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# Clarkson type inequalities and their relations to the concepts of type and cotype

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Abstract

We prove some multi-dimensional Clarkson type inequalities for Banach spaces. The exact relations between such inequalities and the concepts of type and cotype are shown, which gives a conclusion in an extended setting to M. Milman's observation on Clarkson's inequalities and type. A similar investigation concerning the close connection between random Clarkson inequality and the corresponding concepts of type and cotype is also included. The obtained results complement, unify and generalize several classical and some recent results of this type.

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## Introduction

Since they were proved for $L_p$ $(1 < p < \infty)$ in the context of uniform convexity in [4], Clarkson's inequalities have been treated a great deal by many authors (cf. [19]). These investigations were mostly devoted to various proofs and generalizations of these inequalities for $L_p$ and some other concrete Banach spaces ([3, 1, 17, 7, 14, 24, 10, 18, 5, 6, 15, 16, 20, 22, 11, 25, 19]. In particular R. P. Boas [3] and M. Koskela [14] extended these inequalities in parameters involved. On the other hand the operator norms of the Littlewood matrices between $\ell_r^{2^n}$-spaces were calculated (implicitly) in A. Pietsch [23]. M. Kato [10] determined these norms between $L_p$-valued $\ell_r^{2^n}$-spaces. This yielded a multi-dimensional global version of the classical Clarkson inequalities, we call it generalized Clarkson's inequality (GCI), which includes those of Boas [3] and Koskela [14]. Later GCI was investigated in [27, 15, 16, 20, 11, 8, 19, etc.]; A. Tonge [27] presented its another proof, and L. Maligranda and L. E. Persson [15, 16] extended GCI to an almost full range of the parameters. Also in connection with GCI Tonge [27] proved random Clarkson inequality (RCI) for $L_p$.

On the other hand, as far as we know in literature, M. Milman [18] first observed Clarkson's inequalities and (Rademacher) type in the same framework in the general Banach space setting. Recently M. Kato and Y. Takahashi [13] characterized the Banach spaces in which Clarkson's inequalities hold as those of type $p$ with "type $p$ constant" 1, resp., cotype $p'$ with "cotype $p'$ constant" 1 $(1/p+1/p' = 1)$. Using this result, they [25] proved RCI for any Banach space $X$ in which $(p, p')$-Clarkson inequality holds, where the unknown absolute constant $K$ included in Tonge's original inequality was replaced by 1 (see [26] for some further results on RCI).

The aim of this paper is to characterize the Banach spaces in which these multi-dimensional inequalities GCI and RCI hold in terms of type and cotype. In the first section we recall these Clarkson type inequalities for $L_p$. Some figures presented there will illustrate the relation among several variants of GCI. In Section 2 we show that GCI of Kato or more generally that of Maligranda-Persson holds in any Banach space $X$ satisfying $(p, p')$-Clarkson inequality $(1 \le p \le 2)$, and moreover the converse is true (Theorem 2.5). In Section 3 we first show that these GCI's hold in $X$ if and only if $X$ is of type $p$ and the "type $p$ constant" is 1, resp., cotype $p'$ and "cotype $p'$ constant" is 1 (Theorem 3.2). This extends the previous result of Kato and Takahashi [13] stated above and gives a conclusion to the observation of Milman [18] in the extended setting. Secondly, as the general case in type constant, we prove that $X$ is of type $p$ if and only if RCI holds in $X$ with an absolute constant $K$, where the best value of $K$ is estimated by type $p$ constants of $X$ (Theorem 3.4). In the final section we shall present several related results, especially concerning

Lebesgue-Bochner spaces $L_r(X)$. It is shown that if $(t, t')$-Clarkson inequality holds in $X$, then GCI's hold in $L_r(X)$ (Theorem 4.2); hence the original GCI's for $L_p$ are derived directly from the parallelogram law for scalars as the classical Clarkson and random Clarkson inequalities (Takahashi-Kato [26], cf. also [15]). We also have some exact relations between GCI and RCI; in particular GCI in $X$ and RCI ($K = 1$) in $L_{p'}(X)$ is equivalent (Theorem 4.9).

These results in this paper complement, unify and generalize several classical and some recent results of this type.

## 1. Generalized Clarkson and random Clarkson inequalities

In this section we recall and discuss some versions of the classical Clarkson inequalities (CI), their multi-dimensional generalizations GCI's and the random Clarkson inequality RCI. In the following, prime means taking conjugate numbers unless otherwise mentioned, that is, $1/p + 1/p' = 1/q + 1/q' = \cdots = 1$ for $1 \leq p,\, q, ... \leq \infty$.

**Clarkson's inequalities (CI)** ([4])

(i) *Let* $1 < p \leq 2$. *Then for all* $f,\, g \in L_p$

$$\left(\|f+g\|_p^{p'} + \|f-g\|_p^{p'}\right)^{1/p'} \leq 2^{1/p'}\left(\|f\|_p^p + \|g\|_p^p\right)^{1/p}. \tag{1.1}$$

(ii) *Let* $2 \leq q < \infty$. *Then for all* $f,\, g \in L_q$

$$\left(\|f+g\|_q^q + \|f-g\|_q^q\right)^{1/q} \leq 2^{1/q}\left(\|f\|_q^{q'} + \|g\|_q^{q'}\right)^{1/q'}. \tag{1.2}$$

For the case $1/p + 1/q = 1\,(1 < p \leq 2)$ the inequalities (1.1) and (1.2) are equivalent. This fact can be most easily understood as a duality principle which holds in the general Banach space setting (see Lemma 2.1).

