Collect. Math. **52**, 1 (2001), 55–73

Semiperfect countable $\mathbb{C}$-separative C-finite semigroups

Torben Maack Bisgaard

Nandrupsvej 7, st. th., DK-2000 Frederiksberg C, Denmark

E-mail: torben.bisgaard@get2net.dk

Received September 6, 2000

Abstract

Semiperfect semigroups are abelian involution semigroups on which every positive semidefinite function admits a disintegration as an integral of hermitian multiplicative functions. Famous early instances are the group on integers (Herglotz' Theorem) and the semigroup of nonnegative integers (Hamburger's Theorem). In the present paper, semiperfect semigroups are characterized within a certain class of semigroups. The paper ends with a necessary condition for the semiperfectness of a finitely generated involution semigroup, a condition which has since been found to be also sufficient.

1. Introduction

With the possible exception of finite abelian groups, the oldest example of a semiperfect semigroup is the group $\mathbb{Z}$ of integers. Herglotz' Theorem [23] of 1911 asserts that a two-sided sequence $(s_n)_{n=-\infty}^{\infty}$ of complex numbers is a trigonometric moment sequence, in the sense that

$$s_n = \int_{\mathbb{T}} z^n \, d\mu(z), \quad n \in \mathbb{Z}$$

for some measure μ on the complex unit circle $\mathbb{T}$, if and only if (s_n) is positive semidefinite in the sense that

$$\sum_{j,k=0}^{n} c_j \overline{c_k} s_{j-k} \geq 0$$

for every choice of n in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $c_0, \ldots, c_n$ in the complex field $\mathbb{C}$. When the condition is satisfied, there is just one such measure μ.

Keywords: Semigroup, involution, positive definite function.
MSC2000: 43A35, 44A60.

Hamburger's Theorem of 1920 ([22], see the monographs by Akhiezer [1], Shohat and Tamarkin [30], p. 5, or Berg, Christensen, and Ressel [3], 6.2.2) asserts that a sequence $(s_n)_{n=0}^{\infty}$ of reals is a moment sequence, in the sense that

$$s_n = \int_{\mathbb{R}} x^n \, d\mu(x), \quad n \in \mathbb{N}_0 \tag{1}$$

for some measure μ on the real line $\mathbb{R}$, if and only if (s_n) is positive semidefinite in the sense that

$$\sum_{j,k=0}^{n} c_j c_k s_{j+k} \geq 0$$

for every choice of $n \in \mathbb{N}_0$ and $c_0, \ldots, c_n \in \mathbb{R}$.

Sz.-Nagy [32] showed in 1952 that a sequence $(s_n)_{n=0}^{\infty}$ of self-adjoint bounded linear operators on a Hilbert space H admits the disintegration (1) for some measure μ on $\mathbb{R}$ with values that are positive bounded linear operators on H if and only if (s_n) is of positive type in the sense that

$$\sum_{j,k=0}^{n} \langle s_{j+k}\xi_j, \xi_k \rangle \geq 0$$

for every choice of $n \in \mathbb{N}_0$ and $\xi_0, \ldots, \xi_n \in H$ where $\langle \cdot, \cdot \rangle$ denotes the inner product on H. It is not sufficient that (s_n) be positive semidefinite in the sense that for each $\xi \in H$ the scalar sequence $(\langle s_n \xi, \xi \rangle)_{n=0}^{\infty}$ be positive semidefinite; a counterexample exists already for $H = \mathbb{C}^2$ [4].

The moment problems solved by Herglotz, Hamburger, and Sz.-Nagy can be generalized to arbitrary abelian involution semigroups. Suppose $(S, +, *)$ is an abelian semigroup equipped with an *involution*, that is, a mapping $s \mapsto s^* : S \to S$ satisfying $(s^*)^* = s$ and $(s+t)^* = s^* + t^*$ for all $s, t \in S$. Such a structure will be called a $*$-*semigroup*, abbreviated 'semigroup' when confusion is unlikely, such as when applying an adjective which makes sense only in the presence of an involution (e.g., 'semiperfect semigroup'). For subsets H and K of S, write $H + K = \{\, x + y \mid x \in H,\, y \in K \,\}$, abbreviated $a+K$ in case $H = \{a\}$ for some $a \in S$. Suppose D is a complex vector space and let $\mathcal{S}(D)$ be the set of all sesquilinear forms on D. A function $\varphi: S + S \to \mathcal{S}(D)$ is of *positive type* if

$$\sum_{j,k=1}^{n} \varphi(s_j + s_k^*)(\xi_j, \xi_k) \geq 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $\xi_1, \ldots, \xi_n \in D$. Denote by $\mathcal{P}(S, D)$ the set of all such functions. Make the convention that in the notation for an entity in the definition of which a complex vector space D occurs, the symbol 'D' is omitted (together with any comma immediately preceding it) in case $D = \mathbb{C}$. Furthermore, identify $\mathcal{S}(\mathbb{C})$ with $\mathbb{C}$ itself by identifying $a \in \mathbb{C}$ with the sesquilinear form $(\xi, \eta) \mapsto a\xi\overline{\eta}$ on $\mathbb{C}$. Thus, $\mathcal{P}(S)$ is the set of those functions $\varphi: S + S \to \mathbb{C}$ which are *positive semidefinite* in the sense that

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^*) \geq 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{C}$. We reserve the name 'positive definite' for functions such that the same sum is positive whenever the s_j are pairwise distinct and the c_j are not all zero.

A *character* on S is a function $\sigma: S \to \mathbb{C}$, not identically zero, such that $\sigma(s^*) = \overline{\sigma(s)}$ and $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S.

Let $\mathcal{A}(S^*)$ be the least σ-ring of subsets of S^* rendering measurable for each $s \in S$ the function $\widehat{s}: S^* \to \mathbb{C}$ defined by $\widehat{s}(\sigma) = \sigma(s)$ for $\sigma \in S^*$. For $s \in S$ and $n \in \mathbb{N}$, define an $\mathcal{A}(S^*)$-measurable set $G_{s,n}$ as the set of those $\sigma \in S^*$ such that $|\sigma(s)| > 1/n$. Let $\mathcal{A}_0(S^*)$ be the subring of $\mathcal{A}(S^*)$ consisting of those measurable sets which are contained in the union of finitely many $G_{s,n}$. The set of those sets which are contained in the union of countably many $G_{s,n}$ is a σ-ring of subsets of S^* rendering $\widehat{s}$ measurable for each $s \in S$ and so contains $\mathcal{A}(S^*)$ by the definition of the latter. Hence, the subring $\mathcal{A}_0(S^*)$ generates $\mathcal{A}(S^*)$ as a σ-ring. It follows that every measure μ on $\mathcal{A}_0(S^*)$ which is *finite* in the sense that $\mu(A) < \infty$ for all $A \in \mathcal{A}_0(S^*)$ extends to a unique measure on $\mathcal{A}(S^*)$ ([21], Theorem A p. 53). (A *measure* is positive by definition. When we presently introduce measures with values that are sesquilinear forms, those forms will be positive by definition.)

