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A note on the density of the parabolic area integral

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Abstract

The density of the area integral for parabolic functions is defined in analogy with the case of harmonic functions. We prove its equivalence with the local time of the associated martingale. Using probabilistic methods, we show its equivalence in L^p-norm with the parabolic area function for $p>1$.

Introduction

A large part of what is known about the boundary behavior of harmonic or parabolic functions in $\mathbb{R}^{d+1}_+$ relies on the identity $Hu^2 = 2\,|\nabla u|^2$ if $Hu = 0$. (Here H is the Laplacian if u is harmonic, or the heat operator if u is a parabolic function. In the latter case, the gradient is restricted to the horizontal variables in $\mathbb{R}^d$). In both cases, the quantity Hu^2 plays a fundamental role in the study of of the area integral of harmonic or parabolic functions.

The study of harmonic functions, as far as maximal functions, area integrals, H^p spaces are concerned, has been developed in the fundamental papers of Calderón [6], Stein [20], Stein and Weiss [21], Fefferman and Stein [11], and many others. The parabolic analog has been treated by Calderón and Torchinsky [7], Jones and Tu [15], and others.

Another point of view has been introduced by Gundy [12] (see: [13], [14], [4], [5], [2]) for harmonic functions. We consider here the case of parabolic functions. The starting point, valid for H-harmonic functions for the two possible choices of H, is

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that $H(|u-r|)$ is a positive measure on $\mathbb{R}_{+}^{d+1}$ for each $r \in \mathbb{R}$, and, in the sense of distributions, one has the disintegration formula

$$\int_{-\infty}^{+\infty} H\big(|u-r|\big)dr = H(u^2).$$

This is explained in Section 2.

The principal results obtained here for parabolic functions are as follows. We consider the integral

$$D_T^r(\theta) := \int\limits_{\Pi_T(\theta)} H\big(|u-r|\big)t^{-d/2}dydt,$$

where $\Pi_T(\theta)$ is the parabolic region $\Pi_T(\theta) = \Big\{(y,t) : |y-\theta|^2 < t, t \leq T\Big\}$. The functional $D_T^r(\theta)$ is called the *density of the parabolic area integral* at level r. The parabolic area integral is obtained by integration over the parameter r. Calderón and Torchinsky [7] have shown that the parabolic area integral and parabolic maximal function have equivalent norms for all L^p, $0 < p < \infty$. We show here that $D^*(\theta) = \sup\limits_{r,T} D_T^r(\theta)$ is another functional that is norm-equivalent for $p > 1$ to the above functionals.

To do this, we have explored the probabilistic methods used by Gundy and Brossard in the harmonic case, specifically, we have obtained the L^p-inequalities from Barlow-Yor inequalities, using the analogy between the density of the area integral and the local time of the heat martingale. In the construction, we have followed Brossard's paper [4], who considers the Brownian motion starting from some point x instead of the "background radiation" as in the fundamental of Gundy [12].

Let us mention that we leave open the case $0 < p \leq 1$. In the harmonic case these inequalities are proved in [14] using "good lambda inequalities". The probabilistic methods used here seem not be sufficient to prove this case.

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1. Preliminaries: parabolic functions and its functionals associated

We say that a function $u(x,t), x \in \mathbb{R}^d, t > 0$ is a *parabolic function* if it satisfies the heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\Delta u(x,t), \tag{1}$$

where $\Delta = \sum\limits_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$.

We denote by $p(t,x,y)$ the Gaussian density centered at $x \in \mathbb{R}^d$ with variance $t > 0$, that is

$$p(t,x,y) = \left(2\pi t\right)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

The function $p(t,x,y)$ is certainly parabolic as are all the functions $u(x,t)$ arising as the parabolic extension of a Schwartz distribution f on $\mathbb{R}^d$, by

$$u(x,t) = \int_{\mathbb{R}^d} f(y)p\big(t,x,y\big)dy. \tag{2}$$

Moreover, if $u(x,t)$ is parabolic and positive in $\mathbb{R}^{d+1}_+$, then it has a representation as (2).