Let $A_n = (\epsilon_{ij})$ denote the Littlewood matrices, that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}, \quad n = 1, 2....$$

We shall now present and compare some natural multi-dimensional generalizations of Clarkson's inequalities **GCI** in terms of the Littlewood matrices:

**Generalized Clarkson's inequality (GCI)**

(a) **The standard form** ([10]): *Let* $1 \le p \le \infty$ *and let* $t = \min\{p, p'\}$*. Then for any* $n \in \mathbb{N}$ *and for all* $f_1, f_2, ..., f_{2^n} \in L_p$

$$\left\{ \sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} f_j \Big\|_p^{t'} \right\}^{1/t'} \le 2^{n/t'} \left\{ \sum_{j=1}^{2^n} \|f_j\|_p^t \right\}^{1/t}. \tag{1.3}$$

(b) **The Kato form** (see [10] and also [25, 20, 8, 19]): *Let* $1 \le p, r, s \le \infty$ *and* $n \in \mathbb{N}$*. Then for all* $f_1, f_2, ..., f_{2^n} \in L_p$,

$$\left\{ \sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} f_j \Big\|_p^{s} \right\}^{1/s} \le 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|f_j\|_p^r \right\}^{1/r}, \tag{1.4}$$

where, letting $t = \min\{p, p'\}$,

$$c(r, s; p) = \begin{cases} 1/r' + 1/s - 1/t' & \text{if } \ t \le r \le \infty,\ 1 \le s \le t', \\ 1/s & \text{if } \ 1 \le r \le t,\ 1 \le s \le r', \\ 1/r' & \text{if } \ s' \le r \le \infty,\ t' \le s \le \infty. \end{cases}$$

(c) **The Maligranda-Persson form** (see [15] and also [16, 22]): *Let* $0 < p, r, s < \infty$ *and* $n \in \mathbb{N}$*. Then for all* $f_1, f_2, ..., f_{2^n} \in L_p$*, the inequality* (1.4) *holds with the constant* $C(r, s; p) = 1/s - 1/r + 1/q$ *in place of* $c(r, s; p)$*, where* $q = \min\{p, p', r, s'\}$ *with the convention that* $p'$ *is omitted if* $p \le 1$ *and* $s'$ *is omitted if* $s \le 1$.

Above and in the sequel we have the usual (supremum) interpretation of the sums when $s = \infty$ or $r = \infty$ and of the integrals when $p = \infty$.

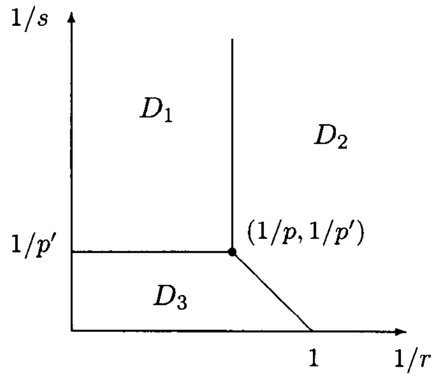
*Remark 1.1.* (i) If $n = 1$, then GCI (a) is just (1.1) or (1.2), and GCI (b) coincides with Koskela's inequality ([14]), a part of which is Boas' inequality ([3]).

(ii) If $p \ge 1$, then GCI on the forms (a)-(c) can be compared: Indeed, in this case the constant $C(r, s; p)$ on the form (c) is rewritten as

$$C(r, s; p) = \begin{cases} 1/r' + 1/s - 1/t' & \text{if } \ t \le r \le \infty,\ 0 < s \le t', \\ 1/s & \text{if } \ 1 \le r \le t,\ 0 < s \le r', \\ & \quad \text{or } \ 0 < r \le 1,\ 0 < s \le \infty, \\ 1/r' & \text{if } \ s' \le r \le \infty,\ t' \le s \le \infty, \end{cases} \tag{1.5}$$

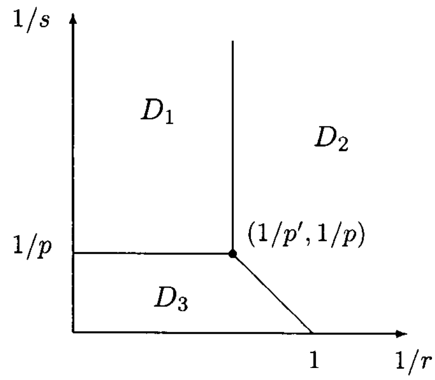
where $t = \min\{p, p'\}$ (see also our Theorem 2.5). Thus GCI(c) includes GCI(b).

In the figures 1a-1b below we illustrate the difference among the sets of parameters in the plane with axes $1/r$ and $1/s$ for which GCI in the above three forms (a)-(c) can be applied, where $C(r, s; p) = 1/r' + 1/s - 1/t'$ on the region $D_1$, and $C(r, s; p) = 1/s$, resp. $1/r'$ on $D_2$, resp. on $D_3$. Note that GCI on the form (a) corresponds to the point $(1/p,\ 1/p')$ in Figure 1a and the point $(1/p',\ 1/p)$ in Figure 1b. GCI on the form (b) corresponds to Figures 1a and 1b restricted to the unit square $\{(1/r,\ 1/s) :\ 0 \le 1/r,\ 1/s \le 1\}$, and GCI on the form (c) corresponds to the first quadrant in Figures 1a-1b.




the case $1 \le p \le 2$

Figure 1a.




the case $2 \le p \le \infty$

Figure 1b.

