Collect. Math. **51**, 3 (2000), 309–326

Sharp L^p estimates for the segment multiplier

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Received March 3, 2000

Abstract

Let S be the segment multiplier on the real line, i.e., the linear operator obtained by taking the inverse Fourier transform of $\hat{f}\chi_{[a,b]}$ where we denote by $\hat{f}$ the Fourier transform of a function f and by $\chi_{[a,b]}$ the characteristic function of the segment $[a, b]$ (finite with positive measure). Our main result consists in computing, for all $1 < p < \infty$, the best constant c_p in the inequality $\|Sf\|_p \leq c_p\|f\|_p$. We obtain along the way some results on the Hilbert Transform and on the "gap Hilbert transform"which might have some independent interest. Also we compute the best constant in the $L^p(\mathbb{R}^n)$ estimate for the "box multiplier", which is a higher dimensional version of the segment multiplier.

1. Introduction

The *Hilbert transform* $(Hf)(\zeta) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x)}{\zeta - x}\, dx$ is well-defined for $f \in C_c^\infty(\mathbb{R})$ and it can be extended to a bounded linear operator on $L^p(\mathbb{R})$ which satisfies the inequality of M. Riesz

$$\|Hf\|_p \leq n_p \|f\|_p \qquad 1 < p < \infty.$$

The (p,p) norm of H is the best constant n_p in this inequality and it is given by

$$n_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } \ 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } \ 2 \leq p < \infty. \end{cases} \tag{1}$$

This is also the norm of the *conjugate function operator*, the analogue of H on the unit circle. The expression (1) was found to be sharp first for a discrete family of p's in [4]. The full result is in [8] and was proven independently also by B. Cole (unpublished).

Let $[a,b]$ be a bounded interval of positive measure. Our main result consists in proving that (1) is also the norm, for $1 < p < \infty$, of the segment multiplier $S \equiv S_{[a,b]}$, i.e., the linear operator defined by

$$Sf(x) = \int_a^b \hat{f}(\xi)\ e^{2\pi i x\xi}\ d\xi. \tag{2}$$

Let us denote by m_p the norm of S. Multipliers norms are invariant by translation and dilation, therefore m_p does not depend on $[a,b]$. Let us choose the symmetric interval $[-r,r]$ with $r > 0$. We claim that m_p does not exceed the norm n_p of H. In fact

$$\begin{aligned} Sf(x) &= \int_{\mathbb{R}} \hat{f}(\xi)\chi_{[-r,r]}(\xi)\ e^{2\pi i x\xi}\ d\xi = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\ \frac{\sin\big(2\pi r(x-t)\big)}{x-t}\ dt \\ &= \operatorname{Im}\left(\frac{1}{\pi}\int_{\mathbb{R}} f(t)\ \frac{e^{2\pi i r(x-t)}}{x-t}\ dt\right) = \operatorname{Im}\left(M_r\ HM_{-r}\ f\right)(x), \end{aligned} \tag{3}$$

where M_r, for every $r \in \mathbb{R}$, is the linear operator defined by

$$(M_r f)(x) = e^{2\pi i r x} f(x). \tag{4}$$

We have

$$||Sf||_p \leq ||M_r H M_{-r} f||_p \leq n_p ||f||_p. \tag{5}$$

The norm of S is given by $m_p = \sup \frac{||Sf||_p}{||f||_p}$, where the supremum is taken over all functions f in $L^p(\mathbb{R})$ which are not identically zero. Therefore (5) implies

$$m_p \leq n_p, \tag{6}$$

and our main task is reduced to show the reverse inequality. Our techniques lead to some corollaries, side-results, and straightforward generalizations to higher dimensional multipliers, which we will state separately.

Note that the (p,p) norms of S and H are originally defined by taking supremums over *real-valued* $L^p(\mathbb{R})$ functions. It turns out that these two norms are unchanged if we consider *complex-valued* $L^p(\mathbb{R})$ functions. This non-trivial fact is the consequence of a general theorem of J. Marcinkiewicz and A. Zygmund about vector-valued linear operators, (see [7]). This theorem implies that a linear operator that maps boundedly a real-valued L^p space into itself also maps the complex-valued version of the same space into itself *with the same norm.* This condition is not true for multipliers in general: an important counterexample is the Riesz projection (half-line multiplier) P , defined by

$$(Pf)(\zeta) = \frac{f(\zeta) + i(Hf)(\zeta)}{2} = \int_0^{+\infty} \hat{f}(\xi)\, e^{2\pi i \zeta \xi}\, d\xi. \tag{7}$$

The (p,p) norm of P does depend on the choice of the domain (real-valued versus complex-valued $L^p(\mathbb{R})$). Note that, by translation invariance, we might reduce the study of the multiplier associated to $[a,b]$ to the case $[-r,r]$ which, by (3), clearly maps real-valued functions into real-valued functions. The same is no longer true for the multiplier associated to the half-line $[0,\infty)$.

