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Fixed point free maps of a closed ball with small measures of noncompactness

Martin Väth

University of Würzburg, Department of Mathematics, Am Hubland

D-97074 Würzburg, Germany

E-mail: vaeth@mathematik.uni-wuerzburg.de

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Abstract

We show that in all infinite-dimensional normed spaces it is possible to construct a fixed point free continuous map of the unit ball whose measure of noncompactness is bounded by 2. Moreover, for a large class of spaces (containing separable spaces, Hilbert spaces, and $\ell_\infty(S)$) even the best possible bound 1 is attained for certain measures of noncompactness.

Throughout, let X be a normed space, and

$$B(X) = \{x \in X : \|x\| \le 1\}, \qquad S(X) = \{x \in X : \|x\| = 1\}.$$

It is well-known that whenever X has infinite dimension, there exists a fixed point free continuous map of $B(X)$. Using the axiom of choice, this has first been proved in [10]. A more constructive proof has been given in [19]:

Roughly speaking, the idea is to find a *closed* subset Γ of $B(X)$ which is homeomorphic to $[0,1)$. By the (scalar-valued!) Tietze-Urysohn extension theorem, one may then construct a retraction onto Γ; the composition of this retraction with a fixed point free map of $[0,1)$ (via the homeomorphism) gives the desired map. The construction of Γ can in the above situation be carried out by a countable (recursive) application of Riesz' lemma—the axiom of choice is not needed in its full generality. More precisely, the so-called principle of dependent choices (see e. g. [17]) suffices which allows countably many recursive or nonrecursive choices. Throughout this paper we shall assume only this axiom in place of the (general) axiom of choice.

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It is an unsolved problem what additional properties a fixed point free map F of $B(X)$ can have: Although it is always possible to find even a Lipschitz continuous such map [5, 20] almost nothing is known about the best possible Lipschitz constant (see [15] for a summary of known results in this direction; see also [12, 13]). A well-known fixed point theorem on nonexpanding maps implies that at least in uniformly convex spaces the Lipschitz constant must be larger than 1 (see e. g. [21, 23] or [15]).

We shall discuss a related question now: What can be said about the best possible γ-Lipschitz constant where γ denotes a measure of noncompactness?

More precisely, we are interested in the following measures of noncompactness:

Definition 1. For a set $M \subseteq X$, we define:

1. The *Kuratowski measure of noncompactness* $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M has a finite covering of sets with diameter at most ε.
2. The *Hausdorff measure of noncompactness* $\chi_H(M)$ (with respect to a set $H \subseteq X$) is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in H.
3. The *lattice measure of noncompactness* $\beta(M)$ (in literature also called *separation measure of noncompactness*) is the supremum of all $\varepsilon > 0$ such that M contains a sequence x_n with $\|x_n - x_k\| \geq \varepsilon$ $(n \neq k)$.

Here, we put $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

The above measures of noncompactness $\gamma \in \{\alpha, \beta, \chi_H\}$ are indeed measures of noncompactness in the sense of Sadovskiĭ, i. e. they satisfy

$$\gamma(M) = \gamma(\overline{\mathrm{conv}} M) \qquad \bigl(M \subseteq H \subseteq X,\ H = \overline{\mathrm{conv}} H\bigr), \tag{1}$$

see e. g. [1, 4]. Moreover, they are equivalent in the sense that

$$\chi_X(M) \leq \chi_H(M) \leq \beta(M) \leq \alpha(M) \leq 2\chi_X(M) \qquad (M \subseteq H \subseteq X).$$

For $\gamma \in \{\alpha, \beta, \chi_H\}$ and a map $F : D \subseteq X \to X$, we define $[F]_\gamma$ as the infimum (actually minimum) of all $L \geq 0$ such that

$$\gamma\bigl(F(M)\bigr) \leq L\gamma(M) \qquad (M \subseteq D).$$

(Put $[F]_\gamma = \infty$ if no such L exists). If $F : B(X) \to B(X)$ is Lipschitz continuous with constant L, it follows that $[F]_\alpha \leq L$. Hence, any fixed point free Lipschitz continuous map of $B(X)$ provides an estimate to our above question. However, we can do much better.

It follows from Darbo's fixed point theorem [6] (and its extension of Sadovskiĭ [24]) that any fixed point free continuous map $F : B(X) \to B(X)$ must satisfy $[F]_\gamma \geq 1$ for $\gamma \in \{\alpha, \beta, \chi_X\}$ (if X is a Banach space). Even estimates for the minimal displacement $\kappa(F) = \inf\{\|x - f(x)\| : x \in B(X)\}$ are known [14, 22]. For example, it has been proved in [14] that $\kappa(F) \leq \max\{1 - 1/\alpha(F), 0\}$ in normed spaces.

As a consequence of Darbo's fixed point theorem, $[F]_\gamma = 1$ is the best possible constant that can be expected for a fixed point free map. We shall see that this constant is actually achieved for Hilbert spaces and for $\gamma = \chi_X$ when X is either separable or if

X at least has a geometry which allows a reduction to the separable case. For general spaces X, we shall find the universal bound $[F]_\gamma \leq 2$.

DEFINITION 2. We say that a normed space X is *δ-separated* (with respect to a set $H \subseteq X$), if there is a sequence of pairwise disjoint points $e_n \in B(X)$ and (continuous) paths $\Gamma_n \subseteq B(X)$ without double points joining e_n with e_{n+1} such that the following holds:

1. The paths Γ_n are pairwise disjoint (up to the end points e_n).
2. For any $x \in H$ and any $\varepsilon > 0$ we have $\operatorname{dist}(x, \Gamma_n) \geq \delta - \varepsilon$ for all except finitely many n.

In case $H = X$, we call X *strongly δ-separated.* If the second condition holds for all points $x \in \Gamma := \bigcup \Gamma_n$, we say that X is *weakly δ-separated.*

It is clear that any strongly δ-separated space is also weakly δ-separated. The converse holds up to the factor 2:

Proposition 1

Any weakly δ-separated space is strongly $\delta/2$-separated.