Let $B_n = (b_{ij})$, $n = 1, 2, ...,$ denote a random $n \times n$-matrix whose coefficients are independent identically distributed random variables taking the values $+1$ or $-1$ with equal probability. We shall close this section by presenting the random Clarkson inequality **RCI** in terms of the matrices $B_n$:

**Random Clarkson inequality (RCI)** (see [27] and also [25, 26]). *Let* $1 \le p, r, s \le \infty$ *and* $n \in \mathbb{N}$. *Then, with* $\mathbf{E}$ *denoting the mathematical expectation, for all* $f_1, f_2, ..., f_n \in L_p$, *we have*

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij} f_j\right\|_p^s\right)^{1/s} \le K\, n^{c(r,s;p)} \left(\sum_{j=1}^{n} \|f_j\|_p^r\right)^{1/r},$$

*where the constant* $c(r, s; p)$ *is the same as in GCI of the form* (b) *and* $K$ *is a fixed constant independent on* $n$, $r$ *and* $s$.

## 2. Clarkson type inequalities in Banach spaces

Let $X = (X, \|\cdot\|)$ denote a Banach space. Let $1 \le p \le 2$. We say that $(p, p')$-*Clarkson inequality* holds in $X$ if for all $x, y \in X$

$$\left(\|x+y\|^{p'} + \|x-y\|^{p'}\right)^{1/p'} \le 2^{1/p'}\left(\|x\|^p + \|y\|^p\right)^{1/p} \tag{2.1}$$

holds. More generally we say that $(p, p'; n)$-*Clarkson inequality*, $n \in \mathbb{N}$, holds in $X$ if for all $x_1, x_2, ..., x_{2^n} \in X$

$$\left\{\sum_{i=1}^{2^n}\Big\|\sum_{j=1}^{2^n}\epsilon_{ij}x_j\Big\|^{p'}\right\}^{1/p'} \le 2^{n/p'}\left\{\sum_{j=1}^{2^n}\|x_j\|^p\right\}^{1/p} \tag{2.2}$$

holds. Here $\epsilon_{ij}$ are the entries of the Littlewood matrices. Clearly $(p, p'; 1)$-Clarkson inequality is precisely (2.1).

The equivalence of (1.1) and (1.2) noted in the preceding section is stated for a general Banach space:

**Lemma 2.1** ([13, Proposition 2.1]; [8, Proposition 2.1])

*Let* $1 \le p \le 2$. *Then the* $(p, p')$*-Clarkson inequality holds in a Banach space* $X$ *if and only if it holds in the dual space* $X'$ : *The same is true for the* $(p, p'; n)$*-Clarkson inequality.*

In view of this lemma it is enough in general to consider Clarkson type inequalities in the case $1 \le p \le 2$.

**Theorem 2.2**

*Let* $1 \le p \le 2$. *Then the following assertions are equivalent.*

*(i)* $(p, p')$*-Clarkson inequality* (2.1) *holds in* $X$.

*(ii)* $(p, p'; n)$*-Clarkson inequality* (2.2) *holds in* $X$ *for any* $n \in \mathbb{N}$.

*(iii)* $(p, p'; n)$*-Clarkson inequality* (2.2) *holds in* $X$ *for some* $n \in \mathbb{N}$.

*In addition to the above, the same assertions* (i)-(iii) *for the dual space* $X'$ *are equivalent.*

*Proof.* (i) $\Rightarrow$ (ii): The statement (ii) is true for $n = 1$ by assumption. Assume that (2.2) holds for a fixed $n$. Then, by using (2.1) and Minkowski's inequality, we obtain that

$$\begin{aligned}
&\left\{\sum_{i=1}^{2^{n+1}} \Big\| \sum_{j=1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'}\right\}^{1/p'} \\
&= \left\{\sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j + \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'} + \sum_{i=2^n+1}^{2^{n+1}} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j + \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'}\right\}^{1/p'} \\
&= \left\{\sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j + \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'} + \sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j - \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'}\right\}^{1/p'} \\
&= \left\{\sum_{i=1}^{2^n} \left( \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j + \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'} + \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j - \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'}\right)\right\}^{1/p'} \\
&\le \left\{\sum_{i=1}^{2^n} 2 \left( \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \Big\|^{p} + \Big\| \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p}\right)^{p'/p}\right\}^{1/p'} \\
&= 2^{1/p'} \left[\left\{\sum_{i=1}^{2^n} \left( \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \Big\|^{p} + \Big\| \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p}\right)^{p'/p}\right\}^{p/p'}\right]^{1/p} \\
&\le 2^{1/p'} \left[\left\{\sum_{i=1}^{2^n} \Big\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \Big\|^{p'}\right\}^{p/p'} + \left\{\sum_{i=1}^{2^n} \Big\| \sum_{j=2^n+1}^{2^{n+1}} \epsilon_{ij} x_j \Big\|^{p'}\right\}^{p/p'}\right]^{1/p} \\
&\le 2^{1/p'} \left[2^{np/p'} \left\{\sum_{j=1}^{2^n} \|x_j\|^p\right\} + 2^{np/p'} \left\{\sum_{j=2^n+1}^{2^{n+1}} \|x_j\|^p\right\}\right]^{1/p} \\
&= 2^{1/p'} \cdot 2^{n/p'} \left\{\sum_{j=1}^{2^{n+1}} \|x_j\|^p\right\}^{1/p} \\
&= 2^{(n+1)/p'} \left\{\sum_{j=1}^{2^{n+1}} \|x_j\|^p\right\}^{1/p} .
\end{aligned}$$

