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Characterization of moment multisequences by a variation of positive definiteness

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Abstract

A function $\varphi\colon \mathbb{N}_0^k \to \mathbb{R}$ is a moment (multi-)sequence if and only if the linear form L on the polynomial algebra $\mathbb{R}[x_1,\dots,x_k]$ defined by $L(x_1^{n_1}\dots x_k^{n_k}) = \varphi(n_1,\dots,n_k)$ is nonnegative on every polynomial p such that for some odd natural number m the polynomial $p(x_1^m,\dots,x_k^m)$ is a sum of squares of real polynomials.

1. Introduction

On abelian groups, functions that admit a disintegration as an integral of characters are characterized by positive definiteness, by the discrete version of the Bochner-Weil Theorem. The same is true on the semigroup $(\mathbb{N}_0,+)$, by Hamburger's Theorem. In contrast, in either the multidimensional moment problem (associated with the semigroup $\mathbb{N}_0^k$) or the complex moment problem (associated with $\mathbb{N}_0^2$ with 'switching' involution), positive definiteness is not sufficient. These moment problems can be generalized to arbitrary abelian involution semigroups. The original aim of this piece of research was to show how an arbitrary $*$-semigroup S that is adapted and $\mathbb{C}$-separative (as defined below) can be embedded in a larger semigroup Q such that moment functions on S are characterized by being 'positive definite with respect to Q' as defined below. In the end, we had to assume that S is of 'class $\mathcal{M}$' as defined in the body of the paper. This is a condition that is stronger than adaptedness, but weaker than assuming that S has an identity. The notion of positive definiteness with respect to a larger semigroup was introduced by Stochel and Szafraniec [18] though they did not use the term. Our main result implies that a function $\varphi\colon \mathbb{N}_0^k \to \mathbb{R}$ is a k-dimensional moment sequence if and only if the linear form L on the polynomial algebra $\mathbb{R}[x_1,\dots,x_k]$

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defined by $L(x_1^{n_1}\dots x_k^{n_k}) = \varphi(n_1,\dots,n_k)$ is nonnegative on every polynomial p such that for some odd integer m the polynomial $p(x_1^m,\dots,x_k^m)$ is a sum of squares of real polynomials.

Suppose $(S,\cdot,*)$ is an abelian (i.e., commutative) semigroup equipped with an *involution*, that is, a mapping $s\mapsto s^*\colon S\to S$ such that $(s^*)^* = s$ and $(st)^* = t^*s^*$ for all $s,t\in S$. Such a structure will be called a $*$-*semigroup*, even abbreviated 'semigroup' when we apply an adjective which makes sense only for $*$-semigroups. For subsets A and B of S, define $AB = \{\, st \mid s\in A,\ t\in B\,\}$. A function $\varphi\colon SS\to\mathbb{C}$ is *positive definite* if $\sum_{j,k=1}^n c_j\overline{c_k}\varphi(s_k^*s_j)\ge 0$ for all $n\in\mathbb{N}$, $s_1,\dots,s_n\in S$, and $c_1,\dots,c_n\in\mathbb{C}$. Let $\mathcal{P}(S)$ be the set of all such functions. A *character* on S is a function $\sigma\colon S\to\mathbb{C}$, not identically zero, such that $\sigma(s^*) = \overline{\sigma(s)}$ and $\sigma(st) = \sigma(s)\sigma(t)$ for all $s,t\in S$. Denote by S^* the set of all such functions. For a measure μ on S^*, we define

$$\mathcal{L}\mu(s) = \int_{S^*} \sigma(s)\,d\mu(\sigma), \quad s\in SS$$

whenever all the integrals exist (which is the case if and only if for each $s\in S$ the function $\sigma\mapsto\sigma(s)\colon S^*\to\mathbb{C}$ is in $L^2(\mu)$). A function $\varphi\colon SS\to\mathbb{C}$ is a *moment function* if $\varphi = \mathcal{L}\mu$ for some measure μ on S^*, and a moment function φ is *determinate* if there is only one such μ among measures defined on the least σ-ring of subsets of S^* rendering $\sigma\mapsto\sigma(s)$ measurable for each $s\in S$. (Halmos calls a function f measurable with respect to a σ-ring $\mathcal{A}$ if $f^{-1}(B)\in\mathcal{A}$ for each Borel set in $\mathbb{C}\setminus\{0\}$. The condition is equivalent to the condition that the set $A = \{\, x \mid f(x)\ne 0\,\}$ is in $\mathcal{A}$ and the function $f|A$ is measurable with respect to the σ-field consisting of those members of $\mathcal{A}$ which are contained in A. We prefer the σ-ring to the σ-field because we want uniqueness of μ (given φ) to be as likely as possible. Note, however, that if S has an identity 1 (as it does in the last Corollary in the paper) then the σ-ring is a σ-field since the function $\sigma\mapsto\sigma(1)\equiv 1$ is measurable.) Denote by $\mathcal{H}(S)$ the set of all moment functions, and by $\mathcal{H}_D(S)$ the subset of determinate ones. We have $\mathcal{H}(S)\subset\mathcal{P}(S)$ since

$$\sum_{j,k=1}^n c_j\overline{c_k}\mathcal{L}\mu(s_k^*s_j) = \int_{S^*}\left|\sum_{j=1}^n c_j\sigma(s_j)\right|^2 d\mu(\sigma)\ge 0.$$

The $*$-semigroup S is *semiperfect* if $\mathcal{H}(S) = \mathcal{P}(S)$, and *perfect* if we even have $\mathcal{H}_D(S) = \mathcal{P}(S)$.

For a study of moment functions on semigroups, we refer to Berg, Christensen, and Ressel [3], especially Chapter 6. For more recent developments, see the review by Berg [1].