Proof. Let X be weakly δ-separated, and let Γ_n denote the corresponding paths. Let $x \in X$ and $\varepsilon > 0$ be given. We claim that $\operatorname{dist}(x, \Gamma_n) \geq \delta/2 - \varepsilon$ for almost all n. To prove this, it is no loss of generality to assume that $\operatorname{dist}(x, \Gamma_n) < \delta/2 - \varepsilon/2$ for some n, since otherwise we are done already. Then we find some $y \in \Gamma_n$ with $\|x - y\| < \delta/2 - \varepsilon/2$. By assumption, we have $\operatorname{dist}(y, \Gamma_k) \geq \delta - \varepsilon/2$ for almost all k, which implies $\operatorname{dist}(x, \Gamma_k) \geq (\delta - \varepsilon/2) - (\delta/2 - \varepsilon/2) = \delta/2 - \varepsilon$ for almost all k, as claimed. □

We intend to prove now that all spaces are weakly 1-separated. A straightforward attempt to prove this fact is to start with an "almost orthogonal" sequence $e_n \in S(X)$ in the sense that the distance of e_{n+1} to the linear hull U_n of $e_1, \ldots, e_n$ tends to 1 as $n \to \infty$ (Riesz' lemma). Then one might try to connect e_n and e_{n+1} by a line segment, multiplied by a positive scalar function such that the corresponding path Γ_n lies in $S(X)$. However, this approach does not provide us sufficient information on the distance of Γ_n to e. g. e_{n-1}. For this reason, we disturb the path Γ_n such that $\operatorname{dist}(\Gamma_n, U_{n-1})$ tends to 1 as $n \to \infty$: We can do this by combining the proof of Riesz' lemma with the following result on a continuous selection of "almost best approximating" maps:

Lemma 1

If $U \neq 0$ is a separable, complete, and convex subset of some normed space X, then we find for each $c > 1$ a continuous retraction $R : X \to U$ onto U which additionally satisfies $\|x - R(x)\| \leq c \operatorname{dist}(x, U)$.

Proof. Let $u_n \in U$ be dense in U. Choose a sequence $c_n > 0$ with $\sum c_n < \infty$ and $\sum c_n \|u_n\| < \infty$. For $x \in U$ put $R(x) = x$, and for $x \in X \setminus U$ put

$$\lambda_n(x) = c_n \max\left\{0, c - \frac{\|x - u_n\|}{\operatorname{dist}(x, U)}\right\}$$

and

$$R(x) = \frac{\sum \lambda_n(x) u_n}{\sum \lambda_n(x)}.$$

By our choice of c_n, both series converge (since U is complete). Moreover, $\lambda_n(x) > 0$ if and only if $\|x - u_n\| < c \operatorname{dist}(x, U)$. Since u_n is dense in U, this implies on the one hand that the denominator does not vanish. On the other hand, this implies also that $\lambda_n(x)\|x - u_n\| \leq \lambda_n(x) c \operatorname{dist}(x, U)$, and so

$$\|x - R(x)\| = \left\| \frac{\sum \lambda_n(x)(x - u_n)}{\sum \lambda_n(x)} \right\| \leq \frac{\sum \lambda_n(x) \|x - u_n\|}{\sum \lambda_n(x)} \leq c \operatorname{dist}(x, U).$$

Since the right-hand side tends to 0 as $x \to u \in \partial U$, the above formula also shows that R is continuous on ∂U. The fact that R is continuous on $X \setminus U$ and that $R(x) \in \overline{\operatorname{conv}} U = U$ can be verified by a straightforward calculation. □

The proof of Lemma 1 is a slight modification of a well-known construction (see e. g. [8, Proof of Proposition 1.1]).

We emphasize that Lemma 1 does not hold for $c = 1$, i. e. the projection onto an element of best approximation has no continuous selection, in general. An example may be given even for $X = \mathbb{R}^3$ (with an appropriate norm) and U being a one-dimensional subspace. The idea of this example is due to G. Dirr (personal communication):

EXAMPLE 1: Consider in $X = \mathbb{R}^3$ the sets

$$B_r = \left\{(\xi_1, \xi_2, \xi_3) : \frac{2}{1 - \xi_3}\xi_1^2 + \xi_2^2 \leq 1,\ \xi_1 > 0,\ \xi_3 \in [-1, 1)\right\},$$
$$P = \overline{B}_r \setminus B_r = \{(0, \xi_2, \xi_3) : \xi_2, \xi_3 \in [-1, 1]\},$$

and $B_\ell = -B_r$. Observe that B_r is convex: Indeed, if $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$ belong to B_r, and $0 < \lambda < 1$ is given, put $\zeta_i = \lambda \xi_i + (1 - \lambda)\eta_i$. Since ellipses are convex, the function $f(t) = 2\zeta_1^2/(1 - t) + \zeta_2^2$ satisfies $f(\xi_3) \leq 1$ and $f(\eta_3) \leq 1$. By the monotonicity of f on $[-1, 1)$, this implies $f(\zeta_3) \leq 1$, and so $\lambda x + (1 - \lambda)y = (\zeta_1, \zeta_2, \zeta_3) \in B_r$, as claimed.

Now it is easily verified that $B = B_\ell \cup P \cup B_r$ is closed, even, convex, and absorbing. Consequently, we may equip $X = \mathbb{R}^3$ with the Minkowski norm $\|\cdot\|$ corresponding to B, i. e. $B = B(X)$.

Consider now the subspace $U = \{(0, 0, s) : s \in \mathbb{R}\}$ and the points $x_t = (t, \sqrt{1 - t^2}, 0)$ with $t \in [-1, 1]$. For $t < 0$, the intersection $(x_t + B) \cap U$ consists of the single point $y_t = (0, 0, -1)$: Indeed, if $x \in B$ is such that $x_t + x = (0, 0, s) \in U$, we must have $x = (-t, -\sqrt{1 - t^2}, s) \in B_r$, because $t < 0$. Hence, $\frac{2}{1-s}t^2 + (1 - t^2) \leq 1$, and so $\left(\frac{2}{1-s} - 1\right)t^2 \leq 0$. In view of $t < 0$ and $s \in [-1, 1)$, this is only possible if $s = -1$.

We may conclude that y_t is the only point satisfying $\|x_t - y_t\| = 1 = \operatorname{dist}(x_t, U)$. For $t > 0$ a similar calculation shows that the only point satisfying this relation is $y_t = (0, 0, 1)$. Now observe that the mapping $t \mapsto x_t$ defines a continuous path in X, but the mapping $t \mapsto y_t$ onto the corresponding points of best approximation in U is discontinuous at 0 (no matter how y_t is chosen for $t = 0$).

Theorem 1

If a normed space X does not have finite dimension, then it is weakly 1-separated (and thus strongly 1/2-separated).

Proof. Fix some sequence $1 > \varepsilon_n \downarrow \emptyset$. Choose $e_1 \in S(X)$ arbitrary. Now we proceed recursively: Assume that $e_1, \ldots, e_n \in S(X)$ and $\Gamma_1, \ldots, \Gamma_{n-1} \subseteq S(X)$ are already defined. Let U_n denote the linear hull of $e_1, \ldots, e_n$. Since X does not have finite dimension, there is some $f_{n+1} \in X \setminus U_n$. Put $c_n = (1 - \varepsilon_n)^{-1}$, and let R_n be a retraction onto U_{n-1} ($U_0 := \{0\}$) with $\|x - R_n(x)\| \leq c_n \operatorname{dist}(x, U_{n-1})$ (Lemma 1). Define $g_n(t) = e_n + t(f_{n+1} - e_n)$ $(0 \leq t \leq 1)$ and $u_n(t) = R(g_n(t))$. Let $\Gamma_n \subseteq S(X)$ be the path

$$\gamma_n(t) = \frac{g_n(t) - u_n(t)}{\|g_n(t) - u_n(t)\|} \qquad (0 \leq t \leq 1),$$

and put $e_{n+1} = \gamma_{n+1}(1)$.

