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# On the computation of the minimal resolution of smooth parametric varieties

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Abstract

In this paper we give some results on the computation of the minimal resolution of the ideal $I(V)$ of a smooth parametric variety $V \subset \mathbb{P}^n_k$ of dimension $m$ represented by homogeneous polynomials of the same degree $r$ and without base points. We show that the Castelnuovo-Mumford regularity of $V$ is $reg(V) = \min\{d \geq m - \lfloor \frac{m}{r} \rfloor \mid H_V(d) = \binom{dr+m}{m}\} + 1$, where $H_V(d)$ is the Hilbert function of $V$. If $V$ has maximal rank and the minimal degree of a generator of $I(V)$ is $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$ then $reg(V) \leq \alpha + 1$. In this case the shifts of the free modules $F_i$ of a minimal free resolution of $I(V)$ are at most two and we show that the Betti numbers are determined by computing the linear part of the resolution. In particular, if $V$ is minimally resolved the $F_i$ have one shift, for all but one $i$. We show that if $(a_1, ..., a_q) \in k^q$ are the coefficients of the polynomials that represent $V$ there is an open subset $U$ of $\mathbb{A}^q_k$ such that, if $(a_1, ..., a_q) \in U$, $V$ is minimally resolved and that it is possible to check that $U$ is non-empty for fixed $m, n, r$ by computer.

## Introduction

In [8] and [3] the number of generators of the ideal of a smooth general parametric variety was studied. In this paper we want to examine the resolution of this ideal. First we determine the Castelnuovo-Mumford regularity $\sigma = reg(V)$ of any smooth parametric variety $V \subset \mathbb{P}^n_k$, over an algebraically closed field $k$, parametrized by a map

$$\Phi\big([t_0, \ldots, t_m]\big) = \big[f_0(t_0, \ldots, t_m), \ldots, f_n(t_0, \ldots, t_m)\big]$$

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where $f_i \in k[t_0, \ldots, t_m]$ are homogeneous polynomials of the same degree $r$ without base points. We prove that $\sigma = \min \left\{ d \geq m - \lfloor \frac{m}{r} \rfloor \mid H_V(d) = \binom{dr+m}{m} \right\} + 1$, where $H_V(d)$ is the Hilbert function of $V$. Then, considering the set $T$ of $N = \binom{r\sigma+m}{m}$ general points $Q_i = \Phi(P_i)$ of $\mathbb{P}^n_k$, we show that if

$$0 \to \bigoplus_{i \geq \alpha+n-1} R(-i)^{b_{ni}} \to \ldots \to \bigoplus_{i \geq \alpha} R(-i)^{b_{1i}} \to I(T) \to 0$$

is a minimal graded free resolution of $I(T)$ then

$$0 \to \bigoplus_{i=\alpha+n-1}^{\sigma+n-1} R(-i)^{b_{ni}} \to \ldots \to \bigoplus_{i=\alpha}^{\sigma} R(-i)^{b_{1i}} \to I(V) \to 0$$

is a minimal graded free resolution of $I(V)$. If $V$ has maximal rank and the minimal degree of a generator of $I(V)$ is $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$ then $reg(V) \leq \alpha + 1$ and the resolution of $I(V)$ is

$$0 \to R(-\alpha - n + 1)^{a_n} \oplus R(-\alpha - n)^{b_n} \to \ldots \to R(-\alpha)^{a_1} \oplus R(-\alpha - 1)^{b_1} \to I(V) \to 0.$$

Furthermore the numbers $a_i$ and $b_i$ are linked by the following relations:

$$\begin{aligned}
a_1 &= \binom{\alpha+n}{n} - \binom{\alpha r + m}{m} \\
a_2 - b_1 &= (n+1)a_1 - \binom{\alpha+1+n}{n} + \binom{(\alpha+1)r+m}{m} \\
a_{d+1} - b_d &= (-1)^d \binom{\alpha+d+n}{n} \\
&\quad + (-1)^{d+1} \binom{(\alpha+d)r+m}{m} + (-1)^{d+1} \binom{n+d}{n} a_1 \\
&\quad + (-1)^{d+1} \sum_{i=1}^{d-1} (-1)^i (a_{i+1} - b_i) \binom{d-i+n}{n}, \quad 2 \leq d \leq n, \quad (a_{n+1} = 0)
\end{aligned}$$

which allow us to determine the Betti numbers $b_1, \ldots, b_n$ if one knows the Betti numbers $a_1, \ldots, a_n$ of the linear part of the resolution of $I(V)$.

If $a_{i+1}b_i = 0$, for any $n \geq i > 0$, $V$ is said to be minimally resolved. The above mentioned result on the resolution of the set $T$ of general points on $V$ allows us to prove that if $(a_1, ..., a_q) \in k^q$ are the coefficients of the polynomials that represent $V$ there is an open subset $U$ of $\mathbb{A}^q_k$ such that, if $(a_1, ..., a_q) \in U$, $V$ is minimally resolved. But for fixed $m, n, r$ the set $U$ can be empty (for example if $V$ is a curve of degree 5 in $\mathbb{P}^3$ or $\mathbb{P}^4$). We show that the non-emptiness of $U$ can be checked over the finite field $\mathbb{Z}_p$ and this allows us to use computers for finding varieties which have maximal rank and are minimally resolved. The computer calculations suggest that, if $2m < n$ and $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$, fixed $m, n$ a general parametric variety $V$ is minimally resolved, except for a few $r$.

In the following $V \subset \mathbb{P}^n_k$ will be a non-degenerate reduced irreducible variety over an algebraically closed field $k$.

We will say that $V$ *is a parametric variety* if there is a polynomial map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ $(m < n)$, defined by homogeneous polynomials of the same degree $r$, whose image is a dense subset of $V$.