Abelian groups with the inverse involution ($s^* = s^{-1}$) are perfect by the discrete version of the Bochner-Weil Theorem. More generally, a $*$-semigroup S is perfect if it is $*$-*divisible* in the sense that for each $s\in S$ there exist $t\in S$ and $m,n\in\mathbb{N}_0$ such that $m+n\ge 2$ and $s = t^m t^{*n}$ ([11], [12]). It has been discovered recently that the last equation can be replaced by $s^*s = s^*t^m t^{*n}$.

The semigroup $(\mathbb{N}_0,+)$ with its unique involution, the identity, is semiperfect by Hamburger's Theorem [13] but is not perfect since there exist indeterminate moment sequences, such as the example $n\mapsto(4n+3)!$ given by Stieltjes [17].

For $k \geq 2$, the semigroup $(\mathbb{N}_0^k, +)$ is not semiperfect for any of its involutions. For the identical involution, see Berg, Christensen, and Jensen [2] or Schmüdgen [16]; for other involutions, see the example of $\mathbb{N}_0^2$ with the involution $(m, n)^* = (n, m)$ in [3], Chapter 6, and note that an arbitrary involution on $\mathbb{N}_0^k$ is given by the product of certain transpositions of pairwise disjoint pairs of elements of the standard basis of $\mathbb{N}_0^k$. (I.e., after a permutation of the set $\{1, \ldots, k\}$, if necessary, there is some k' such that $2k' \leq k$ and $(n_1, n_2, \ldots, n_k)^* = (n_2, n_1, n_4, n_3, \ldots, n_{2k'}, n_{2k'-1}, n_{2k'+1}, n_{2k'+2}, \ldots, n_k)$.)

Haviland [14] characterized k-dimensional moment sequences as those functions $\varphi\colon \mathbb{N}_0^k \to \mathbb{R}$ such that the linear form L on the polynomial algebra $\mathbb{R}[x_1, \ldots, x_k]$ defined by $L(x_1^{n_1} \ldots x_k^{n_k}) = \varphi(n_1, \ldots, n_k)$ is nonnegative on all nonnegative polynomials. For $k = 1$ the condition reduces to positive definiteness since a nonnegative polynomial in one variable is a sum of (two) squares of real polynomials. For $k \geq 2$ such a reduction is impossible since, as shown already by Hilbert [15], there are nonnegative polynomials that are not sums of squares of real polynomials.

We next discuss the extension of Haviland's criterion to arbitrary $*$-semigroups. The concepts introduced will be useful later on. For a $*$-semigroup S, let $\mathbb{C}[S]$ be the *semigroup ring*, that is, the space of finitely supported complex-valued functions on S considered with the multiplication $*$ (convolution) defined by $a * b(u) = \sum_{s,t \in S: st = u} a(s)b(t)$ for $a, b \in \mathbb{C}[S]$ and $u \in S$ and the involution $\widetilde{\ }$ defined by $\widetilde{a}(s) = \overline{a(s^*)}$ for $a \in \mathbb{C}[S]$ and $s \in S$. Then $\mathbb{C}[S]$ is a commutative complex $*$-algebra. The set of elements of the form δ_s $(s \in S)$ where $\delta_s(t) = \delta(s, t)$ (the Kronecker delta) is a linear basis of $\mathbb{C}[S]$ and a $*$-subsemigroup of the multiplicative $*$-semigroup of $\mathbb{C}[S]$ isomorphic to S itself. For every $*$-subsemigroup T of S, identify $\mathbb{C}[T]$ with the set of those elements of $\mathbb{C}[S]$ that vanish off T. Then $\mathbb{C}[T]$ is a $*$-subalgebra of $\mathbb{C}[S]$. The spaces $\mathbb{C}[SS]$ and $\mathbb{C}^{SS}$ are in duality under the bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle a, \varphi \rangle = \sum_s a(s)\varphi(s)$ for $a \in \mathbb{C}[SS]$ and $\varphi \in \mathbb{C}^{SS}$. We consider $\mathbb{C}[SS]$ with the finest locally convex topology, and $\mathbb{C}^{SS}$ with the topology of pointwise convergence. Each topology is compatible with the duality, cf. [3], Chapter 1. For every subset A of $\mathbb{C}[SS]$, define a closed convex cone $A^\perp$ in $\mathbb{C}^{SS}$ as the set of those $\varphi \in \mathbb{C}^{SS}$ such that $\langle a, \varphi \rangle \geq 0$ for all $a \in A$. Similarly with the two spaces interchanged. Let $\Sigma(S)$ be the convex cone in $\mathbb{C}[SS]$ generated by elements of the form $\widetilde{a} * a$ with $a \in \mathbb{C}[S]$. Then clearly

$$\mathcal{P}(S) = \Sigma(S)^\perp. \tag{1}$$

Moment functions are harder to characterize. Indeed, let Ω_S^0 be the set of all positive multiples of characters on S and let Ω_S be the closure of Ω_S^0 in $\mathbb{C}^S \setminus \{0\}$. By [8], equation (2), we have

$$\Omega_S \setminus \Omega_S^0 \subset \overline{\mathcal{H}(S)} \setminus \mathcal{H}(S),$$

so if $\Omega_S \neq \Omega_S^0$ then the convex cone $\mathcal{H}(S)$ is not closed and therefore impossible to characterize as $A^\perp$ for some set A. Call S adapted if $\Omega_S = \Omega_S^0$. As shown in [10], S is adapted if and only if for each $s \in S$ there is some $n \in \mathbb{N}$ such that $(s^*s)^n \in \overbrace{S \ldots S}^{2n+1}$.

