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Fiber cones and the integral closure of ideals

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I \subseteq R$ be an ideal. This paper studies the question of when $\mathfrak{m}I$ is integrally closed. Particular attention is focused on the case R is a regular local ring and I is a reduced ideal. This question arose through a question posed by Eisenbud and Mazur on the existence of evolutions.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I \subseteq R$ be an ideal. In this paper we are interested in the question of when $\mathfrak{m}I$ is integrally closed. To explain our motivation for studying this question, we recall the definition of an evolution [5].

Definition 1.1. Let k be a ring and let T be a local k-algebra essentially of finite type over k. An *evolution* of T over k is a local k-algebra R, essentially of finite type, and a surjection $R \to T$ of k-algebras inducing an isomorphism $\Omega_{R/k} \otimes_R T \cong \Omega_{T/k}$. The evolution is trivial if $R \to T$ is an isomorphism, and T is evolutionarily stable over k if all evolutions are trivial.

It is possible that over a field k of characteristic 0, every reduced local k-algebra T essentially of finite type over k is evolutionarily stable. No counterexamples are known. See [5, 7, 13] for some partial results. The question concerning the existence

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of evolutions first arose in connection with the proof of Mazur's 'Modular lifting conjecture', which is a crucial ingredient of A. Wiles's proof of Fermat's Last Theorem. However, in this case the algebra in question is a reduced flat algebra over a complete DVR of mixed characteristic. It was observed by Mazur that even in equicharacteristic 0, the question of the existence of non-trivial evolutions was open. Kunz [5] first gave counterexamples in positive characteristic.

Whenever $\mathfrak{m}I$ is integrally closed and $(R, \mathfrak{m})$ is smooth over a field k of characteristic 0 with $[R/\mathfrak{m} : k] < \infty$ it follows that R/I is evolutionarily stable as a k-algebra. To study when $\mathfrak{m}I$ is integrally closed, it is easier to study the ideal $\mathfrak{m}\mathcal{R}(I)$ in $\mathcal{R}(I) = R[It] \subseteq R[t]$, the *Rees algebra* of I. This is the approach we take in this paper, finding conditions under which we can prove this ideal is unmixed, or even defines a Cohen-Macaulay quotient ring. We can then apply these results to conclude that $\mathfrak{m}I$ is integrally closed. See Proposition 1.5. This approach introduces the fiber cone

$$F_{\mathfrak{m}}(I) := \bigoplus_{n\in\mathbb{N}} I^n/\mathfrak{m}I^n = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$$

as a vehicle to study when $\mathfrak{m}I$ is integrally closed.

Impetus for doing so comes from the following observation.

Observation ([7]). *Let $(R, \mathfrak{m})$ be a local domain and let $I \subseteq R$ be an integrally closed ideal such that $F_{\mathfrak{m}}(I)$ is reduced in degree 1. Then $\mathfrak{m}I$ is integrally closed. If $(R, \mathfrak{m})$ is smooth over a field k of characteristic 0 with $[R/\mathfrak{m} : k] < \infty$, and if $I \subseteq R$ is a reduced and equidimensional ideal with $\mathfrak{m}I$ integrally closed, then R/I is evolutionarily stable as a k-algebra.*

Whereas the Rees-algebra $\mathcal{R}(I)$ and the associated graded algebra

$$\mathrm{gr}_I(R) := \bigoplus_{n\in\mathbb{N}} I^n/I^{n+1}$$

of I have been studied extensively over the last decades (cf. [22] and the references quoted there), comparatively little is known about $F_{\mathfrak{m}}(I)$, though this object also contains important information about I and the structure of the special fiber of its blow-up (cf. [17] or [4]). The case of $\mathfrak{m}$-primary (or more generally of equimultiple) ideals has been studied by K. Shah [18], [19] and T. Cortadellas and S. Zarzuela [3], who have been interested in particular in the Cohen-Macaulayness of $F_{\mathfrak{m}}(I)$.

Before beginning our study of the fiber cone, we first recall basic definitions.

The integral closure $\overline{\mathfrak{a}}$ of an ideal $\mathfrak{a} \subseteq R$ is defined to be the set of all elements $x \in R$ satisfying an equation of type

$$x^n + a_1x^{n-1} + \ldots + a_n = 0 \qquad \text{with } \ a_l \in \mathfrak{a}^l.$$

Note that $\overline{\mathfrak{a}} \subseteq R$ is an ideal again. The ideal $\mathfrak{a}$ is called integrally closed if $\overline{\mathfrak{a}} = \mathfrak{a}$, and it is called normal it $\overline{\mathfrak{a}^n} = \mathfrak{a}^n$ for all $n \in \mathbb{N}$. In case $(R, \mathfrak{m})$ is a regular local ring of dimension 2, the product of any two integrally closed ideals is integrally closed [23]. However already in the three-dimensional case this fails badly. In [10] the second author constructed examples of integrally closed $\mathfrak{m}$-primary ideals in a regular local ring of dimension 3 such that $\mathfrak{m}I$ is not integrally closed. In general it may be hard to determine whether for a given integrally closed ideal I the product $\mathfrak{m}I$ is integrally closed again, though this is a very interesting question (not only because of its relations to evolutions).

In the context of trying to understand when $\mathfrak{m}I$ is integrally closed, it is reasonable to assume that I is a normal ideal; already in this special case almost nothing is known. What we find in this paper is a surprising bifurcation depending on the cubics in the defining ideal of the Rees algebra of I. If there are no monic cubics, then we are able to prove $\mathfrak{m}I$ is integrally closed quite generally, see Theorem 1.3. While this case is very general, the argument is much easier than the case in which there are monic cubics in the defining ideal of the Rees algebra of I (with respect to a minimal generating set). In this case we are able to prove that if as many monic cubics appear as is possible (meaning the reduction number of I is 2), then we can again prove $\mathfrak{m}I$ is integrally closed in many cases. Our approach in this last case is through the structure of the fiber cone of I.

