Collect. Math. **52**, 2 (2001), 193–203

New variants of Khintchine's inequality

Ioan Şerb

Faculty of Mathematics and Informatics, "Babeş-Bolyai" University

3400 Cluj-Napoca, Romania

E-mail: ivserb@math.ubbcluj.ro

Received January 26, 2001

Abstract

Variants of Khintchine's inequality with coefficients depending on the vector dimension are proved. Equality is attained for different types of extremal vectors. The Schur convexity of certain attached functions and direct estimates in terms of the Haagerup type of functions are also used.

1. Introduction

Denote by $(r_n)_{n\geq 1}$ the sequence of Rademacher functions, defined by

$$r_n(t) = \operatorname{sign}(\sin 2^n \pi t), t \in [0,1], n = 1, 2, \ldots$$

Recall that the classical Khintchine inequality states that for any $p > 0$ there exist constants $A_p, B_p > 0$ such that

$$A_p \sqrt{\sum_{i=1}^n x_i^2} \leq \left\| \sum_{i=1}^n x_i r_i \right\|_{L_p} = \left(\int_0^1 \left| \sum_{i=1}^n x_i r_i(t) \right|^p dt \right)^{1/p} \leq B_p \sqrt{\sum_{i=1}^n x_i^2},$$

for $n \in \mathbb{N}$ and arbitrary $x_1, \ldots, x_n \in \mathbb{R}$, where $\|\cdot\|_{L_p}$ is the norm in $L_p(0,1)$. The problem to find the best possible constants appearing in the above inequalities has a long history; see, for instance, the survey paper [5] and the attached bibliography.

Supposing now that $\sum_{i=1}^n x_i^2 = 1$, it is easy to see that for some $p > 0$, the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_i = 1/\sqrt{n}, i = 1, \ldots, n$, is in a certain sense

Keywords: Khintchine's inequality, Schur convexity, Haagerup estimates for Rademacher averages.
MSC2000: 60E15, 26D15, 26B25.

extremal. Another extreme case is when $x_1 = 1,\ x_2 = ... = x_n = 0$. For $p > 0$ and $n \in \mathbb{N}$ fixed it is also of interest to maximize the difference

$$S_p(x) = \left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_p}^{p} - \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_p}^{p} \|x\|_2^p,$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\|\cdot\|_q$ is the norm in $\ell_q^n,\ q \in [1, \infty]$. Then it is desirable to obtain an inequality of the form $S_p(x) \le C_p(x)$, where $C_p(x)$ is a "simple" function and such that equality holds for extremal vectors $x = e_1 = (1, 0, ..., 0)$ and / or $x = (1/\sqrt{n}, ..., 1/\sqrt{n})$. Such a result was recently obtained by T. Figiel, P. Hitczenko, W.B. Johnson, G. Schechtman, J. Zinn [1] in the case $p \in (2, 3)$, (and in a more general case of Khintchine inequalities for a class of Orlicz functions). In particular one obtains [1, Theorem 4.2] that

$$\left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_p}^{p} - \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_p}^{p} \|x\|_2^p \le \|x\|_p^p - n^{1-p/2} \|x\|_2^p, \tag{1}$$

$p \in (2, 3)$, with equality for $x = (1/\sqrt{n}, ..., 1/\sqrt{n})$.

On the other hand, in order to obtain estimates for the projection constants of symmetric n-dimensional spaces, H. König, C. Schütt and N. Tomczak-Jaegermann [4] have used the following inequalities

$$\begin{aligned} -\Phi(1)\|x\|_\infty \le \left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_1} - \lim_{n\to\infty} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_1} \|x\|_2 \\ = \left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_1} - \sqrt{\frac{2}{\pi}}\, \|x\|_2 \le \left(1 - \sqrt{\frac{2}{\pi}}\right) \|x\|_\infty, \end{aligned} \tag{2}$$

where

$$\Phi(b) := \frac{2}{\pi} \int_{\pi/2}^{\infty} \frac{\cos t \cos(bt)}{t^2}\, dt,\ b \ge 0.$$

These inequalities are obtained by entirely different methods and equality in the second inequality is attained for $x = e_1 = (1, 0, ..., 0)$.

In this paper we obtain inequalities of type (1) (in the particular case $p = 1$) with equality for vectors of the form $x = (1/\sqrt{n}, ..., 1/\sqrt{n})$. Reverse inequalities are also considered. On the other hand an improved form of (2), with bounds depending on n, is given, i.e. we get

$$\left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_1} \le \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_1} \|x\|_2 + \left(1 - \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_1}\right) \|x\|_\infty. \tag{3}$$

Equality in (3) is attained for $x = e_1$ and (2) can be obtained from (3) as a limit case.

