Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **52**, 2 (2001), 127–155

Transference for hypergroups

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Received June 7, 2000

Abstract

A transference theorem for convolution operators is proved for certain families of one-dimensional hypergroups.

1. Introduction

The results in this paper are a natural extension to those in [5]. In that paper, a transference-type theorem for convolution operators was presented in the setting of developments in ultraspherical polynomials. Before going further we should explain what we mean by "transference-type theorem". The method of transference was introduced by Coifman and Weiss in the Seventies (see [2]) and essentially allows to transfer a convolution operator on an L^p space of functions on an amenable group G to an operator on an L^p space of functions on some measure space $\mathcal{M}$, by means of a representation of G acting on $L^p(\mathcal{M})$. If the convolution operator on $L^p(G)$ has norm N_p, then the "transferred" operator on $L^p(\mathcal{M})$ has norm bounded by cN_p, where c is a constant that depends on the representation. In our case we do not have a group nor a representation, but we can still obtain the same kind of preservation of L^p inequalities as long as we deal with certain families of hypergroups.

A hypergroup (see [1], [8] or [11]) is a locally compact Hausdorff space X with a certain convolution structure $*$ on the space of complex Radon measures on X, $M(X)$. Let δ_x be the Dirac measure at a point $x \in X$. Then the convolution $\delta_x * \delta_y$ of the two

Keywords: Hypergroups, transference.

MSC2000: 43A62, 47B38.

This research was supported in part by the European Commission via the TMR network "Harmonic analysis".

point measures δ_x and δ_y is a probability Radon measure on X with compact support and such that $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is a continuous mapping from $X \times X$ into the space of compact subsets of X, with the appropriate topologies which will be described later. Unlike the case of groups, this convolution is not necessarily the point measure $\delta_{x \cdot y}$ for a composition $x \cdot y$ in X. The role played by the natural left translation of a function f by x, in the group case, is assumed by the generalized (left) translation, defined on a hypergroup by

$$T_x f(y) = \int_X f(t)\, d(\delta_x * \delta_y)(t),$$

for all $y \in X$. Here, we denote by $\int f\, d\mu$ the integral of the function f with respect to the measure μ. For a commutative hypergroup (i.e. $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in X$) the convolution of two functions k and f is given by

$$k * f(x) = \int_X k(y) T_x f(y)\, d\eta(y)$$

where η is a translation-invariant measure, called Haar measure of the hypergroup. All these notions will be explained in full detail in the next section.

This convolution has many of the "nice" properties that one may expect. Among others, if $k \in L^1(X, \eta)$ and $f \in L^p(X, \eta)$, then

$$\|k * f\|_p \leq \|k\|_1 \|f\|_p \tag{1.1}$$

It is therefore natural to study convolution operators of the type

$$\begin{array}{rccc} S_k : & L^p(X, \eta) & \rightarrow & L^p(X, \eta) \\ & f & \mapsto & k * f \end{array}$$

since, by inequality (1.1), S_k is bounded with operator norm less than or equal to $\|k\|_1$. In many situations, we deal with families of hypergroups defined on the same space X, indexed by a parameter $\lambda \geq \lambda_0$, each of them having Haar measure $d\eta_\lambda(x) = (g(x))^\lambda d\eta_0(x)$ for some positive function g on X. It is obvious that, if $k \in L^1(X, \eta_\lambda)$ and if we define the function h on X by $h(x) = k(x)(g(x))^\delta$, for some $\delta > 0$, then $h \in L^1(X, \eta_{\lambda-\delta})$, and the two L^1 norms coincide. Inequality (1.1), then, suggests that the two convolution operators S_k on $L^p(X, \eta_\lambda)$ and S_h on $L^p(X, \eta_{\lambda-\delta})$ are strictly related to each other. More precisely, we will show that the operator norm of S_k, $\|S_k\|$, is bounded above by the norm $\|S_h\|$ in many significant examples, namely when the family of hypergroups is one of the following:

1. Continuous Jacobi polynomial hypergroups of index (λ, λ), $\lambda \geq -1/2$.
2. Continuous Jacobi polynomial hypergroups of index (λ, β), $\lambda \geq \beta \geq -1/2$.
3. Bessel-Kingman hypergroups.
4. Jacobi hypergroups of non-compact type of index (λ, λ), $\lambda \geq -1/2$.
5. Jacobi hypergroups of non-compact type of index (λ, β), $\lambda \geq \beta \geq -1/2$.

All these hypergroups will be described in Section 3. The first case is precisely the one we mentioned at the beginning (ultraspherical polynomials), while the third case was in part discussed by R. O. Gandulfo in his Ph.D. thesis (see [4].)

The main result is stated in Section 4 (Theorem 4.6) in a general way, while all the subsequent corollaries describe how the theorem applies to each particular case. In Section 5 we prove all the lemmas that lead to Theorem 4.6.

2. Hermitian hypergroups, definitions and properties

Let X be a locally compact Hausdorff space. Denote by $M(X)$ the complex Radon measures on X, by $M^+(X)$ the complex Radon measures which are non-negative, and by $M_1(X)$ the probability Radon measures. Denote by $\mathcal{C}(X)$ the continuous complex-valued functions on X, by $\mathcal{C}_c(X)$ those with compact support, by $\mathcal{C}_0(X)$ those which are zero at infinity and by $\mathcal{C}_c^+(X)$ those which are non-negative and compactly supported.

Define the *cone topology* on $M^+(X)$ as the weakest topology such that, for each $f \in \mathcal{C}_c^+(X)$, the mapping $\mu \mapsto \int_X f\, d\mu$ is continuous and such that the mapping $\mu \mapsto \mu(X)$ is continuous. From now on, any unspecified topology on $M^+(X)$ is the cone topology.

Let $(\mu, \nu) \mapsto \mu * \nu$ be a bilinear map from $M(X) \times M(X)$ to $M(X)$. This mapping will be called *positive continuous* if

1. $\mu * \nu \geq 0$, if $\mu \geq 0$ and $\nu \geq 0$.
2. The restricted mapping from $M^+(X) \times M^+(X)$ to $M^+(X)$ is continuous.

Let $\mathcal{K}(X)$ denote the collection of all nonvoid compact subsets of X. If A and B are subsets of X, let $\mathcal{K}_A(B)$ be the collection of all C in $\mathcal{K}(X)$ such that $C \cap A$ is nonvoid and $C \subset B$. We give $\mathcal{K}(X)$ the topology generated by the subbasis of all $\mathcal{K}_U(V)$ for which U and V are open subsets of X (see [9].) It is worth noting (see [10]) that, if X is a metric space with distance d, the above defined topology of $\mathcal{K}(X)$ coincides with the more intuitive Hausdorff topology, given by the Hausdorff metric: for $A \in \mathcal{K}(X)$ and $r > 0$, define

$$V_r(A) = \left\{ y : d(x, y) < r \text{ for some } x \in A \right\}$$

and, for $A, B \in \mathcal{K}(X)$, define the Hausdorff distance by

$$d(A, B) = \inf \left\{ r : A \subset V_r(B) \text{ and } B \subset V_r(A) \right\}.$$

Definition 2.1. The pair $(X, *)$ will be called a hypergroup if the following conditions are satisfied.

- H1. The symbol $*$ denotes a binary operation on $M(X)$, and with this operation, $M(X)$ is a complex associative algebra.
- H2. The bilinear mapping $(\mu, \nu) \mapsto \mu * \nu$ is positive-continuous.
- H3. If $x, y \in X$, then $\delta_x * \delta_y$ is a probability measure with compact support.
- H4. The map $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $X \times X$ to $\mathcal{K}(X)$ is continuous.
- H5. There exists a (necessarily unique) element e of X such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in X$.
- H6. There exists a (necessarily unique) involution $x \mapsto x^-$ of X (that is, a homeomorphism of X such that $(x^-)^- = x$) such that, for all $\mu, \nu \in M(X)$, $(\mu * \nu)^- = \nu^- * \mu^-$, where $\mu^-(A) \overset{\text{def}}{=} \mu(A^-)$ for all measurable $A \subset X$.
- H7. For all $x, y \in X$, the element e is in the support of $\delta_x * \delta_y$ if and only if $x = y^-$.

A hypergroup $(X, *)$ is called *commutative* if $(M(X), +, *)$ is a commutative algebra, and *hermitian* if the involution is the identity map. It is easy to prove that every hermitian hypergroup is commutative.

EXAMPLES: **(a)** Every locally compact group G is a hypergroup with its usual convolution structure. More precisely, $\delta_x * \delta_y = \delta_{x\cdot y}$, e is the unit of the group, and $x^- \stackrel{\text{def}}{=} x^{-1}$.

(b) If G is a locally compact group and H is a compact subgroup with normalized Haar measure ω_H, then the space of double cosets $H\backslash G/H \stackrel{\text{def}}{=} \{HxH,\, x \in G\}$ is a hypergroup. The convolution is given, for point measures, by

$$\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HxtyH} d\omega_H(t),$$

the identity is HeH and the involution is $(HxH)^- = Hx^{-1}H$.

All the hypergroups we consider in this paper will be intervals of the real line, either compact ($[0, \pi]$) or non-compact ($[0, \infty]$). It is a fact (see [1], page 190) that these hypergroups are hermitian, that the neutral element of $[0, \infty)$ is 0 and that the neutral element of $[0, \pi]$ is either 0 or π. For this reason, in the following definitions and properties, we will assume that the hypergroup $(X, *)$ be hermitian.

DEFINITION 2.2. Let η be a positive Radon measure on X. We say that η is a *Haar measure* if, for all $x \in X$ and for all $f \in \mathcal{C}_c(X)$, we have

$$\int_X \int_X f\, d(\delta_x * \delta_y)\, d\eta(y) = \int_X f(y)\, d\eta(y).$$

This definition is more easily understood once we notice that $\int_X f\, d(\delta_x * \delta_y)$ is the generalization of the notion of translation of a function $f(x \cdot y)$ that one has in the group case (see Definition 2.4.)

Theorem 2.3 (see [1], pages 28–41)

Every commutative hypergroup possesses a Haar measure η, which is unique modulo a positive multiplicative constant and has support equal to the whole space X. Only compact hypergroups admit bounded Haar measure η.

DEFINITION 2.4. We define the generalized translation operators T_x, $x \in X$, on $\mathcal{C}(X)$ by

$$T_x f(y) = \int_X f\, d(\delta_x * \delta_y)$$

for all $y \in X$.

The following proposition provides an overview of the basic properties of the translation operators. Some of them are easy consequences of the axioms H1–H7, others are rather technical and we will refer the reader to appropriate references.

Proposition 2.5

(i) For all μ, $\nu \in M(X)$, we have $\mu * \nu = \int_X \int_X (\delta_x * \delta_y)\, d\mu(x)\, d\nu(y)$, where the integral is taken in the Pettis sense.

(ii) *For all* x, y, $z \in X$ *and* $f \in \mathcal{C}(X)$ *we have*

$$\begin{aligned} T_e f(x) &= f(x) \\ T_x f(y) &= T_y f(x) \\ T_x T_y f(z) &= T_y T_x f(z). \end{aligned}$$

(iii) *For all bounded* $f \in \mathcal{C}(X)$, *the mapping* $(x, y) \mapsto T_x f(y)$ *is continuous on* $X \times X$.

(iv) *If* $f \in \mathcal{C}_c(X)$, *then for all* $x \in X$, $T_x f \in \mathcal{C}_c(X)$.

(v) *For all* $x, y \in X$, $T_x(1)(y) = 1$.