The paths Γ_n (and the points e_n) obtained in this way are pairwise disjoint, since by construction $\Gamma_n \cap U_n = \{e_n\}$ and $\Gamma_n \subseteq U_{n+1}$ (observe that $U_1 \subseteq U_2 \subseteq \ldots$). Moreover, we have $\operatorname{dist}(\Gamma_n, U_{n-1}) \geq 1 - \varepsilon_n$: Indeed, for any $t \in [0, 1]$ and any $u \in U_{n-1}$ we have

$$\begin{aligned} \|\gamma_n(t) - u\| &= \frac{\|g_n(t) - u_n(t) - \|g_n(t) - u_n(t)\| u\|}{\|g_n(t) - u_n(t)\|} \\ &\geq \frac{\operatorname{dist}(g_n(t), U_{n-1})}{\|g_n(t) - u_n(t)\|} \geq c_n^{-1} = 1 - \varepsilon_n. \end{aligned}$$

Since $\Gamma_n \subseteq U_n$ and $U_1 \subseteq U_2 \subseteq \ldots$, we thus have $\operatorname{dist}(\Gamma_n, \Gamma_k) \geq 1 - \varepsilon_n$ whenever $|n - k| > 1$. Hence, we have proved that the two conditions of Definition 2 are satisfied with $\delta = 1$. $\square$

We do not know whether each infinite-dimensional space is even strongly 1-separated. At least, this is the case for a rather large class of spaces. Let us first show that this is true for separable spaces:

Theorem 2

If a separable normed space X does not have finite dimension, then it is strongly 1-separated.

Proof. Since X is separable, there is a sequence $U_1 \subseteq U_2 \ldots \subseteq$ of finite-dimensional subspaces of X such that $\bigcup U_n$ is dense in X. (For example, if $\{x_1, x_2, \ldots\}$ is dense in X, one may let U_n denote the linear hull of $x_1, \ldots, x_n$). It is no loss of generality to assume that $\dim U_n = n$. Let $b_1, b_2, \ldots \in S(X)$ be such that U_n is the linear hull of $b_1, \ldots, b_n$. Now we define e_n and Γ_n as in the previous proof with the choice $e_1 = b_1$ and $f_{n+1} = b_{n+1}$. Then Γ_n has the desired properties: If $x \in H = X$ and $\varepsilon > 0$ are given, we find some n and some $u \in U_n$ with $\|x - u\| \leq \varepsilon/2$. For all $k > n+1$ we have $\operatorname{dist}(u, \Gamma_k) \geq \operatorname{dist}(\Gamma_k, U_n) \geq 1 - \varepsilon_k$, and so $\operatorname{dist}(x, \Gamma_k) \geq 1 - \varepsilon/2 - \varepsilon_k \geq 1 - \varepsilon$ for all except finitely many $k > n+1$, as claimed. □

Once we know that separable spaces are strongly 1-separated, we may conclude that even more spaces are 1-separated:

Definition 3. A normed space X has the *separable retraction property*, if there is a separable subspace Y which does not have finite dimension such that for each $\varepsilon > 0$ we find a mapping $R : X \to Y$ which satisfies

$$\|R(x) - R(y)\| \leq (1+\varepsilon)\|x - y\| + \varepsilon \qquad \big(x \in X,\ y \in B(Y)\big) \tag{2}$$

and

$$\|R(y) - y\| \leq \varepsilon \qquad \big(y \in B(Y)\big). \tag{3}$$

For example, one may let R be a linear projection of X onto Y with norm 1 (if such a projection exists).

Corollary 1

Any space X with the separable retraction property is strongly 1-separated.

Proof. Let Y be as in Definition 3. By Theorem 2, the space Y is 1-separated. Let $\Gamma_n \subseteq B(Y) \subseteq B(X)$ be the corresponding paths. Given $x \in X$ and $\varepsilon > 0$, we have $\operatorname{dist}(R(x), \Gamma_n) \geq 1 - \varepsilon$ for almost all n, where R denotes the mapping from Definition 3. Now (2) and (3) imply in view of $\Gamma_n \subseteq B(Y)$ that

$$\begin{aligned}\operatorname{dist}(x, \Gamma_n) &\geq (1+\varepsilon)^{-1}\operatorname{dist}\big(R(x), R(\Gamma_n)\big) - \varepsilon \\ &\geq (1+\varepsilon)^{-1}[\operatorname{dist}(R(x), \Gamma_n) - \varepsilon] - \varepsilon \geq (1+\varepsilon)^{-1}(1 - 2\varepsilon) - \varepsilon\end{aligned}$$

for almost all n. Since the last term tends to 1 as $\varepsilon \downarrow 0$, this implies that X is strongly 1-separated. □

The class of spaces X with the separable retraction property is actually rather large: It contains all separable spaces, but also many other spaces. Before we give some examples, let us note that the retraction property is inherited by “sufficiently large” subspaces:

Proposition 2

Let X have the separable retraction property with Y as in Definition 3. If $X_0 \subseteq X$ is some subspace such that $Y_0 = X_0 \cap \overline{Y}$ is dense in $\overline{Y}$, then also X_0 has the separable retraction property and we may use Y_0 as the corresponding subspace in Definition 3.

For $X_0 = X$, we find as a special case:

Lemma 2

If the subspace $Y \subseteq X$ has the property of Definition 3, then the closure $\overline{Y}$ has this property, too.

Proof. Given $\varepsilon > 0$, let $R : X \to Y$ be a mapping satisfying (2) and (3). For any $y \in B(\overline{Y})$ we find some $y_0 \in B(Y)$ such that $\|y - y_0\| < \varepsilon$. Hence, $\|R(y) - R(y_0)\| \leq \varepsilon^2 + \varepsilon$. This implies

$$\begin{aligned}\|R(x) - R(y)\| &= \|(R(x) - R(y_0)) + (R(y_0) - R(y))\| \\ &< (\varepsilon\|(x - y) + (y - y_0)\| + \varepsilon) + (\varepsilon^2 + \varepsilon) < \varepsilon\|x - y\| + 3\varepsilon^2 + \varepsilon\end{aligned}$$

and $\|R(y) - y\| = \|(R(y) - R(y_0)) + (R(y_0) - y_0)\| \leq \varepsilon^2 + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\overline{Y}$ has the required property. $\square$

Proof of Proposition 2. Replacing Y by $\overline{Y}$ (Lemma 2) we may assume that Y is closed. Given $\varepsilon > 0$, let $R : X \to Y$ be a mapping satisfying (2) and (3). Let $C \subseteq Y_0$ be countable and dense in Y_0 (and thus $Y = \overline{C}$). We define inductively a mapping $\rho : Y \to C$ with the property that $\|\rho(y) - y\| \leq \varepsilon$ for all $y \in Y$. If $c_1, c_2, \ldots$ is an enumeration of the elements of C, we just map the ball (in Y) with radius ε and center c_1 onto c_1, then all points of the ball with radius ε and center c_2 which have not yet been mapped onto c_2, and so on.