2. Variants of Khintchine's inequality and Schur convexity

Let $x = (x_1, ..., x_n)$ be a vector in $\mathbb{R}^n$ and let $x_1^*, ..., x_n^*$ be the components of x in decreasing order, $x_1^* \geq ... \geq x_n^*$. For $x, y \in \mathbb{R}^n$ we write $x \prec y$ if $\sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*$, $k = 1, ..., n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. A function $\varphi : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur convex* on D if $\varphi(x) \leq \varphi(y)$ whenever $x, y \in D$ and $x \prec y$, see [6]. Let us denote by D the subset of $\mathbb{R}^3$ given by $D = \{(x, y, a) : x \geq y \geq 0, a \geq 0\}$ and by $f : D \to \mathbb{R}_+$ the function

$$f(x, y, a) = |x + y + a| + |x - y + a| + |x + y - a| + |x - y - a|, (x, y, a) \in D.$$

In order to prove some variants of Khintchine's inequality we need

Lemma 2.1

The functions $g_i : D \to \mathbb{R}_+$, $i = 1, 2, 3$ *defined by*

$$g_1(x, y, a) = f(x, y, a) - 2y, \ (x, y, a) \in D,$$

$$g_2(x, y, a) = f(x, y, z) - 2x - 2y, \ (x, y, a) \in D,$$

$$g_3(x, y, a) = -f(x, y, z) + 4x, \ (x, y, a) \in D,$$

satisfy the property: for all $x, y, x_1, y_1, a > 0$, *with* $x \geq x_1 \geq y_1 \geq y$, *and* $x_1^2 + y_1^2 = x^2 + y^2$ *we have* $g_i(x, y, a) \geq g_i(x_1, y_1, a)$, $i = 1, 2, 3$.

Proof. We consider only the case $i = 1$. The other cases can be treated with similar arguments. It is sufficient to prove that the function $g_1^* : [0, \pi/4] \to \mathbb{R}_+$, given by $g_1^*(t) = g_1(r\cos t, r\sin t, a), t \in [0, \pi/4]$ is decreasing for any fixed $r, a > 0$. From the homogeneity of g_1 we may assume that $r = 1$. We have

$$g_1^*(t) = 2\cos t - 2\sin t + 2a + |\cos t + \sin t - a| + |\cos t - \sin t - a|, \ t \in [0, \pi/4],$$

$a > 0$. If $\cos t - \sin t \geq a$, then $g_1^*(t) = 4\cos t - 2\sin t$, which is decreasing on $[0, \arccos(\sqrt{2}a/2) - \pi/4]$, for $a < 1$. If $\cos t - \sin t < a$ and $\cos t + \sin t \geq a$, then $g_1^*(t) = 2\cos t + 2a$, is also a decreasing function on any subinterval of $[0, \pi/4]$. Finally, if $\cos t + \sin t < a$, then $g_1^*(t) = 4a - 2\sin t$ is a decreasing function. The continuity of g_1^* ensures that g_1^* is decreasing on $[0, \pi/4]$ for any $a \geq 0$. □

For a vector $x = (x_1, ..., x_n) \in \mathbb{R}^n, n \geq 2$, let $x_1^{\#}, ..., x_n^{\#}$ be the sequence of absolute values of the components of x in increasing order, $0 \leq x_1^{\#} \leq ... \leq x_n^{\#}$. Then there exists a natural number $k \in [1, n-1]$ such that $x_k^{\#} \leq (1/\sqrt{n})\|x\|_2 \leq x_{k+1}^{\#}$. With this notation we can state the main result of this section:

Theorem 2.2

For any $n \in \mathbb{N}, n \geq 2$ *and* $x = (x_1, ..., x_n) \in \mathbb{R}^n$ *we have*

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \geq \left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1} \|x\|_2 + \frac{1}{2}\Big(x_1^{\#} + ... + x_k^{\#} - \frac{k}{\sqrt{n}}\|x\|_2\Big); \tag{4}$$

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \geq \left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1} \|x\|_2 + \frac{1}{2}\big(\|x\|_1 - \sqrt{n}\|x\|_2\big); \tag{5}$$

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \leq \left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1} \|x\|_2 + \Big(x_{k+1}^{\#} + ... + x_n^{\#} - \frac{n-k}{\sqrt{n}}\|x\|_2\Big). \tag{6}$$

Equality in (4), (5) *and* (6) *is attained for* $x_1 = x_2 = ... = x_n = \|x\|_2/\sqrt{n}$.