For every mapping $\mu: \mathcal{A}_0(S^*) \to \mathcal{S}(D)$ and for each $\xi \in D$ we define a mapping $\mu(\cdot)(\xi,\xi): \mathcal{A}_0(S^*) \to \mathbb{C}$ by $\mu(\cdot)(\xi,\xi)(A) = \mu(A)(\xi,\xi)$ for $A \in \mathcal{A}_0(S^*)$. The mapping μ is a *measure* if $\mu(\cdot)(\xi,\xi)$ is a measure for each $\xi \in D$. Let $F_+(S^*, D)$ be the set of those measures μ in this sense such that for each $\xi \in D$ the scalar measure $\mu(\cdot)(\xi,\xi)$ integrates the function $|\widehat{s}|^2$ for each $s \in S$. For $\mu \in F_+(S^*, D)$ define $\mathcal{L}\mu: S+S \to \mathcal{S}(D)$ by

$$\mathcal{L}\mu(s)(\xi,\xi) = \int_{S^*} \sigma(s)\, d\mu(\cdot)(\xi,\xi)(\sigma)$$

for $s \in S+S$ and $\xi \in D$. The integral is understood as one with respect to the unique measure on $\mathcal{A}(S^*)$ which extends the measure $\mu(\cdot)(\xi,\xi)$; it exists by Hölder's inequality. The remaining values of $\mathcal{L}\mu(s)$ follow by polarization. A function $\varphi: S+S \to \mathcal{S}(D)$ is a *moment function* if $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*, D)$, and a moment function φ is *determinate* if there is only one such μ. Denote by $\mathcal{H}(S, D)$ the set of all moment functions, and by $\mathcal{H}_D(S, D)$ the subset of determinate moment functions. Using ideas from the paper of Schmüdgen [29] on the matrix version of the multidimensional moment problem, one can show $\mathcal{H}(S, D) \subset \mathcal{P}(S, D)$. The semigroup S is *semiperfect of order* $d \in \mathbb{N}$ if $\mathcal{H}(S, \mathbb{C}^d) = \mathcal{P}(S, \mathbb{C}^d)$, and *completely semiperfect* if this is so for all $d \in \mathbb{N}$. If S is completely semiperfect then we even have $\mathcal{H}(S, D) = \mathcal{P}(S, D)$ for every complex vector space D [10]. The semigroup S is said to be *semiperfect* for brevity if it is semiperfect of order 1. Every semigroup which has ever (to our knowledge) been shown to be semiperfect has even been shown to be completely semiperfect. The exception is the semiperfect semigroup in [5], Example 3, the complete semiperfectness of which could probably easily be established along the same lines as the semiperfectness. The semigroup S is said to be *perfect* if $\mathcal{H}_D(S) = \mathcal{P}(S)$. As remarked by Christian Berg in the late 1980's, if S is perfect then $\mathcal{H}_D(S, D) = \mathcal{P}(S, D)$ for every complex vector space D.

The group $\mathbb{Z}$ with the *inverse involution* ($n^* = -n$) is perfect by Herglotz' Theorem. More generally, every abelian group with the inverse involution is perfect by the discrete version of the Bochner-Weil Theorem. Even more generally, a $*$-semigroup S is perfect if it is an *abelian inverse semigroup* in the sense that $s + s^* + s = s$ for all $s \in S$. (*Warning*: The 'Bochner-Weil Theorem for Locally Compact Abelian Inverse Semigroups' is false, even in the compact metrizable case. Berg, Christensen, and Ressel write ([3], p. 143): "The compact semigroup $S = [0, 1]$ with maximum as semigroup operation has only one continuous semicharacter, namely, the constant semicharacter." On this semigroup, the function φ defined by $\varphi(s) = 1 - s$ is a continuous positive semidefinite function, and the unique measure μ such that $\varphi = \mathcal{L}\mu$ is concentrated on the set of discontinuous characters. The same authors continue: "The right dual object to look at might be the set of semicharacters which are continuous at 0." However, there exist a compact metrizable semigroup S with zero and $s^* = s = s + s$ for all $s \in S$ and a continuous positive semidefinite function φ on S such that the unique measure μ such that $\varphi = \mathcal{L}\mu$ is concentrated on the set of those characters which are discontinuous at 0 [8].)

The perfectness of abelian inverse semigroups has found two generalizations: (1) A $*$-semigroup S is perfect if $2(s + s^*) = s + s^*$ for all $s \in S$; (2) a $*$-semigroup S is perfect if it is $*$-*divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $m, n \in \mathbb{N}_0$ such that $m + n \geq 2$ and $s = mt + nt^*$ (for the case of semigroups with zero, see the paper by Ressel and the author [16]; for the general case, the paper by Sakakibara and the author [17]). So far, no natural result is known which generalizes both (1) and (2).

The semigroup $\mathbb{N}_0$ with its unique involution, the identity, is semiperfect by Hamburger's Theorem and completely semiperfect by the result of Sz.-Nagy cited above. This semigroup is not perfect since there exist indeterminate moment sequences, such as the example $n \mapsto (4n + 3)!$ given by Stieltjes [31] in 1894, 26 years prior to the publication of Hamburger's Theorem.

The group $\mathbb{Z}$, *considered with the identical involution*, is semiperfect as shown by Jones, Njåstad, and Thron [25]; see [3], 6.4.1, for a modern proof. The complete semiperfectness of $\mathbb{Z}$ is an easy consequence of that of $\mathbb{N}_0$. The semigroup $\mathbb{Z}$, like $\mathbb{N}_0$, is non-perfect since there exist indeterminate two-sided moment sequences, such as $n \mapsto e^{n^2/2}$ ([3], 6.4.6).

For $k \geq 2$ the semigroups $\mathbb{N}_0^k$ and $\mathbb{Z}^k$, considered with the identical involution, are non-semiperfect. For $\mathbb{N}_0^k$, this was first shown by Berg, Christensen, and Jensen [2] and, independently and simultaneously, by Schmüdgen [28]. Each set of authors proved that the convex cone Σ generated by sums of squares of polynomials in $\mathbb{R}[x, y]$ is closed in the finest locally convex topology. Since, as shown already by Hilbert [24], the set Σ is not all of the set $\mathbb{R}[x, y]_+$ of nonnegative polynomials, by the Hahn-Banach Theorem it follows that there is a linear form L on $\mathbb{R}[x, y]$ which is nonnegative on Σ but not on all of $\mathbb{R}[x, y]_+$. The function $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ defined by $\varphi(m, n) = L(x^m y^n)$ for $(m, n) \in \mathbb{N}_0^2$ is then positive semidefinite, but not a moment function. Perhaps the simplest example of a polynomial in $\mathbb{R}[x, y]_+ \setminus \Sigma$ is the Motzkin polynomial $1 + x^4y^2 + x^2y^4 - 3x^2y^2$, which is nonnegative by the arithmetic-geometric inequality and which is not a sum of squares of real polynomials by term-inspection.

Thus no explicit example of a function $\varphi \in \mathcal{P}(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$ was produced. The first such example was given by Friedrich [19]. In his example,

$$\varphi(0,n) = \exp\left\{ \left[\binom{n/2+2}{2} + 1 \right]! \log \binom{n/2+2}{2}! \right\}$$

for even $n \geq 8$. This lead to the question: How fast must $\varphi(m,n)$ grow as $m+n \to \infty$ if $\varphi \in \mathcal{P}(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$? It was shown in [7] that there is some $\varphi \in \mathcal{P}(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$ such that

$$\varphi(m,n) = O\big((m+n)^{a(m+n)}\big) \quad \text{as } m+n \to \infty$$

for each $a > 1$, and the constant 1 is the best possible.

The example in [7] involves the integral

$$\int_0^\infty x^n e^{-x/(1+(\log x)^2)}\, dx,$$

which we have not been able to evaluate. Let us describe a function $\varphi \in \mathcal{P}(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$, of growth intermediate between that of Friedrich and the one from [7], which has the merit of being completely explicit. Let S be the semigroup $\mathbb{N}_0 \setminus \{1\}$. The first published proof of the non-semiperfectness of S is due to Nakamura and Sakakibara [26] although the result had been known to the present author 4 years earlier. Let γ be the positive solution to the equation $\sum_{n=1}^\infty \gamma^{n^2} = 1/2$, define $a = \gamma^{-1/4}$, and define $f\colon S \to \mathbb{R}$ by

$$f(n) = \begin{cases} a^{n^2} & \text{if } n \text{ is even and } n \neq 2 \\ 0 & \text{if } n \text{ is odd or } n = 2. \end{cases}$$

Then f is positive semidefinite but not a moment function. If $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ is defined by $\varphi(m,n) = f(2m+3n)$ for $(m,n) \in \mathbb{N}_0^2$ then φ is positive semidefinite but not a moment function.

The case of $\mathbb{Z}^k$ is an exercise in [3].

These negative results are subsumed in the result that a subsemigroup of $\mathbb{Z}^k$ with the identical involution is semiperfect only if it is $\{0\}$ or isomorphic to $\mathbb{Z}$ or $\mathbb{N}_0$. The case of semigroups containing 0 is due to Sakakibara [27]. The general case is a corollary of the characterization of semiperfect countable $\mathbb{R}$-separative C-finite semigroups, to be cited presently.

