^0(C, \mathcal{N}_C) = h^0(X_1, \mathcal{N}_{X_1}) + h^0(X_2, \mathcal{N}_{X_2})$. Likewise for $D = L \cup Y$. Therefore, we can check using standard techniques that both T_C and T_D have dimension 22, and thus both curves C and D are unobstructed. This implies that C is a member of one dimension 22 component of the Hilbert scheme while D is the member of a different dimension 22 component (in particular, neither curve is in the intersection of these two components). Thus, there can be no irreducible family that contains both C and D. □

Remark 4.6. Let $D = L \cup Y$ be as in Proposition 4.5. Let Q be a smooth quadric surface in the hyperplane spanned by Y, containing Y and P. Let Λ be a plane containing L that meets Q in a line. Then $S = \Lambda \cup Q$ is an aCM surface of degree three containing D, and its negative canonical divisor, $-K$, consists of a conic C in Λ, and a twisted cubic on Q meeting the conic in the two points where C meets Q (see [4] Proposition 4.1). That is, we can assume that $-K = C \cup Y$. Now if F is a linear form cutting out L on Λ, then $H_F - K$ is an aG divisor on S linking D to a curve $E = E_1 \cup E_2$, where E is a curve like one of type $(P, 2, 2)$, but here E_2 contains the point P. The proof of Proposition 4.5 shows that these two curves, both having degree 4, genus -1 and deficiency module M_2 are not in the same irreducible family. However, we should note here that it is not known if these curves can be evenly G-linked.

Unfortunately, we were neither able to prove nor disprove that the curves C and D from Proposition 4.5 were in the same G-liaison class, and now it seems quite possible that these two curves may not be able to be G-linked. Likewise, we had limited success linking the curves mentioned in Proposition 4.3 to curves of type (P, d, d). However, in general Question 4.2 (c.) remains open.

5. Comments

While other work has recently been done on G-liaison of curves in $\mathbb{P}^4$, see [1] and [2], the examples done by Hartshorne [4] and the examples worked out here are among the first for non-aCM curves. Already, we see that the picture for G-liaison in codimension ≥ 3 appears to be more complicated than the codimension 2 case. Even in these examples we came across curves with isomorphic deficiency modules that leave us wondering whether or not they can be G-linked.

Along with a theorem that could tell us exactly when two curves in $\mathbb{P}^4$ can be (evenly) G-linked, we are also concerned with the possible structure of an even liaison class. We don't know what the structure should look like, but this paper rules out a possible generalization of the Lazarsfeld-Rao property that had still remained in [4]. This too remains an open question.

Finally, we might consider the importance of the use of G-liaison in this paper. In [7] Migliore conjectured that a pair of skew lines in $\mathbb{P}^4$ can be CI-linked to another pair of skew lines if and only if they are contained in the same hyperplane $H \subset \mathbb{P}^4$. On the other hand, Theorem 3.6 tells us that we can G-link a pair of skew lines in $\mathbb{P}^4$ to *any other* pair of skew lines (in $\mathbb{P}^4$). It is not hard to imagine that we will not be able to replace G-links with CI-links for the general cases of the examples in this paper, although we do not give a concrete reason for this to be so. There is still much work to be done for G-liaison of curves in $\mathbb{P}^4$ and indeed for G-liaison of all schemes of codimension ≥ 3.

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