Since the equivalence is unpublished, in the present paper we take the last condition as the definition of adaptedness. If S is adapted then

$$\mathcal{H}(S) = (S^*)^{\perp\perp}. \tag{2}$$

We shall not make any use of this fact, which is shown in [10]. Note, however, that if S has an identity then it is [6], Proposition 3, item (ii). For finitely generated $*$-semigroups with identity such that S^* separates points, (2) is in [3], Section 6.1. Now Haviland's result is (2) applied to $S = \mathbb{N}_0^k$.

If T is a $*$-subsemigroup of S, say that $\varphi\colon TT \to \mathbb{C}$ is *positive definite with respect to S* if

$$\varphi \in \bigl(\Sigma(S) \cap \mathbb{C}[TT]\bigr)^{\perp}.$$

In more plain terms, φ is positive definite with respect to S if and only if

$$\sum_{k=1}^{n} \langle \widetilde{a}_k * a_k, \varphi \rangle \geq 0$$

whenever $a_1, \ldots, a_n \in \mathbb{C}[S]$ are such that $\sum_{k=1}^{n} \widetilde{a}_k * a_k$ is supported by TT. In even plainer terms, φ is positive definite with respect to S if and only if

$$\sum_{k=1}^{n} \sum_{i,j=1}^{m} c_{k,i}\overline{c_{k,j}}\varphi(s_j^* s_i) \geq 0$$

for all $m, n \in \mathbb{N}$, $s_i \in S$, and $c_{k,i} \in \mathbb{C}$ $(i = 1, \ldots, m,\ k = 1, \ldots, n)$ such that

$$\sum_{k=1}^{n} \sum_{i,j : s_j^* s_i = u} c_{k,i}\overline{c_{k,j}} = 0 \quad \text{if } u \notin TT.$$

For a fourth way of stating the condition, see the paper by Stochel and Szafraniec [18]. Say that T is an *extending subsemigroup of S* if every function on TT which is positive definite with respect to S extends to a positive definite function on S.

For every subset X of a set that carries an involution, denote by X_h the set of those elements of X that are *hermitian* in the sense of being invariant under the involution. Stochel and Szafraniec [18] showed that if $S_h \subset T$ then

$$\mathbb{C}[SS]_h \subset \mathbb{C}[TT]_h + \Sigma(S) \tag{3}$$

and that if this latter inclusion holds then T is an extending subsemigroup of S.

We shall need the fact that a different set of conditions is sufficient for (3) to hold. Call a $*$-semigroup S $\mathbb{C}$-*separative* if S^* separates points in S. If the problem of characterizing moment functions on $\mathbb{C}$-separative semigroups can be solved then one has solved the same problem without $\mathbb{C}$-separativity. Indeed, let χ be the quotient mapping of S onto its greatest $\mathbb{C}$-separative $*$-homomorphic image, that is, the quotient $*$-semigroup $S/{\sim}$ where $\sim$ is the congruence in S defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for all $\sigma \in S^*$. Then a function $\varphi\colon SS \to \mathbb{C}$ is a moment function if and only if, firstly, φ factors via χ, i.e., $\varphi = \Phi \circ \chi$ for some function Φ on $\chi(S)\chi(S)$, and secondly, Φ is a moment function. Thus, it suffices to consider $\mathbb{C}$-separative semigroups. (This is said with the reservation that in order to determine whether φ factors via χ one needs to be able to decide, given $s, t \in S$, whether $s \sim t$.)

For every subset V of SS, denote by $E(V)$ the set of those $v \in V$ such that if $s, t \in S$, $s^*s, t^*t \in V$, and $t^*s = v$ then $s = t$. For every subset U of SS, denote by $C(U)$ the union of all finite subsets V of SS such that $E(V) \subset U$. The letters E and C should remind the reader of extreme points and convex hull. The analogy should not be overemphasized. For example, if S is $(\mathbb{N}_0^2, +)$ with the identical involution and if $U = \{(0,0),(4,2),(2,4)\}$ then the point $v = (2,2)$, which is the midpoint of the triangle with corners in U, is not in $C(U)$. In fact, if we define $V = U \cup \{v\}$ then $v \in E(V)$ although v is not an extreme point of V in any reasonable sense. For a list of elementary properties of the mappings E and C, see [7], Theorem 2. In particular, C is a closure operation, i.e., if U and V are subsets of SS then $U \subset C(U)$, if $U \subset V$ then $C(U) \subset C(V)$, and finally $C\big(C(U)\big) = C(U)$.

We shall show that if S is $\mathbb{C}$-separative and if T is a $*$-subsemigroup of S such that $SS = C(TT)$ then T is an extending subsemigroup of S. The condition $SS = C(TT)$ is satisfied if $S_h \subset T$ since for an arbitrary $*$-semigroup S one has $SS \subset C(\{\, s^*s \mid s \in S\,\}) \subset C\big((SS)_h\big)$, cf. [7], Theorem 2, where item 9 has a natural $*$-analogue.

Our main aim is to characterize moment functions. Given an adapted $\mathbb{C}$-separative semigroup S, we shall construct a perfect semigroup Q containing S such that every moment function on SS extends to a moment function on Q. For semigroups with identity, the prescription is simple. Indeed, let N be any subsemigroup of the multiplicative semigroup of odd natural numbers such that $1 \in N$ and $N \neq \{1\}$. For $n \in N$ define $f_n\colon S \to \mathbb{C}^{S^*}$ by

$$f_n(s)(\sigma) = \sigma(s)|\sigma(s)|^{1/n-1}$$

with the convention $0|0|^{1/n-1} = 0$. Then the set

$$R = \bigcup_{n \in N} f_n(S)$$

is a $*$-semigroup when considered with pointwise multiplication and pointwise complex conjugation, and the mapping f_1 is an embedding of S into R. If S has an identity, just take $Q = R$, suppressing the embedding f_1. In this case, S turns out to be an extending subsemigroup of Q, so a function on SS is a moment function if and only if it is positive definite with respect to Q. The assumption that S have an identity can be replaced by the assumption that for each $s \in S$ there is some $e \in S$ (*not* required to satisfy $e^2 = e$) such that $s = es$.