The sharp norm of this multiplier P, in the real case, has been found by I.E. Verbitsky, and later, independently, by M. Essen (see [2], [9]). It is

$$||P||^{\mathbb{R}}_{p,p} = \frac{1}{2}\sqrt{1+n_p^2}. \tag{8}$$

The sharp norm in the complex case is strictly bigger than (8) because of the lower estimate

$$||P||^{\mathbb{C}}_{p,p} \geq \frac{1}{\sin(\frac{\pi}{p})}, \tag{9}$$

which has been known for a while (see e.g., [4]). Very recently B. Hollenbeck and I.E. Verbitsky (see [6]) have proven that this estimate is sharp, i.e., the equal sign actually holds in (9). Note that P is injective on real-valued L^p functions, while there is a whole Hardy space of complex-valued L^p functions which are mapped into 0.

2. One-parameter families leading to the (p, p) norm and sub-norm of the Hilbert transform

In order to show that $m_p \geq n_p$ we will start with a couple of lemmas about the Hilbert transform H. Let p be a fixed exponent in the range $1 < p < \infty$, let $\delta \in (0, 1)$, and let χ_δ be the characteristic function of the set

$$A_\delta \equiv \{x \in \mathbb{R} : \delta < |x| < 1/\delta\}. \tag{10}$$

We define the following pair of one-parameter functions

$$\begin{aligned} \phi_\delta(x) &= \phi_\delta(x; p) = |x|^{-1/p}\chi_\delta(x), \\ \psi_\delta(x) &= \psi_\delta(x; p) = |x|^{-1/p}\chi_\delta(x)\mathrm{sgn}\,(x). \end{aligned} \tag{11}$$

Observe that ϕ_δ is even, while ψ_δ is odd. They are both real-valued and in $L^p(\mathbb{R})$. For $\delta = 0$ we have, as "limit case", the following pair of weak-L^p functions

$$\phi_0(x) = \phi_0(x; p) = |x|^{-1/p} \quad \text{and} \quad \psi_0(x) = \psi_0(x; p) = |x|^{-1/p}\mathrm{sgn}\,(x). \tag{12}$$

The functions (11) are truncations of the functions (12), and their norm is

$$\|\phi_\delta\|_p = \|\psi_\delta\|_p = (-4\log\delta)^{1/p}. \tag{13}$$

We have the following

Lemma 1

Let H be the Hilbert transform and let $\phi_\delta = \phi_\delta(x;p)$ and $\psi_\delta = \psi_\delta(x;p)$ be defined as in (11) *and* (12) *above. Then*

$$\text{(14}a\text{)} \qquad \lim_{\delta\to 0} \frac{||H\phi_\delta||_p}{||\phi_\delta||_p} = \tan\left(\frac{\pi}{2p}\right) \quad \textit{if } \ 1 < p \le 2,$$

$$\text{(14}b\text{)} \qquad \lim_{\delta\to 0} \frac{||H\psi_\delta||_p}{||\psi_\delta||_p} = \cot\left(\frac{\pi}{2p}\right) \quad \textit{if } \ 2 \le p < \infty.$$

Moreover, for a.e. $\zeta \in \mathbb{R}$,

$$\text{(15}a\text{)} \qquad (H\phi_\delta)(\zeta) = \tan\left(\frac{\pi}{2p}\right)\psi_0(\zeta) + h_\delta(\zeta),$$

$$\text{(15}b\text{)} \qquad (H\psi_\delta)(\zeta) = -\cot\left(\frac{\pi}{2p}\right)\phi_0(\zeta) + k_\delta(\zeta),$$

where $\lim\limits_{\delta\to 0} h_\delta(\zeta) = \lim\limits_{\delta\to 0} k_\delta(\zeta) = 0$ *in the sense of the uniform convergence on the compact subsets of* $(-\infty,0)\bigcup(0,+\infty)$.

Proof. The Hilbert transform of ϕ_0 and ψ_0 is well defined as a Cauchy principal value integral, which can be computed using the theorem of residues. Our proof is based on the choice of a contour in $\mathbb{C}$ which corresponds to the truncation given by χ_δ, followed by an estimate of the remainder terms which arise in the computation.

Let us fix p and let $\zeta \in (\delta, 1/\delta)$. We have

$$\text{(16)} \qquad \begin{aligned} (H\phi_\delta)(\zeta) &= \text{p.v.}\frac{1}{\pi}\left\{\int_\delta^{1/\delta} \frac{x^{-1/p}}{\zeta - x}dx + \int_{-1/\delta}^{-\delta} \frac{(-x)^{-1/p}}{\zeta - x}dx\right\} \\ &= \text{p.v.}\frac{2\zeta}{\pi}\int_\delta^{1/\delta} \frac{x^{-1/p}}{\zeta^2 - x^2}dx. \end{aligned}$$

The function $f(z) = \frac{2\zeta}{\pi}\,\frac{z^{-1/p}}{\zeta^2 - z^2}$ is meromorphic in the complex plane with the positive real axis $x \ge 0$ removed. We choose the branch of $z^{-1/p}$ with $\text{Arg}(z) \in (0, 2\pi)$, and $\epsilon > 0$ so small that $(\zeta - \epsilon, \zeta + \epsilon) \subset (\delta, 1/\delta)$.