Proof. Observe first that if (5) is true then (4) is also true, so we prove (5). From the symmetry of $\|\sum_{i=1}^n x_i r_i\|$ we may suppose that $0 \le x_1 \le ... \le x_n$. For $n = 2$, (5) is equivalent to $x_2 \ge (x_1 + x_2)/2$, which is trivial. Let $n > 2$. If $x_1 = x_2 = ... = x_n$, then in (5) we have equality. We may assume $x_1 < x_n$. It follows that

$$\begin{aligned}\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} &= \frac{1}{2^n}\sum_{\varepsilon_i=\pm 1}\left|\sum_{i=1}^n x_i\varepsilon_i\right| \\ &= \frac{1}{2^{n-2}}\sum_{\varepsilon_i=\pm 1}\frac{1}{4}\left(\left|x_n + x_1 + \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right|\right. \\ &\quad + \left|x_n - x_1 + \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right| + \left|x_n + x_1 - \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right| \\ &\quad \left. + \left|x_n - x_1 - \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right|\right) \\ &= \frac{1}{2^{n-2}}\sum_{\varepsilon_i=\pm 1}\frac{1}{4} g_2\left(x_n, x_1, \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right) + \frac{1}{2}(x_n + x_1).\end{aligned}$$

If $\|x\|_2/\sqrt{n} < \sqrt{(x_n^2 + x_1^2)/2}$, using Lemma 2.1 (for g_2), one obtains

$$\begin{aligned}\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} &\ge \frac{1}{2^{n-2}}\sum_{\varepsilon_i=\pm 1}\frac{1}{4} g_2\left(x_n', \frac{\|x\|_2}{\sqrt{n}}, \left|\sum_{i=2}^{n-1} x_i\varepsilon_i\right|\right) + \frac{1}{2}(x_n + x_1) \\ &= \left\|\frac{\|x\|_2}{\sqrt{n}} r_1 + \sum_{i=2}^{n-1} x_i r_i + x_n' r_n\right\|_{L_1} \\ &\quad + \frac{1}{2}\left(x_n + x_1 - x_n' - \frac{\|x\|_2}{\sqrt{n}}\right),\end{aligned}$$

where $x_n^2 + x_1^2 = (x_n')^2 + \|x\|_2^2/n$. If $\|x\|_2/\sqrt{n} > \sqrt{(x_n^2 + x_1^2)/2}$, again using Lemma 2.1, we have:

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \ge \left\|x_1' r_1 + \sum_{i=2}^{n-1} x_i r_i + \frac{\|x\|_2}{\sqrt{n}} r_n\right\|_{L_1} + \frac{1}{2}\left(x_n + x_1 - x_1' - \frac{\|x\|_2}{\sqrt{n}}\right),$$

with $x_1^2 + x_n^2 = (x_1')^2 + \|x\|_2^2/n$. If $\|x\|_2/\sqrt{n} = \sqrt{(x_n^2 + x_1^2)/2}$, then

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \ge \left\|\frac{\|x\|_2}{\sqrt{n}} r_1 + \sum_{i=2}^{n-1} x_i r_i + \frac{\|x\|_2}{\sqrt{n}} r_n\right\|_{L_1} + \frac{1}{2}\left(x_1 + x_n - 2\frac{\|x\|_2}{\sqrt{n}}\right).$$

Let us observe that applying this procedure once we have

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \ge \left\|\sum_{i=1}^n y_i r_i\right\|_{L_1} + \frac{1}{2}(x_1 + ... + x_n - y_1 - ... - y_n),$$

where $0 \le y_1 \le ... \le y_n$, $y_1^2 + ... + y_n^2 = x_1^2 + ... + x_n^2$, and at least one of y_i's is $\|x\|_2/\sqrt{n}$. After at most $n-1$ steps all the y_i's become equal to $\|x\|_2/\sqrt{n}$, and finally:

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1} \ge \left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1} \|x\|_2 + \frac{1}{2}\big(x_1 + ... + x_n - \sqrt{n}\|x\|_2\big).$$