In the general case, for every subset U of R define the *convex hull of U in R* to be the set of those $r \in R$ such that there exist $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$ such that $|r|^{n+1} = |ru_1 \ldots u_n|$. Now let Q be the convex hull of S in R. In this case, for reasons of proof technique we shall have to assume that N is the set of all odd natural numbers. Then Q is perfect, and every moment function on S extends to a moment function on Q. Unfortunately, in the case that S has no identity we have not been able to show that S is an extending subsemigroup of Q.

At the end of the paper, we shall indicate the form taken by our main result in case $S = \mathbb{N}_0^k$ with the identical involution.

2. A sufficient condition for a subsemigroup to be extending

The purpose of the present section is to prove the following result.

Theorem 1

If S is a $\mathbb{C}$-separative semigroup and T is a $*$-subsemigroup of S such that $SS = C(TT)$ then T is an extending subsemigroup of S.

Proof. If we prove (3) then it follows that T is extending. So suppose $a \in \mathbb{C}[SS]_h$; we have to show $a \in \mathbb{C}[TT]_h + \Sigma(S)$. Since a is hermitian then the restriction of a to any set of the form $\{u, u^*\}$ with $u \in SS$ is again hermitian. Since a is the sum of finitely many such restrictions, it suffices to consider one of these. Thus we may assume that a is supported by $\{u, u^*\}$ for some $u \in SS$. Then $a = \lambda\delta_u + \overline{\lambda}\delta_{u^*}$ for some $\lambda \in \mathbb{C}$. Since $u \in SS$ then we can choose $s, t \in S$ such that $u = t^*s$. The element $b = (\overline{\lambda}\delta_{s^*} + \delta_{t^*}) * (\lambda\delta_s + \delta_t)$ is in $\Sigma(S)$ and is equal to $|\lambda|^2\delta_{s^*s} + \delta_{t^*t} + a$. If we show that $a - b$ is in the right-hand side of (3) then we are done. This can be done by showing that $-\delta_{t^*t}$ is in the right-hand side of (3) since the case of the other term is similar (the right-hand side of (3) being a convex cone).

Since $t^*t \in SS = C(TT)$ then by the definition of C there is a finite subset V of SS such that $t^*t \in V$ and $E(V) \subset TT$. Let W be the set of those elements of V which can be written in the form v^*v with $v \in S$. For every subset A of SS, let $P(A)$ be the set of those $c \in \mathbb{C}[SS]$, supported by A, such that $c(x) \geq 0$ for all x and $\sum_x c(x) = 1$. It should cause no confusion if we call elements of $P(A)$ 'probability measures'. Let $\mathcal{A}$ be the set of those subsets A of W such that for each $x \in W$ there is some $c \in P(A)$ such that $c - \delta_x \in \Sigma(S)$.

It suffices to show that there is some $A \in \mathcal{A}$ such that $A \subset E(V)$. Indeed, choose such an A. Since $t^*t \in W$ then there is some $c \in P(A)$ such that $c - \delta_{t^*t} \in \Sigma(S)$. Then we can write $-\delta_{t^*t} = (-c) + (c - \delta_{t^*t}) \in \mathbb{C}[TT]_h + \Sigma(S)$ where we used the fact that c is supported by the set $A \subset E(V) \subset TT$, which consists of hermitian elements, and that c is real-valued, hence hermitian.

We first note that W itself is in $\mathcal{A}$ since, given any $x \in W$, as the desired c we can use $c = \delta_x$. Thus $\mathcal{A} \neq \emptyset$.

Since the finite set W has only finitely many subsets then we can choose $A \in \mathcal{A}$ minimal with respect to the inclusion ordering. If $A \subset E(V)$ then we are done. Suppose $A \not\subset E(V)$; we shall derive a contradiction. Choose $v \in A \setminus E(V)$ and define $B = A \setminus \{v\}$. We shall show that $B \in \mathcal{A}$, contradicting the minimality of A.

Suppose $x \in W$; we have to show that there is some $c \in P(B)$ such that $c - \delta_x \in \Sigma(S)$. Since $A \in \mathcal{A}$ then there is some $d \in P(A)$ such that

$$d - \delta_x \in \Sigma(S). \tag{4}$$

Since d is supported by the set $A = B \cup \{v\}$ then we can write

$$d = e + \alpha\delta_v \tag{5}$$

where e is nonnegative-valued and supported by B and where $\alpha \geq 0$.

Since $v \notin E(V)$ then there exist $p, q \in S$ such that $p^*p, q^*q \in V$, $q^*p = v$, and $p \neq q$. Now

$$\frac{1}{2}\big(\delta_{p^*p} + \delta_{q^*q}\big) - \delta_v = \frac{1}{2}\big(\delta_{p^*} - \delta_{q^*}\big) * \big(\delta_p - \delta_q\big) \in \Sigma(S). \tag{6}$$

If p^*p and q^*q are not both equal to v then this shows that there is some $f \in P(W)$ such that

$$f \neq \delta_v \quad \text{and} \quad f - \delta_v \in \Sigma(S). \tag{7}$$

But p^*p and q^*q cannot be both equal to v. Indeed, suppose they are. Since v is hermitian then $v = q^*p = p^*q$. Thus $p^*p = q^*p = p^*q = q^*q$. Applying to these identities an arbitrary $\sigma \in S^*$, we obtain $|\sigma(p)|^2 = \overline{\sigma(q)}\sigma(p) = \overline{\sigma(p)}\sigma(q) = |\sigma(q)|^2$. It is trivial to verify that if z and w are complex numbers such that $|z|^2 = \overline{w}z = \overline{z}w = |w|^2$ then $z = w$. Thus $\sigma(p) = \sigma(q)$. This being so for all $\sigma \in S^*$, since S is $\mathbb{C}$-separative it follows that $p = q$, a contradiction. Thus we have (7).