Let $\Sigma_{\delta,\,\epsilon}$ be the positively oriented and closed path defined as the union of the following sub-paths: (i) the circle γ_δ with center $z=0$ and radius δ in the negative sense; (ii) the segment $[\delta, \zeta-\epsilon]$ of the real axis in the positive sense; (iii) the half circle $\gamma^+_{\zeta,\,\epsilon}$ of center ζ and radius ϵ contained in the half plane $\mathrm{Im}\,(z) \geq 0$ in the negative sense; (iv) the segment $[\zeta+\epsilon, \frac{1}{\delta}]$ of the real axis in the positive sense; (v) the circle Γ_δ centered at the origin and of radius $\frac{1}{\delta}$ in the positive sense; (vi) the segment $[\zeta+\epsilon, \frac{1}{\delta}]$ of the real axis in the negative sense; (vii) the half circle $\gamma^-_{\zeta,\,\epsilon}$ of center ζ and radius ϵ contained in the half plane $\mathrm{Im}\,(z) \leq 0$ in the negative sense; (viii) the segment $[\delta, \zeta-\epsilon]$ of the real axis in the negative sense.

Since $\Sigma_{\delta,\,\epsilon}$ goes once around the pole $z=-\zeta$ we have, by the residue theorem, that $\int_{\Sigma_{\delta,\,\epsilon}} f dz = 2ie^{-\frac{\pi i}{p}}\zeta^{-\frac{1}{p}}$. On the other hand $\int_{\Sigma_{\delta,\,\epsilon}} f dz$ is also equal to

$$(1-e^{-(2\pi i)/p})\left(\int_\delta^{\zeta-\epsilon} f(x)dx + \int_{\zeta+\epsilon}^{1/\delta} f(x)dx\right) + \int_{\gamma_{\zeta,\,\epsilon}} f dz + \int_{\gamma_\delta} f dz + \int_{\Gamma_\delta} f dz.$$

We evaluate the integral of f on the circle $\gamma_{\eta,\,\epsilon}$ invoking again the residue theorem, keeping into account that $z^{-1/p}$ changes determination when we go from the upper to the lower half-circle. We obtain

$$\int_{\gamma_{\zeta,\,\epsilon}} f dz = i\zeta^{-1/p}(1+e^{-(2\pi i)/p}),$$

an expression which does not depend on ϵ. Letting $\epsilon \to 0$, by (13) we obtain

$$(1-e^{-(2\pi i)/p})(H\phi_\delta)(\zeta) = i\zeta^{-1/p}(2e^{-(\pi i)/p} - 1 - e^{-(2\pi i)/p}) - \int_{\gamma_\delta} f dz - \int_{\Gamma_\delta} f dz.$$

Using the identity $\frac{1-\cos\alpha}{\sin\alpha} = \tan(\frac{\alpha}{2})$ we finally get

$$H(\phi_\delta)(\zeta) = \tan(\frac{\pi}{2p})\zeta^{-1/p} + h_\delta(\zeta), \tag{17}$$

where

$$h_\delta(\zeta) = \frac{ie^{(i\pi)/p}}{2\sin\frac{\pi}{p}}\Big(\int_{\gamma_\delta} f dz + \int_{\Gamma_\delta} f dz\Big). \tag{18}$$

Although this expression for the remainder term is only valid when $\zeta \in (\delta, 1/\delta)$, it becomes valid for any fixed $\zeta > 0$ as soon as δ is small enough. Moreover, when

$\zeta < 0$ we can simply use the fact that $(H\phi_\delta)(-\zeta) = -(H\phi_\delta)(\zeta)$. Therefore the formula (15a) holds, and we need to check that $h_\delta(\zeta) \to 0$ uniformly on compact subsets of the punctured line.

For $z \in \gamma_\delta$, and $\zeta \in (\delta, 1/\delta)$ we have $|\zeta^2 - z^2| \geq \eta^2 - \delta^2 \geq \zeta(\zeta - \delta)$, and this implies that $|\int_{\gamma_\delta} f dz| \leq 4\frac{\delta^{1-1/p}}{\zeta-\delta}$.

Similarly, if $z \in \Gamma_\delta$, and $\zeta \in (\delta, 1/\delta)$ we have $|\zeta^2 - z^2| \geq \delta^{-2} - \zeta^2 \geq \frac{1}{\delta}(\frac{1}{\delta} - \zeta)$, and therefore $|\int_{\Gamma_\delta} f dz| \leq 4\frac{\delta^{1+1/p}\zeta}{1-\delta\zeta}$. So, if $\zeta \in K$, where K is a compact subset of $(-\infty, 0)\bigcup(0, +\infty)$, for $\delta > 0$ small enough we have

$$|h_\delta(\zeta)| \leq \frac{2}{\sin\frac{\pi}{p}}\left(\frac{\delta^{1-1/p}}{\zeta - \delta} + \frac{\delta^{1+1/p}\zeta}{1 - \delta\zeta}\right), \tag{19}$$

which gives us the uniform bound we needed.

The formula (15b) can be proven using the same path in $\mathbb{C}$ and a meromorphic function constructed from ψ_0 instead of ϕ_0. It can also be deduced from the fact that the Hilbert transform composed with itself is $-I$, where I is the identity operator.