Observe that, if the characteristic of $k$ is zero then any rational map $\mathbb{P}^m_k \to \mathbb{P}^n_k$ can be given by homogeneous polynomials. So, if the characteristic of $k$ is zero, parametric and unirational are synonymous.

If $M = \bigoplus_{d\geq 0} M_d$ is a graded finitely generated module over the polynomial ring $k[x_0, \ldots, x_n]$ then the Hilbert function of $M$ is given by $H(M, d) = \dim_k(M_d)$. If $M$ is the coordinate ring $A(V)$ of a variety $V$ we say that $H_V(d) = \dim_k A(V)_d$ is the Hilbert function of $V$ and that $P_V(d) = H_V(d)$ $(d >> 0)$ is the Hilbert polynomial of $V$. Note that $H_V(d) = \binom{d+r}{r} - \dim_k I(V)_d$.

## 1. Regularity of smooth parametric varieties without base points

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is $d$-regular if $H^i(\mathcal{F}(d-i)) = 0$ for $i > 0$. If $\mathcal{F}$ is $d$-regular, $\mathcal{F}$ is $d+1$-regular [7, Lecture 14].

DEFINITION 1.1. A variety $V \subset \mathbb{P}^n_k$ is *$d$-regular* if the sheaf $\mathcal{I}_V$ associated to the ideal $I(V)$ of $V$ is $d$-regular. We set:

$$reg(V) = \min \{d \mid \mathcal{I}_V \text{ is } d\text{-regular}\}.$$

In the following we say that $V$ *is generated in degree* $d$ if the ideal $I(V)$ of $V$ can be generated by forms of degree $\leq d$.

**Proposition 1.2**

*If $\sigma = reg(V)$ and $\alpha = \min \{d \mid I(V)_d \neq 0\}$ then a minimal graded free resolution of $I(V)$ is given by*

$$0 \to F_n \xrightarrow{\phi_n} \ldots F_i \xrightarrow{\phi_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{\phi_1} I(V) \xrightarrow{\phi_0} 0$$

*where, for $i = 1, \ldots, n$*

$$F_i = \bigoplus_{j=\alpha+i-1}^{\sigma+i-1} R(-j)^{b_{ij}}.$$

*In particular $V$ is generated in degree $\sigma$.*

*Proof.* See [4, Sections 20.5 and 20.6]. □

In the following we will say that $b_{ij}$ are the *Betti numbers* and we will denote with $N_i = ker\phi_i$, $n \geq i \geq 0$, the *modules of the syzygies* of the resolution of Proposition 1.2. We will denote by $(N_i)_d$ the graded part of degree $d$ of $N_i$ and by $(N_i)_{\leq d}$ the submodule of $N_i$ generated by the elements of degree $\leq d$.

**Proposition 1.3**

*Let $V$ be a parametric variety parametrized by a map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ given by $\Phi([t_0,\dots,t_m]) = [f_0(t_0,..,t_m),\dots,f_n(t_0,..,t_m)]$ where $f_i \in k[t_0,\dots,t_m]$ are homogeneous polynomials of degree $r$. Then:*
*(a) $H_V(d) \le \min\left\{\binom{d+n}{n}, \binom{dr+m}{m}\right\}$, for all $d$;*
*(b) $V$ is smooth of dimension $m$ and has degree $r^m$ if and only if $H_V(d) = \binom{dr+m}{m}$, for some $d > 0$ (that is the Hilbert polynomial $P_V(d)$ of $V$ is $\binom{dr+m}{m}$).*

*Proof.* $(a)$ Since $H_V(d)$ is the dimension of $A(V)_d$, the claim follows from the natural map:

$$\alpha : A(V)_d \to K[t_0,\dots,t_m]_{dr}$$

which is easily proved to be injective. In fact for all $F \in A(V)_d = k[x_0,\dots,x_n]$ put

$$\alpha(F) = F\big(f_0(t_0,\dots,t_m),\dots,f_n(t_0,\dots,t_m)\big) \in k[t_0,\dots,t_m]$$

and observe that $\alpha$ vanishes on all polynomials $F$ which are 0 on $V$; conversely, if $F(P) \ne 0$ for some $P \in V$, then the same holds for some point in the image of $\Phi$, i.e. there is a point $Q \in \mathbb{P}^m$ with $\alpha(F)(Q) \ne 0$, so that $\alpha(F)$ is not null. Hence the kernel of $\alpha$ is exactly $I(V)_d$ and $\alpha$ is injective.

(b) The condition that $V$ has dimension $m$ and degree $r^m$ implies that the base locus of the $f_i$ is empty and hence $V$ is isomorphic to $\mathbb{P}^m$. Hence the map $\alpha$ is an isomorphism for $d >> 0$. Thus the Hilbert polynomial of $V$ is $P_V(d) = \binom{dr+m}{m}$ and $H_V(d) = \binom{dr+m}{m}$, for $d >> 0$. Vice versa assume that this last equality is attained for some $d$. Then the polynomials $f_0^{i_0}\cdots f_n^{i_n}$, $i_0+\cdots+i_n = d$, generate all polynomials of degree $dr$ in $t_0,\dots,t_m$ and this is clearly impossible unless the $f_i$ have no base points and separate points and tangent vectors; it follows that $\Phi$ is a regular embedding and $V$ is smooth of dimension $m$ and degree $r^m$. □

**Theorem 1.4**

*Let $V \subset \mathbb{P}^n_k$ be a smooth parametric variety of dimension $m$ and represented by polynomials of the same degree $r$ and without base points. Then:*

$$reg(V) = \min\left\{ d \ge m - \lfloor \frac{m}{r} \rfloor \mid H_V(d) = \binom{dr+m}{m} \right\} + 1.$$