For each y in the support of f, since $A \in \mathcal{A}$ we can find $g_y \in P(A)$ such that $g_y - \delta_y \in \Sigma(S)$. Now the element $g = \sum_y f(y)g_y$ is in $P(A)$ and satisfies

$$g - f \in \Sigma(S). \tag{8}$$

Since g is supported by the set $A = B \cup \{v\}$ then we can write

$$g = h + \beta\delta_v \tag{9}$$

where h is nonnegative-valued and supported by B and where $\beta \geq 0$.

Since g is a probability measure then $\beta \leq 1$. Suppose $\beta = 1$. Again since g is a probability measure then $h = 0$, so $g = \delta_v$. Now (7) and (8) say that $f - g$ and $g - f$ are both in $\Sigma(S)$. If $\sigma \in S^*$ then we have $\langle f - g, \sigma\rangle + \langle g - f, \sigma\rangle = 0$. The terms are nonnegative since a character is, in particular, a positive definite function, cf. (1). Thus $\langle f - g, \sigma\rangle = 0$. Since S is $\mathbb{C}$-separative then the mapping $k \mapsto (\langle k, \sigma\rangle)_{\sigma \in S^*} : \mathbb{C}[SS] \to \mathbb{C}^{S^*}$ is one-to-one, cf. [3], Section 6.1. Thus we can infer $f = g$. That is, $f = \delta_v$, contradicting (7). This proves $\beta < 1$.

Adding (7) and (8) and using (9), we get

$$h - (1 - \beta)\delta_v \in \Sigma(S). \tag{10}$$

Since g is a probability measure then (9) shows that the total mass of h is just $1 - \beta$. Thus, multiplying (10) by $(1 - \beta)^{-1}$ we find some $k \in P(B)$ such that $k - \delta_v \in \Sigma(S)$. Multiplying by α and adding (4), we obtain $c - \delta_x \in \Sigma(S)$ where $c = e + \alpha k$, cf. (5). Now (5) shows that the total mass of e is just $1 - \alpha$. Thus $c \in P(B)$. This completes the proof. □

3. An enveloping perfect semigroup

Throughout this section, S denotes a $*$-semigroup. We shall later make further assumptions that will then hold throughout the rest of the section. Define R as in the Introduction. It was shown in [4] that R is a $*$-semigroup. More precisely, this was shown only in case N is the set of all odd natural numbers. For the general case, note that it was shown in [4] that $f_n(S)$ is $*$-stable and $f_m(S)f_n(S) \subset f_{mn}(S)$. It was also shown in [4] that R is $*$-divisible, hence perfect. It was shown in [4] and [5] that the characters on R are just the functions $r \mapsto r(\sigma)$ with $\sigma \in S^*$. It was shown in [5] that if S has an identity then those measures on S^* whose $\mathcal{L}$-transforms are defined as functions on R are the same as those whose $\mathcal{L}$-transforms are defined as functions on S. We need a similar result in the case that S is only assumed to be adapted. An adapted semigroup need not have an identity, hence the need to introduce the semigroup Q. The problem can be illustrated with the semigroup $S = (]1, \infty[\,, +)$, which is adapted. In this case, $R =]0, \infty[$, and it is easy to define a moment function on S that does not extend to a moment function on R.

Now assume that S is $\mathbb{C}$-separative, and suppress the embedding f_1. Then the mapping f_n $(n \in N)$ is given by $f_n(s) = s|s|^{1/n-1}$ for $s \in S$. Define Q as in the Introduction.

Theorem 2

The set Q is a $*$-subsemigroup of R containing S.

Proof. Clearly Q is $*$-stable. To see that Q is stable under multiplication, suppose $q, r \in Q$; we have to show $qr \in Q$. Choose $j \in \mathbb{N}$ and $s_1, \ldots, s_j \in S$ such that $|q|^{j+1} = |qs_1 \ldots s_j|$. Choose $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in S$ such that $|r|^{k+1} = |rt_1 \ldots t_k|$. With $n = jk$, we may assume that $j = k = n$. (Replace the j-tuple $(s_1, \ldots, s_j)$ by an n-tuple in which each s_i is repeated k times.) With $u_i = s_i t_i$ we have $|qr|^{n+1} = |qru_1 \ldots u_n|$, proving $qr \in Q$. Thus Q is a $*$-subsemigroup of R. To see that it contains S, suppose $s \in S$; we have to show $s \in Q$. With $n = 1$ and $s_1 = s$ we have $|s|^{n+1} = |s|^2 = |ss_1| = |ss_1 \ldots s_n|$, proving $s \in Q$. □

An *ideal* of a $*$-semigroup X is a nonempty $*$-stable subset H of X such that $XH \subset H$. In particular, $HH \subset H$, so H is a semigroup.

A $*$-semigroup H is *quasi-perfect* if for each $\varphi \in \mathcal{P}(H)$ there is a unique measure μ on H^* (on the σ-ring mentioned in the Introduction) such that $\varphi(h) = \mathcal{L}\mu(h)$ for $h \in HHH$. Every perfect semigroup is quasi-perfect. Every ideal of a quasi-perfect semigroup is quasi-perfect. For both facts, see [12].