(vi) *Let* $1 \le p \le \infty$. *For all* $x \in X$, $f \in \mathcal{C}(X) \cap L^p(X, \eta)$, *we have*

$$\|T_x f\|_p \le \|f\|_p.$$

(Notice that the density of $\mathcal{C}_c(X)$ in $L^p(X, \eta)$, $1 \le p < \infty$, and the above property allow us to extend the definition of translation operator to the spaces $L^p(X, \eta)$.)

(vii) *For* $f \in L^p(X, \eta)$, $1 \le p < \infty$, *the mapping* $x \mapsto T_x f$ *is continuous from* X *into* $L^p(X, \eta)$.

(viii) *For all* $g, h \in L^1(X, \eta)$ *and for all* $x \in X$, *we have* $\int_X (T_x g) h \, d\eta = \int_X g(T_x h) \, d\eta$.

Proof. (i) The two bilinear maps R_1 and R_2 from $M^+(X) \times M^+(X)$ to $M^+(X)$ given by $R_1(\mu, \nu) = \mu * \nu$ and $R_2(\mu, \nu) = \int_X \int_X (\delta_x * \delta_y) \, d\mu(x) \, d\nu(y)$ are positive continuous and coincide when μ and ν are point measures. By the density in $M^+(X)$ of the finitely supported measures, R_1 and R_2 coincide on $M^+(X)$, and therefore on $M(X)$.

(ii) The first two equalities are trivial; for the third, observe that, using (i),

$$T_x T_y f(z) = \langle \delta_y * (\delta_x * \delta_z), f \rangle = \langle \delta_x * (\delta_y * \delta_z), f \rangle = T_y T_x f(z).$$

(iii) This follows from the continuity of the map $x \mapsto \delta_x$ and axiom H2.

(iv) Continuity follows again from the continuity of the map $x \mapsto \delta_x$ and axiom H2. As for the compactness of the support, see [1], page 19. In all our examples, though, $X = [0, 2\pi]$ (and the support of $T_x f$ is obviously compact), or $X = [0, \infty)$; in this case the support of $\delta_x * \delta_y$ is always $[|x - y|, x + y]$, and this implies that $T_x f$ is compactly supported.

(v) This is trivial.

(vi) This follows from the definition of Haar measure and the inequality

$$\left|T_x f(y)\right|^p \le T_x\big(|f|^p\big)(y),$$

implied by Jensen's inequality.

(vii) This is a standard "$\varepsilon/3$ argument". One has to approximate f in L^p with a function $g \in \mathcal{C}_c(X)$

(viii) This is rather technical. See [1], page 34. □

Definition 2.6. For g and h in $L^1(X, \eta)$, the convolution of g and h is given by

$$g * h(x) = \int_X T_x g(y) h(y) \, d\eta(y) = \int_X g(y) T_x h(y) \, d\eta(y).$$

Proposition 2.7

Let $1 \le p \le \infty$. If g is in $L^1(X, \eta)$ and h is in $L^p(X, \eta)$, then the function $g * h$ belongs to $L^p(X, \eta)$ and we have

$$\|g * h\|_p \le \|g\|_1 \|h\|_p.$$

Proof. This is an easy application of Minkowski's inequality and Proposition 2.5. (vi). □

3. Examples of one-dimensional hermitian hypergroups

One-dimensional hypergroups are those for which the space X is $\mathbb{R}$, $\mathbb{T}$, $\mathbb{R}_+$ or a compact interval $[0, a]$, $a > 0$. It can be proved that for any hypergroup $(X, *)$ where X is either $\mathbb{R}$ or $\mathbb{T}$, the convolution arises from the group structure of $\mathbb{R}$ or $\mathbb{T}$ respectively (see [1], page 189). We are mainly interested in the remaining two cases: $X = [0, \infty)$ or $X = [0, a]$ (this second case can always be reduced to the case $X = [0, \pi]$). These, as we said in the previous section, are hermitian hypergroups.

Before we present the three main examples that will be studied in this chapter, let us state some preliminary definitions. Recall the Gaussian hypergeometric function

$${}_2F_1(a, b; c; z) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, $(a)_k \overset{\text{def}}{=} a(a+1)\ldots(a+k-1)$ and $|z| < 1$. Considered as a function of z, there is a unique analytic continuation to $\{z \in \mathbb{C} : z \notin [1, \infty)\}$. For any $\alpha \ge \beta \ge -1/2$, define the measures $m^1_{\alpha,\beta}$ on $[0, 1]$ and m^2_β on $[0, \pi]$ by

$$dm^1_{\alpha,\beta}(v) = \begin{cases} q_{\alpha,\beta}(1 - v^2)^{\alpha-\beta-1} v^{2\beta+1} dv & \text{if } \alpha > \beta \ge -\frac{1}{2} \\ d\delta_1(v) & \text{if } \alpha = \beta \ge -\frac{1}{2} \end{cases}$$

where

$$q_{\alpha,\beta} = \left(\int_0^1 (1 - v^2)^{\alpha-\beta-1} v^{2\beta+1} dv \right)^{-1} = \frac{2\Gamma(\alpha + 1)}{\Gamma(\beta + 1)\Gamma(\alpha - \beta)},$$

and

$$dm^2_\beta(\theta) = \begin{cases} c_\beta \sin^{2\beta} \theta \, d\theta & \text{if } \beta > -\frac{1}{2} \\ d\left(\frac{\delta_0 + \delta_\pi}{2}\right)(\theta) & \text{if } \beta = -\frac{1}{2} \end{cases}$$

where

$$c_\beta = \left(\int_0^\pi \sin^{2\beta} \theta \, d\theta \right)^{-1} = \frac{\Gamma(\beta + 1)}{\sqrt{\pi}\,\Gamma\left(\beta + \frac{1}{2}\right)}.$$

EXAMPLE 3.1: The continuous Jacobi polynomial hypergroup.

This is sometimes referred to as the dual Jacobi polynomial hypergroup. The Jacobi polynomials are defined by

$$P_n^{\alpha,\beta}(t) = \binom{n+\alpha}{n} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right)$$

where $\alpha, \beta > -1$, $n = 0, 1, 2, \ldots$ For fixed α and β, $P_n^{\alpha,\beta}$ is a real valued polynomial of degree n, and $\{P_n^{\alpha,\beta}\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $(1-t)^\alpha(1+t)^\beta$. We shall use the normalized Jacobi polynomials

$$R_n^{\alpha,\beta}(t) \stackrel{\text{def}}{=} \frac{P_n^{\alpha,\beta}(t)}{P_n^{\alpha,\beta}(1)} = {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right),$$

so that $R_n^{\alpha,\beta}(1) = 1$.

In the case $\alpha \geq \beta \geq -1/2$, we are in the fortunate situation in which the Jacobi polynomials satisfy the so-called hypergroup-type product formula. Define the function Φ on $[0,\pi] \times [0,\pi] \times [0,1] \times [-1,1]$, taking values on $[0,\pi]$, by

$$\Phi(x,y,v,\cos\theta) \stackrel{\text{def}}{=} \arccos\Bigg(\frac{1}{2}(1-v^2)(\cos x + \cos y) + \frac{1}{2}(1+v^2)\cos x \cos y$$
$$+ \sin x \sin y \, v \cos\theta - \frac{1}{2}(1-v^2)\Bigg).$$

The product formula is (see [6] and [7])

$$R_n^{\alpha,\beta}(\cos x) R_n^{\alpha,\beta}(\cos y) = \int_0^\pi \int_0^1 R_n^{\alpha,\beta}\big(\cos\Phi(x,y,v,\cos\theta)\big)\, dm_{\alpha,\beta}^1(v)\, dm_\beta^2(\theta eta),$$

where $\alpha \geq \beta \geq -1/2$, $n = 1, 2, \ldots$, $x, y \in [0,\pi]$.

Note that, if $\alpha = \beta = -1/2$, $R_n^{\alpha,\beta}$ are the Tchebichef polynomials of first kind, that is $R_n^{-1/2,-1/2}(\cos x) = \cos(nx)$, and the product formula reduces to the well known formula

$$\cos(nx)\cos(ny) = \frac{1}{2}\cos\big(n(x+y)\big) + \frac{1}{2}\cos\big(n(x-y)\big).$$

The Riesz representation theorem guarantees the existence of a probability measure $\mu_{x,y}^{\alpha,\beta} \in M_1([0,\pi])$ such that, for every continuous f on $[0,\pi]$

$$\int_0^\pi f\, d\mu_{x,y}^{\alpha,\beta} = \int_0^\pi \int_0^1 f\big(\Phi(x,y,v,\cos\theta)\big)\, dm_{\alpha,\beta}^1(v)\, dm_\beta^2(\theta),$$

so that, taking $f = R_n^{\alpha,\beta} \circ \cos$, we get

$$R_n^{\alpha,\beta}(\cos x) R_n^{\alpha,\beta}(\cos y) = \int_0^\pi R_n^{\alpha,\beta}(\cos t)\, d\mu_{x,y}^{\alpha,\beta}(t).$$

We can now define a convolution product on $M([0,\pi])$ as follows: if ν and λ belong to $M([0,\pi])$, define $\nu*\lambda$ by its action on $f\in\mathcal{C}([0,\pi])$ by

$$\int_0^\pi f\,d(\nu*\lambda)\stackrel{\text{def}}{=}\int_0^\pi\int_0^\pi\left(\int_0^\pi f\,d\mu_{x,y}^{\alpha,\beta}\right)d\nu(x)\,d\lambda(y).$$

Thus $\delta_x*\delta_y=\mu_{x,y}^{\alpha,\beta}$.

Let us check that $([0,\pi],*)_{\alpha,\beta}$ is a hypergroup. All the axioms H2, H3, H5 and H6 follow easily from the definition of $\mu_{x,y}^{\alpha,\beta}$ and $*$ (here, the identity of the hypergroup is 0 and the involution is the identity map, so that, as we said, the hypergroup is hermitian.) As for axiom H1, all we have to check is the associative property: the equality

$$\int_0^\pi f\,d\big((\nu*\lambda)*\gamma\big)=\int_0^\pi f\,d\big(\nu*(\lambda*\gamma)\big)$$

is easily verified when $f=R_n^{\alpha,\beta}\circ\cos$ and follows, by density, for any continuous f. In order to prove that axioms H4 and H7 hold, we need to find out what the support of $\mu_{x,y}^{\alpha,\beta}$ is. Suppose first that $\alpha>\beta>-1/2$; then $\operatorname{supp}(\mu_{x,y}^{\alpha,\beta})$ coincides with the range of the function $\Phi(x,y,\cdot,\cdot)$. Define $z=ve^{i\theta}$; as v and θ vary in their respective domains, z varies in $\mathbb{D}^+=\{|z|\le 1,\ \mathrm{im}z\ge 0\}$. Also, defining $A=\sqrt{(1+\cos x)(1+\cos y)}$ and $B=\sqrt{(1-\cos x)(1-\cos y)}$, we get

$$\cos\big[\Phi(x,y,v,\cos\theta)\big]=\frac{|A+Bz|^2-2}{2}.$$

If $0\le x+y\le\pi$, then $A\ge B$ and, as z varies in $\mathbb{D}^+$, $\frac{|A+Bz|^2-2}{2}$ varies between $\frac{(A-B)^2-2}{2}=\cos(x+y)$ and $\frac{(A+B)^2-2}{2}=\cos(x-y)$.
If $x+y>\pi$, then $\frac{|A+Bz|^2-2}{2}$ varies between -1 and $\cos(x-y)$.
Thus $\operatorname{supp}(\mu_{x,y}^{\alpha,\beta})=[|x-y|,\min(\pi,x+y)]$.
Similar arguments show that $\operatorname{supp}(\mu_{x,y}^{\alpha,\beta})$ equals

$$\begin{aligned}
&[|x-y|,\min(\pi,x+y)] &&\text{if}\quad \alpha>\beta=-\frac12\\
&[|x-y|,\pi-|x+y-\pi|] &&\text{if}\quad \alpha=\beta>-\frac12\\
&\{|x-y|\}\cup\{[\pi-|x+y-\pi|]\} &&\text{if}\quad \alpha=\beta=-\frac12
\end{aligned}$$

Axioms H4 and H7 follow now easily.