Let $R_0 = \rho \circ R$. Then we have for any $x \in X_0$ and $y \in B(Y_0)$ that

$$\begin{aligned}\|R_0(x) - R_0(y)\| = \|(\rho(R(x)) - R(x)) + (R(x) - R(y)) + (R(y) - \rho(R(y)))\| \leq \varepsilon \\ + (\varepsilon\|x - y\| + \varepsilon) + \varepsilon\end{aligned}$$

and also $\|R_0(y) - y\| = \|(\rho(R(y)) - R(y)) + (R(y) - y)\| < 2\varepsilon$. $\square$

Example 2: Any infinite-dimensional inner product space X has the separable retraction property and we may even use any separable subspace $Y \subseteq X$ of infinite dimension in Definition 3:

Indeed, let $\overline{X}$ and $\overline{Y}$ denote the completion of X and Y. The Hilbert space $\overline{X}$ has the separable retraction property and we may use $\overline{Y} \subseteq \overline{X}$ as the corresponding subspace in Definition 3 (let $R : \overline{X} \to \overline{Y}$ be the orthogonal projection onto the element of nearest distance). Hence, the statement follows from Proposition 2.

Example 3: Any space $X = \ell_p(S)$ ($1 \leq p < \infty$) with an infinite (not necessarily countable!) set S has the separable retraction property: Let $S_0 \subseteq S$ be countable, and Y be the subspace of all functions $x \in X$ which vanish outside S_0. The mapping R can be chosen as the projection which puts all components outside S_0 to 0.

Example 4: Recall that a Banach space X is called *weakly compactly generated* (see e. g. [7, 16]) if there is some weakly compact set $K \subseteq X$ whose linear hull is dense in X. All separable spaces and all reflexive spaces are weakly compactly generated. Using the axiom of choice, it can be proved that for any weakly compactly generated Banach space X and any separable $X_0 \subseteq X$ there is a linear projection $R : X \to Y$ with norm 1 onto a separable subspace $Y \supseteq X_0$, see e. g. [2] or [9, Chapter 5, §2, Theorem 3]. This result implies that any weakly compactly generated Banach space has the separable retraction property (if one assumes the axiom of choice).

Using a refinement of the construction from [19] described in the beginning, we can prove the following result.

We call a set $M \subseteq X$ *relatively compact* if $\overline{M}$ is compact. Observe that $\alpha(M) = 0$ does not imply that M is relatively compact, since X need not be complete (but the converse holds). In this sense, the following statement is even slightly stronger than one might expect after the first reading:

Theorem 3

Let X be weakly δ_1-separated and δ_2-separated with respect to some $H \subseteq X$ (with the same paths Γ_n).

Then there is a fixed point free continuous map $F : X \to B(X)$ with the following property: If the image $F(M)$ is not relatively compact, then $\beta(M) \geq \delta_1$ and $\chi_H(M) \geq \delta_2$.

It can additionally be arranged that $F(X)$ is contained in the set Γ occurring in Definition 2.

Proof. Let e_n and Γ_n be as in Definition 2. For each n, we find some $\varepsilon_n > 0$ such that $\operatorname{dist}(\Gamma_n, \Gamma_k) \geq \varepsilon_n$ ($|k - n| > 1$):

Indeed, let n be fixed. We find a finite $(\delta_1/3)$-net $N \subseteq \Gamma_n$ for Γ_n. For almost all k we have $\operatorname{dist}(N, \Gamma_k) \geq 2\delta_1/3$, and so $\operatorname{dist}(\Gamma_n, \Gamma_k) \geq \delta_1/3$. In particular, the relation $\operatorname{dist}(\Gamma_n, \Gamma_k) < \delta_1/3$ holds at most for finitely many k. Since $\operatorname{dist}(\Gamma_n, \Gamma_k) > 0$ whenever $|k - n| > 1$ (because Γ_n and Γ_k are compact and disjoint), we find indeed some $\varepsilon_n > 0$ with the required properties.

Without loss of generality, we may assume that $\varepsilon_n \to 0$. Then we consider the "half-open line" $\Gamma = \bigcup \Gamma_n$, and the "shrinking tube" $T = \bigcup_n \{x \in X : \operatorname{dist}(x, \Gamma) \leq \varepsilon_n\}$ around Γ. Fix some strictly increasing sequence $\alpha_n \uparrow 1$ with $\alpha_1 = 0$, and define a scalar continuous map $f : \Gamma \to [0, 1)$ such that $f : \Gamma_n \to [\alpha_n, \alpha_{n+1}]$ and $f(e_n) = \alpha_n$.

Now we extend f onto the tube T: For $x \in \partial T$, we put $f(x) = 0$. To define f in the interior of T, we divide T into the chain $T_n = \{x \in X : \operatorname{dist}(x, \Gamma_n) < \varepsilon_n\}$. Observe that $T_n \cap T_k = \emptyset$ for $|n - k| > 1$ and $T_n \cap T_{n+1} \ni e_{n+1}$. On each $T_n \cap T_{n+1}$, we use the Tietze-Urysohn theorem to extend f to a continuous map $f : T_n \cap T_{n+1} \to [0, \alpha_{n+1}]$ in such a way that we need not redefine f on some of the sets where it already was defined. After we have done so, we extend f in a similar manner to a continuous map $f : T_n \to [0, \alpha_{n+1}]$.

The map $f : T \to [0, 1)$ obtained in this way has the property that $f(T_n) \subseteq [0, \alpha_{n+1}]$, $f(\Gamma_n) \subseteq [\alpha_n, \alpha_{n+1}]$, and $f(\partial T) = 0$. Putting $f(x) = 0$ outside T, we have a continuous map $f : X \to [0, 1)$. Now let $g : [0, 1) \to \Gamma$ be continuous with $g(0) = e_3$ such that the interval $[\alpha_n, \alpha_{n+1}]$ is mapped onto Γ_{n+2}. We claim that $F = g \circ f$ has the required properties:

If x is some point in the image of F, we have $x \in \Gamma_n$ for some n. But then $f(x) \in [\alpha_n, \alpha_{n+1}]$ which implies $F(x) \in \Gamma_{n+2}$ which is disjoint from $\Gamma_n \ni x$. Hence, F has no fixed points.

Now, let $M \subseteq X$ be such that $F(M)$ is not relatively compact. Then $s = \sup\ f(M)$ must be 1, because otherwise $F(M)$ were contained in the compact set $g([0,s])$. Since $f(T_k) \subseteq [0, \alpha_{k+1}]$, we may conclude that M intersects infinitely many T_k, say $x_n \in M \cap T_{k_n}$ where $k_1 < k_2 < \ldots$.

Given $\varepsilon > 0$, we may assume by passing to a subsequence that $\varepsilon_{k_n} < \varepsilon$. Hence, for each x_n we find some $y_n \in \Gamma_{k_n}$ with $\|x_n - y_n\| < \varepsilon$.