Recall that the analytic spread $l(I)$ of $I \subseteq R$ is defined to be $l(I) := \dim(F_{\mathfrak{m}}(I))$. If $R/\mathfrak{m}$ is infinite, then I has a (minimal) reduction generated by $l(I)$ elements, i.e. there exists an ideal $J = (a_1, \ldots, a_{l(I)}) \subseteq I$ with

$$JI^n = I^{n+1} \qquad \text{for all } n \text{ sufficiently large.}$$

The least n with this property is called the reduction number of I with respect to J and denoted $r_J(I)$. In general $r_J(I)$ depends on the minimal reduction J, and we write $r(I)$ for the smallest of these numbers and call it the reduction number of I. If $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay, then $r_J(I)$ is independent of the special choice of a minimal reduction $J \subseteq I$, cf. [8], [3].

Assume that $r_1, \ldots, r_m$ is a minimal set of generators and let

$$(*) \qquad R^{\nu} \xrightarrow{\varphi} R^m \xrightarrow{\varepsilon} I \longrightarrow 0$$

be a minimal presentation of I induced by it. The t^{th} Fitting ideal $\text{Fitt}_t(I)$ of I is defined to be the ideal generated by the $(m-t)^{\text{th}}$-rowed subdeterminants of φ (resp. any matrix of it). As $(*)$ is a minimal presentation, we have

(1) $\text{Fitt}_{m-1}(I) \subseteq \mathfrak{m}$.

(2) $\text{Fitt}_m(I) = R$.

Set $P = R[T_1, \ldots, T_m]$, write

$$\varphi(e_i) = (\lambda_{i,1}, \ldots, \lambda_{i,m})$$

and set $l_i(T) = \sum_{j=1}^{m} \lambda_{i,j} T_j$ for $i = 1, \ldots, \nu$ to get

$$\text{Sym}_R(I) = P/(l_1, \ldots, l_\nu) = P/\mathfrak{a}_1$$

with $\mathfrak{a}_1 = (l_1, \ldots, l_\nu)$. Furthermore we have a canonical epimorphism

$$P \longrightarrow \mathcal{R}(I)$$

with $T_l \longmapsto r_l \cdot t$ for $l \in \{1, \ldots, m\}$, so that we can write

$$\mathcal{R}(I) = P/\mathfrak{a}$$

for some ideal $\mathfrak{a}(I) = \mathfrak{a} \subseteq P$ with $\mathfrak{a}_1 \subseteq \mathfrak{a}$ and $\mathfrak{a} \cap P_1 = \mathfrak{a}_1 \cap P_1$ (cf. [22], chapt. 8).

DEFINITION 1.2. i) By $\mathfrak{a}_n(I) = \mathfrak{a}_n \subseteq P$ we denote the ideal generated by all homogeneous forms in $\mathfrak{a}$ of degree at most n. We let $c_n = c_n(I)$ denote the content of the ideal $\mathfrak{a}_n$, namely the ideal in R generated by the coefficients of all elements in $\mathfrak{a}_n$.

ii) The ideal $I \subseteq R$ is called *syzygetic*, if $\mathfrak{a}_1 = \mathfrak{a}_2$, or, equivalently, if $\mathrm{Sym}^2_R(I) = I^2$.

Evidently $c_1 \subseteq c_2 \subseteq \ldots$, and $c_1 = \mathrm{Fitt}_{m-1}(I)$, where $m = \mu(I)$.

Our first theorem gives very general conditions under which $\mathfrak{m}I$ is integrally closed.

Theorem 1.3

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension $d > 1$ and let $I \subseteq R$ be an unmixed ideal. Assume that I is minimally generated by m elements and that $P = R[T_1, \ldots, T_m] \to \mathcal{R}(I)$ is the usual surjective map of a polynomial ring over R onto the Rees algebra of I. Let $\mathfrak{a}$ be equal to the kernel. Assume there exists an integer $n \geq 2$ such that the following hold:

(1) $\mathfrak{a}_{n+1} \subseteq \mathfrak{m}P$,
(2) I^n *and* I^{n+1} *are integrally closed, and*
(3) $\mathrm{depth}(R/I^n) = 0$.

Then $I \cap \overline{\mathfrak{m}I} \subseteq \mathfrak{m}I$. In particular, if I is integrally closed then $\overline{\mathfrak{m}I} = \mathfrak{m}I$.

Proof. Choose an element $d \notin I^n$ such that $\mathfrak{m} \cdot d \subseteq I^n$. If $\mathfrak{m} \cdot d \subseteq \mathfrak{m}I^n$, then the determinant trick proves that $d \in \overline{I^n} = I^n$, a contradiction. Hence we may choose $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$, such that $xd \notin \mathfrak{m}I^n$. Fix a generating set $a_1, \ldots, a_m$ for I, and write $xd = F(a_1, \ldots, a_m)$, where F is a homogeneous polynomial in P of degree n. Moreover, $F \notin \mathfrak{m}P$. Let $f \in I \cap \overline{\mathfrak{m}I}$ with $f \notin \mathfrak{m}I$. Then $fd \in \overline{\mathfrak{m}I(I^n : \mathfrak{m})} \subseteq \overline{I^{n+1}} = I^{n+1}$. Write $fd = G(a_1, \ldots, a_m)$, where $G(T_1, \ldots, T_m) \in P$ is a homogeneous polynomial of degree $n+1$. Also choose a linear polynomial $L \in P$ such that $f = L(a_1, \ldots, a_m)$ and note that $L \notin \mathfrak{m}P$. We can write the element xfd in two ways giving a polynomial $xG - LF \in \mathfrak{a}_{n+1}$. But $xG - LF \notin \mathfrak{m}P$ contradicting our assumption and finishing the proof. $\square$

While the proof of Theorem 1.3 is simple, the conditions of this theorem will often be satisfied, so that it gives a general situation in which we can conclude that $\mathfrak{m}I$ is integrally closed. A first example is given by the following corollary:

Corollary 1.4

Let $(R, \mathfrak{m})$ be a regular local ring of dimension three, and let I be an ideal of height two such that I is normal, R/I is Cohen-Macaulay and I is generically a complete intersection. If $\mathfrak{a}_3(I) \subseteq \mathfrak{m}P$, then $\mathfrak{m}I$ is integrally closed.