Theorem 3

The semigroup Q is an ideal of R, hence quasi-perfect.

Proof. Since Q is a $*$-semigroup then it is, in particular, a nonempty $*$-stable set. Thus we only have to show $RQ \subset Q$. Suppose $r \in R$ and $q \in Q$; we have to show $rq \in Q$. Since $r \in R$ then there exist $n \in N$ and $s \in S$ such that $r = s|s|^{1/n-1}$. Hence $|r| = |s|^{1/n}$. Since $q \in Q$ then there exist $k \in \mathbb{N}$ and $s_1, \ldots, s_k \in S$ such that $|q|^{k+1} = |qs_1 \ldots s_k|$. By induction it follows that $|q|^{nk+1} = |q(s_1 \ldots s_k)^n|$. Now

$$|rq|^{nk+1} = |r|^{nk+1}|q|^{nk+1} = |r|\,|s|^k|q(s_1 \ldots s_k)^n| = |rqt_1 \ldots t_{nk}|$$

where $t_i = ss_i$ and $t_{mk+i} = s_i$ for $m = 1, \ldots, n-1$ and $i = 1, \ldots, k$. Thus $rq \in Q$. This completes the proof. □

A $*$-semigroup H is *flat* if every positive definite function on H which vanishes on HHH vanishes identically. A $*$-semigroup is perfect if and only if it is quasi-perfect and flat [9].

Now assume that S is adapted. Because of problems with number theory, we also now assume that N is the set of all odd natural numbers (though we have no indication that such an assumption should be necessary for the truth of the next result).

Theorem 4

The semigroup Q is flat, hence perfect.

Proof. Suppose φ is a positive definite function on Q that vanishes on QQQ; we have to show that φ vanishes identically. For $u \in QQ$ we can choose $s, t \in Q$ such that $u = t^*s$. By the Cauchy-Schwarz inequality, $|\varphi(u)|^2 \leq \varphi(s^*s)\varphi(t^*t)$. Thus it suffices to show $\varphi(q^*q) = 0$ for $q \in Q$. Since $q \in Q$ then there exist $k \in \mathbb{N}$ and $s_1, \ldots, s_k \in S$ such that $|q|^{k+1} = |qs_1 \ldots s_k|$. For each i, since S is adapted there is some $n \in \mathbb{N}$ such that $(s_i^*s_i)^n \in \overbrace{S \ldots S}^{2n+1}$, that is, $(s_i^*s_i)^n = t_{i,1} \ldots t_{i,2n+1}$ for some $t_{i,j} \in S$ $(j = 1, \ldots, 2n+1)$. This condition for some n obviously implies the corresponding condition for all greater n. Thus we may assume that we have the same n for all i, and we still are at liberty to take any greater n. Now

$$|q|^{2nk+1} = |q(s_1 \ldots s_k)^{2n}| = |qt_{1,1} \ldots t_{1,2n+1} \ldots t_{k,1} \ldots t_{k,2n+1}|.$$

Writing $m = 2nk + 1$, we have an odd integer m such that $2nk < m \leq (2n+1)k$. Collecting some of the factors $t_{i,j}$ into one, if necessary, we find $u_1, \ldots, u_m \in S$ such that $|q|^{2nk+1} = |qu_1 \ldots u_m|$. Since $q \in R$ then there exist $p \in N$ and $v \in S$ such that $q = v|v|^{1/p-1}$. Now define $w = (v^*v)^{nk} \in S$ and $r = w|w|^{1/mp-1} \in R$. Then $|r| = |w|^{1/mp} = |v|^{2nk/mp}$. But $|q| = |v|^{1/p}$, so $|r| = |q|^{2nk/m}$. Now

$$|q|\,|r|^m = |q|^{2nk+1} = |qu_1 \ldots u_m| = |q|\,|u_1 \ldots u_m|.$$

Since the functions q and r have the same support, it follows that $|r|^{m+1} = |ru_1 \ldots u_m|$. Hence $r \in Q$. We now define an infinite sequence $(r_i)_{i=0}^{\infty}$ in Q, informally by $r_i = r|q/r|^i$ for even i and $r_i = (q/r)|q/r|^{i-1}$ for odd i. Since the function r need not be invertible, the formal definition supplements the informal one by stating that these functions vanish off the support of r (which is the same as the support of q). Then $r_i \in Q$ for all $i \in \mathbb{N}_0$. To see this, since we have already established that $r \in Q$, and since Q is an ideal of R, then it suffices to verify that the element q/r, which we formally define by

$$(q/r)(\sigma) = \begin{cases} q(\sigma)/r(\sigma) & \text{if } r(\sigma) \neq 0 \ \big(\text{or equivalently, } q(\sigma) \neq 0\big) \\ 0 & \text{otherwise,} \end{cases}$$

is in R. Since $q = v|v|^{1/p-1}$ and $r = w|w|^{1/mp-1}$ where $w = (v^*v)^{nk}$ then

$$q/r = v|v|^{1/p-2nk/mp-1} = v|v|^{1/mp-1}.$$

But (recalling that we now assume that N is the set of all odd natural numbers) this shows the desired fact. Thus all the r_i are in Q.