We now define a sequence $n_1 < n_2 < \ldots$ inductively as follows: Put $n_1 = 1$. If $n_1, \ldots, n_j$ are already defined, our assumption implies that each of the finitely many relations

$$\|y_n - y_m\| \geq \delta_1 - \varepsilon \qquad (n = n_1, \ldots, n_j)$$

is satisfied for almost all m. Hence, the relations are even satisfied simultaneously for almost all indices m. Let $m > n_j$ be some index with this property, and put $n_{j+1} = m$. For the thus defined sequence we have

$$\begin{aligned}\delta_1 - \varepsilon < \|y_n - y_{n_{j+1}}\| &= \|(y_n - x_n) + (x_n - x_{n_{j+1}}) + (x_{n_{j+1}} - y_{n_{j+1}})\| \\ &\leq \varepsilon + \|x_n - x_{n_{j+1}}\| + \varepsilon \quad (n = n_1, \ldots, n_j).\end{aligned}$$

Hence, the sequence $z_j = x_{n_j} \in M$ satisfies $\|z_j - z_i\| \geq \delta_1 - 3\varepsilon$ $(j \neq i)$, and so $\beta(M) \geq \delta_1 - 3\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\beta(M) \geq \delta_1$, as claimed.

The proof of the estimate $\chi_H(M) \geq \delta_2$ is similar: Given $\varepsilon > 0$, let $x_n \in M \cap T_{k_n}$ and $y_n \in \Gamma_{k_n}$ be as above, and $N \subseteq H$ be an arbitrary finite set. Our assumption implies that for each $x \in N$ the relation $\|x - y_n\| \geq \delta_2 - \varepsilon$ holds for almost all n. Hence, the relation holds for almost all n even simultaneously for all $x \in N$. Fixing some n with this property, we have

$$\delta_2 - \varepsilon \leq \|x - y_n\| = \|(x - x_n) + (x - y_n)\| \leq \|x - x_n\| + \varepsilon.$$

Since $x_n \in M$, the (arbitrary!) finite set $N \subseteq H$ can at most be an $(\delta_2 - 2\varepsilon)$-net for M, i.e. $\chi_H(M) \geq \delta_2 - 2\varepsilon$. Hence, $\chi_H(M) \geq \delta_2$. □

Sometimes one is not only interested in fixed point free continuous maps $F : B(X) \to B(X)$ but even in such maps with the additional property that F vanishes on $\partial B(X) = S(X)$. This can easily be achieved:

Proposition 3

Let $F : B(X) \to B(X)$ be continuous without fixed points. Then there is a continuous map $G : B(X) \to B(X)$ without fixed points with $[G]_\gamma = [F]_\gamma$ $(\gamma \in \{\alpha, \beta, \chi_X\})$ which additionally satisfies $G|_{S(X)} = 0$.

Proof. A map with the required properties is given by

$$G(x) = \begin{cases} \dfrac{1}{2}F(2x) & \text{if } \|x\| \leq \dfrac{1}{2}, \\ (1 - \|x\|)F\Big(\dfrac{x}{\|x\|}\Big) & \text{if } \|x\| > \dfrac{1}{2}. \end{cases}$$

Indeed, let $B_{1/2} = \{x \in X : \|x\| \leq \frac{1}{2}\}$. Since each γ is semi-homogeneous (i. e. $\gamma(\lambda M) = |\lambda|\gamma(M)$), it readily follows that $[G|_{B_{1/2}}]_\gamma = [F]_\gamma$, and so $[G]_\gamma \geq [F]_\gamma$. For the converse equality, we also use the fact that each γ is semi-additive, i. e. $\gamma(M_1 \cup M_2) = \max\{\gamma(M_1), \gamma(M_2)\}$: Given a set $M \subseteq B(X)$ it thus suffices to prove $\gamma(G(M_i)) \leq [F]_\gamma \gamma(M_i)$ $(i = 1, 2)$ where $M_1 = M \cap B_{1/2}$ and $M_2 = M \setminus M_1$, since $G(M) = G(M_1) \cup G(M_2)$. For $i = 1$, the estimate follows from $[G|_{B_{1/2}}] = [F]_\gamma$. For $i = 2$, we apply (1): Letting

$$RM_2 = \Big\{\frac{x}{\|x\|} : x \in M_2\Big\},$$

we have $\gamma(RM_2) \leq \gamma\big(\overline{\operatorname{conv}}(M_2 \cup \{0\})\big) = \gamma(M_2 \cup \{0\}) = \gamma(M_2)$, and so

$$\gamma\big(G(M_2)\big) \leq \gamma\big(\overline{\operatorname{conv}}(F(RM_2) \cup \{0\})\big) \leq \gamma\big(F(RM_2)\big) \leq [F]_\gamma \gamma(RM_2) \leq [F]_\gamma \gamma(M_2),$$

as claimed. □

Let us now summarize the main results:

Corollary 2

If the normed space X does not have finite dimension, then there is a fixed point free continuous map $F : B(X) \to B(X)$ such that $[F]_\gamma \leq 2$ $(\gamma \in \{\alpha, \beta, \chi_X\})$.

Moreover, if X is separable or at least has the separable retraction property (Definition 3) we even have $[F]_{\chi_X} \leq 1$.

It may additionally be arranged that $F|_{S(X)} = 0$.

Proof. By Theorem 1, X is weakly 1-separated; under the additional assumptions, Theorem 2 resp. Corollary 1 imply that X is even strongly 1-separated. Thus, the first statements follow from Theorem 3, observing that $\gamma(F(M)) \leq \gamma(\Gamma) \leq \gamma(B(X))$ for each $M \subseteq X$, and that $\beta(B(X)) \leq \alpha(B(X)) \leq 2\chi_X(B(X)) \leq 2$. The final statement follows from Proposition 3. □

In Hilbert spaces the best possible value is always achieved:

Theorem 4

If an inner product space X does not have finite dimension, then there exists a fixed point free continuous map $F : B(X) \to B(X)$ with $F|_{S(X)} = 0$ such that $[F]_\alpha = [F]_\beta = [F]_{\chi_X} = 1$. Moreover,

$$\alpha\big(F(M)\big) \leq \beta(M) \qquad \big(M \subseteq B(X)\big). \tag{4}$$

Proof. Let $e_1, e_2, \ldots$ be an orthonormal system in X, and Γ_n be given by the path $\gamma_n(t) = g_n(t)/\|g_n(t)\|$ $(0 \leq t \leq 1)$ where $g_n(t) = e_n + t(e_{n+1} - e_n)$. For any $|n - k| > 1$ and any $t, s \in [0, 1]$ the points $\gamma_n(t)$ and $\gamma_k(s)$ are orthonormal to each other, and so $\|\gamma_n(t) - \gamma_k(s)\| = \sqrt{2}$. This implies that X is weakly $\sqrt{2}$-separated. Moreover, we have $\|\gamma_n(t) - \gamma_k(s)\| \leq \sqrt{2}$ for each k, n and each $t, s \in [0, 1]$. Indeed, writing $\gamma_n(t) = \sum \xi_j e_j$ and $\gamma_k(s) = \sum \eta_j e_j$, we have $\xi_j, \eta_j \geq 0$ and $\|(\xi_j)_j\|_2 = \|(\eta_j)_j\|_2 = 1$