Let $\chi_{2\delta}$ be the characteristic function of $A_{2\delta} \equiv \{x \in \mathbb{R} : 2\delta < |x| < \frac{1}{2\delta}\}$. From Minkowski's inequality and (15a) it follows that

$$\frac{||H\phi_\delta||_p}{||\phi_\delta||_p} \geq \frac{||\chi_{2\delta}H\phi_\delta||_p}{||\phi_\delta||_p} \geq \tan\left(\frac{\pi}{2p}\right)\frac{||\phi_{2\delta}||_p}{||\phi_\delta||_p} - \frac{||\chi_{2\delta}h_\delta||_p}{||\phi_\delta||_p}.$$

If $1 < p \leq 2$, because of (1), we have $\tan\left(\frac{\pi}{2p}\right) \geq \frac{||H\phi_\delta||_p}{||\phi_\delta||_p}$ and therefore (14a) follows if we show that $\lim_{\delta\to 0}\frac{||\phi_{2\delta}||_p}{||\phi_\delta||_p} = 1$ and $\lim_{\delta\to 0}\frac{||\chi_{2\delta}h_\delta||_p}{||\phi_\delta||_p} = 0$. The first one of these two limits is immediate, because of (13). The second one, because of the estimate (19), is reduced to proving that

$$\lim_{\delta\to 0}\frac{||\chi_{2\delta}h_1||_p^p}{\log\delta} = \lim_{\delta\to 0}\frac{||\chi_{2\delta}h_2||_p^p}{\log\delta} = 0, \tag{20}$$

where we have set

$$h_1(\zeta) = \frac{\delta^{1-1/p}}{\zeta - \delta} \quad \text{and} \quad h_2(\zeta) = \frac{\delta^{1+1/p}\zeta}{1 - \delta\zeta}. \tag{21}$$

Note that, since $H\phi_\delta$ is an odd function, it suffices to do L^p estimates on the positive half-line $\zeta > 0$.

We have

$$||\chi_{2\delta}h_1||_p^p = \delta^{p-1}\int_{2\delta}^{1/(2\delta)} \frac{d\zeta}{(\zeta-\delta)^p} = \frac{1}{p-1} - \frac{\delta^{p-1}}{p-1}(\frac{2\delta}{1-2\delta^2})^{p-1}.$$

This expression tends to $\frac{1}{p-1}$ when $\delta \to 0$, therefore the first limit in (20) is zero.

Then we have

$$||\chi_{2\delta}h_2||_p^p = \delta^{p+1}\int_{2\delta}^{1/(2\delta)} (\frac{\zeta}{1-\delta\zeta})^p d\zeta = \int_{1/2}^{1-2\delta^2} (\frac{1-t}{t})^p dt.$$

This expression also tends to a finite constant when $\delta \to 0$, therefore the second limit in (20) is zero.

If $2 \le p < \infty$ we have $\cot\left(\frac{\pi}{2p}\right) \ge \frac{||H\phi_\delta||_p}{||\phi_\delta||_p}$ and the formula (14b) follows from (15b) in a very similar fashion. □

Lemma 2

Let H be the Hilbert transform and let ϕ_δ and ψ_δ be defined as in (11) *above, then for a.e. $\zeta \in \mathbb{R}$,*

$$(H\phi_\delta)(\zeta) = \tan\left(\frac{\pi}{2p}\right)\psi_\delta(\zeta) + r_\delta(\zeta), \tag{22a}$$

$$(H\psi_\delta)(\zeta) = -\cot\left(\frac{\pi}{2p}\right)\phi_\delta(\zeta) + s_\delta(\zeta), \tag{22b}$$

where $\lim_{\delta\to 0} r_\delta(\zeta) = \lim_{\delta\to 0} s_\delta(\zeta) = 0$ in the sense of the uniform convergence on the compact subsets of $(-\infty, 0)\bigcup(0, +\infty)$. Moreover, both r_δ and s_δ belong to $L^p(\mathbb{R})$ and their norms are bounded by a positive constant which is independent of δ.

Proof. We denote, as before, by χ_δ the characteristic function of the set A_δ defined in (10). We denote by χ_δ^c the characteristic function of $B_\delta \equiv \mathbb{R}\backslash A_\delta$.

By Lemma 1 we know that

$$(H\phi_\delta)(\zeta) = \tan(\pi/2p)\left[\psi_\delta + \mathrm{sgn}\,(\zeta)|\zeta|^{-1/p}\chi_\delta^c(\zeta)\right] + \mathrm{sgn}\,(\zeta)h_\delta(\zeta),$$

where, for $\zeta \in (\delta, 1/\delta)$ the term $h_\delta(\zeta)$ is given by the expression (18).