*Proof.* By Proposition 1.3 we have $H_V(d) = \binom{dr+m}{m}$, for some $d > 0$. Then let $\sigma = \min\{d \ge m - \lfloor \frac{m}{r} \rfloor \mid H_V(d) = \binom{dr+m}{m}\} + 1$. Since $V$ is isomorphic to $\mathbb{P}^m$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(\sigma-1)) \to H^0(\mathcal{O}_V(\sigma-1))$ surjects; this implies $H^1(\mathcal{I}_V(\sigma-1)) = 0$ and $H^i(\mathcal{I}_V(\sigma-i)) = H^{i-1}(\mathcal{O}_V(\sigma-i)) = H^{i-1}(\mathcal{O}_{\mathbb{P}^m}(r(\sigma-i))$ for $m > i > 1$. Since $\sigma > m - \lfloor \frac{m}{r} \rfloor$ we have $r(\sigma-m-1) \ne -m-1$ and then these cohomology groups vanish for $i > 0$; hence $V$ is $\sigma$-regular, which implies $reg(V) \le \sigma$. If $d = reg(V)$, $H^1(\mathcal{I}_V(d-1)) = 0$ and the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \to H^0(\mathcal{O}_V(d-1))$ surjects. Then $H_V(d-1) = \binom{(d-1)r+m}{m}$. Thus, to prove the result it is enough to show that $d > m - \lfloor \frac{m}{r} \rfloor$. But, if $V$ is $d$-regular $H^{m+1}(\mathcal{I}_V(d-i)) = H^m(\mathcal{O}_V(d-i)) = H^m(\mathcal{O}_{\mathbb{P}^m}(r(d-m-1)) = 0$, which implies $r(d-m-1) > -m-1$. From this inequality it easily follows that $d \ge m+1-\lfloor \frac{m}{r} \rfloor$. □

**Corollary 1.5**

*If $V$ is a Veronese variety, that is $n = \binom{r+m}{m} - 1$, then $reg(V) = m+1-\lfloor \frac{m}{r} \rfloor$.*

*Proof.* It is well known that if $V$ is a Veronese variety $V$ is projectively normal and its Hilbert function is $H_V(d) = \binom{dr+m}{m}$, for any $d \geq 0$. □

In the case of curves we find a result of [8, Proposition 1.5] as a consequence of Theorem 1.4

**Corollary 1.6**

*If $C$ is a smooth rational curve of degree $r$, then*

$$reg(V) = \min\left\{d \mid H_V(d) = dr + 1\right\} + 1.$$

*Proof.* It is well known that a rational curve of degree $r$ can be represented by polynomials of the same degree $r$ without bases points. Then the formula follows from Theorem 1.4 in which $m = 1$. □

## 2. Computing the minimal resolution of a parametric variety via points

Definition 2.1. Let $d$ and $m$ be fixed positive integers. We say that the set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^m$ of $N = \binom{d+m}{m}$ points is in *generic position* if it is not contained in any hypersurface of degree $d$.

*Remarks.* 1) If $m = 1$ any set of points is in generic position.

2) The N-tuples of points $(P_1, \ldots, P_N)$ which are in generic position form a non-empty open subset of $\mathbb{P}^m \times \ldots \times \mathbb{P}^m$, that is almost all sets of $N = \binom{d+m}{m}$ points $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^m$ are in generic position [5, Theorem 4].

3) A systematic way of finding points in generic position in $\mathbb{P}^m$ has been described in [3].

**Proposition 2.2**

*Let $V \subset \mathbb{P}^n_k$ be a parametric variety parametrized by the map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$, given by homogeneous polynomials, $f_0, \ldots, f_n$ of the same degree $r$. Let $\sigma = reg(V)$ and $\alpha = \min\{d \mid I(V)_d \neq 0\}$. Consider $N = \binom{r\sigma+m}{m}$ points $P_1, ..., P_N$ of $\mathbb{P}^m_k$ in generic position such that $Q_i = \Phi(P_i)$ form a set $T$ of $N$ points of $V$. Then $\sigma < \sigma' = reg(T)$. Furthermore if*

$$0 \to \bigoplus_{j=\alpha+n-1}^{\sigma'+n-1} R(-j)^{b_{nj}} \xrightarrow{\phi'_n} \ldots \xrightarrow{\phi'_2} \bigoplus_{j=\alpha}^{\sigma'} R(-j)^{b_{1j}} \xrightarrow{\phi'_1} I(T) \xrightarrow{\phi'_0} 0$$

*is a minimal graded free resolution of $I(T)$ then*

$$0 \to \bigoplus_{j=\alpha+n-1}^{\sigma+n-1} R(-j)^{b_{nj}} \xrightarrow{\phi_n} \ldots \xrightarrow{\phi_2} \bigoplus_{j=\alpha}^{\sigma} R(-j)^{b_{1j}} \xrightarrow{\phi_1} I(V) \xrightarrow{\phi_0} 0$$

*is a minimal graded free resolution of $I(V)$. That is, for every $i = 1, \ldots, n$ and $j = \alpha + i - 1, \ldots, \sigma + i - 1$ the Betti numbers $b_{ij}$, of the minimal resolution of $I(T)$ give all the Betti numbers of the minimal resolution of $I(V)$.*

*In particular a minimal set of generators of $I(V)$ is given by the elements of degree $\leq \sigma$ of a minimal set of generators of $I(T)$ and $H_V(d) = H_T(d)$, for any $d \leq \sigma$.*