(actually, all except at most two of the entries of $(\xi_j)_j$ and $(\eta_j)_j$ vanish). In particular, $\xi_j, \eta_j \geq 0$ implies that

$$\begin{aligned}\|\gamma_n(t) - \gamma_k(s)\| = \|(\xi_j - \eta_j)_j\|_2 &\leq \|(\max\{|\xi_j|, |\eta_j|\})_j\|_2 \\ &\leq \|(|\xi_1|, |\eta_1|, |\xi_2|, |\eta_2|, \dots)\|_2.\end{aligned}$$

Since the vector in the last norm is the sum of two vectors which are orthonormal to each other (in the space ℓ_2), its norm is $\sqrt{2}$. Hence, $\|\gamma_n(t) - \gamma_k(s)\| \leq \sqrt{2}$, as claimed.

We thus have proved that $\|\gamma_n(t) - \gamma_k(s)\| \leq \sqrt{2}$, and so the diameter of the set $\Gamma = \bigcup \Gamma_n$ is bounded by $\sqrt{2}$. In particular, for the map F of Theorem 3, we have

$$\alpha\big(F(M)\big) \leq \alpha(\Gamma) \leq \sqrt{2} \leq \beta(M)$$

for each set $M \subseteq X$ for which $F(M)$ is not relatively compact. This estimate proves (4), and $[F]_\alpha \leq 1$ and $[F]_\beta \leq 1$ follows. To see that $[F]_{\chi X} \leq 1$ holds for the *same* function F, we observe that by Example 2 the linear hull Y of $e_1, e_2, \dots$ can be used to witness that X has the separable retraction property. Hence, the proof of Corollary 1 shows that actually the same paths Γ_n can be used to witness that X is strongly 1-separated, and so the estimate $[F]_{\chi X} \leq 1$ follows from Theorem 3.

To see the converse estimate $[F]_\gamma \geq 1$, we can not use Darbo's fixed point theorem, since we do not assume that X is complete. However, it is clear from the construction that F maps the set $M = \{e_1, e_2, \dots\}$ onto the set $M \setminus M_0$ where M_0 is finite. Hence, $\gamma(F(M)) = \gamma(M)$. Since $\gamma(M) > 0$ (because e. g. $\beta(M) > 0$), this implies $[F]_\gamma \geq 1$, as claimed. $\square$

For $X = \ell_2$, a much simpler example of a fixed point free continuous function $F : B(\ell_2) \to B(\ell_2)$ with $[F]_\gamma = 1$ is due to Kakutani [18] (see e. g. [4, Chapter II, Example 7]): For

$$x = (\xi_1, \xi_2, \dots) \in B(\ell_2) \quad \text{put} \quad F(x) = \Big(\sqrt{1 - \|x\|^2}, \xi_1, \xi_2, \dots\Big).$$

However, as we shall see below, this map does not satisfy the stronger compactness condition (4).

An example of a fixed point free continuous function $F : B(\ell_2) \to B(\ell_2)$ which additionally satisfies $F|_{S(\ell_2)} = 0$ but only the estimate $[F]_\alpha \leq 2$ was presented in [3].

In the space $X = c_0$, the sharp bound $[F]_\alpha = 1$ was already obtained in [3]. The latter is even easier to describe than in the Hilbert space case. This is not too surprising, since one might expect: The "better" the geometry of the space, the harder it is to find fixed point free functions (recall e. g. that in uniformly convex Banach spaces no fixed point free nonexpanding function can exist). In this sense it is extremely surprising that for Hilbert spaces (with the "nicest" geometry), we could construct a fixed point free function with the best possible bounds. Thus, it is a natural conjecture that the best possible bound $[F]_\alpha = 1$ is actually achieved in *every* space X (of infinite dimension). However, it is hard to prove this conjecture, since mappings are very difficult to describe in general spaces.

Let us prove the conjecture for the spaces $X = \ell_p(S)$ (and once more for $X = c_0$): Note that in particular even in the "worst" space $X = \ell_\infty(S)$ (which might be considered as the extreme converse of a Hilbert space) the conjecture $[F]_\alpha = 1$ thus holds true.

Theorem 5

Let X be a subspace of some space $\ell_p(S)$ $(1 \le p \le \infty)$ which contains infinitely many elements with pairwise disjoint support (thus S is infinite but need not necessarily be countable).

Then there is a fixed point free map F of $B(X)$ satisfying $F|_{S(X)} = 0$, $[F]_\alpha = [F]_\beta = [F]_{\chi_X} = 1$, and even (4).

Proof. Let e_n be a sequence of elements with disjoint support with $\|e_n\| = 1$. With these vectors, we may proceed analogously to the proof of Theorem 4. The main difference is that we now get the estimates $\|\gamma_n(t) - \gamma_k(s)\| = 2^{1/p}$ for $|n-k| > 1$ and $\|\gamma_n(t) - \gamma_k(s)\| \le 2^{1/p}$ for all n, k, and so $\alpha(F(M)) \le 2^{1/p} \le \beta(M)$ whenever $F(M)$ is not relatively compact (here, we put $2^{1/\infty} := 1$). Hence (4) holds which implies $[F]_\alpha = [F]_\beta = 1$. In the case $p < \infty$, the proof of $[F]_{\chi_X} = 1$ is similar as in the proof of Theorem 4.

In the case $p = \infty$, the equality $[F]_{\chi_X} = 1$ follows from $[F]_\alpha = 1$ in view of the fact that $\alpha(M) = 2\chi_X(M)$ for any $M \subseteq X = \ell_\infty(S)$. The latter has been observed e. g. in [4, Chapter II, Example 2] (for $S = \mathbb{N}$). The argument is simple: If $A_1, \ldots, A_n \subseteq M$ is a finite covering of M with $\operatorname{diam}(A_i) \le 2\varepsilon$, then $\{(\frac{1}{2}(\xi_{i,s} + \eta_{i,s}))_s : i = 1, \ldots, n\} \subseteq X$ provides a finite ε-net for M where $\xi_{i,s} = \inf\{\xi_s : (\xi_s)_s \in A_i\}$ and $\eta_{i,s} = \sup\{\xi_s : (\xi_s)_s \in A_i\}$. $\square$

We note that Theorem 4 is not a straightforward consequence of Theorem 5 for $p = 2$ unless we assume the (uncountable) axiom of choice (since the proof of the representation theorem $X \cong \ell_2(S)$ for Hilbert spaces X uses the axiom of choice).