Proof. We apply Theorem 1.3 with $n = 2$. All powers of I are integrally closed, and $\operatorname{depth}(R/I^2) = 0$ by [10]. Since $\mathfrak{a}_3(I) \subseteq \mathfrak{m}P$, Theorem 1.3 immediately gives that mI is integrally closed. □

The assumptions of (1.4) imply in any case that $\mathfrak{a}_2(I) \subseteq \mathfrak{m}P$. This Corollary focuses attention on the cubics in the defining ideal of the Rees algebra. More generally, the assumptions in Theorem 1.3 that I^n and I^{n+1} be integrally closed will be automatic if the Rees algebra of I is normal, one of the main cases we are considering. The third assumption of Theorem 1.3 will be the typical case when $\dim(R/I) = 1$. If it is not satisfied, and if $(R, \mathfrak{m})$ is smooth over a field k, then R/I will be evolutionarily stable over k anyway by [7], (2.1). Thus it makes sense to concentrate on the remaining condition, that there are no monic cubics in the defining ideal of the Rees algebra of I (with respect to a minimal set of generators of I). In section two we will handle the other extreme case-the case in which as many monic cubics as possible appear. This is the case when the reduction number is two, that is when $I^3 = JI^2$ for a minimal reduction J of I. We first give a criterion in terms of the fiber cone for $\mathfrak{m}I$ to be integrally closed.

Proposition 1.5

Suppose that $I \subseteq R$ is a normal ideal in a normal local d-dimensional domain $(R, \mathfrak{m})$ with infinite residue class field and suppose that I has analytic spread $l = d$. If $F_{\mathfrak{m}}(I)$ is equidimensional without embedded components then

$$\mathfrak{m}I^n = \overline{\mathfrak{m}I^n} \qquad \text{for all } \ n \in \mathbb{N}\,.$$

Proof. We may view both $\operatorname{gr}_I(R)$ and $F_{\mathfrak{m}}(R)$ as homomorphic images of the Rees-algebra $\mathcal{R}(I)$, and we may write $\operatorname{gr}_I(R) = \mathcal{R}(I)/I\mathcal{R}(I)$ and $F_{\mathfrak{m}}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. As $I\mathcal{R}(I) \subseteq \mathfrak{m}\mathcal{R}(I)$ and as both $\operatorname{gr}_I(R)$ and $F_{\mathfrak{m}}(I)$ are equidimensional of dimension d (by [6], (18.24) and (18.19), and by assumption) we conclude that the minimal prime ideals of $\mathfrak{m}\mathcal{R}(I)$ are contained among the minimal prime ideals of $I\mathcal{R}(I)$. Thus if $\mathfrak{P} \supseteq \mathfrak{m}\mathcal{R}(I)$ is a minimal prime, it is a prime of height 1 in $\mathcal{R}(I)$, implying that $\mathcal{R}(I)_{\mathfrak{P}}$ is a discrete valuation ring, and therefore

$$\overline{\mathfrak{m}\mathcal{R}(I)_{\mathfrak{P}}} = \mathfrak{m}\mathcal{R}(I)_{\mathfrak{P}}$$

As $F_{\mathfrak{m}}(I)$ has no embedded components, we get that[1]

$$\overline{\mathfrak{m}\mathcal{R}(I)} = \mathfrak{m}\mathcal{R}(I).$$

Let $u \in \overline{\mathfrak{m}I^n}$. Then u satisfies an equation

$$u^N + r_1 u^{N-1} + \ldots + r_N = 0$$

[1] This follows from the the well-known fact that if A is a Noetherian ring and $I, J \subseteq A$ are two ideals with $I_{\mathfrak{p}} \subseteq J_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(A/J)$, then $I \subseteq J$. This follows as the associated primes of $(I+J)/J$ are contained in the associated primes of A/J. Since the module $(I+J)/J$ is zero after localizing at all of these primes, it follows it must be zero.

where $r_i \in (\mathfrak{m}I^n)^i$ for $i = 1, ..., N$. Multiply by t^{Nn} to obtain that

$$(ut^n)^N + r_1 t^n (ut^n)^{N-1} + ... + r_N t^{Nn} = 0. \tag{1.3.1}$$

Since $u \in \overline{\mathfrak{m}I^n} \subseteq \overline{I^n} = I^n$, and since $r_i \in I^{ni}$, this equation is an equation with all coefficients in $\mathcal{R}(I)$. Moreover, $r_i \in \overline{(\mathfrak{m}I^n)^i}$ shows that $r_i t^{ni} \in \mathfrak{m}^i \mathcal{R}(I)$. It follows that equation (1.3.1) proves that $ut^n \in \overline{\mathfrak{m}\mathcal{R}(I)} = \mathfrak{m}\mathcal{R}(I)$. □

Corollary 1.6

Let $(R, \mathfrak{m})$ be a normal local Cohen-Macaulay domain with infinite residue class field and let $I \subseteq R$ be a normal $\mathfrak{m}$-primary ideal with reduction number at most 1. Then $\mathfrak{m}I^n$ is integrally closed for all $n \in \mathbb{N}$.