That is, the $(p, p'; n+1)$-Clarkson inequality holds. Thus, according to the induction axiom, we have the conclusion. The assertion (ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i): Assume that the $(p, p'; n+1)$-Clarkson inequality holds for some $n$, and put $x_{2^n+1} = \cdots = x_{2^{n+1}} = 0$. Then we have

$$\left\{ \sum_{i=1}^{2^{n+1}} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^{p'} \right\}^{1/p'} \leq 2^{(n+1)/p'} \left\{ \sum_{j=1}^{2^n} \|x_j\|^p \right\}^{1/p}.$$

Noting that

$$\left\{ \sum_{i=1}^{2^{n+1}} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^{p'} \right\}^{1/p'} = \left\{ 2 \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^{p'} \right\}^{1/p'},$$

we have the $(p, p'; n)$-Clarkson inequality. By iterating this procedure we have (2.1). By Lemma 2.1 we have the latter assertion for $X'$. $\square$

Now the following elementary lemma is useful in our later discussions:

**Lemma 2.3**

*Let $\{a_j\}$ be a sequence of nonnegative numbers and let $n \in \mathbb{N}$.*
*(a) If $0 < \alpha \leq \beta \leq \infty$, then*

$$\left\{ \frac{1}{n} \sum_{j=1}^{n} a_j^{\alpha} \right\}^{1/\alpha} \leq \left\{ \frac{1}{n} \sum_{j=1}^{n} a_j^{\beta} \right\}^{1/\beta}.$$

*(b) If $0 < \beta \leq \alpha \leq \infty$, then*

$$\left\{ \sum_{j=1}^{n} a_j^{\alpha} \right\}^{1/\alpha} \leq \left\{ \sum_{j=1}^{n} a_j^{\beta} \right\}^{1/\beta}.$$

The statement in (a) is only a consequence of the well-known fact that the scale of power means is nondecreasing. (We refer the reader to [21] for some historical remarks and recent developments concerning this fact; even for the more general case with Gini means and when the sums are replaced by integrals or more general isotone linear functionals). The statement in (b) is only another way to write the usual embedding between $\ell_p$-spaces or simply the inequality $(\sum c_j)^b \leq \sum c_j^b,\ 0 < b \leq 1,\ c_j \geq 0$.

**Lemma 2.4** ([8, Proposition 2.2]; see also [26])

*Let $1 \leq t < p \leq 2$. Then $(p, p'; n)$-Clarkson inequality implies $(t, t'; n)$-Clarkson inequality.*

We are now in a position to show that, maybe to be surprising, GCI of Kato form, and more generally that of Maligranda-Persson form holds in any Banach space satisfying the $(p, p')$-Clarkson inequality, and moreover the converse is derived from each of these GCI's for *some* $n$.

**Theorem 2.5**

*Let $1 \leq p \leq 2$. The following assertions are equivalent.*

*(i) $(p, p')$-Clarkson inequality holds in a Banach space $X$.*

*(ii) Let $1 \leq r,\ s \leq \infty$. Then for any, resp., some $n \in \mathbb{N}$ GCI of Kato form* (b) *holds in $X$; that is, for every $x_1, x_2, ..., x_{2^n} \in X$*

$$\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^s \right\}^{1/s} \leq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r} \tag{2.3}$$

*holds, where*

$$c(r, s; p) = \begin{cases} 1/r' + 1/s - 1/p' & \text{if} \quad p \leq r \leq \infty,\ 1 \leq s \leq p', \\ 1/s & \text{if} \quad 1 \leq r \leq p,\ 1 \leq s \leq r', \\ 1/r' & \text{if} \quad s' \leq r \leq \infty,\ p' \leq s \leq \infty. \end{cases}$$

*(iii) Let $0 < r,\ s \leq \infty$. Then for any, resp., some $n \in \mathbb{N}$ GCI of Maligranda-Persson form* (c) *holds in $X$; that is, for every $x_1, x_2, ..., x_{2^n} \in X$*

$$\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^s \right\}^{1/s} \leq 2^{nC(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r} \tag{2.4}$$

*holds, where*

$$C(r, s; p) = \begin{cases} 1/r' + 1/s - 1/p' & \text{if} \quad p \leq r \leq \infty,\ 0 < s \leq p', \\ 1/s & \text{if} \quad 1 \leq r \leq p,\ 0 < s \leq r', \\ & \quad \text{or} \quad 0 < r \leq 1,\ 0 < s \leq \infty, \\ 1/r' & \text{if} \quad s' \leq r \leq \infty,\ p' \leq s \leq \infty. \end{cases}$$

*In addition to the above, these assertions* (i)-(iii) *for the dual space $X'$ are equivalent.*