The translation operators are, by definition, given by

$$\begin{aligned}
T_x^{\alpha,\beta}f(y)&=\int_0^\pi f\,d(\delta_x*\delta_y)=\int_0^\pi f\,d\mu_{x,y}^{\alpha,\beta}\\
&=\int_0^\pi\int_0^1 f\big(\Phi(x,y,v,\cos\theta)\big)dm_{\alpha,\beta}^1(v)dm_\beta^2(\theta).
\end{aligned}$$

Finally, it is easy to check that $d\eta_{\alpha,\beta}(x) \stackrel{\text{def}}{=} (1-\cos x)^{\alpha}(1+\cos x)^{\beta}\sin x\,dx$ is the Haar measure of the hypergroup: the equality

$$\int_0^{\pi} T_x^{\alpha,\beta} f(y)\,d\eta_{\alpha,\beta}(y) = \int_0^{\pi} f(y)\,d\eta_{\alpha,\beta}(y)$$

is trivially true when $f = R_n^{\alpha,\beta} \circ \cos$ (follows from the orthogonality of the Jacobi polynomials) and can be extended by density to the whole $\mathcal{C}([0,\pi])$.

Let us analyze in detail the case $\alpha = \beta = -1/2$. In this case, the translation operator is

$$\begin{aligned} T_x^{-1/2,-1/2} f(y) &= \frac{1}{2} f\big(\Phi(x,y,1,1)\big) + \frac{1}{2} f\big(\Phi(x,y,1,-1)\big) \\ &= \frac{1}{2} f\big(\arccos(\cos x \cos y + \sin x \sin y)\big) \\ &\quad + \frac{1}{2} f\big(\arccos(\cos x \cos y - \sin x \sin y)\big) \\ &= \frac{1}{2} f\big(|x-y|\big) + \frac{1}{2} f\big(\pi - |x+y-\pi|\big) \end{aligned}$$

and the convolution is given by

$$\begin{aligned} k * f(x) &= \int_0^{\pi} k(y) T_x^{-1/2,-1/2} f(y)\,dy \\ &= \frac{1}{2}\int_0^{\pi} k(y)\big[f(|x-y|) + f(\pi - |x+y-\pi|)\big]\,dy. \end{aligned}$$

Let K and F be the 2π-periodic even functions given on $[-\pi,\pi]$ by $K(x) = k(|x|)$ and $F(x) = f(|x|)$. We can write the convolution $K \star F$ in the group $\mathbb{T}$

$$K \star F(x) = \int_{-\pi}^{\pi} K(y) F(x-y)\,dy$$

The function $K \star F$ is an even 2π-periodic function, and for $x \in [-\pi,\pi]$, we have

$$\begin{aligned} K \star F(x) = K \star F(|x|) &= \int_{-\pi}^{\pi} K(y) F(|x|-y)\,dy \\ &= \int_{-\pi}^{0} k(-y) F(|x|-y)\,dy + \int_0^{\pi} k(y) F(|x|-y)\,dy \\ &= \int_0^{\pi} k(y)\big[F(|x|+y) + F(|x|-y)\big]\,dy \\ &= \int_0^{\pi} k(y)\big[f(\pi - ||x|+y-\pi|) + f(||x|-y|)\big]\,dy = 2k * f(|x|). \end{aligned}$$

Thus, the convolution we just defined on $[0,\pi]$ when $\alpha = \beta = -1/2$ coincides, modulo a multiplicative constant, with the convolution on $\mathbb{T}$.

EXAMPLE 3.2: The Bessel-Kingman hypergroup.

Let $X = [0, \infty)$ and $\alpha \geq -1/2$. Recall that the Bessel function of first kind and index α is given by

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{\alpha+2k},$$

a series converging on the complex plane cut along the ray $(-\infty, 0]$.

For any $\lambda \in [0, \infty)$, consider the function

$$j_\lambda^\alpha(x) = \begin{cases} 2^\alpha \Gamma(\alpha+1) \dfrac{J_\alpha(\lambda x)}{(\lambda x)^\alpha} & \text{if } \lambda x \neq 0 \\ 1 & \text{if } \lambda x = 0. \end{cases}$$

The functions $j_\lambda^\alpha(x)$ are real and satisfy

$$\forall \lambda \geq 0, \quad \forall x \geq 0, \quad |j_\lambda^\alpha(x)| \leq 1.$$

Moreover, they satisfy the product formula (see [11], page 5)

$$j_\lambda^\alpha(x) j_\lambda^\alpha(y) = \int_0^\pi j_\lambda^\alpha \left(\sqrt{x^2 + y^2 - 2xy \cos\theta} \right) dm_\alpha^2(\theta).$$

Note that, for $\alpha = -1/2$, $j_\lambda^{-1/2}(x) = \cos(\lambda x)$ and the product formula reduces to

$$\cos(\lambda x)\cos(\lambda y) = \frac{1}{2}\cos\big(\lambda(x-y)\big) + \frac{1}{2}\cos\big(\lambda(x+y)\big).$$

Once again by the Riesz representation theorem, for any $x, y \in [0, \infty)$, there exists a probability measure $\mu_{x,y}^\alpha$ such that, for all $f \in \mathcal{C}_0([0, \infty))$

$$\int_0^\infty f \, d\mu_{x,y}^\alpha = \int_0^\pi f \left(\sqrt{x^2 + y^2 - 2xy\cos\theta} \right) dm_\alpha^2(\theta).$$

Note that, since the range of $\sqrt{x^2 + y^2 - 2xy\cos\theta}$ is $[|x-y|, x+y] = \mathrm{supp}(\mu_{x,y}^\alpha)$, $\mu_{x,y}^\alpha$ is compactly supported. We can now define a hypergroup-type convolution on $M([0, \infty))$: for any $\nu, \gamma \in M([0, \infty))$, define $\nu * \gamma$ by its action on $f \in \mathcal{C}_0([0, \infty))$

$$\int_0^\infty f \, d(\nu * \gamma) \stackrel{\text{def}}{=} \int_0^\infty \int_0^\infty \left(\int_0^\infty f \, d\mu_{x,y}^\alpha \right) d\nu(x) \, d\gamma(y)$$

so that $\delta_x * \delta_y = \mu_{x,y}^\alpha$.

We will indicate this hypergroup by $([0, \infty), *)_\alpha$ or, for reasons that will become clear later, by $([0, \infty), *)_{\alpha,\alpha}$. Just as in the previous example, we can check that $([0, \infty), *)_\alpha$ is a hypergroup with identity equal to 0 and involution equal to the identity map, all the axioms being, at this point, easily verified. The translation operator is, by definition, given by

$$T_x^\alpha f(y) = \int_0^\infty f\, d(\delta_x * \delta_y) = \int_0^\infty f\, d\mu^\alpha_{x,y}$$
$$= \int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right) dm^2_\alpha(\theta).$$

If $\alpha > -1/2$, by a change of variable we obtain that for all $\lambda \geq 0$, for all $x, y \geq 0$ and for all $f \in \mathcal{C}([0,\infty))$,

$$T_x^\alpha f(y) = \int_0^\infty f(z) W_\alpha(x,y,z) z^{2\alpha+1}\, dz$$

where W_α is the function, invariant under any rearrangement of the three variables, given by

$$W_\alpha(x,y,z) = \begin{cases} \frac{2^{1-2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \frac{((x+y)^2-z^2)^{\alpha-1/2}(z^2-(x-y)^2)^{\alpha-1/2}}{(xyz)^{2\alpha}} & \text{if } |x-y| < z < x+y \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mu^\alpha_{x,y}(z)\, dz = W_\alpha(x,y,z) z^{2\alpha+1}\, dz$. It is now easy to see that $d\eta_\alpha(z) \stackrel{\text{def}}{=} z^{2\alpha+1}\, dz$ is a Haar measure:

$$\int_0^\infty T_x^\alpha f(y)\, d\eta(y) = \int_0^\infty \int_0^\infty f(z) W_\alpha(x,y,z)\, d\eta_\alpha(z)\, d\eta_\alpha(y)$$
$$= \int_0^\infty f(z) \int_0^\infty j_0^\alpha(y) W_\alpha(x,z,y)\, d\eta_\alpha(y)\, d\eta_\alpha(z)$$
$$= \int_0^\infty f(z) j_0^\alpha(x) j_0^\alpha(z)\, d\eta_\alpha(z) = \int_0^\infty f(z)\, d\eta_\alpha(z),$$

since $j_0^\alpha(x) = 1$. In the case $\alpha = -1/2$, it is easy to check that the Haar measure is $d\eta_{-1/2}(x) = dx$. In this case, the translation operator is given by

$$T_x^{-1/2} f(y) = \frac{1}{2} f(x+y) + \frac{1}{2} f(|x-y|)$$

and the convolution by

$$k * f(x) = \frac{1}{2}\int_0^\infty k(y)\big[f(x+y) + f(|x-y|)\big]\, dy.$$

Let K and F be the even functions on $\mathbb{R}$ given by $K(x) = k(|x|)$ and $F(x) = f(|x|)$. We can write the convolution (on the group $\mathbb{R}$) of these two functions, obtaining an even function $K \star F$. Note that

$$K \star F(x) = K \star F(|x|) = \int_{-\infty}^\infty K(y) F(|x|-y)\, dy$$
$$= \int_0^\infty k(y) F(|x|-y)\, dy + \int_{-\infty}^0 k(-y) F(|x|-y)\, dy$$
$$= \int_0^\infty k(y)\big[F(|x|-y) + F(|x|+y)\big]\, dy$$
$$= \int_0^\infty k(y)\big[f(||x|-y|) + f(|x|+y)\big]\, dy = 2(k * f)(|x|).$$

Thus, the convolution in the Bessel-Kingman hypergroup of index $\alpha = -1/2$ coincides, modulo a multiplicative constant, with the convolution on $\mathbb{R}$.

EXAMPLE 3.3: Jacobi hypergroups of non-compact type.

These are the non-compact analogues of the hypergroups presented in the first example.