A $*$-*homomorphism* between $*$-semigroups is a homomorphism h satisfying $h(s^*) = h(s)^*$ for all s in the domain.

Given a subset M of $\mathbb{C}$, a $*$-semigroup S is said to be M-*separative* if the M-valued characters on S separate points in S. The *greatest M-separative $*$-homomorphic image* of S is the quotient $*$-semigroup $S/{\sim}$ where $\sim$ is the congruence relation in S defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for every M-valued character σ on S. As the name implies, $S/{\sim}$ is M-separative, and among M-separative $*$-homomorphic images of S it is 'greatest' in the sense of corresponding to the smallest congruence relation. Thus, if f is a $*$-homomorphism of S into an M-separative $*$-semigroup T then there is a unique $*$-homomorphism $h\colon S/{\sim} \to T$ such that $f = h \circ g$ where g is the quotient mapping of S onto $S/\!\sim$. Clifford and Preston [18] use the term 'maximal' where we use 'greatest'.

Suppose S is a $*$-semigroup. For every subset V of S, denote by $E(V)$ the set of those $v \in V$ such that if $s, t \in S$, $s + s^*, t + t^* \in V$, and $s + t^* = v$ then $s = t$. For every subset U of S, denote by $C(U)$ the union of all finite subsets V of S such that $E(V) \subset U$. The $*$-semigroup S is *C-finite* if $C(U)$ is a finite set for every finite subset U of S. In [6] we included in the definition of C-finiteness the condition of $\mathbb{R}$-separativity. We apologize for being inconsistent.

A *$*$-subsemigroup* of a $*$-semigroup is a subsemigroup stable under the involution.

A $*$-semigroup H is *$*$-archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbb{N}$ such that $n(x + x^*) = y + z$. A *$*$-archimedean component* of a $*$-semigroup S is a $*$-archimedean $*$-subsemigroup of S which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components, and every $*$-archimedean $*$-subsemigroup of a $*$-semigroup S is contained in a unique $*$-archimedean component of S. See Clifford and Preston [18], Section 4.3, for the case of the identical involution.

An abelian semigroup H is *archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbb{N}$ such that $nx = y + z$. An *archimedean component* of an abelian semigroup S is an archimedean subsemigroup of S which is maximal for the inclusion ordering. An abelian semigroup considered with the identical involution is $*$-archimedean if and only if it is archimedean, and the $*$-archimedean components of an abelian semigroup considered with the identical involution are precisely its archimedean components.

The main result of [6] states that a countable $\mathbb{R}$-separative C-finite semigroup S satisfying $S = S + S$ is semiperfect if and only if the following condition is satisfied:

(B) Each archimedean component of S is isomorphic to the product of a torsion group of exponent 1 or 2 and one of the semigroups $\{0\}$, $\mathbb{Z}$, $\mathbb{N}$.

Every semigroup satisfying $S = S + S$ and (B) is completely semiperfect.

Just as [6] was going to press, it turned out that a semiperfect countable $\mathbb{R}$-separative C-finite semigroup S automatically satisfies $S = S + S$. It was too late for a revision. See [11].

It was shown near the end of [6] that every $\mathbb{R}$-separative finitely generated semigroup is C-finite. Thus the main result of [6] (as augmented in [11]) implies a characterization of semiperfect (or equivalently, completely semiperfect) $\mathbb{R}$-separative finitely generated semigroups.

It is even possible to characterize semiperfect (or equivalently, completely semiperfect) finitely generated abelian semigroups carrying the identical involution without the condition of $\mathbb{R}$-separativity. Consider an arbitrary $*$-semigroup S. Denote by χ the quotient mapping of S onto its greatest $\mathbb{C}$-separative $*$-homomorphic image. For $d \in \mathbb{N}$ the $*$-semigroup S is semiperfect of order d if and only if, firstly, $\chi(S)$ is semiperfect of order d, and secondly, every positive semidefinite function on S factors via χ ([5], proof of Proposition 5). If S is finitely generated and carries the identical involution then the same is true of its $*$-homomorphic image $\chi(S)$. Since $\chi(S)$ is furthermore $\mathbb{R}$-separative then whether $\chi(S)$ is semiperfect (or equivalently, completely semiperfect) can be determined by previous results. It remains to consider the factoring problem.

A $*$-semigroup S is of *class* $\mathcal{M}$ if for each $x \in S$ there exist $e \in S$ and $n \in \mathbb{N}$ such that $n(x+x^*) = e + n(x+x^*)$. The main result of [9] states that if S is a $*$-semigroup of class $\mathcal{M}$ satisfying $S = S+S$ then every positive semidefinite function on S factors via χ. Every semiperfect finitely generated $*$-semigroup is of class $\mathcal{M}$ ([11], Theorem 17). However, not every semiperfect finitely generated $*$-semigroup S satisfies $S = S+S$. Fortunately, in [9] we also considered the case that S is of class $\mathcal{M}$ but $S \neq S+S$. We quote: *Suppose S is a $*$-semigroup of class $\mathcal{M}$. Define $A = \{\, x \in S \mid x + x^* \in S+S+S \,\}$. Let $\overline{A}$ be the least subset of S, containing A, such that if $x \in S$ and $x + x^* \in \overline{A} + S$ then $x \in \overline{A}$. Define $E = S \setminus \overline{A}$. Every set that generates S as a semigroup without involution contains E. In particular, if S is finitely generated then E is finite. Define an equivalence relation $\sim$ in E^2 by the condition that $(e,f) \sim (g,h)$ if and only if $e + f^* = g + h^*$. In order that every positive semidefinite function on S factor via χ, it is necessary that for every nonempty subset $\mathcal{A}$ of E^2 which is a union of equivalence classes with respect to $\sim$ and which is itself an equivalence relation on some subset of E there exist $(e,f) \in \mathcal{A}$ such that $e + f^* \in \overline{A} + S$. If E is finite, the condition is also sufficient.* This result, which formally contains the case that $S = S+S$ (in which case the set E is empty, so the condition is empty), completes the characterization of semiperfect (or equivalently, completely semiperfect) finitely generated abelian semigroups with the identical involution, cf. [11], Theorem 18. It also shows how the problem of characterizing semiperfect (or, presumably equivalently, completely semiperfect) finitely generated abelian semigroups with arbitrary involution is reduced to solving the $\mathbb{C}$-separative case.

Since semiperfect finitely generated semigroups with the identical involution have been completely characterized, it is natural to try to extend the result to arbitrary involution. Since the first step in the case of the identical involution was the characterization of semiperfect countable $\mathbb{R}$-separative C-finite semigroups, it seems natural to begin by extending this result by replacing the condition of $\mathbb{R}$-separativity by that of $\mathbb{C}$-separativity. It is the first main purpose of the present paper to do this. We shall show.

Theorem 1

A countable $\mathbb{C}$-separative C-finite semigroup S is semiperfect (or equivalently, completely semiperfect) if and only if $S = S+S$ and the following condition is satisfied:

(CT) *For each $*$-archimedean component H of S there exist an abelian torsion group D carrying the inverse involution, a semigroup P, which is $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$ and carries the identical involution, and a $*$-subgroup G of the $*$-group $D \times (P-P)$ such that H is isomorphic to the $*$-semigroup $G \cap (D \times P)$.*

Every $*$-semigroup S satisfying $S = S+S$ and (CT) *is completely semiperfect.*

The sufficiency part follows from results in other sources. Indeed, suppose S is a $*$-semigroup and $d \in \mathbb{N}$. Denoting by $M_d(\mathbb{C})$ the algebra of square complex matrices of order d with the adjoint operation $(*)$ as involution, let $M_d(\mathbb{C})[S]$ be the *semigroup ring*, that is, the space of finitely supported $M_d(\mathbb{C})$-valued functions on S equipped with the multiplication $*$ (convolution) and the involution $\tilde{}$ defined by