Since $r_\delta = r_\delta\chi_\delta + r_\delta\chi_\delta^c$, we can prove our claim separately for the two terms in the right hand side of this equality. Again, since r_δ is an odd function, it suffices to prove our estimates on the positive half-line. For $\zeta > 0$ we have

$$r_\delta(\zeta)\chi_\delta(\zeta) = \frac{ie^{(i\pi)/p}}{\pi \sin\frac{\pi}{p}}\zeta\left(\frac{1}{\zeta^2}\int_{\gamma_\delta}\frac{z^{-1/p}}{1-(z/\zeta)^2}dz - \int_{\Gamma_\delta}\frac{z^{-1/p-2}}{1-(\zeta/z)^2}dz\right)$$

Let $c_p = \frac{1}{\pi\sin\frac{\pi}{p}}$. Since $|z/\zeta|<1$ for $z\in\gamma_\delta$ and $|\zeta/z|<1$ for $z\in\Gamma_\delta$, we obtain

$$\begin{aligned}
|r_\delta(\zeta)\chi_\delta(\zeta)| &= c_p\chi_\delta(\zeta)\Big|\frac{1}{\zeta}\int_{\gamma_\delta} z^{-1/p}\sum_{n=0}^\infty (z/\zeta)^{2n}dz - \zeta\int_{\Gamma_\delta} z^{-1/p-2}\sum_{n=0}^\infty(\zeta/z)^{2n}dz\Big| \\
&\le c_p\chi_\delta(\zeta)\sum_{n=0}^\infty\left(\Big|\zeta^{-1-2n}\int_{\gamma_\delta} z^{2n-1/p}dz\Big| + \Big|\zeta^{2n+1}\int_{\Gamma_\delta} z^{-2n-2-1/p}dz\Big|\right) \\
&\le c_p\chi_\delta(\zeta)\sum_{n=0}^\infty\left(\Big|\zeta^{-1-2n}\delta^{2n+1-1/p}\int_0^{2\pi} e^{i\theta(2n+1-1/p)}d\theta\Big|\right. \\
&\qquad \left. + \Big|\zeta^{2n+1}\delta^{2n+1+1/p}\int_0^{2\pi} e^{-i\theta(2n+1+1/p)}d\theta\Big|\right) \\
&\le 2c_p\chi_\delta(\zeta)\sum_{n=0}^\infty\left(\frac{\zeta^{-1-2n}\delta^{2n+1-1/p}}{2n+1-\frac{1}{p}} + \frac{\zeta^{2n+1}\delta^{2n+1+1/p}}{2n+1+\frac{1}{p}}\right).
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
||r_\delta\chi_\delta||_p &\le 2c_p\sum_{n=0}^\infty\left(\frac{\delta^{2n+1-1/p}}{2n+1-\frac{1}{p}}\Big(\int_\delta^{1/\delta}\zeta^{-p-2np}d\zeta\Big)^{1/p}\right. \\
&\qquad\left. + \frac{\delta^{2n+1+1/p}}{2n+1+\frac{1}{p}}\Big(\int_\delta^{1/\delta}\zeta^{p+2np}d\zeta\Big)^{1/p}\right) \\
&\le 2c_p\sum_{n=0}^\infty\left(\frac{(1-\delta^{4np+2p-2})^{1/p}}{(2n+1-\frac{1}{p})(2np+p-1)^{1/p}}\right. \\
&\qquad\left. + \frac{(1-\delta^{4np+2p+2})^{1/p}}{(2n+1+\frac{1}{p})(2np+p+1)^{1/p}}\right) \le K
\end{aligned}$$

where K is independent of δ.

We now observe that

$$r_\delta\chi_\delta^c = \Big[(H\phi_\delta) - \tan\Big(\frac{\pi}{2p}\Big)\psi_\delta\Big]\chi_\delta^c = (H\phi_\delta)\chi_\delta^c\,.$$

We use the expression (16) for $H\phi_\delta$ and analyze separately the two cases $\zeta \in (-\delta, \delta)$ and $|\zeta| > 1/\delta$. In the first case we have $|\zeta/x| < 1$ and

$$(H\phi_\delta)(\zeta) = -\frac{2\zeta}{\pi}\int_\delta^{1/\delta} \frac{x^{-1/p-2}}{1-(\zeta/x)^2}\,dx = -\frac{2}{\pi}\sum_{n=0}^{\infty}\zeta^{2n+1}\int_\delta^{1/\delta} x^{-(2n+2+1/p)}\,dx$$
$$= -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{2n+1+1/p}-\delta^{-(2n+1+1/p)}}{2n+1+1/p}\,\zeta^{2n+1}.$$

Therefore

$$||H\phi_\delta||_{L^p(-\delta,\delta)} \le \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{|\delta^{2n+1+1/p}-\delta^{-(2n+1+1/p)}|}{2n+1+1/p}\left(\int_{-\delta}^{\delta}|\zeta|^{2np+p}\right)^{1/p}$$
$$= \frac{2^{1+1/p}}{\pi}\sum_{n=0}^{\infty}\frac{(1-\delta^{4np+2p+2})^{1/p}}{(2n+1+1/p)(2np+p+1)^{1/p}} \le K.$$

In the second case we have $|x/\zeta| < 1$ and

$$(H\phi_\delta)(\zeta) = \frac{2}{\pi\zeta}\int_\delta^{1/\delta}\frac{x^{-1/p}}{1-(x/\zeta)^2}\,dx = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{\zeta^{2n+1}}\int_\delta^{1/\delta}x^{2n-1/p}\,dx$$
$$= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{\zeta^{2n+1}}\,\frac{\delta^{-(2n+1-1/p)}-\delta^{2n+1-1/p}}{2n+1-1/p}.$$