Using Lemma 2.1 (for g_3), a similar proof can be given for (6); here instead of $(1/2)(x_1' + \|x\|_2/\sqrt{n})$ we put $\max\{x_1', \|x\|_2/\sqrt{n}\}$, and $(1/2)(x_n' + \|x\|_2/\sqrt{n})$ becomes $\max\{x_n', \|x\|_2/\sqrt{n}\}$. $\square$

Remarks.

a) Define the functions $\Phi_i : \mathbb{R}_+^n \to \mathbb{R}$, $i = 1, 2, 3$ by

$$\Phi_1(x_1, ..., x_n) = \left\|\sum_{i=1}^n \sqrt{x_i} r_i\right\|_{L_1} - \frac{1}{4}\sum_{i=1}^n \left[1 - \operatorname{sign}\left(x_i - \frac{\sum_{j=1}^n x_j}{n}\right)\right]\sqrt{x_i},$$

$$\Phi_2(x_1, ..., x_n) = \left\|\sum_{i=1}^n \sqrt{x_i} r_i\right\|_{L_1} - \frac{1}{2}\sum_{i=1}^n \sqrt{x_i},$$

$$\Phi_3(x_1, ..., x_n) = \left\|\sum_{i=1}^n \sqrt{x_i} r_i\right\|_{L_1} - \frac{1}{2}\sum_{i=1}^n \left[1 + \operatorname{sign}\left(x_i - \frac{\sum_{j=1}^n x_j}{n}\right)\right]\sqrt{x_i}.$$

The proof of Theorem 2.2 shows that Φ_1 and Φ_2 are Schur-convex and that Φ_3 is Schur-concave on $\mathbb{R}_+^n$.

b) Suppose that $\varphi : \mathbb{R} \to \mathbb{R}_+$ is an even function which satisfies the property: there exists a constant M such that for any fixed $a, r > 0$ the functions $\varphi_{a,r}^M : [0, \pi/4] \to \mathbb{R}$, defined by

$$\varphi_{a,r}^M(t) = E\big(\varphi(\varepsilon_1 r\cos t + \varepsilon_2 r \sin t + a)\big) - M\left(\varphi(r\cos t) + \varphi(r \sin t)\right), t \in [0, \pi/4],$$

are all decreasing (increasing). Here E is the expectation with respect to $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Under these conditions we have

$$\left\|\varphi\left(\sum_{i=1}^n x_i r_i\right)\right\|_{L_1} \ge (\le) \left\|\varphi\left(\frac{\|x\|_2}{\sqrt{n}}\sum_{i=1}^n r_i\right)\right\|_{L_1} + M\left(\sum_{i=1}^n \varphi(x_i) - n\varphi\left(\frac{\|x\|_2}{\sqrt{n}}\right)\right),$$

with equality for $x_1 = ... = x_n = \|x\|_2/\sqrt{n}$. The proof is the same as that of (5) in Theorem 2.2, where $\varphi(x) = |x|$ and $\varphi_{a,r}^{1/2}$ is decreasing for all $a, r > 0$. The function $\varphi(x) = |x|^p, p \in \{2\} \cup [3, \infty)$, has the attached functions $\varphi_{a,r}^0$, $a, r > 0$, increasing (by [3]). As a consequence of Lemma 4.1 in [1], an Orlicz function φ such that φ'' is a concave function in $[0, \infty)$ has the associate functions $\varphi_{a,r}^1; a, r > 0$, increasing.

3. Variants of Khintchine's inequality and Haagerup functions

In this section we obtain variants of inequalities (2) with coefficients depending of n. We first consider an improved form of the first inequality in (2), where equality will be attained for $n = 2$ and $x = (1, 1)$. We need some auxiliary results.

Lemma 3.1

Suppose that $0 \le a_1 \le a_2 \le b_2 \le b_1 \le 1$ *and that* $a_1^2 + b_1^2 = a_2^2 + b_2^2$. *Then*

$$\cos(a_1 t)\cos(b_1 t) \le \cos(a_2 t)\cos(b_2 t),\ t \in [0, \pi/2]. \tag{7}$$

Proof. Indeed, using the well-known representation formula:

$$\cos t = \prod_{k=0}^{\infty} \left(1 - 4t^2/((2k+1)^2\pi^2)\right),\ t \in \mathbb{R},$$

our inequality is equivalent to

$$\prod_{k=0}^{\infty}\left(1 - \frac{4(a_1^2+b_1^2)t^2}{(2k+1)^2\pi^2} + \frac{16a_1^2b_1^2t^4}{(2k+1)^4\pi^4}\right)$$
$$\le \prod_{k=0}^{\infty}\left(1 - \frac{4(a_2^2+b_2^2)t^2}{(2k+1)^2\pi^2} + \frac{16a_2^2b_2^2t^4}{(2k+1)^4\pi^4}\right),$$

$t \in [0, \pi/2]$, which is true since, under our hypothesis, $a_1^2b_1^2 \le a_2^2b_2^2$. □

Lemma 3.2

Let Φ *be the function defined as above. Then*

$$\Phi(0) \le \Phi(b) \le \Phi(1),\ b \ge 0.$$

This is Lemma 8 (i) in [4].