For any $\theta \in \mathbb{R}^d$, set $\Pi_\alpha(\theta) = \{(y,t) : |\theta - y|^2 < \alpha^2 t\}$. This is a parabolic cone of vertex θ, aperture α and vertical axis in $\mathbb{R}^{d+1}_+$. For $T > 0$, we denote the parabolic cone truncated at T by $\Pi_{\alpha,T}(\theta) = \{(y,t) : |\theta - y|^2 < \alpha^2 t, t \leq T\}$.

We say that the parabolic function u has a *parabolic limit* at θ if the following limit exists for all α:

$$\widetilde{u}_\alpha(\theta) = \lim_{\substack{(y,t)\to(\theta,0)\\(y,t)\in\Pi_\alpha(\theta)}} u(y,t).$$

Also, we define the *parabolic maximal function* $M_\alpha(\theta)$ by:

$$M_\alpha(\theta) = \sup_{(y,t)\in\Pi_\alpha(\theta)} |u(y,t)|.$$

We know (see: Jones and Tu [15]) that these two quantities exist for the same set, neglecting a set of zero measure. The *parabolic area function* was defined later by Calderón and Torchinsky [7] as follows (we write the definition more conveniently, in our context):

$$A^2_\alpha(\theta) = \iint_{\Pi_\alpha(\theta)} \left|\nabla u(y,t)\right|^2 t^{-d/2} dydt,$$

where $\nabla = \left(\frac{\partial}{\partial y_j}\right)_{j=1}^d$.

From the Good-Lambda inequalities (see: [7], [1]), we can deduce that for every $\alpha > 1$ and $0 < p < \infty$,

$$\left\|A_\alpha\right\|_p \approx \left\|M_\alpha\right\|_p. \tag{3}$$

We will write $M_{\alpha,T}(\theta)$ and $A_{\alpha,T}(\theta)$ when the functionals are restricted to the truncated cone $\Pi_{\alpha,T}(\theta)$.

2. Density of area integral

To simplify the notation we take $\alpha = 1$. We define the *density of the parabolic area integral* of u, which shall be denoted by $D^r(\theta)$. Following the idea of the definition for the harmonic case (Gundy-Silverstein [14]), $D^r(\theta)$ must satisfy the following relation:

$$\iint_{\Pi(\theta)} g\big(u(y,t)\big) \left|\nabla u\right|^2 (y,t) t^{-d/2} dydt = \int_{-\infty}^{\infty} g(r) D^r(\theta) dr, \tag{4}$$

for all measurable and positive real functions g. If, with one particular g, we define $G(s) = \int_{-\infty}^{\infty} g(r)(s-r)^+ dr$, we have

$$\Delta G(u) = G''(u) \left|\nabla u\right|^2 + G'(u)\Delta u = g(u) \left|\nabla u\right|^2 + 2\frac{\partial}{\partial t} G(u).$$

We denote by H the heat operator, that is, $H = \frac{1}{2}\Delta - \frac{\partial}{\partial t}$. The expression above becomes:

$$HG(u) = \frac{1}{2} g(u) \left|\nabla u\right|^2 \quad \text{and} \quad HG(u) = \int_{-\infty}^{\infty} g(r) H(u-r)^+ dr, \tag{5}$$

in the sense of distributions. The fact that $H\left((u-r)^+\right)$ is a positive Radon measure with support in the set $\{u = r\}$ can be proved as in [4]. Using a smooth approximation of the indicator function of $\Pi(\theta)$, and using (5) at the left-hand side of (4), we obtain

$$\begin{aligned}
&\iint_{\Pi(\theta)} g\big(u(y,t)\big) \left|\nabla u\right|^2 (y,t) t^{-d/2} dydt \\
&= 2 \iint_{\Pi(\theta)} HG\big(u(y,t)\big) t^{-d/2} dydt \\
&= 2 \iint_{\Pi(\theta)} \int_{-\infty}^{\infty} g(r) H\left((u(y,t)-r)^+\right) t^{-d/2} drdydt \\
&= 2 \int_{-\infty}^{\infty} g(r) \left(\iint_{\Pi(\theta)} H\left((u(y,t)-r)^+\right) t^{-d/2} dydt \right) dr,
\end{aligned}$$

where, in the last equality, we have used Fubini's theorem. Now we can write the following explicit expressions for $D^r(\theta)$:

$$\begin{aligned}
\frac{1}{2} D^r(\theta) &= \iint_{\Pi(\theta)} H\left((u(y,t)-r)^+\right) t^{-d/2} dydt \\
&= \iint_{\Pi(\theta)} H\left((u(y,t)-r)^-\right) t^{-d/2} dydt.
\end{aligned}$$