Proof. By [3], (1.1) respectively (3.2) the fiber cone $F_\mathfrak{m}(I)$ is Cohen-Macaulay, hence in particular equidimensional without embedded components in this case. The claim follows from Proposition 1.5. □

To study evolutions and evolutionarily stable algebras the case of reduced and equidimensional ideals $I \subseteq R$ in a regular local ring R with $\mathrm{ht}(I) = \dim(R) - 1$ is of particular interest. In this case a result of Cowsik and Nori [4] states that I has a minimal reduction generated by a regular sequence if and only if I itself is generated by a regular sequence (and thus is its own minimal reduction). Obviously $\mathfrak{m}I$ is integrally closed in this latter case. In all other cases we have that $l(I) = \dim(R)$. However we do not know (not even in case $\dim(R) = 3$) whether $F_\mathfrak{m}(I)$ is equidimensional in this case. In connection with the above proposition, this raises the following questions:

Problems 1.7

(i) What conditions on $I \subseteq R$ ensure that $F_\mathfrak{m}(I)$ is equidimensional?
(ii) What conditions on $I \subseteq R$ ensure that $F_\mathfrak{m}(I)$ has no embedded components?
(iii) What conditions on $I \subseteq R$ ensure that $F_\mathfrak{m}(I)$ is Cohen-Macaulay?

We end this section with an example to show that even for reasonably nice ideals, the fiber cone can be equidimensional and reduced without being Cohen-Macaulay.

EXAMPLE 1.8: Let $R = k[x, y, z]$ be a polynomial ring in 3 variables. Let A be the matrix,

$$A = \begin{pmatrix} x & z & y^2 & yz \\ y & x & x^2 & x^2 \\ z & y & z^2 & y^2 \\ 0 & 0 & xy & z^2 \\ 0 & x+y & xz & 0 \end{pmatrix}.$$

Let I be the ideal generated by the maximal minors of this 5 by 4 matrix. Using MACAULAY, we can see that I is height two, and consequently defines a Cohen-Macaulay quotient by the Hilbert-Burch theorem. The ideal I is also generically a complete intersection since the submaximal minors of A have height 3. The Rees algebra of I is defined by 14 equations of degrees $2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 7$, and 7.

We write the Rees algebra as a factor ring of $R[T_1, ..., T_5]$, and the fiber cone as a factor of $k[T_1, ..., T_5] = S$. As a quotient S/J of this ring, the fiber cone is defined by six equations of height two (the analytic spread of I is three). The free resolution of S/J as a graded S-module is

$$0 \to S(-8) \oplus S(-10) \to S(-7)^4 \oplus S(-8)^3 \to S(-5)^4 \oplus S(-7)^2 \to S \to S/J \to 0.$$

It follows that the fiber cone is not Cohen-Macaulay. However, it is unmixed and equidimensional since a computation shows that the maximal minors of the last matrix in the resolution has height 4. Localizing at primes not containing these minors makes S/J Cohen-Macaulay, while localizing at an arbitrary prime containing these minors gives depth at least one by the Auslander-Buchsbaum formula. A further check on MACAULAY shows that the fiber cone is actually reduced.

2. Reduction number two and the fiber cone

In this section we obtain conditions for when the fiber cone is equidimensional and unmixed, or even Cohen-Macaulay. Our first theorem is especially aimed at the case in which the reduction number is two, although the theorem is more general.

We first extend the idea of the fiber cone slightly. If $\mathfrak{b}$ is an ideal such that $I \subseteq \mathfrak{b} \subseteq \mathfrak{m}$, we let $F_{\mathfrak{b}}(I) = \mathcal{R}(I)/\mathfrak{b}\mathcal{R}(I)$. Since the associated graded ring of I maps onto $F_{\mathfrak{b}}(I)$ and $F_{\mathfrak{b}}(I)$ maps onto $F_{\mathfrak{m}}(I)$, the analytic spread of I is at most the dimension of $F_{\mathfrak{b}}(I)$ which is bounded above by the dimension of R.

Theorem 2.1

Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring, and let $I \subseteq R$ be an unmixed ideal of height $d - 1 = g \geq 1$ and analytic spread d. Assume that I is generically a complete intersection and let $\mathfrak{b} \subseteq R$ be an ideal such that for all minimal reductions J of I,

$$c_n(I) \subseteq \mathfrak{b} \subseteq \mathfrak{m} \qquad \text{for all } \; n \leq r_J(I).$$

Furthermore assume that the grade of G_+ is g, where $G = gr_I(R)$ is the associated graded ring of I, and G_+ is the ideal of positive degree elements in G. Then $F_{\mathfrak{b}}(I)$ is Cohen-Macaulay.

Proof. Recall that the dimension of $F_{\mathfrak{b}}(I)$ is d as the dimension of this ring is between the analytic spread of I and the dimension of R as noted earlier. We may assume that $R/\mathfrak{m}$ is infinite (by passing to the faithfully flat extension $R(X) = R[X]_{\mathfrak{m}R[X]}$). We may choose a minimal reduction $J = (a_1, \ldots, a_g, c) \subseteq I$ such that

(1) $a_1, \ldots, a_g$ is a regular sequence of R.
(2) c is a regular element of R.
(3) $(a_1, \ldots, a_g) \cdot R_{\mathfrak{p}} = I \cdot R_{\mathfrak{p}}$ for all minimal primes $\mathfrak{p}$ over I.
(4) $a_1 + I^2, \ldots, a_g + I^2$ is a regular sequence in $\mathrm{gr}_I(R)$.

These conditions can all be satisfied by using prime avoidance: The first two require that each of the a_i (or c) avoid a finite set of primes. Likewise the last condition is satisfied provided the leading forms of the a_i in $\mathrm{gr}_I(R)$ avoid certain finite sets of primes. All these can be accomplished by general linear combinations of fixed generators of I, using the fact that the residue field is infinite. There will be a general choice which also satisfies the third condition as I is assumed to be generically a complete intersection.