For $\alpha \geq \beta \geq -1/2$, for $\mu \in \mathbb{C}$ and for $t \in [1, \infty)$, let the *Jacobi function* $R_\mu^{\alpha,\beta}(t)$ be defined by

$$R_\mu^{\alpha,\beta}(t) \stackrel{\text{def}}{=} {}_2F_1\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right).$$

Once again we have a hypergroup-type product formula for these functions. Define the function Φ on $[0,\infty) \times [0,\infty) \times [0,1] \times [-1,1]$ taking values in $[0,\infty)$ by

$$\Phi(x, y, v, \cos\theta) \stackrel{\text{def}}{=} \operatorname{argcosh}\Big(\frac{1}{2}(1-v^2)(\cosh x + \cosh y) + \frac{1}{2}(1+v^2)\cosh x \cosh y \\ + \sinh x \sinh y v \cos\theta - \frac{1}{2}(1-v^2)\Big).$$

Note that, if we fix x and y, the range of this function (as a function of v and $\cos\theta$), is $[|x-y|, x+y]$. To see this, call $z = ve^{i\theta}$; as v and $\cos\theta$ vary in their respective domains, z varies in $\mathbb{D}^+$. Defining $A = \sqrt{(\cosh x+1)(\cosh y+1)}$ and $B = \sqrt{(\cosh x-1)(\cosh y-1)}$, we get

$$\cosh\big(\Phi(x, y, v, \cos\theta)\big) = \frac{|A+Bz|^2-2}{2}.$$

It is easy to see that, as z varies in $\mathbb{D}^+$, $\frac{|A+Bz|^2-2}{2}$ varies between $\cosh(x-y)$ and $\cosh(x+y)$.

The product formula is (see [3])

$$R_\mu^{\alpha,\beta}(\cosh x) R_\mu^{\alpha,\beta}(\cosh y) = \int_0^\pi \int_0^1 R_\mu^{\alpha,\beta}\big(\cosh\Phi(x, y, v, \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta).$$

For any $x, y \in [0,\infty)$, by the Riesz representation theorem, there is a probability measure $\mu_{x,y}^{\alpha,\beta}$ on $[0,\infty)$, with support equal to $[|x-y|, x+y]$, such that, for all $f \in \mathcal{C}_0([0,\infty))$

$$\int_0^\infty f\, d\mu_{x,y}^{\alpha,\beta} = \int_0^\pi \int_0^1 f\big(\Phi(x, y, v, \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta).$$

We can now define a hypergroup-type convolution on $M([0,\infty))$: for any ν, $\lambda \in M([0,\infty))$, define $\nu * \lambda$ by its action on $f \in \mathcal{C}_0([0,\infty))$

$$\int_0^\infty f\, d(\nu*\lambda) \stackrel{\text{def}}{=} \int_0^\infty \int_0^\infty \left(\int_0^\infty f\, d\mu_{x,y}^{\alpha,\beta}\right) d\nu(x)\, d\lambda(y),$$

so that $\delta_x * \delta_y = \mu^{\alpha,\beta}_{x,y}$. As usual, this structure makes $([0,\infty), *)_{\alpha,\beta}$ into a hermitian hypergroup (all axioms are now easily verified) with identity equal to 0. The translation operators are, by definition, given by

$$\begin{aligned} T^{\alpha,\beta}_x f(y) &= \int_0^\infty f\, d(\delta_x * \delta_y) = \int_0^\infty f\, d\mu^{\alpha,\beta}_{x,y} \\ &= \int_0^\pi \int_0^1 f\big(\Phi(x,y,v,\cos\theta)\big) dm^1_{\alpha,\beta}(v) dm^2_\beta(\theta). \end{aligned}$$

The Haar measure is

$$\begin{aligned} d\eta_{\alpha,\beta}(x) &\stackrel{\text{def}}{=} (\cosh x - 1)^\alpha (\cosh x + 1)^\beta \sinh x\, dx \\ &= (\cosh x - 1)^{\alpha-\beta} (\sinh x)^{2\beta+1}\, dx. \end{aligned}$$

This is easy to prove in the case $\alpha = \beta = -1/2$. Otherwise the proof follows in a similar way as in Example 2, thanks to the existence (see [3]) of a function $W_{\alpha,\beta}(x,y,z)$ invariant under any rearrangement of the variables x, y, z, and such that

$$T^{\alpha,\beta}_x f(y) = \int_0^\infty f(z) W_{\alpha,\beta}(x,y,z)\, d\eta_{\alpha,\beta}(z).$$

As a final remark, note that if $\alpha = \beta = -1/2$, this hypergroup coincides with the Bessel-Kingman hypergroup of index $-1/2$. Thus, the convolution between two functions in this hypergroup coincides with the convolution on $\mathbb{R}$ of the associated even functions.

4. A transference theorem

Define the family $\mathcal{B} \stackrel{\text{def}}{=} \{(\alpha,\beta) : \alpha \geq \beta \geq -1/2\}$. Suppose that (α',β') and (α,β) belong to $\mathcal{B}$. We say that $(\alpha',\beta') > (\alpha,\beta)$ if and only if

- $\alpha' > \alpha \geq \beta = \beta' \geq -1/2$, or
- $\alpha' = \beta' > \alpha = \beta \geq -1/2$.

Let $(X,*)_{\alpha',\beta'}$ and $(X,*)_{\alpha,\beta}$ be two hypergroups, both taken from the same family among those introduced in the previous section and suppose that $(\alpha',\beta') > (\alpha,\beta)$ (note that if two Bessel-Kingman hypergroups are considered, then the only possible choice is $\alpha' = \beta' > \alpha = \beta \geq -1/2$.)

Recall that, in any possible case, the translation operators are given by

$$T^{\alpha,\beta}_x f(y) = \int_0^\pi \int_0^1 f\big(\Phi(x,y,v,\cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta)$$

for appropriate choices of Φ which, we repeat, are

$$\begin{aligned} \Phi(x,y,v,\cos\theta) = \arccos\Big(&\frac{1}{2}(1-v^2)(\cos x + \cos y) + \frac{1}{2}(1+v^2)\cos x \cos y \\ &+ \sin x \sin y\, v \cos\theta - \frac{1}{2}(1-v^2)\Big) \end{aligned}$$

for the continuous Jacobi polynomial hypergroup,

$$\Phi(x,y,v,\cos\theta)=\sqrt{x^2+y^2-2xy\cos\theta}$$

for the Bessel-Kingman hypergroup, and

$$\Phi(x,y,v,\cos\theta)=\operatorname{argcosh}\Big(\frac{1}{2}(1-v^2)(\cosh x+\cosh y)+\frac{1}{2}(1+v^2)\cosh x\cosh y + \sinh x\sinh y v\cos\theta-\frac{1}{2}(1-v^2)\Big)$$

for the Jacobi hypergroup of non-compact type.

DEFINITION 4.1. With the above assumptions, for any $\psi\in[0,\pi/2]$, define the (α,β,ψ)-pseudo-translation operators by

$$T_x^{\alpha,\beta,\psi}f(y)\stackrel{\text{def}}{=}\int_0^\pi\int_0^1 f\big(\Phi(x,y,v\sin\psi,\cos\theta)\big)\,dm^1_{\alpha,\beta}(v)dm^2_\beta(\theta)$$

if $\alpha'>\alpha\geq\beta=\beta'\geq-1/2$, and

$$\begin{aligned}T_x^{\alpha,\beta,\psi}f(y)&=\int_0^\pi\int_0^1 f\big(\Phi(x,y,v,\cos\theta\sin\psi)\big)dm^1_{\alpha,\beta}(v)dm^2_\beta(\theta)\\&=\int_0^\pi f\big(\Phi(x,y,1,\cos\theta\sin\psi)\big)dm^2_\beta(\theta)\end{aligned}$$

if $\alpha'=\beta'>\alpha=\beta\geq-1/2$.

The next lemma allows to obtain a translation operator associated to (α',β') as an average of (α,β,ψ)-pseudo-translation operators.

Lemma 4.2

With the above assumptions, we have

$$T_x^{\alpha',\beta'}f(y)=\int_0^{\pi/2}T_x^{\alpha,\beta,\psi}f(y)d\widetilde{m}_{\alpha,\alpha'}(\psi)$$

for any $f\in\mathcal{C}(X)$. Here $\widetilde{m}_{\alpha,\alpha'}$ is a probability measure on $[0,\pi/2]$ defined by

$$d\widetilde{m}_{\alpha,\alpha'}(\psi)\stackrel{\text{def}}{=}q_{\alpha',\alpha}(\sin\psi)^{2\alpha+1}(\cos\psi)^{2(\alpha'-\alpha)-1}\,d\psi.$$

The proof of this lemma, as well as the proof of the next two, will be given in the next section.

DEFINITION 4.3. For any $f\in\mathcal{C}_c(X)$ and for any $b\in X$, define the function f_b on X by

$$f_b(x)\stackrel{\text{def}}{=}f\big(\Psi(x,b)\big)$$

where

$$\Psi(x,b) \stackrel{\text{def}}{=} \begin{cases} \Phi(x,b,0,1) & \text{if } \alpha' > \alpha \geq \beta = \beta' \geq -1/2 \\ \\ \Phi(x,b,1,0) & \text{if } \alpha' = \beta' > \alpha = \beta \geq -1/2. \end{cases}$$

The next lemma explains how one can write an L^p norm in $(X,*)_{\alpha',\beta'}$ as an average of L^p norms in $(X,*)_{\alpha,\beta}$.

Lemma 4.4

There exists a positive measure $\mu_{\alpha,\alpha',\beta}$ on an interval $I \subset X$ such that, for all $f \in \mathcal{C}_c(X)$,

$$\|f\|^p_{L^p(X,\eta_{\alpha',\beta'})} = \int_I \|f_b\|^p_{L^p(X,\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b).$$

Lemma 4.5

There is a change of variables $Q = (Q_1, Q_2)$,

$$Q : X \times I \longrightarrow X \times [0,\pi/2]$$
$$(a,b) \longmapsto (x,\psi)$$

with $Q_1(a,b) = \Psi(a,b)$, such that, for any $G \in L^1(X \times [0,\pi/2], \eta_{\alpha',\beta'} \otimes \widetilde{m}_{\alpha,\alpha'})$, we have

$$\int_0^{\pi/2}\int_X G(x,\psi)\, d\eta_{\alpha',\beta'}(x)\, d\widetilde{m}_{\alpha,\alpha'}(\psi) = \int_I\int_X G(Q(a,b))\, d\eta_{\alpha,\beta}(a)\, d\mu_{\alpha,\alpha',\beta}(b)$$

and

$$T^{\alpha,\beta,Q_2(a,b)}_{Q_1(a,b)} f(y) = T^{\alpha,\beta}_a f_b(y).$$

We can now state and prove the main result of this paper.

Theorem 4.6

Let $k \in L^1(X,\eta_{\alpha',\beta'})$. Define the function h on X by

$$h(x) = k(x)\frac{d\eta_{\alpha',\beta'}}{d\eta_{\alpha,\beta}}(x)$$

Obviously $h \in L^1(X,\eta_{\alpha,\beta})$. Suppose that the operator norm of h, as a convolution operator on $L^p(X,\eta_{\alpha,\beta}), 1 \leq p < \infty$, equals $N_p(h)$. Then the operator norm of k, as a convolution operator on $L^p(X,\eta_{\alpha',\beta'})$, is bounded above by $N_p(h)$.