The map constructed in the previous proofs (more precisely in the proof of Theorem 3) is of course not very explicit. At least for $p < \infty$, we can give a simpler map with the same properties (only the *proof* that the map in the following example has the required properties is lengthy):

EXAMPLE 5: Let X be a subspace of some $\ell_p(S)$ $(1 \le p < \infty)$ which contains infinitely many elements with pairwise disjoint support. To simplify notation we may by an obvious identification assume that $\mathbb{N} \subseteq S$ and that X contains the vectors e_n where $e_n = (\delta_{sn})_s$ (with the Kronecker symbol δ_{sn}). Choose a sequence $\alpha_n \uparrow 1$ with $\alpha_1 = 0$, and put $\alpha_s = 0$ for $s \in S \setminus \mathbb{N}$. For $x = (\xi_s)_s$, let $f(x) = \|(\alpha_s \xi_s)_s\|$. Put

$$g(t) = e_{n+2} + \frac{t - \alpha_n}{\alpha_{n+1} - \alpha_n}(e_{n+3} - e_{n+2}) \qquad \big(\alpha_n \le t \le \alpha_{n+1}\big)$$

and $G(t) = g(t)/\|g(t)\|$. Then $F(x) = G(f(x))$ maps $B(X)$ continuously into $S(X)$ and does not have any fixed points. Indeed, if $x \in F(X)$, then $x = y/\|y\|$ where

$y = e_n + \lambda(e_{n+1} - e_n)$ for some n and $0 \leq \lambda \leq 1$ which implies $f(x) \in [\alpha_n, \alpha_{n+1}]$, and so $g(f(x))$ is linear independent from x. Consequently, $F(x) \neq x$.

As we have seen in the proof of Theorems 4/5, the range of G (which is the range of F) has diameter $2^{1/p}$, and so $\alpha(F(X)) \leq 2^{1/p}$. Moreover, if $F(M)$ is not relatively compact, then $s = \sup f(M)$ must be 1, since otherwise $F(M)$ is contained in the compact set $G([0, s])$. We shall prove that $\sup f(M) = 1$ for some $M \subseteq B(X)$ in turn implies $\chi_X(M) \geq \chi_{l_p}(M) \geq 1$ and $\beta(M) \geq 2^{1/p}$ which then shows that we actually have (4) and $[F]_\alpha = [F]_\beta = [F]_{\chi_X} = 1$ (for the converse estimates $[F]_\gamma \geq 1$ we may argue as in the proof of Theorems 4/5).

Indeed, the relation $\sup f(M) = 1$ implies that for each n and each ε we find some $x = (\xi_s)_s \in M$ such that $\sum_{k>n} |\xi_k|^p > 1 - \varepsilon$:

Otherwise, we would have for all $x = (\xi_s)_s \in M$ that

$$\begin{aligned} f(x)^p \leq \alpha_n^p \sum_{k=1}^{n} |\xi_k|^p + \sum_{k=n+1}^{\infty} |\xi_k|^p \leq \alpha_n^p \|x\|^p \\ + (1 - \alpha_n^p) \sum_{k=n+1}^{\infty} |\xi_k|^p \leq \alpha_n^p + (1 - \alpha_n^p)(1 - \varepsilon), \end{aligned}$$

and so $f(x)^p$ were bounded by $1 - \varepsilon(1 - \alpha_n^p) < 1$, contradicting $\sup f(M) = 1$.

Given $\varepsilon > 0$, we may thus inductively choose elements $x_k = (\xi_s^{(k)})_s \in M$ and indices $n_k \in \mathbb{N}$ with $n_1 < n_2 < \dots$ such that $\sum_{n=n_k}^{\infty} |\xi_n^{(k)}|^p > 1 - \varepsilon$ and $\sum_{n=n_{k+1}}^{\infty} |\xi_n^{(k)}|^p < \varepsilon$. Since $\sum_{n=1}^{\infty} |\xi_n^{(k)}|^p \leq \|x\|^p \leq 1$, this implies

$$\sum_{n=n_k}^{n_{k+1}-1} |\xi_n^{(j)}|^p \begin{cases} > 1 - 2\varepsilon & \text{if } j = k, \\ < \varepsilon & \text{if } j \neq k. \end{cases}$$

Consequently, we have for $j \neq k$ that

$$\begin{aligned} \sum_{n=n_k}^{n_{k+1}-1} |\xi_n^{(k)} - \xi_n^{(j)}|^p &= \|(\xi_{n_k}^{(k)}, \dots, \xi_{n_{k+1}-1}^{(k)}) - (\xi_{n_k}^{(j)}, \dots, \xi_{n_{k+1}-1}^{(j)})\|_p^p \\ &\geq \Big| \|(\xi_{n_k}^{(k)}, \dots, \xi_{n_{k+1}-1}^{(k)})\|_p - \|(\xi_{n_k}^{(j)}, \dots, \xi_{n_{k+1}-1}^{(j)})\|_p \Big|^p \\ &> \big((1 - 2\varepsilon)^{1/p} - \varepsilon^{1/p}\big)^p \end{aligned}$$

(we may assume that ε is sufficiently small). We may conclude that for $j \neq k$ the relation

$$\|x_k - x_j\|_p^p \geq \sum_{n=n_k}^{n_{k+1}-1} |\xi_n^{(k)} - \xi_n^{(j)}|^p + \sum_{n=n_j}^{n_{j+1}-1} |\xi_n^{(k)} - \xi_n^{(j)}|^p > 2\big((1 - 2\varepsilon)^{1/p} - \varepsilon^{1/p}\big)^p$$

holds. Since the last expression tends to 2 as $\varepsilon \to 0$, we thus have proved that indeed $\beta(M) \geq 2^{1/p}$.

The proof that $\chi_{\ell_p}(M) \geq 1$ is similar: If $N \subseteq X$ is a finite set, we find for each $\varepsilon > 0$ some n such that for each $y = (\eta_s)_s \in N$ the estimate $\sum_{k=n}^{\infty} |\eta_k|^p < \varepsilon$ holds. As we have seen above, we find some $x = (\xi_s)_s \in M$ such that $\sum_{k>n} |\xi_k|^p > 1 - \varepsilon$. A similar calculation as before shows that $\|x - y\| > (1-2\varepsilon)^{1/p} - \varepsilon^{1/p}$. Hence, $\chi_X(M) \geq (1 - 2\varepsilon)^{1/p} - \varepsilon^{1/p}$ which for $\varepsilon \downarrow 0$ implies $\chi_X(M) \geq 1$, as claimed.

The map F in the previous example has a rather strange property: Any ball $B_r = \{x \in X : \|x\| \leq r\}$ with $r < 1$ is mapped into a relatively compact set. However, for $r = 1$ the image has a rather large measure of noncompactness. An inspection of the proof of Theorem 3 shows that the map F constructed there has the same property (since the tube T in the proof becomes "arbitrary small", and so B_r intersects at most finitely many of the sets T_n which implies $\sup f(B_r) \leq \alpha_{n+1} < 1$ for some n).