$a * b(u) = \sum_{s,t \in S: s+t=u} a(s)b(t)$ and $\widetilde{a}(u) = a(u^*)^*$ for $a, b \in M_d(\mathbb{C})[S]$ and $u \in S$. For $a \in M_d(\mathbb{C})[S]$ define $\widehat{a}: S^* \to M_d(\mathbb{C})$ by $\widehat{a}(\sigma) = \sum_{s \in S} \sigma(s)a(s)$ for $\sigma \in S^*$. Let $M_d(\mathbb{C})[S+S]_+$ be the set of those $a \in M_d(\mathbb{C})[S+S]$ (i.e., those $a \in M_d(\mathbb{C})[S]$ supported by $S+S$) which are *nonnegative* in the sense that $\widehat{a}(\sigma)$ is a positive semidefinite matrix for each $\sigma \in S^*$, and let $M_d(\mathbb{C})[S+S]_+^{\mathrm{sa}}$ be the subset consisting of those which are furthermore *self-adjoint* in the sense that $a = \widetilde{a}$. The $*$-semigroup S is *adapted* if for each $x \in S$ there exist $n \in \mathbb{N}$ and $y_1, \ldots, y_{n+1} \in S$ such that $n(x + x^*) = y_1 + y_1^* + \cdots + y_{n+1} + y_{n+1}^*$. By a result in [10], the $*$-semigroup S is semiperfect of order d if and only if S is adapted and the convex cone

$$\Sigma_d(S) = \{\, \widetilde{a}_1 * a_1 + \cdots + \widetilde{a}_n * a_n \mid a_1, \ldots, a_n \in M_d(\mathbb{C}) \,\}$$

is dense in $M_d(\mathbb{C})[S+S]_+^{\mathrm{sa}}$.

By [13], Theorem 2, if S is an arbitrary $*$-semigroup satisfying (CT) then not only is $\Sigma_d(S)$ dense in $M_d(\mathbb{C})[S]_+$, but for each $a \in M_d(\mathbb{C})[S]_+$ there is even some $b \in M_d(\mathbb{C})[S]$ such that $a = \widetilde{b} * b$. If furthermore $S = S + S$ then S is obviously adapted, hence semiperfect of order d. This being so for all $d \in \mathbb{N}$, S is completely semiperfect. This proves the sufficiency of the condition in Theorem 1. We shall prove the necessity of the condition in the next section.

Unlike the case of the identical involution, the step from Theorem 1 to a characterization of semiperfect $\mathbb{C}$-separative finitely generated $*$-semigroups is not easy and has not been completed at the time of writing. This is because a semiperfect $\mathbb{C}$-separative finitely generated $*$-semigroup need not be C-finite, cf. the example of $\mathbb{Z}$ with the inverse involution. Let (C) be the condition which is as (CT) except that the word 'torsion' is omitted. We shall show in Section 3 that in order for a $\mathbb{C}$-separative finitely generated $*$-semigroup S to be semiperfect it is necessary that $S = S + S$ and that (C) hold. Again, the example of $\mathbb{Z}$ shows that we could not retain the word 'torsion'. It is unknown whether every $*$-semigroup S satisfying $S = S + S$ and (C) is semiperfect (even if S is assumed to be finitely generated). One special case in which the answer is known to be affirmative is the case that for each $*$-archimedean component H of S we have $H = D \times P$ (see [12]).

Note added in proof: Since the above was written, it has turned out that as long as one stays within the realm of finitely generated abelian semigroups with involution, the condition of which we show the necessity at the end of this paper is also sufficient. However, the proof exists only in hand-written form.

2. Necessity

This section contains the proof of the necessity of the condition in Theorem 1. We assume from the outset that S is a semiperfect countable $\mathbb{C}$-separative C-finite semigroup. We have to show that $S = S + S$ and that (CT) holds. Define $\Sigma(S) = \Sigma_1(S)$, identifying a square matrix of order 1 with its unique entry. Saying that S is *normal* means that $C(\emptyset) = \emptyset$.

Lemma 1

We have $\Sigma(S) = \mathbb{C}[S + S]_+$.

Proof. Since S is $\mathbb{C}$-separative then S is normal ([11], Corollary 2). Since S is countable, normal, and C-finite then by [11], Theorem 10, the convex cone $\Sigma(S)$ is closed in the finest locally convex topology, hence equal to $\mathbb{C}[S+S]_+$ by the fact that S is semiperfect. $\square$

Denote by ρ the quotient mapping of S onto its greatest $\mathbb{R}_+$-separative $*$-homomorphic image. Say that a mapping f of a set X into a set Y is *proper* if $f^{-1}(y)$ is a finite set for each $y \in Y$. This use of the term is in accordance with its use in general topology if the sets involved are considered with the discrete topology. Since S is $\mathbb{C}$-separative then S is, in particular, $*$-*separative* in the sense that if $x, y \in S$ are such that

$$x + x^* = y + x^* = x + y^* = y + y^* \tag{2}$$

then $x = y$. (Proof: Apply to (2) an arbitrary character σ on S, see that it follows that $\sigma(x) = \sigma(y)$, and use $\mathbb{C}$-separativity.) Hence, each $*$-archimedean component of S is cancellative [9].

Lemma 2

The semigroup $\rho(S)$ is C-finite and the mapping ρ is proper.

Proof. Define a $*$-subsemigroup $S^\#$ of S by $S^\# = \{\, s+s^* \mid s \in S \,\}$. By [11], Theorem 4, the semigroup $\rho(S)$ is isomorphic to $S^\#$ which, being a $*$-subsemigroup of the C-finite semigroup S, is C-finite. Thus $\rho(S)$ is C-finite. To see that the mapping ρ is proper, suppose $y \in \rho(S)$; we have to show that the set $A = \rho^{-1}(y)$ is finite. Let K be the archimedean component of $\rho(S)$ containing y. The set $H = \rho^{-1}(K)$ is a $*$-archimedean component of S ([5], Lemma 3) which clearly contains A. Recall that H is cancellative. Choose $a \in A$. For $s \in A$ we have $s + a^* \in C(\{a + a^*, s + s^*\})$ by [11], Theorem 1, item (viii). Now $a + a^* = s + s^*$. To see this, note that $\rho(a) = y = \rho(s)$. By [11], equation (3), it follows that there is some $n \in \mathbb{N}$ such that $n(a + a^*) = n(y + y^*)$. For $\sigma \in S^*$ we have $|\sigma(a)|^{2n} = \sigma\bigl(n(a + a^*)\bigr) = \sigma\bigl(n(s + s^*)\bigr) = |\sigma(s)|^{2n}$, hence $|\sigma(a)| = |\sigma(s)|$, so $\sigma(a + a^*) = |\sigma(a)|^2 = |\sigma(s)|^2 = \sigma(s + s^*)$. This being so for all $\sigma \in S^*$, since S is $\mathbb{C}$-separative it follows that $a + a^* = s + s^*$, as claimed. Thus $s + a^* \in C(\{a + a^*\})$. Thus the mapping $s \mapsto s + a^*$ maps A into the set $C(\{a + a^*\})$ which is finite since S is C-finite. That mapping is one-to-one since H is cancellative, so the set A is likewise finite. $\square$

Saying that S is a *Z-semigroup* means that each archimedean component of $\rho(S)$ is isomorphic to a subsemigroup of $\mathbb{Z}$.

Lemma 3

Each archimedean component of $\rho(S)$ is isomorphic to $\{0\}$, $\mathbb{Z}$, *or* $\mathbb{N}$. *In particular, S is a Z-semigroup.*

Proof. Being a $*$-homomorphic image of the countable semiperfect semigroup S, $\rho(S)$ is countable and (by [16], Proposition 1) semiperfect. Thus $\rho(S)$ is a semiperfect countable $\mathbb{R}_+$-separative C-finite semigroup. By [11], Theorem 15, it follows that each archimedean component of $\rho(S)$ is isomorphic to the product of a torsion group and one of the semigroups $\{0\}$, $\mathbb{Z}$, $\mathbb{N}$. The torsion group involved must be the trivial group since $\rho(S)$ (hence each archimedean component of it) is $\mathbb{R}_+$-separative. Thus each archimedean component of $\rho(S)$ is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$. □

Lemma 4

We have $S = S + S$, and S is of class $\mathcal{M}$. For each $x \in S$ there is some $e \in S$ such that $x = e + x$.