The last expression justifies the following definition:

DEFINITION 1. For a parabolic function $u(x,t)$ defined in $\mathbb{R}_+^{d+1}$and $r \in \mathbb{R}$, we define the density of the parabolic area integral by

$$D^r(\theta) = \iint_{\Pi(\theta)} H\big(|u(y,t) - r|\big) t^{-d/2} dydt.$$

We define also,

$$D^*(\theta) = \sup_{r \in \mathbb{R}} D^r(\theta).$$

We denote $D_T^r(\theta)$ the density of the area integral when the integral is restricted to $\Pi_T(\theta)$.

3. Probabilistic approach. Local time

Let T be a fixed time. Let $B = (B_t; t > 0)$ be a Brownian motion in $\mathbb{R}^d$ starting at $x \in \mathbb{R}^d$. We denote by P_x and E_x the probability and the expected value associated to B. An application of Itô's formula gives us:

$$u(B_t, T-t) = u(x,T) + \int_0^t \nabla u(B_t, T-s) dB_s + \int_0^t H\big(u(B_s, T-s)\big) ds.$$

The last integral is zero since u is parabolic; hence, we deduce that the process $U = (U_s = u(B_s, T-s); 0 < s < T)$ is a continuous local martingale with increasing process

$$A_t^2 = \int_0^t |\nabla u|^2 (B_s, T-s) ds, \qquad 0 \le t \le T.$$

We shall denote

$$U_T = \lim_{t \to T} U_t \qquad \text{and} \qquad M_T = \sup_{0 \le t \le T} |U_t|.$$

Because u is parabolic, U_T exist P_x-a.s. (see Doob [9]). If $M(\theta)$ is bounded a.e., then $U_T = \widetilde{u}(B_T)$, P_x-a.s., where $\widetilde{u}$ denotes the parabolic limit of u.

In fact,

$$u(x,T) = E_x[U_t] = \int u(y, T-t)\, p(t,x,y) dy.$$

Because $\lim\limits_{t \to T} u(y, T-t) = \widetilde{u}(y)$ and $\lim\limits_{t \to T} p(t,x,y) = p(T,x,y)$, by the bounded convergence theorem we have

$$\int u(y, T-t)\, p(t,x,y) dy \to \int \widetilde{u}(y)\, p(T,x,y) dy,$$

when t tends to T. Then

$$u(x,T) = E_x[U_T] = \lim_{t \to T} E_x[U_t] = E_x\big[\widetilde{u}(B_T)\big],$$

and

$$U_T = \widetilde{u}(B_T) \qquad P_x\text{-a.s.}$$

Let $(L_t^r; 0 < t < T, r \in \mathbb{R})$ be the local time family associated with the local martingale defined above. We also denote

$$L_t^* = \sup_{r\in\mathbb{R}} L_t^r.$$

We have the following occupation formula for the local time:

$$\int_0^\sigma g(U_t)\,|\nabla u|^2\,(B_t, T-t)dt = \int_{-\infty}^{\infty} g(r)L_\sigma^r dr, \tag{6}$$

where g is a measurable, real, positive function and σ is a stopping time.

We shall denote by $P_{x\to\theta}^T$ the probability associated to the Brownian bridge between x and θ during the time interval $[0,T]$, and $E_{x\to\theta}^T$ the expected value with respect to $P_{x\to\theta}^T$. Consider W a parabolic domain, that is, for a Borel set E in $\mathbb{R}^d$, $W = \bigcup_{\theta\in E} \Pi_T(\theta)$ we define the stopping time

$$\sigma = \inf\big\{0 \le t \le T; (B_t, T-t) \notin W\big\}.$$

We denote $\mathcal{G}_W(T,t,x,y)$ the Green function of the Dirichlet problem for the heat equation in W, that is