*Proof.* (i) $\Rightarrow$ (iii): For an arbitrarily fixed $n$ put

$$A_s := \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \epsilon_{ij} x_j \right\|^s \right\}^{1/s} \quad \text{and} \quad B_r := \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r}.$$

According to the assumption and Theorem 2.2 it holds that

$$A_{p'} \leq 2^{n/p'} B_p. \tag{2.5}$$

Thus, by using (2.5) and Lemmas 2.3 and 2.4, we have the following:

(a) If $p \leq r \leq \infty$ and $0 < s \leq p'$, then by (2.5)

$$\begin{aligned} A_s \leq 2^{n(1/s-1/p')} A_{p'} &\leq 2^{n/s} B_p \\ &\leq 2^{n/s} 2^{n(1/p-1/r)} B_r = 2^{n(1/s-1/r+1/p)} B_r. \end{aligned}$$

(b) Let $1 \leq r \leq p$ and $0 < s \leq r'$. Then by Lemma 2.4 with $t = r$,

$$A_s \leq 2^{n(1/s-1/r')} A_{r'} \leq 2^{n(1/s-1/r')} 2^{n/r'} B_r = 2^{n/s} B_r.$$

(c) If $0 < r \leq 1$ and $0 < s \leq \infty$, then

$$A_s \leq 2^{n/s} B_1 \leq 2^{n/s} B_r.$$

(d) Let $s' \leq r \leq \infty$ and $p' \leq s \leq \infty$. Then by Lemma 2.4 with $t = s'$.

$$A_s \leq 2^{n/s} B_{s'} \leq 2^{n/s} 2^{n(1/s'-1/r)} B_r = 2^{n/r'} B_r.$$

This completes the proof of (2.4). The other implications are clear by Theorem 2.2. We have the latter assertion for $X'$ by Lemma 2.1. $\square$

## 3. Exact relations between Clarkson type inequalities and the notions of type and cotype

Let $1 \le p \le 2$. A Banach space $X$ is said to be *of* (*Rademacher*) *type* $p$ provided there exist a constant $M$ and some $s$, $1 \le s < \infty$, such that

$$\left\{ \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^s dt \right\}^{1/s} \le M \left\{ \sum_{j=1}^n \|x_j\|^p \right\}^{1/p} \tag{3.1}$$

holds for all finite systems $\{x_j\}$ in $X$, where $r_j(t)$ are the Rademacher functions, that is, $r_j(t) = \operatorname{sgn}(\sin 2^j \pi t)$. Let $2 \le q \le \infty$. $X$ is said to be *of* (*Rademacher*) *cotype* $q$ provided there exist a constant $M$ and some $s$, $1 \le s < \infty$, such that

$$\left\{ \sum_{j=1}^n \|x_j\|^q \right\}^{1/q} \le M \left\{ \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^s dt \right\}^{1/s} \tag{3.2}$$

holds for all finite systems $\{x_j\}$ in $X$. We denote by $T_{p(s)}(X)$ resp. $C_{q(s)}(X)$ the smallest constant $M$ satisfying (3.1) resp. (3.2) for all finite systems $\{x_j\}$ in $X$.

*Remark 3.1.* (a) According to the well-known Khintchin-Kahane inequality (see [28]) each of the above definitions is equivalent if we take any fixed $s$, $1 \le s < \infty$.

(b) It is clear that $1 \le T_{p(s_1)}(X) \le T_{p(s_2)}(X)$ and $C_{q(s_1)}(X) \ge C_{q(s_2)}(X) \ge 1$ if $1 \le s_1 \le s_2$.

Now we introduce the *Rademacher matrices* $R_n = \left(r_{ij}^{(n)}\right)$ ($2^n \times n$ matrices) recursively as follows:

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \quad (n = 1, 2 \ldots).$$

Note here that $r_{ij}^{(n)} = r_j\left((2i-1)/2^{n+1}\right)$. The following relations are crucial for our investigation in this section:

$$\begin{aligned} \left\{ \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^s dt \right\}^{1/s} &= \left\{ \frac{1}{2^n} \sum_{\theta_j = \pm 1} \Big\| \sum_{j=1}^n \theta_j x_j \Big\|^s \right\}^{1/s} \\ &= \left\{ \frac{1}{2^n} \sum_{i=1}^{2^n} \Big\| \sum_{j=1}^n r_{ij}^{(n)} x_j \Big\|^s \right\}^{1/s}, \end{aligned} \tag{3.3}$$

and

$$\mathbf{E}\Big\|\sum_{j=1}^{n} b_{ij}x_j\Big\|^s = \frac{1}{2^n}\sum_{\theta_j=\pm 1}\Big\|\sum_{j=1}^{n}\theta_j x_j\Big\|^s = \int_0^1 \Big\|\sum_{j=1}^{n} r_j(t)x_j\Big\|^s dt \quad (1 \le i \le n), \tag{3.4}$$

where $B_n = (b_{ij})$ are the stochastic matrices defined in Section 1, and $\mathbf{E}$ denotes the mathematical expectation. In particular, (3.3) implies that the definitions of type and cotype can be given in terms of the operator norms of the Rademacher matrices between $\ell_r^n(X)$-spaces (see [12]); i.e., $X$ is of type $p$ if and only if there exist a constant $M > 0$ and some $s$, $1 \le s < \infty$ such that

$$\big\|R_n : \ell_p^n(X) \to \ell_s^{2^n}(X)\big\| \le M\, 2^{n/s} \quad \text{for } n = 1, 2, \dots. \tag{3.5}$$

Here, as usual, $\ell_r^n(X)$ denotes the $X$-valued $\ell_r^n$-space, i.e., the direct sum of $n$ copies of $X$ with the norm

$$\big\|\{x_j\}\big\|_{\ell_r^n(X)} := \left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r}.$$

A similar characterization can be made concerning the notion of cotype (see [12]).