*Proof.* First we prove the last statement. Since $I(V)$ is generated in degree $\sigma$ we have to prove that $I(V)_d = I(T)_d$ for $d \leq \sigma$. It is obvious that $I(V)_d \subset I(T)_d$, for any $d$, so we have to prove the opposite inclusion, for $d \leq \sigma$. If $f \notin I(V)_d$ then there is a point $P \in \mathbb{P}^m$ such that $f(\Phi(P)) \neq 0$, so that $f(f_0, \ldots, f_n)$ is a non-zero polynomial of degree $dr \leq \sigma r$; on the other hand, if $f \in I(T)_d$ then $f(f_1, \ldots, f_n)$ vanishes on the $N = \binom{\sigma r+m}{m}$ points $P_i$ and this contradicts the assumption of generic position. The equality of the Hilbert functions follows from their definition $H_V(d) = \binom{d+r}{r} - \dim_k I(V)_d = H_T(d) = \binom{d+r}{r} - \dim_k I(T)_d$. Now we prove that $\sigma < \sigma'$. It is well known that $\sigma' = \min\{d \mid H_T(d) = \binom{\sigma r+m}{m}\}+1$ and that $H_T(d) < \binom{r\sigma+m}{m}\}$ for $d < \sigma' - 1$. By Proposition 1.3 $H_T(\sigma-1) = H_V(\sigma-1) \leq \binom{r(\sigma-1)+m}{m}$ and then $H_T(\sigma-1) \neq \binom{r\sigma+m}{m}$. thus $\sigma < \sigma'$. The claim on the resolutions is equivalent to saying that if $N_i = Ker\phi_i$ and $N_i' = Ker\phi_i'$ then $N_i = (N_i')_{\leq \sigma+i}$, $i = 0, \ldots, n$. We prove this by induction. By the previous considerations we have $N_0 = I(V) = (I(T))_{\leq\sigma} = (N_0')_{\leq\sigma}$. Now if $N_i'$ is minimally generated by $m_1, \ldots, m_h, \ldots, m_l$ where $m_1, \ldots, m_h$ are the generators of $N_i = (N_i')_{\leq\sigma+i}$, the elements of $N_{i+1}$ are the syzygies of $m_1, \ldots, m_h$ and $N_{i+1} = (N_{i+1}')_{\leq\sigma+i}$. $\square$

Let $T = \{P_1, ..., P_h\} \in \mathbb{P}_k^n$ be a set of points and $d$ be a positive integer. The vector space $I(T)_d$ is easily seen to be the null space of a matrix with elements in $k$. In fact if $R_n = \{F \in k[X_0, ..., X_n] \mid deg(F) = d\}$ then:

$$I(T)_d = \left\{F \in R_d \mid F(P_i) = 0,\ i = 1, ..., h\right\}.$$

If we denote with $\mathcal{T}_i$, $i = 1, ..., u$, the terms of degree $d$ in the indeterminates $X_0, ..., X_r$ then $S = \{\mathcal{T}_1, ..., \mathcal{T}_u\}$ is a basis of the $k$-vector space $R_d$. Let $G_d(T)$ be the following $\binom{d+r}{r} \times h$ matrix:

$$G_d(T) = (b_{ij}) \text{ where } b_{ij} = \mathcal{T}_i(P_j).$$

If $F = a_1\mathcal{T}_1 + ... + a_u\mathcal{T}_u \in k[X_0, ..., X_n]$, then the vector of the coefficients of $F$ is

$$(F)_S = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_u \end{pmatrix}.$$

Then, $I(T)_d = \left\{F \in R_d \mid G_d(T)(F)_S = 0\right\}$, that is $F \in I(T)_d$ if and only if $(F)_S$ is a vector of the null space of the matrix $G_d(T)$.

**Proposition 2.3**

*Let $V \subset \mathbb{P}_k^n$ be a parametric variety parametrized by the map $\Phi : \mathbb{P}_k^m \to \mathbb{P}_k^n$, given by homogeneous polynomials, $f_0, \ldots, f_n$ of the same degree $r$. Let $a_1, \ldots, a_q$ be all the coefficients of $f_1, \ldots, f_n$. For any $i = 0, \ldots, n$ and $d = \alpha + i, \ldots, \sigma + i$, $H(N_i, d)$ (and then $H_V(d)$) is the rank of a matrix whose elements are polynomials in $a_1, \ldots, a_q$.*

*Proof.* Clearly it is enough to show that the elements of a basis of $(N_i)_d$ have coefficients which are polynomials in $a_1, \ldots, a_q$. We proceed by induction. By the previous considerations on sets of points, for any $d$, if $F \in (N_0)_d = I(T)_d$ ($T$ as in Proposition 2.2) the vector of the coefficients of $F$ belongs the null space of the matrix $G_d(T)$ whose elements are monomials evaluated at the coordinates of the points of $T$. Now the points of $T$ correspond to the values of parameters $t_0, \ldots, t_m$ and then the coordinates of the points of $T$ are linear functions of the coefficients $a_0, \ldots, a_q$. Then the coefficients of $F$ are polynomials in $a_1, \ldots, a_q$. Thus the claim is proved for $N_0$. We assume now the claim true for $N_i$ that is that the elements of a basis of $(N_i)_d$ have coefficients which are polynomials in $a_1, \ldots, a_q$. Then the elements of a minimal set $G_1, ...., G_h$ of generators of $N_i$ have coefficients which are polynomials in $a_1, \ldots, a_q$. But a basis of $(N_{i+1})_d$ consists of syzygies of $G_1, ...., G_h$. These syzygies have coefficients which belong to the null space of a matrix whose elements are, by induction, polynomials in $a_1, \ldots, a_q$. Then a basis of this null space has elements whose coefficients are polynomials in $a_1, \ldots, a_q$. Thus the claim. $\square$

## 3. Parametric varieties with maximal rank and minimally resolved

DEFINITION 3.1. Let $V$ be a variety and $\rho(d) : H^0(O_{\mathbb{P}^n_k}(d)) \to H^0(O_V(d))$ be the natural restriction map. We say that $V$ has *maximal rank* if, for every integer $d \geq 0$, $\rho(d)$ has maximal rank as a map of vector spaces i.e. it is injective or surjective.