The rest of the proof is now easy. Indeed, for $i = 1, 2, \ldots$ the element $|r_i|^2$ is equal to $r_{i-1}^* r_{i+1}$ or its adjoint according as i is even or odd, so by the Cauchy-Schwarz inequality, $\varphi(|r_i|^2)^2 \leq \varphi(|r_{i-1}|^2 \varphi(|r_{i+1}|^2)$. By induction it follows that if $\varphi(|r_i|^2)$ vanishes for some i then it vanishes for all positive i and in particular for $i = 1$, that is, $\varphi(|q|^2) = 0$, as desired. Thus it suffices to show that $\varphi(|r_i|^2)$ vanishes for some i. Since φ vanishes on QQQ by hypothesis then it suffices to show that $|r_i|^2 \in QQQ$ for some i. A short computation shows $|r_i|^2 = (v^*v)^{(2nk+i)/mp}$, so if we take $i = 2p + 2nk(2p-1)$ then we have $|r_i|^2 = (v^*v)^2 \in SSSS \subset SSS \subset QQQ$, as desired. This completes the proof. □

Theorem 5

Every moment function on S extends to a moment function on Q and so is positive definite with respect to Q.

Proof. Suppose φ is a moment function on S; we have to show that φ extends to a moment function on Q. Choose a measure μ on S^* such that $\varphi = \mathcal{L}\mu$. For $\sigma \in S^*$ the function $q \mapsto q(\sigma)$ is a character on Q. Thus, if we can define a function Φ on QQ by $\Phi(q) = \int q \, d\mu$ then Φ is an extension as desired. We only have to verify that if $q \in Q$ then $q \in L^2(\mu)$. Since $q \in Q$ then we can choose $n \in \mathbb{N}$ and $s_1, \ldots, s_n$ such that $|q|^{n+1} = |qs_1 \ldots s_n|$. Hence, on the support of q we have $|q| = \sqrt[n]{|s_1 \ldots s_n|}$, so the desired fact follows by Hölder's inequality.

Since Φ is a moment function then it is positive definite. For $a \in \Sigma(Q) \cap \mathbb{C}[SS]$ we have $\langle \varphi, a \rangle = \langle \Phi, a \rangle \geq 0$. Thus φ is positive definite with respect to Q. □

4. The main result

Only two Lemmas are needed before we can prove our main result. We continue with some of the assumptions from the preceding section, viz., S is a $\mathbb{C}$-separative semigroup, N is a subsemigroup of the multiplicative semigroup of odd natural numbers such that $1 \in N$ and $N \neq \{1\}$, and Q is defined as in the Introduction. We do not assume that

S is adapted since we shall introduce a stronger assumption. We assume that S is of *class* $\mathcal{M}$, that is, for each $s \in S$ there exist $e \in S$ and $n \in \mathbb{N}$ such that

$$(s^*s)^n = e(s^*s)^n. \tag{11}$$

It follows that $s = es$. Indeed, since S is $\mathbb{C}$-separative then we only have to verify $\sigma(s) = \sigma(es)$ for $\sigma \in S^*$. Since $\sigma(es) = \sigma(e)\sigma(s)$ then this is trivial if $\sigma(s) = 0$. Thus we may assume $\sigma(s) \neq 0$. We then have to show $\sigma(e) = 1$. But this follows by applying σ to (11) and dividing by the nonzero number $|\sigma(s)|^{2n}$. Similarly, $s^* = es^*$.

Note that even if S were not assumed to be $\mathbb{C}$-separative, (11) would show $(s^*s)^n \in \overbrace{S \ldots S}^{2n+1}$, proving that S is adapted.

Lemma 1

$Q = R$.

Proof. Suppose $r \in R$; we have to show $r \in Q$. Since $r \in R$ then we can choose $s \in S$ and $n \in N$ such that $r = s|s|^{1/n-1}$. Since S is of class $\mathcal{M}$ then there is some $e \in S$ such that $s = es$ and $s^* = es^*$. With $s_1 = \cdots = s_{n-1} = e$ and $s_n = s$ we now have $|r|^{n+1} = |r|\,|r|^n = |r|\,|s| = |rs_1 \ldots s_n|$, showing $r \in Q$. □

Lemma 2

The semigroup S is an extending subsemigroup of Q.

Proof. Since S is $\mathbb{C}$-separative then it suffices to show $QQ = C(SS)$. One inclusion being trivial, we only have to show $QQ \subset C(SS)$. For $u \in QQ$ we can choose $s, t \in Q$ such that $u = t^*s$. By the $*$-analogue of [7], Theorem 2, item 9, we have $t^*s \in C(\{s^*s, t^*t\})$. Hence, if we show that all elements of QQ of the form q^*q for some $q \in Q$ are in $C(SS)$ then it follows that the latter set contains QQ. (Indeed, from $\{s^*s, t^*t\} \subset \{\, q^*q \mid q \in Q \,\}$ we can infer $C(\{s^*s, t^*t\}) \subset C(\{\, q^*q \mid q \in Q \,\})$ since C is a closure operation. By the same token, from $\{\, q^*q \mid q \in Q \,\} \subset C(SS)$ we can infer $C(\{\, q^*q \mid q \in Q \,\}) \subset C(C(SS)) = C(SS)$.)