Proof. Let $f \in \mathcal{C}_c(X)$. For the sake of clarity, denote by $*_{\alpha,\beta}$ the convolution on $(X,*)_{\alpha,\beta}$ and by $L^p_{\alpha,\beta}$ the space $L^p(X,\eta_{\alpha,\beta})$. Then, using Lemma 4.2,

$$\begin{aligned}(k *_{\alpha',\beta'} f)(x) &= \int_X k(y) T_x^{\alpha',\beta'} f(y)\, d\eta_{\alpha',\beta'}(y)\\ &= \int_X k(y) \int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi)\, d\eta_{\alpha',\beta'}(y)\\ &= \int_X h(y) \int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi)\, d\eta_{\alpha,\beta}(y).\end{aligned}$$

Thus, using Jensen's inequality and Lemmas 4.4 and 4.5, we get

$$\begin{aligned}\|k *_{\alpha',\beta'} f\|^p_{L^p_{\alpha',\beta'}} &= \int_X \left| \int_X h(y) \int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi)\, d\eta_{\alpha,\beta}(y) \right|^p d\eta_{\alpha',\beta'}(x)\\ &= \int_X \left| \int_0^{\pi/2} \int_X h(y) T_x^{\alpha,\beta,\psi} f(y)\, d\eta_{\alpha,\beta}(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi) \right|^p d\eta_{\alpha',\beta'}(x)\\ &\le \int_X \left[\int_0^{\pi/2} \left| \int_X h(y) T_x^{\alpha,\beta,\psi} f(y)\, d\eta_{\alpha,\beta}(y) \right| d\widetilde{m}_{\alpha,\alpha'}(\psi) \right]^p d\eta_{\alpha',\beta'}(x)\\ &\le \int_X \int_0^{\pi/2} \left| \int_X h(y) T_x^{\alpha,\beta,\psi} f(y)\, d\eta_{\alpha,\beta}(y) \right|^p d\widetilde{m}_{\alpha,\alpha'}(\psi)\, d\eta_{\alpha',\beta'}(x)\\ &= \int_I \int_X \left| \int_X h(y) T_{Q_1(a,b)}^{\alpha,\beta,Q_2(a,b)} f(y)\, d\eta_{\alpha,\beta}(y) \right|^p d\eta_{\alpha,\beta}(a)\, d\mu_{\alpha,\alpha',\beta}(b)\\ &= \int_I \int_X \left| \int_X h(y) T_a^{\alpha,\beta} f_b(y)\, d\eta_{\alpha,\beta}(y) \right|^p d\eta_{\alpha,\beta}(a)\, d\mu_{\alpha,\alpha'\beta}(b)\\ &= \int_I \|h *_{\alpha,\beta} f_b\|^p_{L^p_{\alpha,\beta}} d\mu_{\alpha,\alpha',\beta}(b) \le [N_p(h)]^p \int_I \|f_b\|^p_{L^p_{\alpha,\beta}} d\mu_{\alpha,\alpha',\beta}(b)\\ &= [N_p(h)]^p\, \|f\|^p_{L^p_{\alpha',\beta'}}.\end{aligned}$$

By the density of $\mathcal{C}_c(X)$ in $L^p_{\alpha',\beta'}$, the theorem is proved. □

The following corollaries show how Theorem 4.6 applies specifically to the various examples studied in Section 3. The next two corollaries deal with the continuous Jacobi polynomial hypergroups. Here p is a real number, $1 \le p < \infty$ and $L^p_{\alpha,\beta}$ denotes the space $L^p([0,\pi],(1-\cos x)^\alpha(1+\cos x)^\beta \sin x\, dx)$.

Corollary 4.7

Let $\alpha' \ge \beta' \ge -1/2$.

(a) Let $\alpha \in [\beta',\alpha')$. Suppose $k \in L^1_{\alpha',\beta'}$ and define $h(x) \stackrel{\text{def}}{=} k(x)(1-\cos x)^{\alpha'-\alpha}$. Obviously $h \in L^1_{\alpha,\beta'}$. Suppose that h, as a convolution operator on $L^p_{\alpha,\beta'}$ has norm $N_p(h)$. Then k, as a convolution operator on $L^p_{\alpha',\beta'}$, has norm bounded above by $N_p(h)$.

(b) Let $\alpha \in [-1/2, \beta')$. Suppose $k \in L^1_{\alpha',\beta'}$ and define

$$h(x) \stackrel{\text{def}}{=} k(x)(1-\cos x)^{\alpha'-\alpha}(1+\cos x)^{\beta'-\alpha}.$$

Obviously $h \in L^1_{\alpha,\alpha}$. Suppose that h, as a convolution operator on $L^p_{\alpha,\alpha}$ has norm $N_p(h)$. Then k, as a convolution operator on $L^p_{\alpha',\beta'}$ has norm less than or equal to $N_p(h)$.

Proof. Part (a) coincides with Theorem 4.6, since $(\alpha', \beta') > (\alpha, \beta')$. As for part (b), apply Theorem 4.6 twice; first to the case $(\beta', \beta') > (\alpha, \alpha)$, and then to $(\alpha', \beta') > (\beta', \beta')$. □

Corollary 4.8

Let $\alpha' \geq \beta' \geq -1/2$, and let $k \in L^1_{\alpha',\beta'}$. Let H be the 2π-periodic even function on $\mathbb{R}$, given on $[-\pi, \pi]$ by

$$H(x) \stackrel{\text{def}}{=} k(|x|)(1-\cos x)^{\alpha'}(1+\cos x)^{\beta'}|\sin x|.$$

Let $N_p(H)$ be the smallest constant such that, for all even functions $F \in L^p(\mathbb{T})$,

$$\|H \star F\|_{L^p(\mathbb{T})} \leq N_p(H)\|F\|_{L^p(\mathbb{T})}.$$

Then k, as a convolution operator on $L^p_{\alpha',\beta'}$, has norm less than or equal to $N_p(H)/2$.

Proof. Apply Corollary 4.7. (b) with $\alpha = -1/2$, and get

$$\|k * f\|^p_{L^p_{\alpha',\beta'}} \leq [N_p(h)]^p \, \|f\|^p_{L^p_{\alpha',\beta'}}$$

where $h(x) = k(x)(1-\cos x)^{\alpha'}(1+\cos x)^{\beta'}\sin x$. By the observations at the end of the presentation of the Jacobi polynomial hypergroups, we see that $N_p(h) \leq N_p(H)/2$, since

$$\int_0^\pi |h * f(x)|^p dx = \frac{1}{2}\int_{-\pi}^{\pi}\left|\frac{H \star F(x)}{2}\right|^p dx \leq \left[\frac{N_p(H)}{2}\right]^p \|f\|^p_{L^p_{-1/2,-1/2}}.$$

This proves the corollary. □

The next two corollaries deal with the Bessel-Kingman hypergroups. Here we have $1 \leq p < \infty$ and L^p_α will denote $L^p([0,\infty), x^{2\alpha+1}dx)$.

Corollary 4.9

Let $\alpha' \geq \alpha \geq -1/2$, and suppose $k \in L^1_{\alpha'}$. Define $h(x) \stackrel{\text{def}}{=} k(x)x^{2(\alpha'-\alpha)}$. Obviously $h \in L^1_\alpha$. If h, as a convolution operator on L^p_α, has norm $N_p(h)$, then k, as a convolution operator on $L^P_{\alpha'}$, has norm less than or equal to $N_p(h)$.

Proof. This corollary follows directly from Theorem 4.6, just put $\beta' = \alpha'$ and $\beta = \alpha$. □

Corollary 4.10

Let $\alpha' \geq -1/2$ and let $k \in L^1_{\alpha'}$. Define, on $\mathbb{R}$, the even function $H(x) = k(|x|)|x|^{2\alpha'+1}$. Obviously $H \in L^1(\mathbb{R})$. Let $N_p(H)$ be the smallest constant such that, for all even functions $F \in L^p(\mathbb{R})$,

$$\|H \star F\|^p_{L^p(\mathbb{R})} \leq [N_p(H)]^p \, \|F\|^p_{L^p(\mathbb{R})}.$$

Then k, as a convolution operator on $L^p_{\alpha'}$, has norm less than or equal to $N_p(H)/2$.

Proof. Apply Corollary 4.9 with $\alpha = -1/2$ and then proceed as in the proof of Corollary 4.8. □

The next two corollaries involve the Jacobi hypergroups of non-compact type. Here we have $1 \leq p < \infty$ and $L^p_{\alpha,\beta}$ denotes the space $L^p([0,\infty], (\cosh x - 1)^\alpha (\cosh x + 1)^\beta \sinh x \, dx)$.

Corollary 4.11

Let $\alpha' \geq \beta' \geq -1/2$.

(a) Let $\alpha \in [\beta', \alpha')$. Suppose $k \in L^1_{\alpha',\beta'}$ and define $h(x) \stackrel{\text{def}}{=} k(x)(\cosh x - 1)^{\alpha'-\alpha}$. Obviously $h \in L^1_{\alpha,\beta'}$. Suppose that h, as a convolution operator on $L^p_{\alpha,\beta'}$ has norm $N_p(h)$. Then k, as a convolution operator on $L^p_{\alpha',\beta'}$, has norm bounded above by $N_p(h)$.

(b) Let $\alpha \in [-1/2, \beta')$. Suppose $k \in L^1_{\alpha',\beta'}$ and define

$$h(x) \stackrel{\text{def}}{=} k(x)(\cosh x - 1)^{\alpha'-\alpha}(\cosh x + 1)^{\beta'-\alpha}.$$

Obviously $h \in L^1_{\alpha,\alpha}$. Suppose that h, as a convolution operator on $L^p_{\alpha,\alpha}$ has norm $N_p(h)$. Then k, as a convolution operator on $L^p_{\alpha',\beta'}$ has norm less than or equal to $N_p(h)$.

Proof. Part (a) coincides with Theorem 4.6, since $(\alpha', \beta') > (\alpha, \beta')$. As for part (b), apply Theorem 4.6 twice; first to the case $(\beta', \beta') > (\alpha, \alpha)$, and then to $(\alpha', \beta') > (\beta', \beta')$. □

Corollary 4.12

Let $\alpha' \geq \beta' \geq -1/2$, and let $k \in L^1_{\alpha',\beta'}$. Let H be the even function defined on $\mathbb{R}$ by

$$H(x) \stackrel{\text{def}}{=} k(|x|)(\cosh x - 1)^{\alpha'}(\cosh x + 1)^{\beta'} |\sinh x|.$$

Let $N_p(H)$ be the smallest constant such that, for all even functions $F \in L^p(\mathbb{R})$,

$$\|H \star F\|_{L^p(\mathbb{R})} \leq N_p(H) \|F\|_{L^p(\mathbb{R})}.$$

Then k, as a convolution operator on $L^p_{\alpha',\beta'}$, has norm less than or equal to $N_p(H)/2$.

Proof. Apply Corollary 4.11. (b) with $\alpha = -1/2$, and then proceed as in the proof of Corollary 4.8. □

5. Proofs of the lemmas

In this section we will prove Lemmas 4.2, 4.4 and 4.5. Let us begin with a technical lemma.

Lemma 5.1

For any $A > 0$, $\gamma \geq 0$ and $\delta > 0$, we have

$$\int_0^A \left(A^2 - v^2\right)^\gamma v^{2\delta-1} dv = A^{2\gamma+2\delta} q^{-1}_{\gamma+\delta,\delta-1}.$$

Proof. By substitution, $v = Au$, we obtain

$$\int_0^A \left(A^2 - v^2\right)^\gamma v^{2\delta-1} dv = A^{2\gamma+2\delta} \int_0^1 \left(1 - u^2\right)^\gamma u^{2\delta-1} du = A^{2\gamma+2\delta} q^{-1}_{\gamma+\delta,\delta-1}. \qquad \square$$

Proof of Lemma 4.2. The proof of this lemma does not depend on the choice of the function Φ, it only depends on the indices (α, β) and (α', β'); we have therefore five possible situations.