**Theorem 3.2**

*Let $V$ be a parametric variety parametrized by a map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ given by polynomials $f_i$ of degree $r$ and $\alpha = \min\{d \in \mathbb{N} \mid I(V)_d \neq 0\}$. The following conditions are equivalent:*

*(a) $V$ has maximal rank;*

*(b) $H_V(d) = \min\left\{\binom{d+n}{n}, \binom{dr+m}{m}\right\}$;*

*(c) $H_V(\alpha - 1) = \binom{\alpha+n-1}{n}$ and $H_V(\alpha) = \binom{\alpha r+m}{m}$;*

*furthermore if $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$ the previous conditions are equivalent to*

*(d) $V$ is smooth, the $f_i$ have no base points and $reg(V) \leq \alpha + 1$ (in particular the ideal $I(V)$ of $V$ is generated by forms of degree $\alpha$ and $\alpha + 1$ ).*

*Proof.* $(a) \Leftrightarrow (b)$ Clear since $h^0(O_{\mathbb{P}^n_k}(d)) = \binom{d+n}{n}$ and $h^0(O_V(d)) = \binom{dr+m}{m}$ and by Proposition 1.3, $(b)$.

$(b) \Leftrightarrow (c)$ Since $H_V(d) = \binom{d+n}{n} - \dim_k I(V)_d$, then $H_V(d) = \binom{d+n}{n}$ is equivalent to $I(V)_d = 0$ and then $H_V(\alpha - 1) = \binom{\alpha+n-1}{n}$ implies $H_V(d) = \binom{d+n}{n}$, for $d < \alpha$. Furthermore by Theorem 1.4 $H_V(\alpha) = \binom{\alpha r+m}{m}$ implies that $V$ is $\alpha + 1$ regular and then $H_V(d) = \binom{dr+m}{m}$ for any $d \geq \alpha$.

$(c) \Rightarrow (d)$ Since $H_V(\alpha) = \binom{\alpha r+m}{m}$, by Theorem 1.4, $reg(V) \leq \alpha + 1$ and, by Proposition 1.2, $V$ is generated in degree $\alpha + 1$.

$(d) \Rightarrow (c)$ By Theorem 1.4 $H_V(\alpha) = \binom{\alpha r+m}{m}$ and since $V$ has no generator of degree $< \alpha$ we have $H_V(\alpha-1) = \binom{\alpha+n-1}{n}$. $\square$

*Remarks.* (1) If $V$ has maximal rank then $\alpha = \min\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\}$.

(2) The assumption $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$ of Theorem 3.2 is necessary. For example if $V \subset \mathbb{P}^n_k$ is a Veronese variety with $r \geq m > 3$, $V$ has maximal rank since $H_V(d) = \binom{dr+m}{m}$, for any $d \geq 0$, and $reg(V) = m + 1 - \lfloor \frac{m}{r} \rfloor = m$ [Corollary 1.5] but it is well known that $\alpha = 2$.

By Theorem 3.2, if $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$ and $V$ has maximal rank then the resolution of $I(V)$ is of the form :

$$(*) \quad 0 \to R(-\alpha-n+1)^{a_n} \oplus R(-\alpha-n)^{b_n} \xrightarrow{\phi_n} \ldots \xrightarrow{\phi_2} R(-\alpha)^{a_1} \oplus R(-\alpha-1)^{b_1} \xrightarrow{\phi_1} I(V) \xrightarrow{\phi_0} 0.$$

We have $H(N_i, \alpha+i) = a_i$. Note that $a_1 = \dim_k(I(V)_\alpha) \neq 0$.

Now, for every $i = 0, \ldots, n$, let $N_i = ker\ \phi_i$ and $W(N_i)$ denote the vector subspace of $(N_i)_{\alpha+i+1}$ generated by $(N_i)_{\alpha+i}$ under multiplication by $X_0, ..., X_n$, that is

$$W(N_i) = X_0(N_i)_{\alpha+i} + \ldots + X_n(N_i)_{\alpha+i} \subseteq (N_i)_{\alpha+i+1}.$$

Clearly $b_i = H(N_i, \alpha+i+1) - \dim_k(W(N_i))$ and $\dim_k(W(N_i)) \leq \min\{(n+1)a_i, H(N_i, \alpha+i+1)\}$

DEFINITION 3.3. Let $V$ be a parametric variety of maximal rank parametrized by a map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ given by polynomials $f_i$ of degree $r$. Let $\min \left\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\right\} = \alpha \geq m - \lfloor \frac{m}{r} \rfloor$. $V$ is said *minimally resolved* if, for any $i = 0, \ldots, n$, the vector spaces $W(N_i)$ have maximal dimension

$$(**) \quad \dim_k(W(N_i)) = \min \left\{(n+1)a_i, H(N_i, \alpha+i+1)\right\}$$

that is $b_i = \max\{0, H(N_i, \alpha+i+1) - (n+1)a_i\}$. If $(**)$ holds for $i = 0$, $V$ is said *minimally generated.*

Minimally generated curves have been studied in [8]. Minimally generated surfaces and threefolds have been studied in [3].

**Proposition 3.4**

*Let $V \subset \mathbb{P}^n_k$ be a non-degenerate parametric variety represented by polynomials $f_i \in k[t_0, \ldots, t_m]$, $i = 1, \ldots, n$. Assume that $m < 2n$ and* $\min\{d \mid I(V)_d \neq 0\} = \alpha \geq m - \lfloor \frac{m}{r} \rfloor$. *Let $a_1, ..., a_q$ be all the coefficients of all the polynomial $f_i$. Then:*

*(a) the set $U \subset \mathbb{A}^q$ of the $q$-tuples $(a_1, ..., a_q)$ for which $V$ is smooth of dimension $m$ and degree $m^r$ is open and non-empty;*
*(b) the set $U_1 \subset \mathbb{A}^q$ of the $q$-tuples $(a_1, ..., a_q) \in U$ for which $V$ has maximal rank is open;*
*(c) the set $U_2 \subset \mathbb{A}^q$, of the $q$-tuples $(a_1, ..., a_q) \in U_1$ for which $V$ is minimally generated is open;*
*(d) the set $U_3 \subset \mathbb{A}^q$, of the $q$-tuples $(a_1, ..., a_q) \in U_2$ for which $V$ is minimally resolved is open.*

*Proof.* (a) First we prove that for any $m, r > 0$ and $n > 2m$ there exists a $V_0$ which is smooth, of dimension $m$ and degree $r^m$: it is enough to take a general projection in $\mathbb{P}^n$ of the Veronese embedding of $\mathbb{P}^m$, which is defined by a basis for the space of homogeneous polynomials of degree $n$. Then by Proposition 1.3, (b) there exists a $d_0$ such that $H_{V_0}(d) = \binom{dr+m}{m}$.