We will prove that $a_1+\mathfrak{b}I, \ldots, a_g+\mathfrak{b}I, c+\mathfrak{b}I$ is a regular sequence in $F_{\mathfrak{b}}(I)$, proving it is Cohen-Macaulay ($d = g+1$). Fix a minimal generating set $a_1, \ldots, a_m$ of I, where we may assume $a_{g+1} = c$. Set $a_i^* := a_i + \mathfrak{b} \cdot I$ ($i = 1, \ldots, g$), $c^* := c + \mathfrak{b} \cdot I \in F_{\mathfrak{b}}(I)$ and $q = r_J(I)$.

Claim 1: $a_1^*, \ldots, a_g^*$ is a regular sequence in $F_{\mathfrak{b}}(I)$.

Proof of Claim 1: By the generalized Valabrega-Valla criterion of Cortadellas and Zarzuela [3], (2.3) iiii), we have to show

$$(a_1, \ldots, a_g) \cap \mathfrak{b} \cdot I^{n+1} = (a_1, \ldots, a_g) \cdot \mathfrak{b} \cdot I^n \qquad \text{for all } \ n \in \mathbb{N}$$

Here "$\supseteq$" is obvious and it remains to show the inclusion "$\subseteq$". This we will prove by induction on n:

$n = 0$: Let $u \in (a_1, \ldots, a_g) \cap \mathfrak{b} \cdot I$ and write

$$\sum_{i=1}^{g} r_i a_i = u = \sum_{j=1}^{m} s_j a_j$$

with $r_i \in R$ and $s_j \in \mathfrak{b}$. From this we obtain a relation

$$\sum_{i=1}^{g} (s_i - r_i) a_i + \sum_{j=g+1}^{m} s_j a_j = 0$$

implying that

$$s_i - r_i \in \mathrm{Fitt}_{m-1}(I) \subseteq c_1 \subseteq \mathfrak{b} \qquad \text{for all } \ i \in \{1, \ldots, g\}$$

As $s_i \in \mathfrak{b}$, we conclude $r_i \in \mathfrak{b}$, hence

$$(a_1, \ldots, a_g) \cap \mathfrak{b} \cdot I \subseteq (a_1, \ldots, a_g) \cdot \mathfrak{b}$$

$1 \leq n < q$: By the choice of $a_1, ..., a_g$, we have that

$$(a_1, \ldots, a_g) \cap I^{n+1} = (a_1, \ldots, a_g) \cdot I^n$$

Let $u \in (a_1, \ldots, a_g) \cap \mathfrak{b} \cdot I^{n+1}$. Then

$$u \in \big((a_1, \ldots, a_g) \cap I^{n+1}\big) \cap \mathfrak{b} \cdot I^{n+1} = (a_1, \ldots, a_g) \cdot I^n \cap \mathfrak{b} \cdot I^{n+1}.$$

Write $u = \sum_{i=1}^{g} a_i w_i$ with $w_i \in I^n$, and also $u = U(a_1, ..., a_m)$, where $U = U(T_1, ..., T_m)$ is a homogeneous polynomial in the T_i and coefficients in $\mathfrak{b}$ of degree $n+1$. Choose homogeneous polynomials $W_i \in P$ of degree n such that $W_i(a_1, ..., a_m) = w_i$.

It follows that the polynomial $\sum_{i=1}^{g} T_i W_i - U \in \mathfrak{a}$, and has degree $n + 1 \leq q$. By assumption, all the nonzero coefficients of this polynomial are in $\mathfrak{b}$. As the coefficients of U are already in $\mathfrak{b}$, it follows that the coefficients of $\sum_{i=1}^{g} T_i W_i$ are also in $\mathfrak{b}$. Specializing T_i to a_i we obtain that $u \in (a_1, ..., a_g) \cdot \mathfrak{b} \cdot I^n$ as required.

$n \geq q$: As $n \geq q$ and $r(I) = q$, we have

$$I^{n+1} = J \cdot I^n = (a_1, \ldots, a_g, c) \cdot I^n$$

Thus we have to show

$$(a_1, \ldots, a_g) \cap \mathfrak{b} \cdot (a_1, \ldots, a_g, c) \cdot I^n \subseteq (a_1, \ldots, a_g) \cdot \mathfrak{b} \cdot I^n$$

Let $u \in (a_1, \ldots, a_g) \cap \mathfrak{b} \cdot (a_1, \ldots, a_g, c) \cdot I^n$ and write

$$\sum_{i=1}^{g} u_i a_i = u = \sum_{i=1}^{g} v_i a_i + wc$$

with elements $u_i \in R$ and $v_i, w \in \mathfrak{b} \cdot I^n$ to get

$$wc = \sum_{i=1}^{g} (u_i - v_i) a_i$$

thus

$$\begin{aligned} w &\in \big((a_1, \ldots, a_g) : c\big) \cap \mathfrak{b} \cdot I^n \\ &= \Big(\big((a_1, \ldots, a_g) : c\big) \cap I\Big) \cap \mathfrak{b} \cdot I^n \\ &= (a_1, \ldots, a_g) \cap \mathfrak{b} \cdot I^n \\ &= (a_1, \ldots, a_g) \cdot \mathfrak{b} \cdot I^{n-1} \end{aligned}$$

where we used [9], (2.1) iii) and, for the last equality, the inductive assumption. As $c \in I$, this implies

$$u \in (a_1, \ldots, a_g) \cdot \mathfrak{b} \cdot I^n$$

and proves Claim 1.

By Claim 1 we know already that $\operatorname{depth}(F_{\mathfrak{b}}(I)) \geq g$ and that $a_1^*, \ldots, a_g^*$ is a regular sequence in $F_{\mathfrak{b}}(I)$. Set

$$\begin{aligned} S &:= F_{\mathfrak{b}}(I)/(a_1^*, \ldots, a_g^*) \\ &= R/\mathfrak{b} \oplus I/\big(\mathfrak{b} \cdot I + (a_1, \ldots, a_g)\big) \oplus I^2/\big(\mathfrak{b} \cdot I^2 + (a_1, \ldots, a_g) \cdot I\big) \oplus \cdots \end{aligned}$$

As $F_{\mathfrak{b}}(I)$ is a graded local ring, it suffices to show that S is Cohen-Macaulay, and as S is a graded local ring of dimension 1, it suffices to prove

Claim 2: $c^* \in S_1$ is a nonzerodivisor of S.