$$\mathcal{G}_W(T,t,x,y)dy = P_x\big\{B_t \in dy, t \le \sigma\big\}.$$

Note that the right-hand side depends on T as follows from the definition of σ. We have the following absolute continuity relation of $P_{x\to\theta}^T$ with respect to P_x (see Revuz-Yor [19]). Let φ_t be a bounded $\mathcal{F}_t$-measurable functional on $\Omega\times[0,T]$ and σ a stopping time. Then,

$$E_{x\to\theta}^T(F_\sigma) = \frac{1}{p(T,x,\theta)} E_x\big[\varphi_\sigma p(T-\sigma, B_\sigma, \theta); \sigma < T\big]. \tag{7}$$

4. Relation between the density of the area integral and the local time

The following results will be used in the proofs given in the next section. They are quite similar to results due to Brossard [4] for the harmonic case.

Proposition 1

Let W be the parabolic region, and σ the stopping time defined before. If $|x-\theta|^2 < T$ we have:

$$E_{x\to\theta}^T\left[L_\sigma^r\right] = \iint_W \frac{\mathcal{G}_W(T,t,x,y)\,p(T-t,y,\theta)}{p(T,x,\theta)} H\big(|u(y,T-t) - r|\big)dydt. \tag{8}$$

Proof. We have from (6) and (7) that

$$\int_{-\infty}^{\infty} g(r) E^T_{x\to\theta}\left[L^r_\sigma\right] dr = E^T_{x\to\theta}\left[\int_0^\sigma g(U_t)\left|\nabla u\right|^2(B_t, T-t)dt\right]$$

$$= \frac{1}{p(T,x,\theta)} E_x\left[p(T-\sigma, B_\sigma, \theta)\int_0^\sigma g(U_t)\left|\nabla u\right|^2(B_t, T-t)dt; \sigma < T\right].$$

Itô's formula, applied to $p(T-\sigma, B_\sigma, \theta)$, allows us to write this expression as

$$\frac{1}{p(T,x,\theta)} E_x\left[\left(\int_0^\sigma \nabla p(T-t, B_t, \theta) dB_t\right)\left(\int_0^\sigma g(U_t)\left|\nabla u\right|^2(B_t, T-t)dt\right);\ \sigma < T\right].$$

The integration by parts formula of stochastic calculus, implies that this quantity is equal to

$$\frac{1}{p(T,x,\theta)} E_x\left[\int_0^\sigma p\big(T-t, B_t, \theta\big)\, g(U_t)\left|\nabla u\right|^2\big(B_t, T-t\big)dt; \sigma < T\right]$$

$$= \int_0^T E_x\left[\frac{p(T-t, B_t, \theta)}{p(T,x,\theta)} g(U_t)\left|\nabla u\right|^2(B_t, T-t); \sigma \geq t\right] dt$$

$$= \iint \frac{\mathcal{G}_W(T,t,x,y)p(T-t,y,\theta)}{p(T,x,\theta)} g\big(u(y,T-t)\big)\left|\nabla u\right|^2(y, T-t)\, dydt$$

$$= \int_{-\infty}^{\infty} g(r) \iint_W \frac{\mathcal{G}_W(T,t,x,y)p(T-t,y,\theta)}{p(T,x,\theta)} H\big(\left|u(y,T-t) - r\right|\big)\, dydtdr,$$

where, in the last equality, we have used (4) and (5). □

In the next proposition, we restrict to the case $W = \mathbb{R}^{d+1}_+$ where the Green function is known.

Proposition 2

If $|x-\theta|^2 < T$, *then we have*

$$E^T_{x\to\theta}\left[L^r_T\right] \geq C(d) D^r_{T/2}(u, \theta), \tag{9}$$

for some constant $C(d)$ *and in consequence*

$$E^T_{x\to\theta}\left[L^*_T\right] \geq C(d) D^*_{T/2}(u, \theta).$$

Proof. In the following $C(d)$ denotes a constant which may change its value. From the Proposition 1, we know that

$$E^T_{x\to\theta}\left[L^r_T\right] = \iint_{\mathbb{R}^d\times[0,T]} \frac{p(t,x,y)p(T-t,y,\theta)}{p(T,x,\theta)} H\big(\left|u(y,T-t) - r\right|\big) dydt.$$