Now by Theorems 2.2 and 2.5, combined with a previous result of Kato and Takahashi [13], we obtain the following exact relations between various variants of Clarkson's inequalities and the notions of type and cotype with their constant one:

**Theorem 3.2**

*Let $1 \le p \le 2$. Then the following statements are equivalent:*

*(i) $X$ is of type $p$ and $T_{p(p')}(X) = 1$.*
*(ii) $X$ is of cotype $p'$ and $C_{p'(p)}(X) = 1$.*
*(iii) $(p, p')$-Clarkson inequality holds in $X$.*
*(iv) $(p, p'; n)$-Clarkson inequality holds in $X$ for any, resp., some $n \in \mathbb{N}$.*
*(v) Let $1 \le r$, $s \le \infty$. Then for any, resp., some $n \in \mathbb{N}$ GCI of Kato form* (2.3) *holds in $X$.*
*(vi) Let $0 < r$, $s \le \infty$. Then for any, resp., some $n \in \mathbb{N}$ GCI of Maligranda-Persson form* (2.4) *holds in $X$.*

*In addition to the above, these assertions* (i)-(vi) *for the dual space $X'$ in place of $X$ are equivalent.*

*Proof.* The equivalence of the assertions (i)-(iii) is proved in Kato and Takahashi [13, Theorems 2.2, 2.4, Corollary 2.11]. Theorems 2.2, 2.5 and Lemma 2.1 give the rest of our assertion. □

We note that Theorem 3.2 extends the result of Kato and Takahashi [13] mentioned in the above proof. Our next aim is to state a similar equivalence theorem for the general case in type constant, but first we prove the following lemma of independent interest:

**Lemma 3.3**

*Let $1 \le u < p \le 2$. Then if $X$ is of type $p$, $X$ is of type $u$ and*

$$T_{u(u')}(X)^{u'} \le T_{p(p')}(X)^{p'}, \tag{3.6}$$

*a fortiori, $T_{u(u')}(X) \le T_{p(p')}(X)$.*

*Proof.* Put $\theta = p'/u'$ $(0 \le \theta \le 1)$. We note that $(1-\theta)/1 + \theta/p = 1/u$, $(1-\theta)/\infty + \theta/p' = 1/u'$, and

$$M_1 = \left\| R_n : \ell_1^n(X) \to \ell_\infty^{2^n}(X) \right\| = 1,$$
$$M_2 = \left\| R_n : \ell_p^n(X) \to \ell_{p'}^{2^n}(X) \right\| \le T_{p(p')}(X) 2^{n/p'}.$$

Thus by using the standard complex interpolation method (see e.g. [2]), we obtain that

$$\begin{aligned} \left\| R_n : \ell_u^n(X) \to \ell_{u'}^{2^n}(X) \right\| &\le M_1^{1-\theta} M_2^{\theta} \\ &\le T_{p(p')}(X)^{p'/u'} 2^{n/u'}, \end{aligned}$$

and, according to our discussion above (cf. (3.5)), this means that $X$ is of type $u$ and (3.6) holds. □

**Theorem 3.4**

*Let $1 \le p \le 2$. The following statements are equivalent:*

*(i) $X$ is of type $p$.*

*(ii) For any $n$ the standard $(p, p'; n)$-random Clarkson inequality*

$$\mathbf{E} \left( \sum_{i=1}^n \left\| \sum_{j=1}^n b_{ij} x_j \right\|^{p'} \right)^{1/p'} \le K n^{1/p'} \left( \sum_{j=1}^n \|x_j\|^p \right)^{1/p} \tag{3.7}$$

*holds in $X$ with some constant $K$ independent on $n$.*

*(iii) For any $n$ the $(p,1;n)$-random Clarkson inequality*

$$\mathbf{E}\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|\right)\le Kn\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p} \tag{3.8}$$

*holds in $X$ with some constant $K$ independent on $n$.*
*(iv) Let $1\le r,\ s\le\infty$. For any $n$ the random Clarkson inequality*

$$\mathbf{E}\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|^s\right)^{1/s}\le Kn^{c(r,s;p)}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r} \tag{3.9}$$

*holds in $X$ with some constant $K$ independent on $n$, $r$ and $s$, where $c(r,s;p)$ is the constant given in GCI* (2.3).

*Moreover, let $K_p(X)$ be the smallest value of such a $K$. Then*

$$T_{p(1)}(X)\le K_p(X)\le T_{p(p')}(X).$$

*Proof.* The implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. (In each inequality to be proved we can take the same constant K as in the inequality assumed.) Indeed, concerning the latter implication we note that, according to Lemma 2.3,

$$\mathbf{E}\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|\right)\le n^{1/p}\ \mathbf{E}\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|^{p'}\right)^{1/p'}.$$

(iii) $\Rightarrow$ (i): Assume that (3.8) holds with some $K$ in $X$. Then by (3.4)

$$\begin{aligned}\int_0^1\Big\|\sum_{j=1}^{n}r_j(t)x_j\Big\|\,dt&=\mathbf{E}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|\\&=\frac{1}{n}\,\mathbf{E}\left(\sum_{i=1}^{n}\Big\|\sum_{j=1}^{n}b_{ij}x_j\Big\|\right)\\&\le K\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}.\end{aligned}$$

Thus $X$ is of type $p$ and $T_{p(1)}(X)\le K$.