Lemma 3.3

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $n \ge 2$ *be such that* $1 = x_1 \ge ... \ge x_n \ge 0$. *Then*

$$\prod_{k=1}^{n} \cos(x_k t) \le \cos^n\left(\frac{\|x\|_2}{\sqrt{n}} t\right),\ t \in [0, \pi/2]. \tag{8}$$

Proof. The case $n = 2$ follows immediately from Lemma 3.1. Suppose that $n > 2$. If $x_1 = ... = x_n = \|x\|_2/\sqrt{n}$, then (8) becomes equality. Assume that $x_1 > x_n$. With arguments as in the proof of Theorem 2.2, by (7) we have:

$$\prod_{k=1}^{n} \cos(x_k t) \le \prod_{k=1}^{n} \cos(y_k t),\ t \in [0, \pi/2],$$

where $1 \ge y_1 \ge ... \ge y_n \ge 0$, and at least one of y_i's is $\|x\|_2/\sqrt{n}$. Applying at most $n - 1$ times the preceding procedure one obtains (8). □

Lemma 3.4

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, *be such that* $1 = x_1 \ge ... \ge x_n \ge 0$. *Denote by*

$$h_n(x) = -\frac{2}{\pi}\int_{\pi/2}^{\infty} \frac{\prod_{k=1}^{n}\cos(x_k t)}{t^2}\, dt\,.$$

Then $h_n(e_1 + e_2) \le h_n(x) \le h_n(e_1)$, *where* $e_1 = (1, 0, ..., 0)$ *and* $e_1 + e_2 = (1, 1, 0, ..., 0)$.

This Lemma is proved in [4, p.18].

Proposition 3.5

For any $n \in \mathbb{N}$ *and* $x = (x_1, ..., x_n) \in \mathbb{R}^n$ *we have:*

$$\left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_1} \geq \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_1} \|x\|_2 + \frac{2\|x\|_\infty}{\pi} \int_{\pi/2}^{\infty} \frac{\cos^n \left(\frac{\|x\|_2}{\sqrt{n}\|x\|_\infty} t \right)}{t^2} \, dt - \Phi(1)\|x\|_\infty. \tag{9}$$

The inequality becomes equality for $n = 2$ *and* $x = (1, 1)$.

Proof. We may suppose, by homogeneity, that $\|x\|_\infty = 1$, and by the symmetry of r_i that $1 = x_1 \geq ... \geq x_n \geq 0$. Using the integral form of Rademacher averages from [2] and applying successively Lemma 3.3, Lemma 3.2, and Lemma 3.4, it follows

$$\begin{aligned}
\left\| \sum_{i=1}^{n} x_i r_i \right\|_{L_1} &= \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{k=1}^{n} \cos(x_k t)}{t^2} \, dt \\
&\geq \frac{2}{\pi} \int_0^{\pi/2} \frac{1 - \cos^n \left(\frac{\|x\|_2}{\sqrt{n}} t \right)}{t^2} \, dt + \frac{2}{\pi} \int_{\pi/2}^{\infty} \frac{1 - \prod_{k=1}^{n} \cos(x_k t)}{t^2} \, dt \\
&\geq \frac{2}{\pi} \int_0^\infty \frac{1 - \cos^n \left(\frac{\|x\|_2}{\sqrt{n}} t \right)}{t^2} \, dt + \frac{2}{\pi} \int_{\pi/2}^{\infty} \frac{\cos^n \left(\frac{\|x\|_2}{\sqrt{n}} t \right)}{t^2} \, dt \\
&\quad - \Phi(1) = \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i \right\|_{L_1} \|x\|_2 \\
&\quad + \frac{2}{\pi} \int_{\pi/2}^{\infty} \frac{\cos^n \left(\frac{\|x\|_2}{\sqrt{n}} t \right)}{t^2} \, dt - \Phi(1). \quad \square
\end{aligned}$$

We turn now to the main result of this section. We first need some technical lemmas. The following result is stated, without proof, in [4].