Therefore

$$||H\phi_\delta||_{L^p(|\zeta|>1/\delta)} \le \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{-(2n+1-1/p)}-\delta^{2n+1-1/p}}{2n+1-1/p}\left(\int_{|\zeta|>1/\delta}|\zeta|^{-2np-p}\right)^{1/p}$$
$$= \frac{2^{1+1/p}}{\pi}\sum_{n=0}^{\infty}\frac{(1-\delta^{4np+2p-2})^{1/p}}{(2n+1-1/p)(2np+p-1)^{1/p}} \le K.$$

We have proven (22a) together with the claim about the uniform L^p boundedness of the "error term" r_δ. The proof of (22b) and the estimates for s_δ can be obtained following exactly the same pattern. □

The following two corollaries will not be applied to our main result, but might have some independent interest.

Corollary 1

The normalized functions

$$\phi_\delta^* = (-4\log\delta)^{-1/p}\phi_\delta \quad \text{and} \quad \psi_\delta^* = (-4\log\delta)^{-1/p}\psi_\delta, \tag{23}$$

have both L^p norm equal to 1. They satisfy

$$(H\phi_\delta^*)(\zeta) = \tan\left(\frac{\pi}{2p}\right)\psi_\delta^*(\zeta) + r_\delta^*(\zeta), \tag{24a}$$

$$(H\psi_\delta^*)(\zeta) = -\cot\left(\frac{\pi}{2p}\right)\phi_\delta^*(\zeta) + s_\delta^*(\zeta), \tag{24b}$$

with

$$\lim_{\delta\to 0}||r_\delta^*||_p = \lim_{\delta\to 0}||s_\delta^*||_p = 0.$$

Proof. It follows immediately from the fact that r_δ and s_δ in lemma 2 are bounded by a constant independent of δ. $\square$

Let us now define the (p,p) *subnorm* of a linear operator T in the following way

$$\inf \frac{||Tf||_p}{||f||_p} \tag{25}$$

where the infimum is taken over all L^p functions f which are not identically zero.

If T is not invertible then its subnorm is equal to zero, but invertible operators like the Hilbert transform H have positive subnorm. In fact it is easy to see that the subnorm of H is the reciprocal $1/n_p$ of its norm. This is a consequence of the fact that $H^2 = -I$, where I is the identity operator.

Corollary 2

Let ϕ_δ^* and ψ_δ^* be defined as in (23)*. Let us consider the following two limits*

$$\lim_{\delta\to 0}||H\phi_\delta^*||_p \quad \text{and} \quad \lim_{\delta\to 0}||H\psi_\delta^*||_p. \tag{26}$$

The first one coincides with the norm n_p of H for $1 < p \le 2$ and with the subnorm $1/n_p$ of H for $2 \le p < \infty$. Exactly the opposite happens for the second one.

Proof. It follows from the fact that r_δ^* and s_δ^* of corollary 1 tend to 0 in the L^p sense as $\delta \to 0$. $\square$

3. The (p,p) norm of the segment multiplier

Let us consider the following one-parameter family of functions

$$h_\delta = -in_p M_\delta H M_{-\delta}\psi_\delta, \tag{27}$$

where n_p, M_δ, ψ_δ, have been defined in (1), (4), (11). Denoting by S_δ the segment multiplier associated to $[-\delta,\delta]$ and remembering (3) we get that

$$S_\delta h_\delta = \mathrm{Im}\,(-in_p M_\delta H M_{-\delta} M_\delta H M_{-\delta}\psi_\delta) = \mathrm{Im}\,(in_p\psi_\delta) = n_p\psi_\delta,$$

and therefore

$$||S_\delta h_\delta||_p = n_p||\psi_\delta||_p = n_p(-4\log\delta)^{1/p}. \tag{28}$$

Both $||h_\delta||_p$ and $||S_\delta h_\delta||_p$ blow up as $\delta \to 0$, but the following cancellation property holds

Lemma 3

Let $1 < p \leq 2$, then for every $\delta \in (0,1)$ we have

$$||h_\delta - iM_\delta\phi_\delta||_p \leq K, \tag{29}$$

where K is a constant independent of δ.

Proof. From (22b) of Lemma 2 we obtain that

$$\phi_\delta = \tan\left(\frac{\pi}{2p}\right)(s_\delta - H\psi_\delta).$$

Using the fact that $\tan(\pi/2p) = n_p$ for $1 < p \leq 2$ we get

$$\begin{aligned} h_\delta(\zeta) - ie^{2\pi i\delta\zeta}\ \phi_\delta(\zeta) &= -in_p\ e^{2\pi i\delta\zeta}\ H\big(e^{-2\pi i\delta x}\ \psi_\delta(x)\big)\ (\zeta) \\ &\quad - ie^{2\pi i\delta\zeta}\ n_p\big(s_\delta(\zeta) - H\psi_\delta(\zeta)\big) \\ &= -in_p\ e^{2\pi i\delta\zeta}\big[H\big((e^{-2\pi i\delta x} - 1)\ \psi_\delta(x)\big)\ (\zeta) + s_\delta(\zeta)\big]. \end{aligned}$$