Proof of Claim 2: Assume there exists an $\overline{r} \in S$, $\overline{r} \neq 0$, with $c^* \cdot \overline{r} = 0$. We may assume that $\overline{r}$ is homogeneous.

$\deg(\overline{r}) = 0$: Then $\overline{r} = r + \mathfrak{b}$ for some $r \in R$ with $rc \in \mathfrak{b} \cdot I + (a_1, \ldots, a_g)$. Write

$$rc + \sum_{i=1}^{g} \lambda_i a_i = \sum_{i=1}^{g} \gamma_i a_i + \gamma_{g+1} c + \sum_{j=g+2}^{m} \gamma_j a_j$$

with $\lambda_i \in R$ and $\gamma_i \in \mathfrak{b}$ to get a relation

$$\sum_{i=1}^{g} (\lambda_i - \gamma_i) a_i + (r - \gamma_{g+1}) c - \sum_{j=g+2}^{n} \gamma_j a_j = 0$$

As $a_1, \ldots, a_m$ is a minimal set of generators of I, this implies

$$r - \gamma_{g+1} \in \operatorname{Fitt}_{m-1}(I) \subseteq \mathfrak{b}$$

hence

$$r \in \mathfrak{b}, \quad \text{i.e.} \quad \overline{r} = 0$$

a contradiction.

$1 \leq \deg(\overline{r}) = n < q$. Let $r \in I^n$ with $\overline{r} = r + \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^{n-1}$. Then

$$rc \in \mathfrak{b} \cdot I^{n+1} + (a_1, \ldots, a_g) \cdot I^n$$

and we have to show that

$$r \in \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^{n-1}$$

to obtain a contradiction in this case as well. Therefore we may assume that

$$r = F(a_{g+1}, a_{g+2}, \ldots, a_m)$$

for some homogeneous polynomial $F(T_{g+1}, \ldots, T_m) \in P$ of degree n. Here we assume again that $a_{g+1} = c$. From this we get a relation

$$\begin{aligned} &a_{g+1} \cdot F(a_{g+1}, a_{g+2}, \ldots, a_m) \\ &\quad = \sum_{i=1}^{g} a_i \cdot G_i(a_1, \ldots, a_m) + H(a_{g+1}, a_{g+2}, \ldots, a_m) \end{aligned}$$

for some homogeneous polynomials $G(T_1, \ldots, T_m) \in P$ of degree n and a homogeneous polynomial $H(T_{g+1}, \ldots, T_m) \in P$ of degree $n+1$ with coefficients in $\mathfrak{b}$. Thus we obtain a polynomial relation

$$Q := T_{g+1} \cdot F(T_{g+1}, \ldots, T_m) - \sum_{i=1}^{g} T_i \cdot G_i(T_1, \ldots, T_m) - H(T_{g+1}, \ldots, T_m) \in \mathfrak{a}_{n+1}$$

By assumption this implies that all the coefficients of Q are in $c_{n+1} \subseteq \mathfrak{b}$. As the coefficients of H are in $\mathfrak{b}$, we conclude that also the coefficient of F are in $\mathfrak{b}$, implying that

$$r \in \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^{n-1}$$

as desired.

$\deg(\overline{r}) = n \geq q$: Let $r \in I^n$ with $\overline{r} = r + \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^{n-1}$. Then

$$rc \in \mathfrak{b} \cdot I^{n+1} + (a_1, \ldots, a_g) \cdot I^n$$

As $n \geq q$ and $r(I) = q$ this implies

$$rc \in \mathfrak{b} \cdot J \cdot I^n + (a_1, \ldots, a_g) \cdot I^n = c \cdot \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^n$$

Thus there exists an $x \in \mathfrak{b} \cdot I^n$ with

$$c(r - x) \in (a_1, \ldots, a_g) \cdot I^n$$

implying that

$$\begin{aligned} r - x &\in \big((a_1, \ldots, a_g) : c\big) \cap I^n \\ &= (a_1, \ldots, a_g) \cap I^n \\ &= (a_1, \ldots, a_g) \cdot I^{n-1} \end{aligned}$$

where again we use [9], (2.1) iii) and, for the last equality, the fact that $a_1 + I^2, \ldots, a_g + I^2$ is a regular sequence in $\mathrm{gr}_I(R)$. This implies

$$r \in \mathfrak{b} \cdot I^n + (a_1, \ldots, a_g) \cdot I^{n-1}$$

i.e. $\overline{r} = 0$, a contradiction, and this completes the proof of Claim 2 and thus of the theorem. □

We next seek for ideals which satisfy the hypothesis of Theorem 2.1. We are particularly interested in the case $\mathfrak{b} = \mathfrak{m}$ since we wish to apply the theorem together with Proposition 1.5 to conclude that $\mathfrak{m}I$ is integrally closed. As we mentioned before, if $c_3(I) \subseteq \mathfrak{m}$, then Theorem 1.3 gives very general conditions under which $\mathfrak{m}I$ is integrally closed. This allows us to focus on the case in which $c_2(I) \subseteq \mathfrak{m}$, but $c_3(I) \not\subseteq \mathfrak{m}$. Thus we'd like to apply Theorem 2.1 in the case $n = 2$, and so need to assume that reduction number is 2. In this case if J is a minimal reduction of I, then $J \cap I^{n+1} = J \cdot I^n$ for all $n \geq 2$. We use this weaker condition in the next proposition to prove that the associated graded ring is Cohen-Macaulay. This proposition is due to M. Johnson and B. Ulrich [15] in the case in which R is Gorenstein. In our application of this result in Corollary 2.3, we don't actually need it (see (2.3)), but the method of proof is similar to that of Theorem 2.1, and it seems of independent interest.