Lemma 3.6

The function $g : [1, \infty) \to \mathbb{R}$ *defined by:*

$$g(\alpha) = \sqrt{\alpha} \int_0^{\pi/2} \frac{\cos^\alpha t \ln \frac{1}{\cos t}}{t^2} \, dt,$$

is increasing on $[1, \infty)$.

Writing g' as a difference of two integrals, $I_1 - I_2$ and integrating by parts I_2, after a straightforward but tedious computation one obtains the positivity of g'.

Let us denote by $S_n := \|\sum_{i=1}^n r_i\|_{L_1}$.

Lemma 3.7

Let $f_n : [1, n] \to \mathbb{R}$ *be the function defined by*

$$f_n(\alpha) = -\frac{S_n}{\sqrt{n}}\sqrt{\alpha} + \frac{2}{\pi}\int_0^{\pi/2} \frac{1-\cos^\alpha t}{t^2}\,dt,\ \ \alpha \in [1, n].$$

Then $f_n(\alpha) \le \max\{f_n(1), f_n(n)\}$.

Proof. A straightforward computation yields

$$\begin{aligned} f_n'(\alpha) &= \frac{1}{\sqrt{\alpha}}\left(-\frac{S_n}{2\sqrt{n}} + \frac{2}{\pi}\sqrt{\alpha}\int_0^{\pi/2}\frac{\cos^\alpha t \ln\dfrac{1}{\cos t}}{t^2}\,dt\right) \\ &= \frac{1}{\sqrt{\alpha}}\left(-\frac{S_n}{2\sqrt{n}} + \frac{2}{\pi}g(\alpha)\right),\ \ \alpha \in [1, n]. \end{aligned}$$

By Lemma 3.6 the function g is increasing, which implies that $f_n'(\alpha) \le 0$ for all $\alpha \in [1, n]$, or $f_n'(\alpha) \ge 0$ for all $\alpha \in [1, n]$, or there exists an $\alpha_0 \in (1, n)$ such that $f_n'(\alpha) \le 0$ for all $\alpha \in [1, \alpha_0]$ and $f_n'(\alpha) > 0$ for all $\alpha \in (\alpha_0, n]$. In all of these cases $f_n(\alpha) \le \max\{f_n(1), f_n(n)\}$. □

Lemma 3.8

If the sequence $(s(n))_{n\ge 1}$ *is defined by*

$$s(n) = f_n(1) - f_n(n), n = 1, 2, \dots,$$

then the sequence $(s(2n))_{n\ge 2}$ *is increasing.*

Proof. We have

$$\begin{aligned} s(2n) = f_{2n}(1) - f_{2n}(2n) &= \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n}{2}\right)}\cdot\frac{1}{\sqrt{2n}}(\sqrt{2n}-1) \\ &- \frac{2}{\pi}\int_0^{\pi/2}\frac{\cos t - \cos^{2n} t}{t^2}\,dt = F(2n)(\sqrt{2n}-1) \\ &- \frac{2}{\pi}\int_0^{\pi/2}\frac{\cos t - \cos^{2n} t}{t^2}\,dt, \end{aligned}$$

where F is the Haagerup function defined in [2] by

$$F(s) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{s+1}{2})}{\sqrt{s}\,\Gamma(\frac{s}{2})},\ \ s > 0.$$

Recall that, by [2, p.238], we have

$$F(s+2)=\sqrt{\frac{s}{s+2}}\,\frac{s+1}{s}F(s),\quad s>0\,. \tag{10}$$

Then

$$\begin{aligned} s(2n+2)-s(2n) &= F(2n+2)\big(\sqrt{2n+2}-1\big)-F(2n)\big(\sqrt{2n}-1\big)\\ &\quad -\frac{2}{\pi}\int_0^{\pi/2}\cos^{2n}t(1+\cos t)\frac{1-\cos t}{t^2}\,dt\,. \end{aligned}$$