Now by Proposition 2.3 $H_V(d_0)$ is equal to the rank $\rho(M)$ of a matrix $M$ whose elements are polynomials in $a_0, \ldots, a_q$ and by Proposition 1.3, (a) $\rho(M) \leq \binom{d_0r+m}{m}$. Then $\rho(M)$ is maximal if and only if $\rho(M) = \binom{d_0r+m}{m}$ that is $(a_0, ..., a_q)$ is not the solution of a finite set of polynomials, i.e. it belongs to an open set of $\mathbb{A}^q$.

(b) By Theorem 3.2, (1) $\Leftrightarrow$ (3) and Proposition 2.3 this condition is equivalent to the maximality of the ranks of two matrices whose elements are polynomials in $a_1, \ldots, a_n$ that is by the non vanishing of a finite number of polynomials in $a_1, \ldots, a_n$. Thus the claim.

(d) This condition is given by the maximality of the dimension of the vector spaces $W(N_i)$. But $W(N_i) \subseteq (N_i)_{\alpha+i+1}$ is generated by the the elements of a basis of $(N_i)_{\alpha+i}$ multiplied by the indeterminates $X_0, \ldots, X_n$. Now the elements of a basis of $(N_i)_{\alpha+i}$ have coefficients which are polynomials in $a_1, \ldots, a_q$ [Proposition 2.3] so $\dim_k(W(N_i))$ is the rank of a matrix with coefficients which are polynomials in $a_1, \ldots, a_n$ and the claim follows as in point $(a)$.

(c) Same argument as in (d) for $i = 0$. $\square$

**Theorem 3.5**

*Let $V$ be a parametric variety parametrized by a map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$ given by polynomials $f_i$ of degree $r$ and $\alpha = \min\{d \in \mathbb{N} \mid I(V)_d \neq 0\}$. Suppose that $V$ has maximal rank and $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$. Set $a_{n+1} = 0$.*
*$(a)$ We have:*

$$a_1 = \binom{\alpha+n}{n} - \binom{\alpha r+m}{m}, \qquad a_2 - b_1 = (n+1)a_1 - \binom{\alpha+1+n}{n} + \binom{(\alpha+1)r+m}{m}$$

*and for $d = 2, ..., n$*

$$(-1)^{d+1}(a_{d+1} - b_d) = -\binom{\alpha+d+n}{n} + \binom{(\alpha+d)r+m}{m} + \binom{n+d}{n}a_1 + \sum_{i=1}^{d-1}(-1)^i(a_{i+1} - b_i)\binom{d-i+n}{n}.$$

*Then to determine all the Betti numbers of the resolution of $V$ it is enough to determine the Betti numbers $a_1, ..., a_n$ of the linear part of the resolution.*
*$(b)$ $V$ is minimally resolved if and only $a_{i+1}b_i = 0$, for any $n \geq i > 0$.*

*Proof.* a) By computing the dimensions of the homogeneous components of (*) we get:

$$\dim_k(I(V))_{\alpha+d} = a_1\binom{n+d}{n} + \sum_{i=1}^{d}(-1)^i(a_{i+1}-b_i)\binom{d-i+n}{n}$$

for $d = 1,\dots,n-1$. We have $\dim_k(I(V))_{\alpha+d} = \binom{\alpha+d+n}{n} - \binom{(\alpha+d)r+m}{m}$. Thus $a_1 = \dim_k(I(V))_\alpha = \binom{\alpha+n}{n} - \binom{\alpha r+m}{m}$ and $a_2 - b_1 = \binom{n+1}{n}a_1 - \dim_k(I(V))_{\alpha+1} = (n+1)a_1 - \binom{\alpha+1+n}{n} + \binom{(\alpha+1)r+m}{m}$ and the third claim easily follows.

(*b*) If $n > i \geq 1$ consider the graded short exact sequence:

$$0 \to N_{i+1} \to R(-\alpha - i+1)^{a_i} \oplus R(-\alpha - i)^{b_i} \to N_i \to 0.$$

Then, from the additivity of the Hilbert function, we get that

$a_{i+1} = H(N_{i+1}, \alpha+i) = (n+1)a_i + b_i - H(N_i, \alpha+i) = (n+1)a_i - \dim_k(W(N_i))$.

Then $a_i = 0$ implies $a_{i+1} = 0$ and

$$h = \min\big\{i \mid \dim_k(W(N_i)) = (n+1)a_i\big\} = \min\big\{i \mid a_{i+1} = 0\big\} > 0.$$

Hence $V$ is minimally resolved if and only if $a_i = 0$ for $i > h$ and $b_i = 0$ for $i \leq h-1$. □

**Corollary 3.6**

*Under the assumptions of Theorem 3.5 if*

$$\binom{\alpha+1+n}{n} - \binom{(\alpha+1)r+m}{m} - (n+1)a_1 \geq 0$$

*and $V$ is minimally generated then $V$ is minimally resolved.*

*Proof.* It is enough to observe that under the assumptions $\binom{\alpha+1+n}{n} - \binom{(\alpha+1)r+m}{m} = H(N_0, \alpha+1) \geq (n+1)a_1$ and then, by Definition 3.3, $\dim_k(W(N_i)) = (n+1)a_i$ that is $a_2 = 0$ which by Theorem 3.5 implies that $V$ is minimally resolved. □