Applying Minkowski's inequality to the two terms into square brackets and keeping into account that, by lemma 2, $||s_\delta||_p \leq K$, we see that our claim is proven once we show that the L^p norm of $(e^{-2\pi i\delta x} - 1)\ \psi_\delta(x)$ is also bounded by a constant independent of δ. We have

$$\begin{aligned} \int_{\mathbb{R}} |e^{-2\pi i\delta x} - 1|^p\ |\psi_\delta(x)|^p\ dx &= 2\int_\delta^{1/\delta} |2\sin\pi\delta x|^p\ \frac{1}{x}\ dx \\ &\leq 2^{p+1}\ (\pi\delta)^p \int_\delta^{1/\delta} x^{p-1}\ dx = \frac{2^{p+1}\pi^p}{p}\ (1-\delta^{2p}) \leq K, \end{aligned}$$

and our lemma is proven. □

Theorem 1

The (p,p) norm m_p of S coincides with the (p,p) norm n_p of H.

Proof. Because of (6) we only need to show that $m_p \geq n_p$. By duality it suffices to consider the case $1 < p \leq 2$. We claim that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{||S_\delta h_\delta||_p}{||h_\delta||_p} \geq n_p(1-\epsilon), \tag{30}$$

where S_δ is the segment multiplier associated to $[-\delta, \delta]$ and h_δ are the functions defined in (27). Note that

$$m_p = \sup_{\substack{h \in L^p(\mathbb{R}) \\ h \neq 0}} \frac{||S_\delta h||_p}{||h||_p}$$

does not depend on the particular $\delta > 0$ chosen. Therefore, assuming (30), we obtain

$$m_p \geq \frac{||S_\delta h_\delta||_p}{||h_\delta||_p} \geq n_p(1-\epsilon),$$

and since $\epsilon > 0$ can be arbitrarily small it follows that $m_p \geq n_p$.

To prove (30) we invoke Lemma 3 and observe that

$$||h_\delta||_p = ||h_\delta - iM_\delta\phi_\delta + iM_\delta\phi_\delta||_p \leq K + (-4\log\delta)^{1/p}.$$

By (28) we get

$$\frac{||S_\delta h_\delta||_p}{||h_\delta||_p} \geq n_p \, \frac{(-4\log\delta)^{1/p}}{K + (-4\log\delta)^{1/p}} = n_p \, \frac{1}{1 + K(-4\log\delta)^{-1/p}},$$

and our proof is complete. $\square$

4. Sharp $L^p(\mathbb{R}^n)$ estimates for the box multiplier

Let $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ be the Cartesian product of n bounded intervals of positive measure. We say that B is a *box* in $\mathbb{R}^n$ if B coincides with this set, or if it is obtained from it via translations and rotations. We can associate to any box $B \subset \mathbb{R}^n$ the multiplier operator S_B which maps boundedly $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The well-known behavior of the n-dimensional Fourier Transform under rotations and dilations implies that the (p,p) norm $m_{p,n}$ of S_B does not depend on the particular choice of the box B. We will choose $B = B_n = [-r,r]^n$ obtaining

$$(S_B f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\chi_B(\xi)e^{2\pi i x\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} f(t)\hat{\chi}_B(x-t)dt, \tag{31}$$

where

$$\hat{\chi}_B(x) = \int_{[-r,r]^n} e^{2\pi i x\cdot\xi}\, d\xi = \prod_{k=1}^{n}\int_{-r}^{r} e^{2\pi i x_k\xi_k}\, d\xi_k = \frac{1}{\pi^n}\prod_{k=1}^{n}\frac{\sin\big(2\pi r(x_k-t_k)\big)}{x_k-t_k}.$$

This means that

$$(S_B f)(x) = \int_{\mathbb{R}^n} f(t_1,\dots,t_n)\prod_{k=1}^{n}\sigma(x_k-t_k)dt_1\dots dt_n, \tag{32}$$

where $\sigma(x) = \frac{1}{\pi}\frac{\sin(2\pi r x)}{x}$ is the convolution kernel associated to the one-dimensional segment multiplier S.

Theorem 2

The (p,p) norm $m_{p,n}$ of the n-dimensional box multiplier S_B is equal to $(n_p)^n$, where n_p, given in (1), *is the norm of the Hilbert transform H.*

Proof. First we show that $||S_B f||_{L^p(\mathbb{R}^n)} \le (n_p)^n ||f||_{L^p(\mathbb{R}^n)}$.