So suppose $q \in Q$; we have to show $q^*q \in C(SS)$. Since $q \in R$ then we can choose $s \in S$ and $n \in N$ such that $q = s|s|^{1/n-1}$. Since S is of class $\mathcal{M}$ then there is some $e \in S$ such that $s = es$ and $s^* = es^*$. Hence, if we define $q^0 = {q^*}^0 = e$ then the laws of powers continue to hold when zeroth powers are included, except that it is not certain that $q^{0+0} = q^0q^0$, an identity that we shall not need. Now define

$$V = \{\, (q^*q)^k \mid k = 0, \ldots, n \,\}.$$

We claim that if $1 \leq k \leq n-1$ then $(q^*q)^k \notin E(V)$. To see that this is so, define $x = q^{k-1}$ and $y = q^kq^*$. Then $x^*x = (q^*q)^{k-1} \in V$, $y^*y = (q^*q)^{k+1} \in V$, and $y^*x = ({q^*}^kq)(q^{k-1}) = (q^*q)^k$. To conclude that $(q^*q)^k \notin E(V)$, we now need only verify that $x \neq y$. There is an exceptional case that will be treated at the end of the proof. Suppose $x = y$, that is, $q^{k-1} = q^kq^*$. It follows that on the support of q

the function q^*q is equal to 1. This is the exceptional case that will be treated later. Disregarding it for the present, since $E(V) \subset V$ (by the definition of E) then we have shown $E(V) \subset \{(q^*q)^0, (q^*q)^n\} = \{e^*e, s^*s\} \subset SS$, and since V is a finite set then by the definition of C it follows that $V \subset C(SS)$. Since in particular $q^*q = (q^*q)^1 \in V$ then it follows that $q^*q \in C(SS)$, as desired.

We can now complete the proof by considering the exceptional case, which is the case that the function q^*q is the identity on the support of q (which is the same as the support of s). But then

$$q^*q = q^*q(q^*q)^{n-1} = (q^*q)^n = s^*s \in SS \subset C(SS)$$

since C is a closure operation. This completes the proof. □

Theorem 6

A function $\varphi: SS \to \mathbb{C}$ is a moment function if and only if it is positive definite with respect to Q.

Proof. We have already seen that every moment function on S is positive definite with respect to Q. Conversely, suppose φ is positive definite with respect to Q. Since S is an extending subsemigroup of Q then φ extends to a positive definite function on Q, say Φ. Now Q is perfect. In the preceding section, this was shown only in the case that N is the set of all odd natural numbers. Note, however, that we now have $Q = R$ and that it was remarked earlier that it follows from published results that R is perfect for arbitrary N. Since Φ is positive definite and since Q is perfect then Φ is a moment function. Thus $\Phi = \mathcal{L}\nu$ for some measure ν on Q^*. We have not identified Q^*. We conjecture that it is just the set of functions of the form $q \mapsto q(\sigma)$ with $\sigma \in S^*$, but this will not be needed. Indeed, denoting by μ the image measure of ν under the mapping $\varrho \mapsto \varrho|S: Q^* \to S^*$, it is trivial to verify that $\varphi = \mathcal{L}\mu$. So φ is a moment function. This completes the proof. □

5. Application to the multidimensional moment problem

The purpose of the present section is to deduce the following result.

Corollary 1

For $k \in \mathbb{N}$, a function $\varphi: \mathbb{N}_0^k \to \mathbb{R}$ is a moment sequence if and only if the linear form L on $\mathbb{R}[x_1, \ldots, x_k]$ defined by $L(x_1^{n_1} \ldots x_k^{n_k}) = \varphi(n_1, \ldots, n_k)$ is nonnegative on every polynomial p such that for some odd natural number m the polynomial $p(x_1^m, \ldots, x_k^m)$ is a sum of squares of real polynomials.

Proof. We are indebted to the anonymous referee for the suggestion to prove the easy part independently. Suppose φ is a moment sequence, that is, there is a positive measure μ on $\mathbb{R}^k$, with moments of all orders, such that

$$\varphi(n_1, \ldots, n_k) = \int_{\mathbb{R}_k} x_1^{n_1} \ldots x_k^{n_k} \, d\mu(x_1, \ldots, x_k)$$

for all $(n_1, \ldots, n_k) \in \mathbb{N}_0^k$. Then L is given by $L(p) = \int_{\mathbb{R}^k} p \, d\mu$ for each polynomial p. Suppose p is a real polynomial in k variables such that for some odd natural number m the polynomial $p(x_1^m, \ldots, x_k^m)$ is a sum of squares of real polynomials. In particular, the latter polynomial is nonnegative on $\mathbb{R}^k$. But the mapping $(x_1, \ldots, x_k) \mapsto (x_1^m, \ldots, x_k^m)$ is a bijection of $\mathbb{R}^k$ onto itself, the inverse being $(x_1, \ldots, x_k) \mapsto (x_1^{1/m}, \ldots, x_k^{1/m})$ which is well-defined since m is odd. Thus $p \geq 0$ everywhere, so $L(p) = \int p \, d\mu \geq 0$.

Let S be the semigroup $(\mathbb{N}_0^k, +)$ considered with the identical involution. Then φ is a moment sequence if and only if it is a moment function on S. With N equal to the set of all odd natural numbers, the semigroup Q defined as in the Introduction can be identified with $\{\, n/m \mid n \in S,\, m \in N \,\}$. Now φ is a moment function if and only if it is positive definite with respect to Q. We can identify $\mathbb{C}[Q]$ with the algebra of 'polynomials with fractional exponents', i.e., linear combinations of the 'fractional monomials' $x_1^{n_1/m} \ldots x_k^{n_k/m}$ where $(n_1, \ldots, n_k) \in S$ and $m \in N$. Since Q carries the identical involution then $\mathbb{C}[Q]_h = \mathbb{R}[Q]$. Saying that φ is positive definite with respect to Q is equivalent to saying that L is nonnegative on every polynomial p which is a sum of squares of (real) fractional polynomials. But this condition on p is equivalent to the condition that for some $m \in N$ the polynomial $p(x_1^m, \ldots, x_k^m)$ is a sum of squares of ordinary polynomials. This is because each of the fractional polynomials whose squares enter into the sum is a linear combination of fractional monomials $x_1^{n_1/m'} \ldots x_1^{n_k/m'}$. Taking m to be a common multiple of the finitely many m' involved, we get the desired fact. □

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