Proposition 2.2

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d > 1$ and let $I \subseteq R$ be an unmixed ideal of height $g := d - 1$ (i.e. all associated primes of I have height g). Assume that I is generically a complete intersection. Furthermore assume that $J = (a_1, \ldots, a_g, c) \subseteq I$ is a reduction of I satisfying

(1) *$a_1, \ldots, a_g$ is a regular sequence of R.*
(2) *$I \cdot R_{\mathfrak{p}} = (a_1, \ldots, a_g) \cdot R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Min}(R/I)$.*
(3) *$J \cap I^{n+1} = J \cdot I^n$ for all $n \geq 2$.*

Then $G := \mathrm{gr}_I(R)$ is Cohen-Macaulay, and $a_1 + I^2, \ldots, a_g + I^2$ is a regular sequence in G. In particular we have

$$(a_1, \ldots, a_g) \cap I^{n+1} = (a_1, \ldots, a_g) \cdot I^n \quad \text{for all } n \in \mathbb{N}.$$

Remark. If $R/\mathfrak{m}$ is infinite, then a reduction $J = (a_1, \ldots, a_g, c)$ of I satisfying (1) and (2) always can be found by a generic position argument.

Proof of the Proposition. For $f \in I^n \setminus I^{n+1}$ we denote by $f^* := f + I^{n+1} \in \mathrm{gr}_I(R)$ its leading form in $\mathrm{gr}_I(R)$.

i) $a_1^*, \ldots, a_g^*$ is a regular sequence in $\mathrm{gr}_I(R)$:

By the Valabrega-Valla criterion [21] it suffices to show that

$$(a_1, \ldots, a_g) \cap I^{n+1} = (a_1, \ldots, a_g) I^n \quad \text{for all } n \in \mathbb{N}$$

as $a_1, \ldots, a_g$ is a regular sequence in R. In any case the inclusion "$\supseteq$" is trivial, and we prove the inclusion "$\subseteq$" by induction on n:

$n = 0$: In this case there is nothing to show.

$n = 1$: First we note that $(a_1, \ldots, a_g) \cdot I$ is an unmixed ideal. In fact by [9], (2.1) ii)

$$(a_1, \ldots, a_g)/(a_1, \ldots, a_g) \cdot I = (R/I)^g$$

is a free R/I-module of rank g. This and the short exact sequence

$$0 \longrightarrow (a_1, \ldots, a_g)/(a_1, \ldots, a_g) \cdot I \longrightarrow R/(a_1, \ldots, a_g) \cdot I \longrightarrow R/(a_1, \ldots, a_g) \longrightarrow 0$$

imply that

$$\begin{aligned} \mathrm{Ass}\big(R/(a_1, \ldots, a_g) \cdot I\big) &\subseteq \mathrm{Ass}\big((a_1, \ldots, a_g)/(a_1, \ldots, a_g) \cdot I\big) \cup \mathrm{Ass}\big(R/(a_1, \ldots, a_g)\big) \\ &= \mathrm{Min}\,(R/I) \cup \mathrm{Min}\big(R/(a_1, \ldots, a_g)\big) \end{aligned}$$

and all these primes have height g as $a_1, \ldots, a_g$ is a regular sequence and as R/I is Cohen-Macaulay.

We have to show:

$$(a_1, \ldots, a_g) \cdot I \supseteq (a_1, \ldots, a_g) \cap I^2$$

As $(a_1,\ldots,a_g)\cdot I$ is unmixed, we only have to check this at the minimal primes of $(a_1,\ldots,a_g)\cdot I$.

Let $\mathfrak{p}\in\operatorname{Spec}(R)$ be minimal over $(a_1,\ldots,a_g)\cdot I$. If $I\subseteq\mathfrak{p}$, then $\mathfrak{p}\in\operatorname{Min}(R/I)$, and therefore

$$(a_1,\ldots,a_g)\cdot R_{\mathfrak{p}} = I\cdot R_{\mathfrak{p}}$$

by (2). Hence

$$\big((a_1,\ldots,a_g)\cdot I\big)\cdot R_{\mathfrak{p}} = I^2\cdot R_{\mathfrak{p}} \supseteq \big((a_1,\ldots,a_g)\cap I^2\big)\cdot R_{\mathfrak{p}}$$

If $I\not\subseteq\mathfrak{p}$, then $IR_{\mathfrak{p}}=R_{\mathfrak{p}}$ and therefore

$$\big((a_1,\ldots,a_g)\cdot I\big)\cdot R_{\mathfrak{p}} = (a_1,\ldots,a_g)\cdot R_{\mathfrak{p}} \supseteq \big((a_1,\ldots,a_g)\cap I^2\big)\cdot R_{\mathfrak{p}}$$

and the claim follows in this case.

$n\geq 2$: We need to show that

$$(a_1,\ldots,a_g)\cdot I^n \supseteq (a_1,\ldots,a_g)\cap I^{n+1}$$

By assumption we have

$$(a_1,\ldots,a_g)\cap I^{n+1}\subseteq J\cap I^{n+1} = J\cdot I^n$$

Let $u\in(a_1,\ldots,a_g)\cap I^{n+1}$ and write

(1) $u=\sum\limits_{i=1}^{g} u_i a_i\in(a_1,\ldots,a_g)$.

(2) $u=\sum\limits_{i=1}^{g} v_i a_i + cw$.

with $u_i\in R$ and $v_i,w\in I^n$. Hence

$$cw=\sum_{i=1}^{g}(u_i-v_i)a_i\in(a_1,\ldots,a_g)$$

implying that

$$w\in\big((a_1,\ldots,a_g):c\big)\cap I^n = (a_1,\ldots,a_g)\cap I^n$$

by [9], (2.1) iii). By the inductive assumption this gives

$$w\in(a_1,\ldots,a_g)\cdot I^{n-1}$$

and, as $c\in I$, the claim follows and i) is proved.