Proof. Since S is semiperfect, it is adapted. Since S is an adapted $*$-separative Z-semigroup, by [11], Theorem 14, it follows that $S = S + S$ and that S is of class $\mathcal{M}$. The remaining statement follows by [11], Corollary 3. □

It remains to be shown that (CT) holds. In the remainder of this section, we assume that H is a $*$-archimedean component of S. We have to show that there exist D, P, and G as in (CT). For every $*$-semigroup T, denote by π the quotient mapping of T onto its greatest $(\mathbb{T} \cup \{0\})$-separative $*$-homomorphic image. Define $D = \pi(H)$ and $P = \rho(H)$.

Lemma 5

The semigroup P is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$.

Proof. By [5], Lemma 3, P is an archimedean component of $\rho(S)$. The claim follows by Lemma 3. □

Lemma 6

The $*$-semigroup D is an abelian group carrying the inverse involution.

Proof. From the fact that D is $(\mathbb{T} \cup \{0\})$-separative it follows that D is an abelian inverse semigroup. Being a $*$-homomorphic image of the $*$-archimedean semigroup H, D is $*$-archimedean. The claim follows. □

Since S is $\mathbb{C}$-separative, so is H. If η is an arbitrary character on H then the function $|\eta|$ is a nonnegative character on H and so factors via P. Since H is $*$-archimedean then η is nowhere zero ([5], Lemma 2), so the function $\eta/|\eta|$ is well-defined. This function being a $\mathbb{T}$-valued character on H, it factors via π. From the $\mathbb{C}$-separativity of H we can therefore infer that the mapping $x \mapsto \big(\pi(x), \rho(x)\big) \colon H \to D \times P$ is one-to-one. We suppress this mapping, thus identifying H with a $*$-subsemigroup of $D \times P$. Since P is $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$ then $P - P$ is a group, and we consider the product group $D \times (P - P)$ with the natural involution, i.e., $(x, n)^* = (-x, n)$ for $x \in D$ and $n \in P - P$. Then the set $G = H - H$ is a $*$-subgroup of $D \times (P - P)$. What remains to be shown is that D is a torsion group and $H = G \cap (D \times P)$. For $n \in P - P$ denote by G_n the set of those $x \in D$ such that $(x, n) \in G$. The set G_0 is a subgroup of D of index 1 or 2, and each G_n is a coset of G_0, equal to G_0 for even n and to the other coset for odd n. For these facts, see [13], proof of Lemma 6.

Theorem 2

If P is a group then so is H, and D is a torsion group. Thus the conclusions of the necessity part of Theorem 1 hold in this case.

Proof. Suppose P is a group. So is H, by [14], Theorem 12. It remains to be shown that D is a torsion group. Since the group G_0 has finite index in D, it suffices to show that G_0 is a torsion group. It obviously suffices to show that G_0 is finite. But this follows from the fact that the set $G_0 \times \{0\}$ is finite, being equal to the set $\rho^{-1}(0)$ which is finite since the mapping ρ is proper. $\square$

In the remainder of this section, we assume that $P = \mathbb{N}$.

A *face* of S is a $*$-subsemigroup X of S such that if $x, y \in S$ and $x + y \in X$ then $x, y \in X$. Every intersection of faces of S, if nonempty, is a face of S, so if A is a nonempty subset of S then there is a least face of S containing A, viz., the intersection of all faces of S containing A, the set of such faces being nonempty since S itself is such a face. If A is a $*$-subsemigroup of S then the least face X of S containing A is equal to the set Y of those $x \in S$ such that $x + y \in A$ for some $y \in S$. The proof consists in noting that it is clear that $Y \subset X$, and for the converse inclusion it suffices to verify that Y is a face of S containing A; which is trivial.

Let X be the least face of S containing H. Then $X + H \subset H$ (see, e.g., the Introduction in [11]). Define $g\colon X \to G$ by the condition that given $x \in X$ one chooses $y \in H$ and sets $g(x) = (x+y) - y$ (difference in the group G). It is easily seen that the definition of $g(x)$ is independent of the choice of y and that the mapping g so defined is a $*$-homomorphism. Note that $g(X) + H = X + H \subset H$. Define $Y = H \cup (\rho \circ g)^{-1}(0) \subset X$. By [14], Lemma 4, Y is a $*$-subsemigroup of S.

Lemma 7

We have $\Sigma(Y) = \mathbb{C}[Y]_+$. Moreover, Y is adapted, so Y is semiperfect.

Proof. Since $\Sigma(S) = \mathbb{C}[S]_+$ then by the proof of [14], Lemma 5, we have $\Sigma(Y) = \mathbb{C}[Y]_+$. To see that Y is semiperfect it therefore suffices to verify that Y is adapted. It clearly suffices to show $Y = Y + Y$. So suppose $y \in Y$. By Lemma 4 there is some $e \in S$ such that $y = e + y$. Since $e + y = y \in Y \subset X$ then $e \in X$ by the definition of a face. Now $g(y) = g(e + y) = g(e) + g(y)$. Since G is a group it follows that $g(e) = 0$. So much the more is $\rho\big(g(e)\big) = 0$. Now $e \in (\rho \circ g)^{-1}(0) \subset Y$. Thus $y = e + y \in Y + Y$. Since $y \in Y$ was arbitrary, we have shown $Y = Y + Y$, as desired. $\square$

We have already once used the *homomorphism theorem*, which asserts that every $*$-homomorphic image of a semiperfect semigroup is semiperfect. See the paper by Ressel and the author [16], Proposition 1.

Corollary 1

The $*$-semigroup $g(Y)$ is semiperfect.

Note that $g\big((\rho \circ g)^{-1}(0)\big) \subset \rho^{-1}(0) = G_0 \times \{0\}$. Since the set to the left is a $*$-semigroup and since G_0 carries the inverse involution, it follows that $g(Y) = (B \times \{0\}) \cup H$ for a certain subgroup B of G_0.

For $n \in \mathbb{N}$ let H_n be the set of those $x \in D$ such that $(x, n) \in H$. Clearly $H_n \subset G_n$. Since $P = \rho(H)$ then

$$H_n \neq \emptyset. \tag{3}$$

Since H is $*$-stable then H_n is symmetric, i.e.,

$$H_n = -H_n. \tag{4}$$

Since H is a semigroup then

$$H_j + H_k \subset H_{j+k} \tag{5}$$

for $j, k \in \mathbb{N}$. Since $g(Y) + H \subset H$ then

$$B + H_n \subset H_n. \tag{6}$$

Lemma 8

The group B is finite, hence a torsion group.

Proof. By (3) and (6), for each $n \in \mathbb{N}$ the set H_n contains a coset of B. Thus it suffices to show that H_n is finite. But this follows from the fact that the set $H_n \times \{n\} = \rho^{-1}(n)$ so is, the mapping ρ being proper. □

By [14], Theorem 11, we have

$$\Sigma(\mathbb{C}[H_n]) = \mathbb{C}[H_{2n}]_+, \tag{7}$$

hence

$$H_n + H_n = H_{2n}, \tag{8}$$

for all $n \in \mathbb{N}$.

Our next aim is to show that D is a torsion group. In the following portion of the present section, we assume that D is *not* a torsion group. That portion will end when we have arrived at a contradiction.

Denote by D_t the torsion of D. Since we now are assuming that D is not a torsion group then the torsion-free group D/D_t is nonzero, so its enveloping rational vector space is of positive dimension. Hence we can choose a nonzero linear form on that space. We see from this that there is a nonzero homomorphism of D into $\mathbb{Q}$, which we denote by $x \mapsto \overline{x}$. The mapping $(x, n) \mapsto (\overline{x}, n)\colon D \times \mathbb{N} \to \mathbb{Q} \times \mathbb{N}$ will also be denoted by $x \mapsto \overline{x}$. For every subset A of D or $D \times \mathbb{N}$ we denote by $\overline{A}$ the image of A under the mapping $x \mapsto \overline{x}$.

Lemma 9

The analogues of (3) *through* (8) *hold with bars over.*

Proof. Left as an exercise. $\square$

Lemma 10

For each $n \in \mathbb{N}$ the set $\overline{H}_n$ is a finite arithmetic progression, i.e., there exist $p_n \in \mathbb{Q}$, $d_n > 0$, and $m_n \in \mathbb{N}_0$ such that $\overline{H}_n = \{-p_n, -p_n + d_n, -p_n + 2d_n, \ldots, -p_n + m_n d_n\}$. Moreover, $-p_n + m_n d_n = p_n$.