**Proposition 3.7**

*Let $V \subset \mathbb{P}^n_k$ be a parametric variety of maximal rank parametrized by the map $\Phi : \mathbb{P}^m_k \to \mathbb{P}^n_k$, given by homogeneous polynomials, $f_0,\dots,f_n$ of the same degree $r$. Let $\alpha = \min\{d \mid I(V)_d \neq 0\} = \min\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\}$. Consider $N = \binom{r\alpha+m}{m}$ points $P_1,\dots,P_N$ of $\mathbb{P}^m$ in generic position such that $Q_i = \Phi(P_i)$ form a set $T$ of $N$ points of $V$. Then the linear part of the resolution of $I(V)$ is also the linear part of the resolution of $I(T)$.*

*Proof.* If in the first part of the proof of Proposition 2.2 we substitute $\alpha$ for $\sigma$ we get that $I(V)_\alpha = I(T)_\alpha$. Now the claim on the resolutions is equivalent to saying that $(N_i)_\alpha = (N'_i)_\alpha$, $i = 0,\dots,n$. We prove this by induction. We have $N_0 = I(V)_\alpha = (I(T))_\sigma = (N'_0)_\sigma$. Now if the linear syzygies of $N'_i$ coincide with the linear syzygies of of $N_i$, the linear syzygies of $N_{i+1}$ and of $N'_{i+1}$ are both syzygies of $m_1,\dots,m_h$ and then coincide. □

## 4. Proving that a parametric variety has maximal rank or is minimally resolved by computer

In the following we want to show how the computer can be used to find parametric smooth varieties $V$ of maximal rank, minimally generated and minimally resolved.

Let $\mathbb{Z}_p$ be the residue field of the integers modulo a prime $p$. If $\overline{f} = \overline{b}_1\mathcal{T}_1 + ... + \overline{b}_u\mathcal{T}_u$ is a polynomial of $\mathbb{Z}_p[t_1, \dots, t_m]$ ($\mathcal{T}_i$ are the terms in $t_1, \dots, t_m$ and $\overline{b}_i \in \mathbb{Z}_p$ are the coefficients of $\overline{f}$) then $f = b_1\mathcal{T}_1 + ... + b_u\mathcal{T}_u$ is the corresponding polynomial in $\mathbb{Z}[t_1, t_2, \dots, t_m]$.

**Proposition 4.1**

*Let $\overline{V} \subset \mathbb{P}^n_{\mathbb{Z}_p^*}$ ($\mathbb{Z}_p^*$= algebraic closure of $\mathbb{Z}_p^*$) be a parametric variety represented by homogeneous polynomials $\overline{f}_i \in \mathbb{Z}_p[t_0, \dots, t_m]$, $i = 0, \dots, n$ of degree $r$. Let $V \subset \mathbb{P}^r_k$ (char(k)=0) be the corresponding parametric variety represented by the polynomials $f_i \in \mathbb{Z}[t_0, \dots, t_m] \subset k[t_0, \dots, t_m]$. Then:*
*(a) if $\overline{V}$ has maximal rank also $V$ has maximal rank;*
*(b) if $\overline{V}$ is minimally generated also $V$ is minimally generated;*
*(c) if $\overline{V}$ is minimally resolved also $V$ is minimally resolved.*

*Proof.* First we recall the following straightforward fact: if $Q = (a_{ij})$ is a $c \times d$ matrix with entries $a_{ij} \in \mathbb{Z}$ and $\overline{Q} = \overline{a}_{ij}$ is the $c \times d$ matrix whose entries are the integers $a_{ij}$ modulo a prime $p$ then $rk(\overline{Q}) \leq rk(Q)$ (rk=rank). Hence if $rk(\overline{Q})$ is maximal, that is $rk(\overline{Q}) = \min\{c, d\}$, also $rk(Q) = \min\{c, d\}$. Now, by Proposition 2.3 and by the proof of Proposition 3.4, the conditions of $(a), (b), (c)$ are all conditions on the maximality of ranks of matrices whose elements are polynomials in the coefficients of the polynomials $f_i$. The proposition then follows. □

**Theorem 4.2**

*Fix integers $m, n, r$ such that $\min\left\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\right\} \geq m - \lfloor \frac{m}{r} \rfloor$. If, for a prime $p$, there exists a parametric variety of $\mathbb{P}^n_{\mathbb{Z}_p^*}$ represented by homogeneous polynomials $f_i \in \mathbb{Z}_p[t_0, \dots, t_m]$, $i = 0, \dots, n$ of degree $r$ which has maximal rank or (respectively) is minimally generated or (respectively) is minimally resolved then, for any field $k$ of characteristic zero, a generic parametric variety $V \subset \mathbb{P}^n_k$ of dimension $m$ represented by polynomials of degree $r$ has maximal rank or (respectively) is minimally generated or (respectively) is minimally resolved.*

*Proof.* The claim follows from Propositions 3.4 and 4.1. □

Using the previous results of the paper it is now easy to construct algorithms that for fixed $m, n, r$ produce examples (if they exist) of parametric varieties $V$ with maximal rank or minimally generated (Algorithm 4.3) or minimally resolved (Algorithm 4.4). These algorithms are based on existing algorithms that compute the Hilbert function, generators and the linear part of the resolution of ideals of a finite set of points (see [1], [2]).

**Algorithm 4.3.** (*For checking maximal rank and minimal generation.*)

**Step 1**. Input the integers $m, n, r$, the prime $p$ and the the coefficients (in $\mathbb{Z}_p$) of the polynomials $\overline{f}_i$ that represent $\overline{V}$.

**Step 2**. If $\alpha = \min\left\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\right\}$, determine the set $T$, of $N = \binom{r(\alpha+1)+m}{m}$ points of $V$, described in Proposition 2.2 (this can always be done, if $p$ is large enough).