Now if $|x-\theta|^2 < T$ and $t > \frac{T}{2}$ we have:

$$\frac{p(t,x,y)p(T-t,y,\theta)}{p(T,x,\theta)} \geq \left[2\pi(T-t)\right]^{-d/2} C(d)\mathbf{1}_{\{|y-\theta|^2<(T-t)\}}.$$

Then,

$$\begin{aligned} E^T_{x\to\theta}\left[L^r_T\right] &\geq C(d) \int_{T/2}^{T}\!\!\int (T-t)^{-d/2}\,\mathbf{1}_{\{|y-\theta|^2<(T-t)\}} H\big(|u(y,T-t)-r|\big)dydt \\ &= C(d) \iint_{\Pi_{T/2}(\theta)} t^{-d/2} H\big(|u(y,t)-r|\big)\,dydt. \ \square \end{aligned}$$

5. Main result: L^p-Inequality

Because $D^r(\theta)$ satisfies (4), for $g \equiv 1$ we have

$$A^2(\theta) = \iint_{\Pi(\theta)} |\nabla u|^2 (y,t) t^{-d/2} dydt = \int_{-\infty}^{\infty} D^r(\theta)dr.$$

The function $D^r(\theta)$ is non zero in the range of u, so we have

$$A^2(\theta) = \int_{-M(\theta)}^{M(\theta)} D^r(\theta)dr \leq 2M(\theta)D^*(\theta). \tag{10}$$

Theorem 1

For every $1 < p < \infty$,

$$C_p \|A\|_p \leq \|D^*\|_p \leq C'_p \|A\|_p .$$

Proof. From (10), Cauchy-Schwarz inequality and the equivalence (3), we obtain

$$\|A\|_p \leq C_p \|D^*\|_p^{1/2} \|M\|_p^{1/2} \leq C_p \|D^*\|_p^{1/2} \|A\|_p^{1/2} .$$

In the other direction: let $I(x)$ denote the set $\left\{\theta : |x-\theta|^2 < T\right\}$. On this set, $p(T,x,\theta) \geq CT^{-d/2}$. Then, using (9),

$$\begin{aligned} CT^{-d/2} \int_{I(x)} \left(D^*_{T/2}(\theta)\right)^p d\theta &\leq \int_{I(x)} \left(E^T_{x\to\theta}\left[L^*_T\right]\right)^p p(T,x,\theta)\,d\theta \\ &\leq \int \left(E^T_{x\to\theta}\left[L^*_T\right]\right)^p p(T,x,\theta)d\theta = E_x\left[L^*_T\right]^p , \end{aligned}$$

we deduce that

$$CT^{-d/2} \int\limits_{I(x)} \left(D^*_{T/2}(\theta)\right)^p d\theta \leq E_x \left[L^*_T\right]^p \leq C' E_x \left[M_T\right]^p .$$

The right-hand side inequality is a consequence of the Barlow-Yor inequality [3]. Now, Doob's inequality implies that

$$E_x \left[M_T\right]^p \leq \left(\frac{p}{p-1}\right)^p E_x\left[\,|U_T|^p\,\right].$$

We can use Doob's inequality since

$$\begin{aligned} \sup_{0\leq t\leq T} E_x\, |U_t|^p &= \sup_{0\leq t\leq T} \int p(t,x,y)\, |u(y,T-t)|^p\, dy \\ &\leq C \sup_{0\leq t\leq T} \int |u(y,T-t)|^p\, dy \leq C \int |M(y)|^p\, dy < \infty. \end{aligned}$$

Now, we integrate in the variable x. Using Fubini's theorem and the remarks at the beginning of Section 3, we have that

$$\begin{aligned} \int \left(D^*_{T/2}(\theta)\right)^p d\theta &\leq C_p \int E_x\left[|U_T|^p\right] dx \\ &= C_p \int |\widetilde{u}(y)|^p \int p(T,x,y)dxdy = C_p \int |\widetilde{u}(y)|^p\, dy \\ &\leq C_p \int |M(y)|^p\, dy. \end{aligned}$$

The proof is finished using the L^p-equivalence (3) between M and A and by making T tend to infinity. □

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