Let $S=\operatorname{gr}_I(R)/(a_1^*,\ldots,a_g^*)$. To complete the proof of the proposition, it suffices to show:

ii) S is a Cohen-Macaulay ring:

As $\dim(S)=1$ and S is graded local, it suffices to show that

$$\text{h-soc}(S)=(0)$$

where h-soc denotes the homogeneous socle of S. Assume otherwise and let $w^* \in S$ be a homogeneous element in the socle of S. Note that

$$(\mathfrak{m}_R, S_+) \cdot w^* = (0)$$

If $\deg(w^*) = 0$, then we may write $w^* = w + I$ for some $w \in R$, and we have in particular that $\mathfrak{m}_R \cdot w \subseteq I$, contradicting the fact that R/I is Cohen-Macaulay.

Thus we may assume that $\deg(w^*) \geq 1$. We may replace R by $\overline{R} := R/(a_1, \ldots, a_g)$ and I by $\overline{I} = I/(a_1, \ldots, a_g)$. As $a_1^*, \ldots, a_g^*$ is a $\mathrm{gr}_I(R)$-regular sequence by i), we have

$$\mathrm{gr}_{\overline{I}}(\overline{R}) = \mathrm{gr}_I(R)/(a_1^*, \ldots, a_g^*) = S$$

and obviously $\overline{J} := (\overline{c})$ is a reduction of $\overline{I}$ with

$$\overline{J} \cap \overline{I}^{n+1} = \overline{J} \cdot \overline{I^n}$$

for all $n \geq 2$ (as $(a_1, \ldots, a_g) \subseteq J$). Hence we also may assume that $g = 0$. Write $w^* := w + I^{n+1}$ for some $w \in I^n \setminus I^{n+1}$. By assumption

$$c \cdot w \in I^{n+2} \cap (c) = I^{n+2} \cap J = J \cdot I^{n+1}$$

as $n \geq 1$. Thus there exists a $v \in I^{n+1}$ with

$$c \cdot (w - v) = 0$$

Hence

$$w - v \in ((0) : c) \cap I = (0)$$

by [9], (2.1) iii) and therefore

$$w \in I^{n+1}$$

a contradiction. This completes the proof of ii) and thus the proof of the proposition.

Corollary 2.3

Let $(R, \mathfrak{m})$ be a normal local Cohen-Macaulay domain of dimension d, and let $I \subseteq R$ be a normal unmixed syzygetic ideal of height $g = d - 1$ and analytic spread d. If I is generically a complete intersection and if I has reduction number 2, then

$$\mathfrak{m} \cdot I^n = \overline{\mathfrak{m} \cdot I^n} \qquad \textit{for all } \; n \in \mathbb{N}$$

Proof. We first use Proposition 2.2 to show that the associated graded ring of I is Cohen-Macaulay. The conditions there are all satisfied except possibly the condition that there is a minimal reduction J such that $I^{n+1} \cap J = JI^n$ for all $n \geq 2$. However, since $JI^n = I^{n+1}$ for all $n \geq 2$, we can apply (2.2). The fact that the associated graded ring is Cohen-Macaulay also follows immediately from Theorem 3.1 in [1]. Next we check the conditions of Theorem 2.1. The graded ring of I is Cohen-Macaulay by the preceding argument. Since I is syzygetic, $c_1(I) = c_2(I) \subseteq \mathfrak{m}$. It follows from Theorem 2.1 that $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay. In particular it is equidimensional and without embedded components. Applying Proposition 1.5 now finishes the proof. □

Our next Corollary answers one of our original questions: Let R be a regular local ring of dimension three, and suppose that $\mathfrak{p}$ is a height two prime such that the Rees algebra of $\mathfrak{p}$ is Cohen-Macaulay and normal. Then is $\mathfrak{m}\mathfrak{p}$ integrally closed? It seems a reasonable class to study in general are those ideals I whose Rees algebra is Cohen-Macaulay and normal. Even with these strong assumptions, it is not clear to us whether $\mathfrak{m}I$ will be integrally closed.

Corollary 2.4

Let $(R, \mathfrak{m})$ be a three-dimensional regular local ring, and let I be a height two ideal of R having analytic spread three. Assume that I is generically a complete intersection, is unmixed, and assume that the Rees algebra of I is Cohen-Macaulay and normal. Then $\mathfrak{m}I^n$ is integrally closed for all $n \geq 1$.

Proof. The ideal I is syzygetic since it is strongly Cohen-Macaulay and generically a complete intersection. Furthermore the fact that the Rees algebra of I is Cohen-Macaulay implies that the reduction number of I is at most 2, using either [14, Theorem 2.3] or [2]. It follows that we may apply Corollary 2.3 to obtain the claim. □

Corollary 2.5

Let $(R, \mathfrak{m})$ be a local Gorenstein ring of dimension d with infinite residue class field, let $I \subseteq R$ be an unmixed ideal of height $g = d - 1$ and analytic spread $l = d$, minimally generated by m elements. Assume

(1) *I is generically a complete intersection.*

(2) *I has reduction number 2.*

(3) *The Koszul cohomology modules $H_j(I)$ (with respect to some set of generators of the ideal I) are Cohen-Macaulay for $j \in \{0, 1\}$.*

Then for any ideal $\mathfrak{b} \subseteq R$ with $\mathrm{Fitt}_{m-1}(I) \subseteq \mathfrak{b} \subseteq \mathfrak{m}$ the fiber cone $F_{\mathfrak{b}}(I)$ is Cohen-Macaulay.

Proof. Write

$$\mathcal{R}(I) = P/\mathfrak{a} = \mathrm{Sym}_R(I)/\overline{\mathfrak{a}}$$

as above. By [15], (4.10), $\overline{\mathfrak{a}}$ is generated by forms of degree 3. Hence I is syzygetic and the corollary follows from Theorem 2.1 as above. □

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