Proof. The set $\overline{H}_n$, being a finite subset of $\mathbb{Q}$, can be identified with a subset of $\mathbb{Z}$. From the bar-analogue of (7), by a result of Gabardo [20] it follows that $\overline{H}_n$ is an arithmetic progression. The remaining statement follows from the fact that $\overline{H}_n$ is symmetric. $\square$

By the Lemma and the bar-analogue of (5) it is clear that

$$p_j + p_k = p_{j+k} \tag{9}$$

for $j, k \in \mathbb{N}$.

Lemma 11

There is some $d > 0$ such that $d_n = d$ for all $n \in \mathbb{N}$.

Proof. It is clear from (5) that d_j and d_k are integral multiples of d_{j+k}. Hence, the sequence $(d_n)_{n=1}^{\infty}$ is nonincreasing. Now that sequence must be constant since by (8) we have $d_n = d_{2n}$ for all n. $\square$

It is clear from (9) and the Lemmas that $\overline{H}_j + \overline{H}_k = \overline{H}_{j+k}$ for all $j, k \in \mathbb{N}$. By induction it follows that

$$\overline{H}_n = \overbrace{\overline{H}_1 + \cdots + \overline{H}_1}^{n} \tag{10}$$

for all $n \in \mathbb{N}$.

Theorem 3

The group D is a torsion group.

Proof. We continue with the assumption that D is *not* a torsion group. The set $\overline{H}_1$ consists of $m_1 + 1$ points. Hence, by (10), the set $\overline{H}_n$ consists of $nm_1 + 1$ points. If we had $m_1 = 0$ then all the $\overline{H}_n$ would be singletons, contradicting the fact that we chose a nonzero homomorphism of D into $\mathbb{Q}$. Thus $m_1 \geq 1$. We leave it as an exercise to describe a $*$-isomorphism f of $\overline{H} \cup \{0\}$ into the $*$-semigroup $(\mathbb{N}_0^2, *)$ where $*$ is the *switching involution*, $(p, q)^* = (q, p)$. (Hint: Map $(-p_1, 1)$ to $(m_1, 0)$.) By Corollary 1 and the homomorphism theorem, the semigroup $\overline{H} \cup \{0\}$ is semiperfect. Hence so is its image by f, which we denote by T. Note that T is the set of those $(p, q) \in (\mathbb{N}_0^2, *)$ such that $p+q$ is a multiple of m_1. Now the property of semiperfectness is clearly a property of the semigroup ring. See [3], proof of 6.3.5, for a $*$-algebra isomorphism between the semigroup rings of the $*$-semigroups $(\mathbb{N}_0^2, *)$ and $\mathbb{N}_0^2$, where by '$\mathbb{N}_0^2$' (without '$*$') we understand the semigroup $\mathbb{N}_0^2$ with the identical involution. Under this isomorphism, the semigroup ring of T, which is a $*$-subalgebra of $\mathbb{C}[\mathbb{N}_0^2, *]$, corresponds to the semigroup ring of a semigroup U which is the same *set* as T, but is now considered as a subsemigroup of $\mathbb{N}_0^2$ with the identical involution. Since T is semiperfect, so is U. But this contradicts the fact that every semiperfect subsemigroup of $\mathbb{N}_0^2$ is $\{0\}$ or isomorphic to $\mathbb{N}_0$. This is the desired contradiction which shows that D is a torsion group. □

We can now quickly complete the proof of Theorem 1. Indeed, by [15], Theorem 4, the group G_0 is contained in H_n for all even n, and the set H_1 contains a coset of every finite subgroup of G_0. Since the sets H_n are finite then G_0 is finite, so H_1 contains a coset of G_0 itself. On the other hand, H_1 is *contained* in such a coset, viz., the set G_1. It follows that we must have $H_1 = G_1$. It easily follows that $H_n = G_n$ for all n, that is, $H = G \cap (D \times \mathbb{N})$, as desired.

3. On semiperfect finitely generated $*$-semigroups

The purpose of the present section is to prove the following result.

Theorem 4

If S is a semiperfect $\mathbb{C}$-separative finitely generated $*$-semigroup then $S = S + S$ and the following condition is satisfied:

(C) *For each $*$-archimedean component H of S there exist an abelian group D carrying the inverse involution, a semigroup P, which is $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$ and carries the identical involution, and a $*$-subgroup G of the $*$-group $D \times (P - P)$ such that H is isomorphic to the $*$-semigroup $G \cap (D \times P)$.*

The proof is the remainder of this section. As we mentioned in the Introduction, a main problem is that the groups D cannot be assumed to be torsion groups. The idea of proof is to 'factor away the non-torsion'. Indeed, we shall describe a certain $*$-homomorphic image $\phi(S)$ of S which is C-finite. Then we know that $\phi(S)$ satisfies (CT), and pulling this information back along ϕ we shall obtain (C).

Let $\mathcal{J}(S)$ be the set of all $*$-archimedean components of S. For $H, K \in \mathcal{J}(S)$ the $*$-subsemigroup $H + K$ of S is $*$-archimedean, hence contained in a unique $*$-archimedean component of S, which we denote by $H \vee K$. The pair $\big(\mathcal{J}(S), \vee\big)$ is an abelian semigroup. Consider $\mathcal{J}(S)$ with the partial ordering $\leq$ defined by the condition that $H \leq K$ if and only if $H \vee K = K$ (that is, $H + K \subset K$).

Since S is $\mathbb{C}$-separative then each $H \in \mathcal{J}(S)$ is cancellative, and we denote by G_H the group $H - H$. For $H, K \in \mathcal{J}(S)$ such that $H \leq K$, we define a mapping $g_{H,K}\colon H \to G_K$ by choosing $y \in K$ and setting

$$g_{H,K}(x) = (x + y) - y \quad \text{(difference in the group } G_K\text{)}$$

for $x \in H$. It is easily seen that the definition of $g_{H,K}(x)$ is independent of the choice of y and that the mapping $g_{H,K}$ so defined is a $*$-homomorphism. Being a

$*$-homomorphism of H into the group G_K, the mapping $g_{H,K}$ extends to a unique $*$-homomorphism of G_H into G_K, also denoted by $g_{H,K}$. We leave it as an exercise to verify that the family $(g_{H,K})$, indexed by those $H, K \in \mathcal{J}(S)$ such that $H \leq K$, is *compatible* in the sense that $g_{H,H}$ is the identity on G_H and $g_{H,L} = g_{K,L} \circ g_{H,K}$ for all $H, K, L \in \mathcal{J}(S)$ such that $H \leq K \leq L$. Hence, the disjoint union

$$G = \bigcup_{H \in \mathcal{J}(S)} G_H$$

becomes a $*$-semigroup when considered with the addition defined by

$$x + y = g_{H,H\vee K}(x) + g_{K,H\vee K}(y) \quad \text{(sum in the group } G_{H\vee K})$$

for $H, K \in \mathcal{J}(S)$, $x \in G_H$, and $y \in G_K$, and the unique involution which extends the given one on each G_H. For the case of no involution, see Clifford and Preston [18], Theorem 4.11. We leave the bit about the involution as an exercise.

For $H \in \mathcal{J}(S)$ we write $D_H = \pi(H)$ and $P_H = \rho(H)$ and denote the mappings $\pi|H$ and $\rho|H$ by π_H and ρ_H, respectively. Since S is $\mathbb{C}$-separative, so is H, so the mapping $x \mapsto \big(\pi(x), \rho(x)\big)\colon H \to D_H \times P_H$ is one-to-one. We suppress this mapping, thus identifying H with a $*$-subsemigroup of $D_H \times P_H$. Since H is $*$-archimedean then D_H is a group. The mapping π_H, being a $*$-homomorphism of H into the group D_H, extends to a unique $*$-homomorphism of G_H into D_H, also denoted by π_H. The semigroup P, being a $*$-homomorphic image of the $*$-archimedean semigroup H, is archimedean. Being also $\mathbb{R}_+$-separative, it is cancellative and the group $P_H - P_H$ is torsion-free. The mapping ρ_H, being a $*$-homomorphism of H into the group $P_H - P_H$, extends to a unique $*$-homomorphism of G_H into $P_H - P_H$, also denoted by ρ_H.