Case 1. Suppose $\alpha' > \alpha > \beta = \beta' \geq -1/2$. In this case, according to Definition 4.1, we have

$$\begin{aligned}
&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi) \\
&= \int_0^{\pi/2} \int_0^\pi \int_0^1 f\big(\Phi(x, y, v\sin\psi, \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta)\, d\widetilde{m}_{\alpha,\alpha'}(\psi) \\
&= q_{\alpha,\beta} q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^\pi \int_0^1 f\big(\Phi(x, y, v\sin\psi, \cos\theta)\big)(1 - v^2)^{\alpha-\beta-1} v^{2\beta+1} dv \\
&\qquad\qquad dm^2_\beta(\theta)(\cos\psi)^{2(\alpha'-\alpha)-1}(\sin\psi)^{2\alpha+1} d\psi \\
&= q_{\alpha,\beta} q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^\pi \int_0^{\sin\psi} f\big(\Phi(x, y, u, \cos\theta)\big)(\sin^2\psi - u^2)^{\alpha-\beta-1} u^{2\beta+1} du \\
&\qquad\qquad dm^2_\beta(\theta)(\cos\psi)^{2(\alpha'-\alpha)-1} \sin\psi\, d\psi \\
&= q_{\alpha,\beta} q_{\alpha',\alpha} \int_0^1 \int_0^\pi f\big(\Phi(x, y, u, \cos\theta)\big) \int_{\arcsin u}^{\pi/2} \left(\sin^2\psi - u^2\right)^{\alpha-\beta-1} (\cos\psi)^{2(\alpha'-\alpha)-1} \\
&\qquad\qquad \sin\psi\, d\psi\, dm^2_\beta(\theta) u^{2\beta+1} du.
\end{aligned}$$

Notice that the innermost integral equals $\int_0^{\sqrt{1-u^2}} (1 - u^2 - t^2)^{\alpha-\beta-1} t^{2(\alpha'-\alpha)-1} dt$, which, by Lemma 5.1, equals $(1 - u^2)^{\alpha'-\beta-1} q^{-1}_{\alpha'-\beta-1,\alpha'-\alpha-1}$.

Thus, we may conclude that

$$\begin{aligned}
&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y)\, d\widetilde{m}_{\alpha,\alpha'}(\psi) \\
&= \frac{q_{\alpha,\beta}\, q_{\alpha',\alpha}}{q_{\alpha'-\beta-1,\alpha'-\alpha-1}} \int_0^1 \int_0^\pi f\big(\Phi(x, y, u, \cos\theta)\big)\, dm^2_\beta(\theta)(1 - u^2)^{\alpha'-\beta-1} u^{2\beta+1} du \\
&= \frac{q_{\alpha,\beta}\, q_{\alpha',\alpha}}{q_{\alpha'-\beta-1,\alpha'-\alpha-1}\, q_{\alpha',\beta}} T_x^{\alpha',\beta} f(y) = T_x^{\alpha',\beta'} f(y).
\end{aligned}$$

Case 2. Suppose $\alpha' > \alpha = \beta = \beta' > -1/2$. In this case we have

$$\begin{aligned}&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y) d\widetilde{m}_{\alpha,\alpha'}(\psi)\\ &= \int_0^{\pi/2}\int_0^{\pi}\int_0^1 f\big(\Phi(x,y,v\sin\psi,\cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta)\, d\widetilde{m}_{\alpha,\alpha'}(\psi)\\ &= \int_0^{\pi/2}\int_0^{\pi} f\big(\Phi(x,y,\sin\psi,\cos\theta)\big) dm^2_\beta(\theta) d\widetilde{m}_{\alpha,\alpha'}(\psi) = T_x^{\alpha',\beta'} f(y),\end{aligned}$$

where the last equality is obtained by means of the substitution $\sin\psi = v$.

Case 3. Suppose $\alpha' > \alpha = \beta = \beta' = -1/2$. Then

$$\begin{aligned}&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y) d\widetilde{m}_{\alpha,\alpha'}(\psi)\\ &= q_{\alpha',-1/2}\int_0^{\pi/2} \frac{1}{2}\big[f(\Phi(x,y,\sin\psi,1)) + f(\Phi(x,y,\sin\psi,-1))\big](\cos\psi)^{2\alpha'} d\psi\\ &= q_{\alpha',-1/2}\int_0^{1} \frac{1}{2}\big[f(\Phi(x,y,v,1)) + f(\Phi(x,y,v,-1))\big](1-v^2)^{\alpha'-1/2} dv = T_x^{\alpha',\beta'} f(y).\end{aligned}$$

Case 4. Let $\alpha' = \beta' > \alpha = \beta > -1/2$. In this case, according to Definition 4.1, we have

$$\begin{aligned}&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y) d\widetilde{m}_{\alpha,\alpha'}(\psi)\\ &= \int_0^{\pi/2}\int_0^{\pi} f\big(\Phi(x,y,1,\cos\theta\sin\psi)\big)\, dm^2_\beta(\theta)\, d\widetilde{m}_{\alpha,\alpha'}(\psi)\\ &= c_\beta\, q_{\alpha',\alpha}\int_0^{\pi/2}\int_0^{\pi} f\big(\Phi(x,y,1,\cos\theta\sin\psi)\big)(\sin\theta)^{2\beta}\\ &\qquad d\theta(\cos\psi)^{2(\alpha'-\alpha)-1}(\sin\psi)^{2\alpha+1} d\psi.\end{aligned}$$

Changing variables in the innermost integral, namely letting $\cos\theta\sin\psi = \cos t$, we obtain that the last integral equals

$$\begin{aligned}&c_\beta\, q_{\alpha',\alpha}\int_0^{\pi/2}\int_{\pi/2-\psi}^{\pi/2+\psi} f\big(\Phi(x,y,1,\cos t)\big)\big(\sin^2\psi - \cos^2 t\big)^{\beta-1/2}\\ &\qquad \sin t\, dt(\cos\psi)^{2(\alpha'-\alpha)-1}\sin\psi\, d\psi\\ &= c_\beta\, q_{\alpha',\alpha}\int_0^{\pi} f\big(\Phi(x,y,1,\cos t)\big)\sin t\int_{|\pi/2-t|}^{\pi/2}(\sin^2\psi - \cos^2 t)^{\beta-1/2}\\ &\qquad (\cos\psi)^{2(\alpha'-\alpha)-1}\sin\psi\, d\psi\, dt\\ &= c_\beta\, q_{\alpha',\alpha}\int_0^{\pi} f\big(\Phi(x,y,1,\cos t)\big)\sin t\int_0^{\sin t}(\sin^2 t - u^2)^{\beta-1/2} u^{2(\alpha'-\alpha)-1} du\, dt\\ &= \frac{c_\beta\, q_{\alpha',\alpha}}{q_{\alpha'-\frac{1}{2},\alpha'-\alpha-1}}\int_0^{\pi} f\big(\Phi(x,y,1,\cos t)\big)(\sin t)^{2\alpha'} dt = T_x^{\alpha',\beta'} f(y).\end{aligned}$$

Case 5. Suppose finally that $\alpha' = \beta' > \alpha = \beta = -1/2$. Then

$$\begin{aligned}
&\int_0^{\pi/2} T_x^{\alpha,\beta,\psi} f(y) d\widetilde{m}_{\alpha,\alpha'}(\psi) \\
&= q_{\alpha',-1/2} \int_0^{\pi/2} \frac{1}{2}\big[f(\Phi(x,y,1,\sin\psi)) + f(\Phi(x,y,1,-\sin\psi))\big](\cos\psi)^{2\alpha'} d\psi \\
&= \frac{q_{\alpha',-1/2}}{2}\Big[\int_0^{\pi/2} f(\Phi(x,y,1,\cos\psi))(\sin\psi)^{2\alpha'} d\psi \\
&\quad + \int_{\pi/2}^{\pi} f\big(\Phi(x,y,1,\cos\psi)\big)(\sin\psi)^{2\alpha'} d\psi\Big] \\
&= \frac{q_{\alpha',-1/2}}{2} \int_0^{\pi} f\big(\Phi(x,y,1,\cos\psi)\big)(\sin\psi)^{2\alpha'} d\psi \\
&= \frac{q_{\alpha',-1/2}}{2c_{\alpha'}} T_x^{\alpha',\beta'} f(y) = T_x^{\alpha',\beta'} f(y). \ \square
\end{aligned}$$

Unlike the case of the above proof, the proofs of Lemmas 4.4 and 4.5 depend heavily on the functions Φ. For this reason, we will prove these two lemmas separately for each of the families of hypergroups we are studying.

5.2. Continuous Jacobi polynomial hypergroups

Case (a). Suppose first that $\alpha' > \alpha \geq \beta = \beta' \geq -1/2$. In this case we have

$$\begin{aligned}
d\eta_{\alpha,\beta}(x) &= (1-\cos x)^{\alpha}(1+\cos x)^{\beta}\sin x\, dx = (1-\cos x)^{\alpha-\beta}(\sin x)^{2\beta+1} dx \\
\Psi(x,b) &= \Phi(x,b,0,1) = \arccos\left((1+\cos x)\frac{1+\cos b}{2} - 1\right) \\
I &= [0,\pi] \\
d\mu_{\alpha,\alpha',\beta}(b) &= \frac{q_{\alpha',\alpha}}{2}\left(\frac{1+\cos b}{2}\right)^{\alpha+\beta+1}(1-\cos b)^{\alpha'-\alpha-1}\sin b\, db \\
x = Q_1(a,b) &= \Psi(a,b) = \arccos\left((1+\cos a)\frac{1+\cos b}{2} - 1\right) \\
\psi = Q_2(a,b) &= \arcsin\left(\frac{1+\cos b}{2}\frac{\sin a}{\sqrt{(1+\cos a)\frac{1+\cos b}{2}\left(2-(1+\cos a)\frac{1+\cos b}{2}\right)}}\right).
\end{aligned}$$

Proof of Lemma 4.4. With the above definitions we have

$$\begin{aligned}
&\int_I \|f_b\|^p_{L^p([0,\pi],\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b) \\
&= \frac{q_{\alpha',\alpha}}{2}\int_0^{\pi}\int_0^{\pi}\left|f\left(\arccos\left((1+\cos x)\frac{1+\cos b}{2} - 1\right)\right)\right|^p (1-\cos x)^{\alpha-\beta} \\
&(\sin x)^{2\beta+1} dx \left(\frac{1+\cos b}{2}\right)^{\alpha+\beta+1}(1-\cos b)^{\alpha'-\alpha-1}\sin b\, db.
\end{aligned}$$

Applying the substitution $\cos t = (1+\cos x)\frac{1+\cos b}{2} - 1$ to the innermost integral, we obtain that the last expression equals

$$\begin{aligned}&\frac{q_{\alpha',\alpha}}{2}\int_0^\pi\int_b^\pi |f(t)|^p(1+\cos t)^\beta(\cos b-\cos t)^\alpha \sin t(1-\cos b)^{\alpha'-\alpha-1}\sin b\,dt\,db\\ &=\frac{q_{\alpha',\alpha}}{2}\int_0^\pi |f(t)|^p(1+\cos t)^\beta \sin t\left[\int_0^t(\cos b-\cos t)^\alpha(1-\cos b)^{\alpha'-\alpha-1}\sin b\,db\right]dt.\end{aligned}$$

Note that the expression within square parentheses equals

$$\begin{aligned}&\int_0^{1-\cos t}(1-\cos t-u)^\alpha u^{\alpha'-\alpha-1}du\\ &=(1-\cos t)^{\alpha'}\int_0^1(1-v)^\alpha v^{\alpha'-\alpha-1}dv=(1-\cos t)^{\alpha'}2q_{\alpha',\alpha}^{-1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_I \|f_b\|^p_{L^p([0,\pi],\eta_{\alpha,\beta})}d\mu_{\alpha,\alpha',\beta}(b)\\ &=\int_0^\pi |f(t)|^p(1+\cos t)^\beta(1-\cos t)^{\alpha'}\sin t\,dt=\|f\|^p_{L^p([0,\pi],\eta_{\alpha',\beta'})}. \ \square\end{aligned}$$

Proof of Lemma 4.5. We can look at the change of variables Q as a double change of variables, as follows

$$\begin{cases}\cos x = X = (1+\cos a)\dfrac{1+\cos b}{2}-1\\ \sin x\sin\psi = Y = \sin a\,\dfrac{1+\cos b}{2}\end{cases}$$

where $(X,Y)\in\mathbb{D}^+$. The first change of variables is easily understood; as for the second, for a fixed a, as b varies in $[0,\pi]$, the point (X,Y) moves along the segment joining $(-1,0)$ with $(\cos a,\sin a)$, and as a varies in $[0,\pi]$, the above segment sweeps the whole $\mathbb{D}^+$.