**Step 3**. Compute the Hilbert functions $H_T(\alpha-1)$, $H_T(\alpha)$.

**Step 4**. If $H_T(\alpha-1) = \binom{\alpha+n-1}{n}$ and $H_T(\alpha) = \binom{\alpha r+m}{m}$ then output: $V$ has maximal rank, otherwise output: it is not possible to decide if $V$ has maximal rank, and stop.

**Step 5**. Compute the elements of degree $\leq \alpha+1$ of a minimal set of generators of $I(T)$. This gives the Betti number $b_1$.

**Step 6**. If $b_1 = \max\left\{0, \binom{\alpha+1+n}{n} - \binom{(\alpha+1)r+m}{m} - (n+1)[\binom{\alpha+n}{n} - \binom{\alpha r+m}{m}]\right\}$ output: $V$ is minimally generated, otherwise stop.

**Algorithm 4.4.** (*For checking maximal rank and minimal resolution.*)

**Step 1**. Input the integers $m, n, r$, the prime $p$ and the the coefficients (in $\mathbb{Z}_p$) of the polynomials $\overline{f}_i$ that represent $\overline{V}$.

**Step 2**. If $\alpha = \min\{d \in \mathbb{N} \mid \binom{d+n}{n} > \binom{rd+m}{m}\}$, determine the set $T$, of $N = \binom{r\alpha+m}{m}$ points of $V$, described in Proposition 3.7 (this can always be done, if $p$ is large enough).

**Step 3**. Compute the Hilbert functions $H_T(\alpha-1)$, $H_T(\alpha)$.

**Step 4**. If $H_T(\alpha-1) = \binom{\alpha+n-1}{n}$ and $H_T(\alpha) = \binom{\alpha r+m}{m}$ then output: $V$ has maximal rank, otherwise output: it is not possible to decide if $V$ has maximal rank, and stop.

**Step 5**. Compute the Betti numbers $a_i, 1 \leq i \leq n$, of the linear part of the resolution of the ideal $I(T)$.

**Step 6**. Compute the Betti numbers $b_i$ of $I(V)$ by using the formulas of Theorem 3.5.

**Step 7**. If $a_{i+1}b_i = 0$ for any $i$ output: $V$ is minimally resolved, otherwise output: it is not possible to decide if $V$ is minimally resolved.

*Remark.* By Proposition 3.4 with a random choice of coefficients in *Step* 1 of Algorithms 4.3 and 4.4 one always finds a variety $V$ of maximal rank or minimally generated or minimally resolved unless for fixed $m, n, r$ there is *no* variety $V$ of maximal rank or minimally generated or minimally resolved.

It is possible to implement Algorithm 4.3 and 4.4 in the language C++ by using a combination of the softwares *Points*99 [9] and *linsyz* [6]. This has been done for us by F. Cioffi on an Intel Pentium running Linux. The examples we have run on the computer suggest that, if $2m < n$ and $\alpha \geq m - \lfloor \frac{m}{r} \rfloor$, for fixed $m, n$ a general parametric variety has maximal rank and is minimally resolved for any $r$ , except for a few exceptions. We quote, the following cases which we proved by computer.

**Theorem 4.5** ([8])

*Let the base field $k$ have characteristic $char(k) = 0$. A general parametric curve $V \subset \mathbb{P}^n_k$, $n \geq 3$, of degree $\delta \leq 100$, has maximal rank and is minimally generated if and only if $n \notin \{3, 4\}$ or $\delta \neq 5$.*

**Theorem 4.6** ([3])

*Let $char(k) = 0$ and $V \subset \mathbb{P}^n_k$ $(n \geq 5)$ be a general parametric surface of degree $\delta \leq 49$. If $n \neq 7$ or $\delta \neq 9$ a general $V$ has maximal rank and is minimally generated.*

A general surface of degree 9 in $\mathbb{P}^7$ has maximal rank but we do not know if it is minimally generated (the computer suggests that it is not minimally generated).

**Theorem 4.7** ([3])

*Let $char(k) = 0$ and $V \subset \mathbb{P}^n_k$ $(n \geq 7)$ be a parametric non-degenerate smooth threefold of degree $\delta \leq 27$. A general $V$ has maximal rank and is minimally generated if and only if $n \neq 7$ or $\delta \neq 8$.*

**Theorem 4.8**

*Let $char(k) = 0$ and $V \subset \mathbb{P}^n_k$ $(3 \leq n \leq 7)$ be a general parametric curve which has degree $\delta \neq 5$ in $\mathbb{P}^3_k$ or in $\mathbb{P}^4_k$ or degree $\delta \neq 6$ in $\mathbb{P}^5_k$ or degree $\delta \notin \{7, 8\}$ in $\mathbb{P}^6_k$ or degree $\delta \notin \{9, 10\}$ in $\mathbb{P}^7_k$. Then if $\delta \leq 20$, $V$ is minimally resolved.*

We do not know if a general parametric curve of degree $\delta = 6$ in $\mathbb{P}^5_k$ or degree $\delta \in \{6, 7\}$ in $\mathbb{P}^6_k$ is minimally resolved (the computer suggests that it is not minimally resolved).

**Theorem 4.9**

*Let $char(k) = 0$ and $V \subset \mathbb{P}^n_k$ $(5 \leq n \leq 7)$ be a general parametric surface of degree $\delta \leq 9$ with $(\delta, n) \neq (9, 7)$. Then $V$ is minimally resolved.*

*Remark.* By using Theorems 4.5, 4.6, 4.7 and Corollary 3.6 it is possible to find other case of minimally resolved varieties. For example a general parametric curve of degree 22 or 24 in $\mathbb{P}^4$, or of degree 26 in $\mathbb{P}^6$ is minimally generated by Theorem 4.5 and then minimally resolved, since it is easily shown that it satisfies the condition of Corollary 3.6. The same happens for a general parametric surface of degree 25 in $\mathbb{P}^8$.

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