An even simpler example of a fixed point free map with $[F]_\gamma = 1$ in $X = \ell_p(S)$ $(1 \leq p < \infty)$ is given by the map mentioned after Theorem 4:

EXAMPLE 6: With the notation as in Example 5, define a map $F : B(X) \to B(X)$ in the following way: For $x = (\xi_s)_s \in X$ let $F(x) = (\eta_s)_s$ where $\eta_s = \xi_s$ for $s \in S \setminus \mathbb{N}$, $\eta_1 = (1 - \|x\|)^{1/p}$, and $\eta_{n+1} = \xi_n$ $(n = 1, 2, \ldots)$. Then F has no fixed points, but $[F]_\alpha = [F]_\beta = [F]_{\chi_X} = 1$.

The simpler map F from Example 6 does not satisfy the stronger compactness condition (4) if $p > 1$, even for $M = B(X)$: Indeed, since $\alpha(B(X)) = 2$ for any infinite-dimensional space X (see e. g. [1, 4]), we evidently have $\alpha(F(M)) = \alpha(M) = 2$ for this map. In contrast, $\beta(M) = 2^{1/p} < 2$ for $1 < p < \infty$ by [4, Theorem 3.10 and 3.13].

Let us finally note that our results are related to the *measure of solvability* introduced in [11]:

Let $B_r = \{x \in X : \|x\| \leq r\}$. Given $F : X \to X$ with $F(x) \neq 0$ $(x \neq 0)$ one defines

$$\begin{aligned}\nu_r(F) = \inf \big\{ k \geq 0 : &\text{ there exists continuous } G : B_r \to X \text{with } G|_{\partial B_r} = 0,\ [G]_\alpha \leq k, \\ &\text{and } F(x) \neq G(x) \text{ for all } x \in B_r(X) \big\}\end{aligned}$$

and calls

$$\nu(F) = \inf_{r>0} \nu_r(F)$$

the *measure of solvability of* F. This measure is related to the so-called *measure of non-solvability* introduced in [25] and has some applications in the spectral theory for nonlinear operators, see e. g. [3, 11]. The explicit calculation of $\nu(F)$ even for simple operators F is rather complicated. Our above results imply:

Corollary 3

In any infinite-dimensional Banach space X, the identity operator I satisfies $1 \leq \nu(I) \leq 2$. Moreover, we have $\nu(I) = 1$ if either X is a Hilbert space or if X is a (closed) subspace of some $\ell_p(S)$ $(1 \leq p \leq \infty)$ which contains infinitely many elements with disjoint support.

Proof. The estimate $\nu(I) \geq 1$ for Banach spaces is a consequence of Rothe's variant of Darbo's fixed point theorem (any continuous map $G : B_r \to X$ with $G(\partial B_r) \subseteq B_r$ and $[G]_\alpha < 1$ has a fixed point; for a proof see e. g. [8, Section 18.1]). For $r = 1$, we have $\nu_1(I) \leq 2$, since by Corollary 2 we find a function $G : B_1 \to B_1$ with $G_{\partial B_1} = 0$, $[G]_\alpha \leq 2$ such that $I(x) \neq G(x)$ for all $x \in B_1$. Hence, $\nu(I) \leq \nu_1(I) \leq 2$. Theorem 4 resp. 5 implies analogously that $\nu(I) \leq \nu_1(I) \leq 1$ if X is a Hilbert space resp. X is a subspace of $\ell_p(S)$ as in the statement. $\square$

The previous argument shows together with Proposition 3: The earlier mentioned conjecture that in any infinite-dimensional Banach space X there is a fixed point free continuous map $F : B(X) \to B(X)$ with $[F]_\alpha \leq 1$ is equivalent to the equality $\nu(I) = 1$ in any infinite-dimensional Banach space.

References

1. R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, Boston, Berlin, 1992.
2. D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math. (2)* **88** (1968), 35–46.
3. J. Appell, E. Giorgieri, and M. Väth, On a class of maps related to the Furi-Martelli-Vignoli spectrum, *Ann. Mat. Pura Appl.*, to appear.
4. J. M. Ayerbe Toledano, T. Domínguez Benavides, and G. López Acedo, *Measures of noncompactness in metric fixed point theory*, Birkhäuser Verlag, Basel, Boston, Berlin, 1997.
5. Y. Benyamini and Y. Sternfeld, Spheres in infinite-dimensional normed spaces are Lipschitz contractible, *Proc. Amer. Math. Soc.* **38**(3) (1983), 439–445.
6. G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.
7. M. M. Day, *Normed linear spaces*, 3rd ed., Springer-Verlag, Berlin, Heidelberg, New York, 1973.
8. K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, Heidelberg, 1985.
9. J. Diestel, *Geometry of Banach spaces–selected topics*, Lectures Notes in Mathematics, **485**, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
10. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367.
11. W. Feng, A new spectral theory for nonlinear operators and its applications, *Abstr. Appl. Anal.* **2** (1997), 163–183.
12. C. Franchetti, Lipschitz maps on the unit ball of normed spaces, *Confer. Sem. Mat. Univ. Bari* **202** (1985), 11 p.
13. C. Franchetti, Lipschitz maps and the geometry of the unit ball in normed spaces, *Arch. Math. (Basel)* **46**(1) (1986), 76–84.
14. M. Furi and M. Martelli, On the minimal displacement of points under α-Lipschitz maps in normed spaces, *Boll. Un. Mat. Ital. (4)* **9** (1974), 791–799.
15. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
16. P. Habala, P. Hájek, and V. Zizlev, *Introduction to Banach spaces II*, Matfyzpress, Karlovy, 1996.
17. T. J. Jech, *The Axiom of Choice*, North-Holland Publ. Company, Amsterdam, 1973.
18. S. Kakutani, Topological properties of the unit sphere of a Hilbert space, *Proc. Imp. Acad. Tokyo* **19** (1943), 269–271.

19. V. L. Klee, Jr., Some topological properties of convex sets, *Trans. Amer. Math. Soc.* **78** (1955), 30–45.
20. P. K. Lin and Y. Sternfeld, Convex sets with the Lipschitz fixed point property are compact, *Proc. Amer. Math. Soc.* **93**(4) (1985), 633–639.
21. S. Reich, The fixed point property for nonexpansive mappings I, *Amer. Math. Monthly* **83** (1976), 266–268.
22. S. Reich, A minimal displacement problem, *Comment. Math. Univ. St. Pauli* **26**(2) (1977/78), 131–135.
23. S. Reich, The fixed point property for nonexpansive mappings II, *Amer. Math. Monthly* **87** (1980), 292–294.
24. B. N. Sadovskiĭ, Limit-compact and condensing operators (Russian), *Uspehi Mat. Nauk* **27**(1) (1972), 81–146.
25. E. U. Tarafdar and H. B. Thompson, On the solvability of nonlinear, noncompact operator equations, *J. Austral. Math. Soc. Ser. A* **43**(1) (1987), 103–126.