Note that

$$dX\,dY=\sin^2 x\cos\psi\,dx\,d\psi=(1+\cos a)\frac{\sin b}{2}\,\frac{1+\cos b}{2}\,da\,db.$$

Thus,

$$\begin{aligned}&q_{\alpha',\alpha}\int_0^{\pi/2}\int_0^\pi G(x,\psi)(1-\cos x)^{\alpha'}(1+\cos x)^{\beta'}\sin x\,dx\,(\sin\psi)^{2\alpha+1}(\cos\psi)^{2(\alpha'-\alpha)-1}d\psi\\ &=\frac{q_{\alpha',\alpha}}{2}\int_0^\pi\int_0^\pi G(Q(a,b))(1-\cos a)^\alpha(1+\cos a)^\beta\sin a\,da\\ &\qquad\qquad\left(\frac{1+\cos b}{2}\right)^{\alpha+\beta+1}(1-\cos b)^{\alpha'-\alpha-1}\sin b\,db,\end{aligned}$$

which proves the first part of the lemma. As for the second, note that by definition

$$T^{\alpha,\beta,Q_2(a,b)}_{Q_1(a,b)} f(y) = \int_0^\pi \int_0^1 f\big(\Phi(Q_1(a,b), y, v\sin(Q_2(a,b)), \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta),$$

whereas

$$\begin{aligned} T^{\alpha,\beta}_a f_b(y) &= \int_0^\pi \int_0^1 f_b\big(\Phi(a, y, v, \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta) \\ &= \int_0^\pi \int_0^1 f\big(\Psi(\Phi(a, y, v, \cos\theta), b)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta). \end{aligned}$$

All there is to prove is that

$$\Phi\big(Q_1(a,b), y, v\sin(Q_2(a,b)), \cos\theta\big) = \Psi\big(\Phi(a, y, v, \cos\theta), b\big)$$

for all a, b, y, v, θ in their respective domains. This is a long but elementary calculation that we shall not show here. In fact, the function Q_2 has been chosen in such a way that the above equality is satisfied. □

Case (b). Suppose now that $\alpha' = \beta' > \alpha = \beta \geq -1/2$. Then we have

$$\begin{aligned} d\eta_{\alpha,\beta}(x) &= (1-\cos x)^\alpha (1+\cos x)^\beta \sin x\, dx = (\sin x)^{2\alpha+1} dx \\ d\eta_{\alpha',\beta'}(x) &= (1-\cos x)^{\alpha'} (1+\cos x)^{\beta'} \sin x\, dx = (\sin x)^{2\alpha'+1} dx \\ \Psi(x,b) &= \Phi(x,b,1,0) = \arccos(\cos x \cos b) \\ I &= \left[0, \frac{\pi}{2}\right] \\ d\mu_{\alpha,\alpha',\beta}(b) &= q_{\alpha',\alpha} (\cos b)^{2(\alpha+1)} (\sin b)^{2(\alpha'-\alpha)-1} db \\ x = Q_1(a,b) &= \Psi(a,b) = \arccos(\cos a \cos b) \\ \psi = Q_2(a,b) &= \arcsin\left(\frac{\sin a \cos b}{\sqrt{1-\cos^2 b \cos^2 a}}\right). \end{aligned}$$

Proof of Lemma 4.4. With the above definitions we have

$$\begin{aligned} &\int_0^{\pi/2} \|f_b\|^p_{L^p([0,\pi],\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b) \\ &= q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^\pi \big|f(\arccos(\cos x \cos b)\big|^p (\sin x)^{2\alpha+1} dx (\cos b)^{2(\alpha+1)} (\sin b)^{2(\alpha'-\alpha)-1} db. \end{aligned}$$

Applying the substitution $\cos t = \cos x \cos b$ to the innermost integral, we obtain that the last expression equals

$$\begin{aligned} &q_{\alpha',\alpha} \int_0^{\pi/2} \int_b^{\pi-b} |f(t)|^p (\cos^2 b - \cos^2 t)^\alpha \sin t\, dt (\sin b)^{2(\alpha'-\alpha)-1} \cos b\, db \\ &= q_{\alpha',\alpha} \int_0^\pi |f(t)|^p \sin t \left[\int_0^{\pi/2-|t-\pi/2|} (\cos^2 b - \cos^2 t)^\alpha (\sin b)^{2(\alpha'-\alpha)-1} \cos b\, db\right] dt \\ &= q_{\alpha',\alpha} \int_0^\pi |f(t)|^p \sin t \left[\int_0^{\sin t} (\sin^2 t - u^2)^\alpha u^{2(\alpha'-\alpha)-1} du\right] dt \\ &= \frac{q_{\alpha',\alpha}}{q_{\alpha',\alpha'-\alpha-1}} \int_0^\pi |f(t)|^p (\sin t)^{2\alpha'+1} dt = \int_0^\pi |f(t)|^p (\sin t)^{2\alpha'+1} dt, \end{aligned}$$

where the second to last equality follows from Lemma 5.1. □

Proof of Lemma 4.5. Once again, we can look at the change of variables Q as a double change of variables

$$\begin{cases} \cos x &= X &= \cos a \cos b \\ \sin x \sin \psi &= Y &= \sin a \cos b \end{cases}$$

where $(X, Y) \in \mathbb{D}^+$. Thus

$$dX\, dY = \sin^2 x \cos\psi\, dx\, d\psi = \cos b \sin b\, da\, db$$

and we have

$$q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^{\pi} G(x, \psi)(\sin x)^{2\alpha'+1} dx (\sin\psi)^{2\alpha+1} (\cos\psi)^{2(\alpha'-\alpha)-1} d\psi$$
$$= q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^{\pi} G(Q(a, b))(\sin a)^{2\alpha+1} da (\cos b)^{2(\alpha+1)} (\sin b)^{2(\alpha'-\alpha)-1} db.$$

which proves the first part. Just as in the previous case, in order to prove the second part of the lemma it is enough to show that

$$\Phi\big(Q_1(a, b), y, 1, \cos\theta \sin(Q_2(a, b))\big) = \Psi\big(\Phi(a, y, 1, \cos\theta), b\big).$$

The left-hand side equals

$$\arccos\left(\cos a \cos b \cos y + \sin(\arccos(\cos a \cos b)) \sin y \cos\theta \frac{\sin a \cos b}{\sqrt{1 - \cos^2 b \cos^2 a}}\right)$$
$$= \arccos(\cos a \cos b \cos y + \sin y \sin a \cos b \cos\theta),$$

which equals the right-hand side. □

5.3. Bessel-Kingman hypergroups

We only have the case $\alpha' = \beta' > \alpha = \beta \geq -1/2$. Therefore

$$\begin{aligned} d\eta_{\alpha,\beta}(x) &= d\eta_{\alpha,\alpha}(x) = x^{2\alpha+1} dx \\ d\eta_{\alpha',\beta'}(x) &= d\eta_{\alpha',\alpha'}(x) = x^{2\alpha'+1} dx \\ \Psi(x, b) &= \Phi(x, b, 1, 0) = \sqrt{x^2 + b^2} \\ I &= [0, \infty) \\ d\mu_{\alpha,\alpha',\beta}(b) &= q_{\alpha',\alpha} b^{2(\alpha'-\alpha)-1} db \\ x = Q_1(a, b) &= \Psi(a, b) = \sqrt{a^2 + b^2} \\ \psi = Q_2(a, b) &= \arcsin\left(\frac{a}{\sqrt{a^2 + b^2}}\right) \end{aligned}$$

Proof of Lemma 4.4. With the above definitions, we obtain

$$\begin{aligned}
&\int_0^\infty \|f_b\|^p_{L^p([0,\infty),\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b) \\
&= q_{\alpha',\alpha} \int_0^\infty \int_0^\infty \left| f\left(\sqrt{x^2+b^2}\right)\right|^p x^{2\alpha+1} dx\, b^{2(\alpha'-\alpha)-1} db \\
&= q_{\alpha',\alpha} \int_0^\infty \int_b^\infty |f(t)|^p (t^2-b^2)^\alpha t\, dt\, b^{2(\alpha'-\alpha)-1} db \\
&= q_{\alpha',\alpha} \int_0^\infty |f(t)|^p t \left[\int_0^t (t^2-b^2)^\alpha b^{2(\alpha'-\alpha)-1} db\right] dt \\
&= \frac{q_{\alpha',\alpha}}{q_{\alpha',\alpha'-\alpha-1}} \int_0^\infty |f(t)|^p t^{2\alpha'+1} dt = \int_0^\infty |f(t)|^p t^{2\alpha'+1} dt,
\end{aligned}$$

where the second to last equality follows from Lemma 5.1. □

Proof of Lemma 4.5. The change of variables is better understood in terms of Q^{-1}:

$$(a,b) = Q^{-1}(x,\psi) = (x\sin\psi, x\cos\psi),$$

thus $da\, db = x\, dx\, d\psi$. Therefore,

$$\begin{aligned}
&q_{\alpha',\alpha} \int_0^{\pi/2} \int_0^\infty G(x,\psi) x^{2\alpha'+1} dx\, (\sin\psi)^{2\alpha+1} (\cos\psi)^{2(\alpha'-\alpha)-1} d\psi \\
&= q_{\alpha',\alpha} \int_0^\infty \int_0^\infty G(Q(a,b)) a^{2\alpha+1} da\, b^{2(\alpha'-\alpha)-1} db.
\end{aligned}$$

As for the second part of the lemma, we have that $T^{\alpha,\beta,Q_2(a,b)}_{Q_1(a,b)} f(y)$ equals

$$\begin{aligned}
&\int_0^\pi f\left(\sqrt{(Q_1(a,b))^2 + y^2 + 2Q_1(a,b) y\cos\theta \sin(Q_2(a,b))}\right) dm^2_\alpha(\theta) \\
&= \int_0^\pi f\left(\sqrt{a^2+b^2+y^2+2ya\cos\theta}\right) dm^2_\alpha(\theta) \\
&= \int_0^\pi f_b\left(\sqrt{a^2+y^2+2ya\cos\theta}\right) dm^2_\alpha(\theta) = T^{\alpha,\beta}_a f_b(y). \ \square
\end{aligned}$$