Since $(1-\cos t)/t^2\le 1/2,\ t\in(0,\pi/2],$ by (10) and the well-known formula

$$\int_0^{\pi/2}\cos^\alpha t\,dt=\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha+2}{2})},\quad \alpha>-1,$$

it follows that

$$\begin{aligned} s(2n+2)-s(2n) &\ge F(2n)\left(1+\frac{1}{\sqrt{2n}}-\frac{2n+1}{2\sqrt{n(n+1)}}\right)\\ &\quad -\frac{1}{\pi}\int_0^{\pi/2}(\cos^{2n}t+\cos^{2n+1}t)\,dt\\ &= F(2n)\left(1+\frac{1}{\sqrt{2n}}-\frac{2n+1}{2\sqrt{n(n+1)}}\right)\\ &\quad -\frac{1}{2\sqrt{2n}}F(2n)-\frac{1}{2\sqrt{2n+1}}F(2n+1)\\ &\ge F(2n)\left(1+\frac{1}{2\sqrt{2n}}-\frac{2n+1}{2\sqrt{n(n+1)}}\right)-\frac{F(2n+2)}{2\sqrt{2n+1}}. \end{aligned}$$

Using again the recurrence formula (10) one obtains

$$\begin{aligned} &s(2n+2)-s(2n)\\ &\ge \frac{F(2n)}{4\sqrt{n(n+1)}}\cdot\frac{\sqrt{2n+1}\big(\sqrt{2n+1}-2\big)+2\sqrt{n+1}\big(\sqrt{n}-\sqrt{2}\big)}{\big(2\sqrt{n(n+1)}+2n+1\big)\big(\sqrt{2n+2}+\sqrt{2n+1}\big)}>0, \end{aligned}$$

for $n\ge 2$. □

Theorem 3.9

For any $n\in\mathbb{N}$ *and* $x=(x_1,...,x_n)\in\mathbb{R}^n$

$$\left\|\sum_{i=1}^n x_i r_i\right\|_{L_1}\le\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1}\|x\|_2+\left(1-\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n r_i\right\|_{L_1}\right)\|x\|_\infty. \tag{11}$$

Proof. Observe that if $0\le a<b\le 1,$ then $a\|x\|_2+(1-a)\|x\|_\infty\le b\|x\|_2+(1-b)\|x\|_\infty$. Using the well-known factorial representation of S_{2n}, (S_{2n-1}) we obtain

that the sequence $(S_{2n}/\sqrt{2n})_{n\geq 1}$, (respectively $(S_{2n-1}/\sqrt{2n-1})_{n\geq 1}$), is increasing (decreasing) and

$$\lim_{n\to\infty} S_{2n}/\sqrt{2n} = \lim_{n\to\infty} S_{2n-1}/\sqrt{2n-1} = \sqrt{2/\pi}.$$

Then we have $S_{2n-1}/\sqrt{2n-1} > \sqrt{2/\pi}$, and in this case (11) follows (for n odd) from the inequality (2), proved in [4].

Let $x = (x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$ be given. We may assume that $1 = x_1 \geq ... \geq x_{2n} \geq 0$. Denoting $\alpha = x_1^2 + ... + x_{2n}^2$, it follows that $\alpha \in [1, 2n]$. By Lemma 5 in [4] we have: $\cos(xt) \geq (\cos t)^{x^2}$, $0 \leq x \leq 1$, and $0 \leq t \leq \pi/2$. This fact, Lemma 3.4 and Lemma 3.7 imply successively that

$$\begin{aligned}
\left\| \sum_{i=1}^{2n} x_i r_i \right\|_{L_1} &= \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{i=1}^{2n} \cos(x_i t)}{t^2} \\
&\leq \frac{S_{2n}}{\sqrt{2n}} \sqrt{\alpha} + f_{2n}(\alpha) + h_{2n}(x) + \frac{2}{\pi} \int_{\pi/2}^\infty \frac{dt}{t^2} \\
&\leq \frac{S_{2n}}{\sqrt{2n}} \sqrt{\alpha} + f_{2n}(\alpha) + h_{2n}(e_1) + \frac{4}{\pi^2} \\
&\leq \frac{S_{2n}}{\sqrt{2n}} \sqrt{\alpha} + \max\left\{f_{2n}(1), f_{2n}(2n)\right\} + h_{2n}(e_1) + \frac{4}{\pi^2}\,.
\end{aligned}$$