Moreover, according to the above proofs of (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) we see that if (3.8) holds with some $K$, then $T_{p(1)}(X) \le K$, and hence $T_{p(1)}(X) \le K_p(X)$.

(i) $\Rightarrow$ (iv): First we prove that (3.7) holds with $K = T_{p(p')}(X)$. Let $x_1, x_2, ..., x_n$ be any finite system in $X$. Then by assumption and (3.4), we have for each $i,\ i = 1, 2, ..., n$,

$$\left( \mathbf{E} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^{p'} \right)^{1/p'} = \left( \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^{p'} \right)^{1/p'} \le T_{p(p')}(X) \left( \sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}, \tag{3.10}$$

so that

$$\begin{aligned} \mathbf{E} \left( \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^{p'} \right)^{1/p'} &\le \left( \mathbf{E} \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^{p'} \right)^{1/p'} \\ &= n^{1/p'} \left( \mathbf{E} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^{p'} \right)^{1/p'} \\ &\le T_{p(p')}(X) n^{1/p'} \left( \sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}, \end{aligned}$$

i.e., (3.7) holds. Hence, according to Lemma 3.3 and the result just proved we have for any $u,\ 1 \le u \le p \le 2$,

$$\begin{aligned} \mathbf{E} \left( \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^{u'} \right)^{1/u'} &\le T_{p(p')}(X)^{p'/u'} n^{1/u'} \left( \sum_{j=1}^{n} \|x_j\|^u \right)^{1/u} \\ &\le T_{p(p')}(X) n^{1/u'} \left( \sum_{j=1}^{n} \|x_j\|^u \right)^{1/u}, \end{aligned} \tag{3.11}$$

which is a special case of (3.9). The remaining part of the proof can be carried out by using the technique used in [25] (cf. [26]), but for the readers convenience we present the details: Let $p \le r \le \infty$ and $1 \le s \le p'$. Then, by (3.4), Lemma 2.3 and (3.10), we have for each $i$, $i = 1, 2, ..., n$,

$$\begin{aligned}\left(\mathbf{E}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} &\le \left(\mathbf{E}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{p'}\right)^{1/p'} \\ &\le T_{p(p')}(X)\left(\sum_{j=1}^{n}\|x_j\|^{p}\right)^{1/p} \\ &\le T_{p(p')}(X)n^{1/p-1/r}\left(\sum_{j=1}^{n}\|x_j\|^{r}\right)^{1/r}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} &\le \left(\mathbf{E}\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} \\ &\le n^{1/s}\left(\mathbf{E}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} \\ &\le T_{p(p')}(X)n^{1/s+1/p-1/r}\left(\sum_{j=1}^{n}\|x_j\|^{r}\right)^{1/r}.\end{aligned}$$

Let $1 \le r \le p$ and $1 \le s \le r'$. Then, according to Lemma 2.3 and (3.11) with $u = r$, we obtain

$$\begin{aligned}\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} &\le n^{1/s-1/r'}\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{r'}\right)^{1/r'} \\ &\le T_{p(p')}(X)n^{1/s}\left(\sum_{j=1}^{n}\|x_j\|^{r}\right)^{1/r}.\end{aligned}$$

Let $s' \le r \le \infty$ and $p' \le s \le \infty$. Then, by again using Lemma 2.3 and (3.11) now with $u = s'$, we obtain that

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} b_{ij}x_j\right\|^{s}\right)^{1/s} \le T_{p(p')}(X)n^{1/s}\left(\sum_{j=1}^{n}\|x_j\|^{s'}\right)^{1/s'}$$

$$\le T_{p(p')}(X)n^{1/s}n^{1/s'-1/r}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r}$$

$$= T_{p(p')}(X)n^{1/r'}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r}.$$

Thus (3.9) holds with $K = T_{p(p')}(X)$ and hence $K_p(X) \le T_{p(p')}(X)$. This completes the proof. □

*Remark 3.5.* (i) If RCI (3.9) holds with some $K$ in $X$, we can take the type $p(p')$ constant $T_{p(p')}(X)$ as such a $K$.

(ii) Consider the constant $K$ in RCI (3.9) depending on $p$, and $r$, $s$; write such a $K$ as $K_p(X; r, s)$. From the above proof of the last implication we see that our upper estimate of $K_p(X)$ can be improved to an estimate of the following type:

$$K_p(X; r, s) \le \begin{cases} T_{p(p')}(X) & \text{if} \quad p \le r \le \infty,\ 1 \le s \le p', \\ T_{p(p')}(X)^{p'/r'} & \text{if} \quad 1 \le r \le p,\ 1 \le s \le r', \\ T_{p(p')}(X)^{p'/s} & \text{if} \quad s' \le r \le \infty,\ p' \le s \le \infty, \end{cases}$$

namely, $K_p(X; r, s) \le T_{p(p')}(X)^{\min\{1, p'/r', p'/s\}}$.