5.4. Jacobi hypergroups of non-compact type

Case (a). Suppose first that $\alpha' > \alpha \geq \beta = \beta' \geq -1/2$. In this case we have

$$\begin{aligned} d\eta_{\alpha,\beta}(x) &= (\cosh x - 1)^{\alpha}(\cosh x + 1)^{\beta} \sinh x\, dx \\ &= (\cosh x - 1)^{\alpha-\beta}(\sinh x)^{2\beta+1} dx \\ \Psi(x,b) &= \Phi(x,b,0,1) = \operatorname{argcosh}\Big((1+\cosh x)\frac{1+\cosh b}{2} - 1\Big) \\ I &= [0,\infty) \\ d\mu_{\alpha,\alpha',\beta}(b) &= \frac{q_{\alpha',\alpha}}{2}\Big(\frac{1+\cosh b}{2}\Big)^{\alpha+\beta+1}(\cosh b - 1)^{\alpha'-\alpha-1}\sinh b\, db \\ x = Q_1(a,b) &= \Psi(a,b) = \operatorname{argcosh}\Big((1+\cosh a)\frac{1+\cosh b}{2} - 1\Big) \\ \psi = Q_2(a,b) &= \arcsin\left(\frac{\frac{1+\cosh b}{2}\sinh a}{\sqrt{(1+\cosh a)\frac{1+\cosh b}{2}\Big((1+\cosh a)\frac{1+\cosh b}{2} - 2\Big)}}\right). \end{aligned}$$

Proof of Lemma 4.4. With the above definitions we have

$$\begin{aligned} &\int_I \|f_b\|^p_{L^p([0,\infty),\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b) \\ &= \frac{q_{\alpha',\alpha}}{2}\int_0^\infty\int_0^\infty \left| f\left(\operatorname{argcosh}\Big((1+\cosh x)\frac{1+\cosh b}{2} - 1\Big)\right)\right|^p (\cosh x - 1)^{\alpha-\beta} \\ &(\sinh x)^{2\beta+1} dx \Big(\frac{1+\cosh b}{2}\Big)^{\alpha+\beta+1}(\cosh b - 1)^{\alpha'-\alpha-1}\sinh b\, db. \end{aligned}$$

Applying the substitution $\cosh t = (1+\cosh x)\frac{1+\cosh b}{2} - 1$ to the innermost integral, we obtain that the last expression equals

$$\begin{aligned} &\frac{q_{\alpha',\alpha}}{2}\int_0^\infty\int_0^b |f(t)|^p(1+\cosh t)^{\beta}(\cosh t - \cosh b)^{\alpha}\sinh t(\cosh b - 1)^{\alpha'-\alpha-1}\sinh b\, dt\, db \\ &= \frac{q_{\alpha',\alpha}}{2}\int_0^\infty |f(t)|^p(1+\cosh t)^{\beta}\sinh t \\ &\quad\times\left[\int_0^t (\cosh t - \cosh b)^{\alpha}(\cosh b - 1)^{\alpha'-\alpha-1}\sinh b\, db\right] dt. \end{aligned}$$

Note that the expression within square parentheses equals

$$\begin{aligned} &\int_0^{\cosh t - 1}(\cosh t - 1 - u)^{\alpha} u^{\alpha'-\alpha-1} du \\ &= (\cosh t - 1)^{\alpha'}\int_0^1 (1-v)^{\alpha} v^{\alpha'-\alpha-1} dv = (\cosh t - 1)^{\alpha'} 2q^{-1}_{\alpha',\alpha}. \end{aligned}$$

Therefore, we have

$$\int_I \|f_b\|^p_{L^p([0,\infty),\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b)$$
$$= \int_0^\infty |f(t)|^p (1+\cosh t)^\beta (\cosh t - 1)^{\alpha'} \sinh t\, dt = \|f\|^p_{L^p([0,\infty),\eta_{\alpha',\beta'})}. \ \square$$

Proof of Lemma 4.5. We can look at the change of variables Q as a double change of variables, as follows

$$\begin{cases} \cosh x &= X = (1+\cosh a)\frac{1+\cosh b}{2} - 1 \\ \sinh x \sin\psi &= Y = \sinh a\, \frac{1+\cosh b}{2} \end{cases}$$

where $(X,Y) \in \mathbb{H}^+ \overset{\text{def}}{=} \{X^2 - Y^2 \geq 1,\ X \geq 0,\ Y \geq 0\}$. The first change of variables needs no explanation; as for the second, for a fixed a, as b varies between 0 and ∞, the point (X,Y) moves inside $\mathbb{H}^+$ along the line passing through $(-1,0)$ and $(\cosh a, \sinh a)$, and as a varies in $[0,\infty)$, the above ray sweeps the whole $\mathbb{H}^+$.

Note that

$$dX\, dY = \sinh^2 x \cos\psi\, dx\, d\psi = (1+\cosh a)\frac{\sinh b}{2}\,\frac{1+\cosh b}{2}\, da\, db.$$

Thus,

$$q_{\alpha',\alpha}\int_0^{\pi/2}\int_0^\infty G(x,\psi)(\cosh x - 1)^{\alpha'}(\cosh x + 1)^{\beta'} \sinh x\, dx (\sin\psi)^{2\alpha+1}$$
$$(\cos\psi)^{2(\alpha'-\alpha)-1} d\psi$$
$$= \frac{q_{\alpha',\alpha}}{2}\int_0^\infty\int_0^\infty G(Q(a,b))(\cosh a - 1)^\alpha(\cosh a + 1)^\beta \sinh a\, da$$
$$\left(\tfrac{1+\cosh b}{2}\right)^{\alpha+\beta+1}(\cosh b - 1)^{\alpha'-\alpha-1}\sinh b\, db,$$

which proves the first part of the lemma. As for the second, note that by definition

$$T^{\alpha,\beta,Q_2(a,b)}_{Q_1(a,b)} f(y) = \int_0^\pi\int_0^1 f\big(\Phi(Q_1(a,b), y, v\sin(Q_2(a,b)), \cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta),$$

whereas

$$T^{\alpha,\beta}_a f_b(y) = \int_0^\pi\int_0^1 f_b\big(\Phi(a,y,v,\cos\theta)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta)$$
$$= \int_0^\pi\int_0^1 f\big(\Psi(\Phi(a,y,v,\cos\theta),b)\big)\, dm^1_{\alpha,\beta}(v)\, dm^2_\beta(\theta).$$

All there is to prove is that

$$\Phi\big(Q_1(a,b), y, v\sin(Q_2(a,b)\big), \cos\theta) = \Psi\big(\Phi(a,y,v,\cos\theta), b\big)$$

for all a, b, y, v, θ in their respective domains. We leave the proof to the reader. □

Case (b). Suppose now that $\alpha' = \beta' > \alpha = \beta \geq -1/2$. Then we have

$$\begin{aligned}
d\eta_{\alpha,\beta}(x) &= (\cosh x - 1)^{\alpha}(\cosh x + 1)^{\beta} \sinh x \, dx = (\sinh x)^{2\alpha+1} dx \\
d\eta_{\alpha',\beta'}(x) &= (\cosh x - 1)^{\alpha'}(\cosh x + 1)^{\beta'} \sinh x \, dx = (\sinh x)^{2\alpha'+1} dx \\
\Psi(x, b) &= \Phi(x, b, 1, 0) = \operatorname{argcosh}(\cosh x \cosh b) \\
I &= [0, \infty) \\
d\mu_{\alpha,\alpha',\beta}(b) &= q_{\alpha',\alpha}(\cosh b)^{2(\alpha+1)}(\sinh b)^{2(\alpha'-\alpha)-1} db \\
x &= Q_1(a, b) = \Psi(a, b) = \operatorname{argcosh}(\cosh a \cosh b) \\
\psi &= Q_2(a, b) = \arcsin\left(\frac{\sinh a \cosh b}{\sqrt{\cosh^2 b \cosh^2 a - 1}}\right).
\end{aligned}$$

Proof of Lemma 4.4. With the above definitions we have

$$\begin{aligned}
&\int_0^{\infty} \|f_b\|^p_{L^p([0,\infty),\eta_{\alpha,\beta})} d\mu_{\alpha,\alpha',\beta}(b) \\
&= q_{\alpha',\alpha} \int_0^{\infty} \int_0^{\infty} \left|f\big(\operatorname{argcosh}(\cosh x \cosh b)\big)\right|^p (\sinh x)^{2\alpha+1} dx (\cosh b)^{2(\alpha+1)} \\
&\qquad\qquad (\sinh b)^{2(\alpha'-\alpha)-1} db.
\end{aligned}$$

Applying the substitution $\cosh t = \cosh x \cosh b$ to the innermost integral, we obtain that the last expression equals

$$\begin{aligned}
&q_{\alpha',\alpha} \int_0^{\infty} \int_b^{\infty} |f(t)|^p (\cosh^2 t - \cosh^2 b)^{\alpha} \sinh t \, dt (\sinh b)^{2(\alpha'-\alpha)-1} \cosh b \, db \\
&= q_{\alpha',\alpha} \int_0^{\infty} |f(t)|^p \sinh t \left[\int_0^t (\cosh^2 t - \cosh^2 b)^{\alpha} (\sinh b)^{2(\alpha'-\alpha)-1} \cosh b \, db\right] dt \\
&= q_{\alpha',\alpha} \int_0^{\infty} |f(t)|^p \sinh t \left[\int_0^{\sinh t} (\sinh^2 t - u^2)^{\alpha} u^{2(\alpha'-\alpha)-1} du\right] dt \\
&= \frac{q_{\alpha',\alpha}}{q_{\alpha',\alpha'-\alpha-1}} \int_0^{\infty} |f(t)|^p (\sinh t)^{2\alpha'+1} dt = \int_0^{\infty} |f(t)|^p (\sinh t)^{2\alpha'+1} dt,
\end{aligned}$$

where the second to last equality follows from Lemma 5.1. □

Proof of Lemma 4.5. Once again, we can look at the change of variables Q as a double change of variables

$$\begin{cases}
\cosh x = X = \cosh a \cosh b \\
\sinh x \sin \psi = Y = \sinh a \cosh b
\end{cases}$$

where $(X, Y) \in \mathbb{H}^+$. Thus

$$dX \, dY = \sinh^2 x \cos \psi \, dx \, d\psi = \cosh b \sinh b \, da \, db$$

and we have

$$q_{\alpha',\alpha}\int_0^{\pi/2}\int_0^{\infty} G(x,\psi)(\sinh x)^{2\alpha'+1}dx(\sin\psi)^{2\alpha+1}(\cos\psi)^{2(\alpha'-\alpha)-1}d\psi$$
$$= q_{\alpha',\alpha}\int_0^{\infty}\int_0^{\infty} G(Q(a,b))(\sinh a)^{2\alpha+1}da(\cosh b)^{2(\alpha+1)}(\sinh b)^{2(\alpha'-\alpha)-1}db.$$

which proves the first part. Just as in the previous case, in order to prove the second part of the lemma it is enough to show that

$$\Phi(Q_1(a,b),y,1,\cos\theta\sin(Q_2(a,b))) = \Psi(\Phi(a,y,1,\cos\theta),b).$$

The left-hand side equals

$$\operatorname{argcosh}\Big(\cosh a\cosh b\cosh y + \sinh(\operatorname{argcosh}(\cosh a\cosh b)) \sinh y\cos\theta\frac{\sinh a\cosh b}{\sqrt{\cosh^2 b\cosh^2 a - 1}}\Big)$$
$$= \operatorname{argcosh}(\cosh a\cosh b\cosh y + \sinh y\sinh a\cosh b\cos\theta),$$

which equals the right-hand side. □

Acknowledgement.

The author thanks Professor Guido Weiss for his advice and encouragement during the preparation of this paper.

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