Since $s(8) = f_8(1) - f_8(8) = \frac{35}{64}\sqrt{2}(2\sqrt{2}-1) - \frac{2}{\pi}(-\mathrm{Si}(\frac{\pi}{2}) + \frac{1}{16}\mathrm{Si}(4\pi) + \frac{7}{8}\mathrm{Si}(2\pi) + \frac{7}{8}\mathrm{Si}(\pi) + \frac{3}{8}\mathrm{Si}(3\pi)) \approx 0.005987... > 0$, where $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$, is the sinus integral, by Lemma 3.8 it follows that $s(2n) \geq 0$, for all $n \geq 4$. Then $f_{2n}(\alpha) \leq \max\{f_{2n}(1), f_{2n}(2n)\} = f_{2n}(1)$, $n \geq 4$. But $\alpha = 1$ if and only if $x = e_1$ and this yields

$$\begin{aligned}
\left\| \sum_{i=1}^{2n} x_i r_i \right\|_{L_1} - \frac{S_{2n}}{\sqrt{2n}} \|x\|_2 &= f_{2n}\left(\|x\|_2^2\right) + h_{2n}(x) + \frac{4}{\pi^2} \\
&\leq f_{2n}(\|e_1\|^2) + h_{2n}(e_1) + \frac{4}{\pi^2} \\
&= \|r_1\|_{L_1} - \frac{S_{2n}}{\sqrt{2n}} \|e_1\|_2 = 1 - \frac{S_{2n}}{\sqrt{2n}},\ n \geq 4\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\| \sum_{i=1}^{2n} x_i r_i \right\|_{L_1} \leq{}& \left\| \frac{1}{\sqrt{2n}} \sum_{i=1}^{2n} r_i \right\|_{L_1} \|x\|_2 \\
&+ \left(1 - \left\| \frac{1}{\sqrt{2n}} \sum_{i=1}^{2n} r_i \right\|_{L_1}\right) \|x\|_\infty,\ n \geq 4\,.
\end{aligned} \tag{12}$$

In the case $n = 1$, the preceding inequality is equivalent to

$$\max\left\{|x_1|, |x_2|\right\} \leq \frac{1}{\sqrt{2}}\sqrt{x_1^2 + x_2^2} + \left(1 - \frac{1}{\sqrt{2}}\right) \max\left\{|x_1|, |x_2|\right\}, x_1, x_2 \in \mathbb{R}\,,$$

i.e. $\sqrt{x_1^2+x_2^2} \geq \max\{|x_1|+|x_2|\}$, which is true. If $n=2$ and $x=(x_1,x_2,x_3,x_4)$, with $1=x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0$, then the inequality (12) becomes

$$4+2x_2+2x_3+|1-x_2-x_3+x_4|+|-1+x_2+x_3+x_4| \leq 6\sqrt{1+x_2^2+x_3^2+x_4^2}\,.$$

If $x_2+x_3 \leq 1-x_4$, then (12) is equivalent to $x_2^2+x_3^2+x_4^2 \geq 0$; if $1-x_4 \leq x_2+x_3 \leq 1+x_4$ then (12) is equivalent to

$$\begin{aligned}2(x_2+x_3+x_4)^2 &- 4(x_2+x_3+x_4)+5\\ &+6(x_2^2+x_3^2+x_4^2-x_2x_3-x_2x_4-x_3x_4) \geq 0\,,\end{aligned}$$

which is true and finally if $x_2+x_3 \geq 1+x_4$ then (12) becomes

$$4(x_2-x_3)^2+(x_2-2)^2+(x_3-2)^2+9x_4^2 \geq 0\,,$$

which is also true. Unfortunately, in the remaining case $n=3$,

$$\begin{aligned}f_6(1)-f_6(6) = \frac{5\sqrt{6}}{16}\big(\sqrt{6}-1\big)-\frac{2}{\pi}\Bigg(&-\mathrm{Si}\Big(\frac{\pi}{2}\Big)+\frac{3}{16}\mathrm{Si}(3\pi)\\ &+\frac{15}{16}\mathrm{Si}(\pi)+\frac{3}{4}\mathrm{Si}(2\pi)\Bigg) \approx -0.00013.. < 0\,.\end{aligned}$$

However, for $x=(x_1,...,x_6)$, with $1=x_1 \geq ... \geq x_6 \geq 0$, using the computer one obtains that the maximum of the following function of five variables,

$$\frac{1}{32}\sum_{\varepsilon_i=\pm 1}\left|1+\sum_{i=2}^{6}\varepsilon_i x_i\right| - \left(1-\frac{15}{8\sqrt{6}}\right)-\frac{15}{8\sqrt{6}}\sqrt{1+x_2^2+...+x_6^2}\,,$$

is zero. This means that (12) is also true for $n=3$. □

Finally we remark that using the inequality (11) one obtains (with the same proofs as in [4]) slightly improved estimates for the absolute projection constants of ℓ_p^n-spaces, $p \in (1,2)$, for large n.

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