## 4. Concluding results and remarks

By combining the results from the preceding sections with some resent ones we can obtain several new results of independent interest. We start with stating the following result in [26]:

**Theorem 4.1** (Takahashi-Kato [26, Theorem 2.3])

*Let $1 \le t \le 2$, $1 \le u \le \infty$ and let $p = \min\{t, u, u'\}$. Then if the $(t, t')$-Clarkson inequality holds in a Banach space $X$, the $(p, p')$-Clarkson inequality holds in $L_u(X)$. If $t \le u \le t'$, the converse is true.*

By combining Theorem 3.2 with Theorem 4.1, and using Lemma 2.1 we have

**Theorem 4.2**

*Let $1 \le t \le 2$, $1 \le u \le \infty$ and let $p = \min\{t, u, u'\}$. Assume that the $(t, t')$-Clarkson inequality holds in a Banach space $X$. Then all the assertions* (i)-(vi) *in Theorem 3.2 are valid for $L_u(X)$ and $L_u(X')$, in place of $X$. In particular, GCI's* (2.3) *and* (2.4) *hold in $L_u(X)$ and $L_u(X')$.*

*Remark 4.3.* According to the above theorem the original GCI's in $L_p$ of Kato and of Maligranda-Persson ($p \geq 1$) are both directly derived from the parallelogram law ($(2,2)$-Clarkson inequality) for scalars. The same was proved for the $(p,p')$-Clarkson inequality and RCI in [26, Corollary 2.5 and Remark 3.3, respectively] (cf. also [15]).

According to Theorem 3.4 we have immediately

**Theorem 4.4**
*(i) Let $X$ be of type $p$ and $T_{p(p')}(X) = 1$. Then the random Clarkson inequality* (3.9) *with $K = 1$ holds in $X$ for any $n$; and conversely*
*(ii) If RCI* (3.9) *with $K = 1$ holds in $X$ for any $n$, then $X$ is of type $p$ and $T_{p(1)}(X) = 1$.*

Theorem 4.4 yields the following previous result of Takahashi-Kato [25] and its weak converse:

**Corollary 4.5**
*(i) Let $1 \leq p \leq 2$. Let $X$ satisfy the $(p,p')$-Clarkson inequality. Then the random Clarkson inequality* (3.9) *with $K = 1$ holds in $X$ for any $n$* ([25]), *and conversely;*
*(ii) If RCI* (3.9) *holds for any $n$ in $X$ with $K = 1$, then the* $(p,1)$*-Clarkson inequality*

$$\|x+y\| + \|x-y\| \leq 2\big(\|x\|^p + \|y\|^p\big)^{1/p} \tag{4.1}$$

*holds in $X$.*

Indeed concerning (ii), we merely note that $(p,1)$-random Clarkson inequality for $n = 2$ is precisely (4.1) (note also that $(p,1)$-Clarkson inequality is weaker than $(p,p')$-one by Lemma 2.3(b)).

*Remark 4.6.* (i) Tonge's original RCI for $L_p$ (with $K = 1$) is a direct consequence of Theorem 4.4 since $T_{t(t')}(L_p) = 1$, where $t = \min\{p,p'\}$.

(ii) There are fairly many Banach spaces satisfying $(p,p')$-Clarkson inequality and hence RCI with $K = 1$; we refer the reader to [19] for such examples (see also [4, 3, 5, 6, 11, 15, 17, 26]).

Finally we consider the relation between GCI and RCI. As an immediate consequence of Theorems 3.2 and 3.4 we have

**Corollary 4.7**
*Let $1 \leq p \leq 2$. If GCI* (2.3) *or GCI* (2.4) *holds in $X$ for any, resp., some $n$, then RCI* (3.9) *with $K = 1$ holds in $X$ for any $n$.*

As exact relations between them, we have that GCI in $X$ and RCI in $L_{p'}(X)$ are equivalent, and also they are equivalent in $L_{p'}(X)$. We need the next lemma in [26].

**Lemma 4.8** (Takahashi-Kato [26, Lemma 3.1])

*Let $1 \leq p \leq 2$. If the $(p,1)$-Clarkson inequality* (4.1) *holds in $L_{p'}(X)$, then the $(p,p')$-Clarkson inequality holds in $X$.*

**Theorem 4.9**

*Let $1 \leq p \leq 2$ and $1/p + 1/p' = 1$. Then the following are equivalent.*
*(i) $X$ is of type $p$ and $T_{p(p')}(X) = 1$.*
*(ii) GCI* (2.3) *(resp.* (2.4)) *holds in $X$.*
*(iii) GCI* (2.3) *(resp.* (2.4)) *holds in $L_r(X)$ for any, resp., some $p \leq r \leq p'$.*
*(iv) RCI* (3.9) *holds in $L_{p'}(X)$ with $K = 1$.*

*In addition to these* (i)-(iv), *the same assertions for $X'$ in place of $X$ are equivalent.*

*Proof.* By Theorems 3.2 and 4.1 we have the equivalence of (i)-(iii). The implication (iii) $\Rightarrow$ (iv) follows from Corollary 4.7 (owing to Theorem 4.1 we may assume that GCI (2.3) (resp. (2.4)) holds in $L_{p'}(X)$). Suppose that (iv) is valid. Then by Corollary 4.5 (ii) and Lemma 4.8 we obtain (i). The latter assertion follows from Lemma 2.1. □

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