For $H, K \in \mathcal{J}(S)$ such that $H \leq K$, the mapping $\pi_K \circ g_{H,K}$ is a $*$-homomorphism of G_H into the group D_K, which carries the inverse involution, and so has the form $\pi_K \circ g_{H,K} = \pi_{H,K} \circ \pi_H$ for a unique homomorphism $\pi_{H,K}\colon D_H \to D_K$. The mapping $\rho_K \circ g_{H,K}\colon G_H \to P_K - P_K$ is a $*$-homomorphism of G_H into the $\mathbb{R}_+$-separative group $P_K - P_K$ and so has the form $\rho_K \circ g_{H,K} = \rho_{H,K} \circ \rho_H$ for a unique homomorphism $\rho_{H,K}\colon P_H - P_H \to P_K - P_K$. We extend the mapping $g_{H,K}\colon G_H \to G_K$ to a $*$-homomorphism of $D_H \times (P_H - P_H)$ into $D_K \times (P_K - P_K)$, also denoted by $g_{H,K}$, by setting

$$g_{H,K}(x, y) = \big(\pi_{H,K}(x), \rho_{H,K}(y)\big)$$

for $(x, y) \in D_H \times (P_H - P_H)$.

For $H \in \mathcal{J}(S)$, let X_H be the least face of S containing H. It is easily seen that

$$X_H = \bigcup_{I \in \mathcal{J}(S) : I \leq H} I.$$

In particular, $X_H + H \subset H$. Since S is finitely generated, so is X_H. Indeed, suppose E is a finite set that generates S as a semigroup without involution. Each element of X_H can be written as the sum of certain elements of E, repetitions allowed, and these must be in X_H by the definition of a face. Thus X_H is generated by the finite set

$E \cap X_H$. Let $g_H: X_H \to G_H$ be the union of the mappings $g_{I,H}$ ($I \in \mathcal{J}(S)$, $I \leq H$). The set $\{0\} \cup \big(g_H(X_H) \cap (D_H \times \{0\})\big)$ has the form $B_H \times \{0\}$ for a unique subgroup B_H of D_H. Let $\phi_H: D_H \to D_H/B_H$ be the quotient mapping and denote by the same symbol the mapping $(x, y) \mapsto \big(\phi_H(x), y\big): D_H \times (P_H - P_H) \to (D_H/B_H) \times (P_H - P_H)$.

For $H, K \in \mathcal{J}(S)$ such that $H \leq K$, we have $X_H \subset X_K$, hence $\pi_{H,K}(B_H) \subset B_K$. Thus the mapping $\phi_K \circ \pi_{H,K}: D_H \to D_K/B_K$ vanishes on B_H and so has the form $\phi_K \circ \pi_{H,K} = \phi_{H,K} \circ \phi_H$ for a unique homomorphism $\phi_{H,K}: D_H/B_H \to D_K/B_K$. We denote by the same symbol the mapping $(x, y) \mapsto \big(\phi_{H,K}(x), \rho_{H,K}(y)\big): (D_H/B_H) \times (P_H - P_H) \to (D_K/B_K) \times (P_K - P_K)$. We leave it as an exercise to verify that the family $(\phi_{H,K})$, indexed by those $H, K \in \mathcal{J}(S)$ such that $H \leq K$, is compatible. It follows that the disjoint union

$$\phi(G) = \bigcup_{H \in \mathcal{J}(S)} \phi_H(G_H)$$

is a $*$-semigroup with the natural operations, cf. [18], Theorem 4.11. We now define a mapping $\phi: G \to \phi(G)$ by $\phi|G_H = \phi_H$ for $H \in \mathcal{J}(S)$. We leave it as an exercise to verify that ϕ is a $*$-homomorphism. Being a $*$-homomorphic image of the semiperfect semigroup S, the semigroup $\phi(S)$ is semiperfect. Since S is finitely generated, so is $\phi(S)$. In particular, this semigroup is countable. We leave it as an exercise to verify that $\phi(S)$ is $\mathbb{C}$-separative. Now the point is that $\phi(S)$ is C-finite. In order not to interrupt the flow of the present proof, we postpone the proof of this to a Lemma below. Believing that it is true, we see that $\phi(S)$ is a semiperfect countable $\mathbb{C}$-separative C-finite semigroup. By Theorem 1 it follows that if K is a $*$-archimedean component of $\phi(S)$ then there exist an abelian group A carrying the inverse involution, a semigroup Q, which is $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$ and carries the identical involution, and a $*$-subgroup F of the $*$-group $A \times (Q - Q)$ such that

$$K = F \cap (A \times Q). \tag{11}$$

Moreover, $\phi(S) = \phi(S) + \phi(S)$. From the latter fact it easily follows that $S = S + S$. It remains to be shown that (C) holds. Suppose H is a $*$-archimedean component of S. We leave it as an exercise to verify that the set $K = \phi(H)$ is a $*$-archimedean component of $\phi(S)$. Hence (11) holds, and applying ϕ^{-1} we get the desired fact.

To make the proof of Theorem 4 complete, it remains to show a few Lemmas. For $n \in \mathbb{N}$ consider $\mathbb{N}_0^n$ with the product ordering, i.e., if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are elements of $\mathbb{N}_0^n$ then we write $x \leq y$ if $x_i \leq y_i$ for $i = 1, \ldots, n$. Say that x and y are *incomparable* if neither $x \leq y$ nor $y \leq x$.

Lemma 12

If X is a subset of $\mathbb{N}_0^n$ such that every two distinct elements of X are incomparable then X is finite.

Proof. If X is empty, there is nothing to prove. Suppose X is nonempty and choose $x = (x_1, \ldots, x_n) \in X$. It suffices to show that the set $Y = X \setminus \{x\}$ is finite. If $y = (y_1, \ldots, y_n) \in Y$ then by hypothesis it is not the case that $y \geq x$. Hence we have $y \in Y_1 \cup \cdots \cup Y_n$ where Y_i, for $i = 1, \ldots, n$, is the set of those $z = (z_1, \ldots, z_n) \in Y$ such that $z_i < x_i$. It suffices to show that Y_i is finite for each i. Let us show that the set $Z = Y_1$ is finite, the other cases being similar. We have $Z = Z_0 \cup \cdots \cup Z_{x_1 - 1}$ where Z_m is the set of those $(z_1, \ldots, z_n) \in Z$ such that $z_1 = m$. It suffices to show that each Z_m is finite. Now $Z_m = \{m\} \times W$ for a certain subset W of $\mathbb{N}_0^{n-1}$. Since every two elements of Z_m are incomparable, the same is true of W. By an obvious induction argument (the beginning of which is trivial), we may assume that it follows that W is finite. This completes the proof. $\square$

Lemma 13

Suppose G is an abelian group, A is a subgroup of G, and S is a finitely generated subsemigroup of G. If the set $A \cap S$ is finite then so is the set $B \cap S$ for every coset B of A in G.

Proof. First suppose S contains 0 and the set $A \cap S$ is reduced to $\{0\}$. Let B be a coset of A in G; we have to show that the set $B \cap S$ is finite. Choose a finite set $\{e_1, \ldots, e_n\}$ which generates S. Define $h\colon \mathbb{N}_0^n \to S$ by $h(x) = \sum_{i=1}^n x_i e_i$ for $x = (x_1, \ldots, x_n) \in \mathbb{N}_0^n$. Then h maps $\mathbb{N}_0^n$ onto S. Hence, we can choose a subset X of $\mathbb{N}_0^n$ such that the mapping $h|X$ is a bijection of X onto S. Now we precisely have to show that the set $Y = h^{-1}(B \cap S)$ is finite. Suppose it is infinite. By the preceding Lemma there exist $x, y \in Y$ such that $x \leq y$ but $x \neq y$. Thus the element $z = y - x$ is a well-defined nonzero element of $\mathbb{N}_0^n$ and $h(z) = h(y) - h(x) \in B - B = A$, hence $h(z) \in A \cap S = \{0\}$, that is, $h(z) = 0$, whence $h(x) = h(y)$, contradicting the fact that $x \neq y$, cf. the choice of Y.

In a second step, still suppose $0 \in S$, but now let $A \cap S$ be arbitrary (except that it should of course be finite). Then the set $C = A \cap S$, being a finite cancellative semigroup, is a finite group. Denoting by a prime the passage to the quotient group G/C, the set $A' \cap S'$ is $\{0\}$, so by what we already proved, if B is a coset of A in G then B' is a coset of A' in G', so the set $B' \cap S'$ is finite. Hence so is its inverse image under the quotient mapping since that mapping is proper, C being finite. That is, the set $B \cap S$ is finite.

In a third and last step, remove the assumption that S contains 0. Let T be the semigroup obtained by adjoining 0 to S. Then the sets $A \cap S$ and $A \cap T$ differ by only one point, and for every coset B of A distinct from A itself the sets $B \cap S$ and $B \cap T$ are the same. Thus, the statement about S follows from the corresponding one for T, which is true by what we already proved. $\square$

Lemma 14

Among $\mathbb{C}$-separative semigroups, the C-finite ones are characterized by the properties mentioned in Lemma 2.

Proof. Suppose S is a $\mathbb{C}$-separative semigroup; we have to show that S is C-finite if and only if $\rho(S)$ is C-finite and the mapping ρ is proper. The 'only if' part is Lemma 2. Conversely, suppose $\rho(S)$ is C-finite and the mapping ρ is proper; we have to show that S is C-finite. Suppose U is a finite subset of S; we have to show that the set $C(U)$ is finite. The set $\rho(U)$ is obviously finite. So is the set $C\big(\rho(U)\big)$ since $\rho(S)$ is C-finite. The semigroup S, being $\mathbb{C}$-separative, is normal. Now ρ is a $*$-homomorphism of the normal $*$-semigroup into $\rho(S)$. It follows that $\rho\big(C(U)\big) \subset C\big(\rho(U)\big)$ ([14], Theorem 8). Thus the set $\rho\big(C(U)\big)$ is finite. Since the mapping ρ is proper, so is the set $\rho^{-1}\big[\rho\big(C(U)\big)\big]$, which clearly contains $C(U)$. $\square$

We can now clear up the remaining point in the proof of Theorem 4. We have to show that $\phi(S)$ is C-finite. By Lemma 14, it suffices to show that $\rho\big(\phi(S)\big)$ is C-finite and the mapping ρ is proper. Since ρ already denotes the quotient mapping of S onto its greatest $\mathbb{R}_+$-separative $*$-homomorphic image, the corresponding mapping for $\phi(S)$ will be denoted by another symbol. Indeed, it is easily seen that there is a unique mapping $\rho' \colon \phi(S) \to \rho(S)$ such that $\rho = \rho' \circ \phi$. We leave it as an exercise to verify that ρ' can be identified with the quotient mapping of $\phi(S)$ onto its greatest $\mathbb{R}_+$-separative $*$-homomorphic image. Thus the greatest $\mathbb{R}_+$-separative $*$-homomorphic image of $\phi(S)$ is (canonically isomorphic to) $\rho(S)$, which is C-finite. It remains to be shown that the mapping ρ' is proper. We leave this as an exercise.

Acknowledgments. Running expenses connected with this piece of research were covered by the Carlsberg Foundation. Joan Bødker at the Institute of Mathematics at the University of Copenhagen kindly sent us journal addresses and copies of articles. Søren Klitgård Nielsen provided vital technical support. My parents, Ingeborg and Knud Maack Bisgård, provided shelter, food, a computer, and music sources.

References

1. N.I. Akhiezer, *The classical moment problem*, Oliver & Boyd, Edinburgh, 1965.
2. C. Berg, J.P.R. Christensen, and C.U. Jensen, A remark on the multidimensional moment problem, *Math. Ann.* **243** (1979), 163–169.
3. C. Berg, J.P.R. Christensen, and P. Ressel, *Harmonic analysis on semigroups*, Springer-Verlag, Berlin, 1984.
4. T.M. Bisgaard, Positive definite operator sequences, *Proc. Amer. Math. Soc.* **121** (1994), 1185–1191.
5. T.M. Bisgaard, Extensions of Hamburger's theorem, *Semigroup Forum* **57** (1998), 397–429.
6. T.M. Bisgaard, Semiperfect countable C-finite semigroups S satisfying $S = S + S$, *Math. Ann.* **315** (1999), 141–168.
7. T.M. Bisgaard, On the growth of positive definite double sequences which are not moment sequences, *Math. Nachr.* **210** (2000), 67–83.
8. T.M. Bisgaard, Bochner's theorem for semigroups: A counterexample, *Math. Scand.*, to appear.
9. T.M. Bisgaard, Factoring of positive definite functions on semigroups, *Semigroup Forum*, to appear.
10. T.M. Bisgaard, The method of moments for adapted semigroups, (preprint).
11. T.M. Bisgaard, Semiperfect finitely generated abelian semigroups without involution, (preprint).
12. T.M. Bisgaard, New examples of semiperfect semigroups based on a converse homomorphism theorem, (preprint).

13. T.M. Bisgaard, On the representation of a nonnegative element of a semigroup ring as an 'absolute square' I. Sufficiency, (preprint).
14. T.M. Bisgaard, On the representation of a nonnegative element of a semigroup ring as an 'absolute square' II. Necessity: The group case, (preprint).
15. T.M. Bisgaard, On the representation of a nonnegative element of a semigroup ring as an 'absolute square' III. Necessity: The general case, (preprint).
16. T.M. Bisgaard and P. Ressel, Unique disintegration of arbitrary positive definite functions on $*$-divisible semigroups, *Math. Z.* **200** (1989), 511–525.
17. T.M. Bisgaard and N. Sakakibara, A reduction of the problem of characterizing perfect semigroups, *Math. Scand.*, to appear.
18. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups. Vol. I*, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1961.
19. J. Friedrich, A note on the two-dimensional moment problem, *Math. Nachr.* **121** (1985), 285–286.
20. J.-P. Gabardo, Trigonometric moment problems for arbitrary finite subsets of $\mathbb{Z}^n$, *Trans. Amer. Math. Soc.* **350** (1998), 4473–4498.
21. P.R. Halmos, *Measure theory*, Springer-Verlag, Berlin, 1974.
22. H.L. Hamburger, Über eine Erweiterung des Stieltjesschen Momentenproblemes, *Math. Ann.* **81** (1920), 235–319; **82** (1921), 120–164, 168–187.
23. G. Herglotz, Über Potenzreihen mit positivem, reellen Teil im Einheitskreis, *Ber. Verh. Kgl. Sachs. Ges. Leipzig, Math.-Phys. Kl.* **63** (1911), 501–511.
24. D. Hilbert, Über die Darstellung definiter Formen als Summe von Formenquadraten, *Math. Ann.* **32** (1888), 342–350.
25. W.B. Jones, O. Njåstad, and W.J. Thron, Orthogonal Laurent polynomials and the strong Hamburger moment problem, *J. Math. Anal. Appl.* **98** (1984), 528–554.
26. Y. Nakamura and N. Sakakibara, Perfectness of certain subsemigroups of a perfect semigroup, *Math. Ann.* **287** (1990), 213–220.
27. N. Sakakibara, Moment problems on semigroups of $\mathbb{N}_0^k$ and $\mathbb{Z}^k$, *Semigroup Forum* **45** (1992), 241–248.
28. K. Schmüdgen, An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional, *Math. Nachr.* **88** (1979), 385–390.
29. K. Schmüdgen, On a generalization of the classical moment problem, *J. Math. Anal. Appl.* **125** (1987), 461–470.
30. J.A. Shohat and J.D. Tamarkin, *The Problem of moments,* American Mathematical Society Mathematical surveys, vol. II, American Mathematical Society, New York, 1943.
31. T.J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse* **8** (1894), 1–122; **9**, 1–47.
32. B. Sz.-Nagy, A moment problem for self-adjoint operators, *Acta Math. Acad. Sci. Hungar.* **3** (1952), 285–293.