When $n=2$ we can write

$$\begin{aligned} ||S_B f||^p_{L^p(\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} |(S_B f)(x_1,x_2)|^p\, dx_1\, dx_2 \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} f(t_1,t_2)\ \sigma(x_1-t_1)\ \sigma(x_2-t_2)\ dt_1\ dt_2\Big|^p dx_1\, dx_2. \end{aligned}$$

Let $S^{(1)}$ and $S^{(2)}$ denote the application of the one-dimensional segment multiplier to the first and second variable of f. Namely

$$S^{(1)}f(x_1,t_2) = \int_{\mathbb{R}} f(t_1,t_2)\ \sigma(x_1-t_1)\ dt_1$$

and

$$S^{(2)}f(t_1,x_2) = \int_{\mathbb{R}} f(t_1,t_2)\ \sigma(x_2-t_2)\ dt_2.$$

We obtain

$$
\begin{aligned}
||S_B f||^p_{L^p(\mathbb{R}^2)} &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |S^{(2)}\ S^{(1)}\ f(x_1,x_2)|^p\ dx_2 \Big)\ dx_1 \\
&\leq n_p^p \int_{\mathbb{R}} ||S^{(1)}\ f(x_1,\cdot)||^p_{L^p(\mathbb{R}^1)}\ dx_1 \\
&= n_p^p \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |S^{(1)}\ f(x_1,t_2)|^p\ dt_2 \Big)\ dx_1 \\
&= n_p^p \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |S^{(1)}\ f(x_1,t_2)|^p\ dx_1 \Big)\ dt_2 \\
&\leq n_p^{2p} \int_{\mathbb{R}} ||f(\cdot,t_2)||^p_{L^p(\mathbb{R}^1)}\ dt_2 = n_p^{2p} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t_1,t_2)|^p\ dt_1\ dt_2 \\
&= n_p^{2p}\ ||f(\cdot,\cdot)||^p_{L^p(\mathbb{R}^2)}.
\end{aligned}
$$

The cases $n > 2$ can be proven following recursively the same pattern.

We now need to show that the constant $(n_p)^n$ is best possible. In fact, by Theorem 1 we know that there exists a sequence g_k of one-variable functions such that $||g_k||_{L^p(\mathbb{R}^1)} = 1$ and $\lim\limits_{k\to\infty} ||Sg_k||_{L^p(\mathbb{R}^1)} = n_p$. It is easy to verify that the sequence of two-variable functions $f_k(x_1,x_2) = g_k(x_1)\ g_k(x_2)$ then satisfy $||f_k||_{L^p(\mathbb{R}^2)} = 1$ and $\lim\limits_{k\to\infty} ||S_B f_k||_{L^p(\mathbb{R}^2)} = (n_p)^2$. It is also clear that this generalizes in a straightforward way to the case $n > 2$. $\square$

Remark. A minor variation of the same proof actually works for the more general space $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times \ldots \times L^{p_n}(\mathbb{R})$, with $p_1,\ldots,p_n$ distinct exponents chosen in the range between 1 and ∞. The norm of the box multiplier S_B, which maps this space into itself, is $n_{p_1} n_{p_2} \cdots n_{p_n}$.

5. Related problems and final observations

There are essentially three kinds of segment multipliers, after dilations and translations are factored out. One is the multiplier associated to the whole real line $\mathbb{R}$, which coincides with the identity operator I. Another one is the multiplier P (Riesz projection) associated to the half-line $[0,\infty)$ or, more in general to $[r,\infty)$ or to $(-\infty,r]$ for some r real. Finally we have the multiplier S associated to a bounded and nonempty interval $[a,b]$.

In the introduction we have pointed out a major difference regarding the (p,p) norms of P and S. There is actually another important difference: the periodic analogue of P does not behave like the periodic analogue of S. In [6] it is shown that $1/\sin(\pi/p)$, the expression on the right hand side of (9), is indeed the norm of the Riesz projection both on the line and on the circle. In the periodic case this projection is obtained just discarding one half of the Fourier coefficients, typically those with negative index.

The analogue of the segment multiplier on the circle is the operator $S_{[h,k]}$, defined for all integers $h < k$ by

$$S_{[h,k]}\left(\sum_{j=-\infty}^{+\infty} c_j\ e^{2\pi ijx}\right) = \sum_{j=k}^{h} c_j\ e^{2\pi ijx}. \tag{33}$$

The (p,p) norms of these operators can be estimated but are not, to the best of our knowledge, sharply known. They do depend on the particular choice of the "frequency window" $[h,k]$. This is in stark contrast with the situation on the line, where the norm is n_p regardless of this window.

We do not know yet whether it is possible to apply similar techniques to a larger set of multipliers obtaining sharp norms, but we do know that good estimates can be obtained in a variety of cases. Consider for example the "Haar multiplier" A, associated to the function $a(\xi) = \chi_{[0,1]}(\xi) - \chi_{[-1,0]}(\xi)$. Let a_p be the (p,p) norm of A. We claim that, for $1 < p < \infty$, we have

$$n_p \leq a_p \leq \min\{2n_p, n_p^2\}. \tag{34}$$

In fact an infinite dilation of $a(\xi)$ yields the function $\operatorname{sgn}(\xi)$ which is, except for the factor $-i$, the multiplier associated to the Hilbert transform H. This observation, together with Fatou's Lemma, gives us the lower estimate. The first of the two upper estimates just follows from our Theorem 1 and Minkowski's inequality, while the second follows from the same theorem plus the observation that $a(\xi) = \operatorname{sgn}(\xi)\chi_{[-1,1]}(\xi)$. Note that n_p^2 is better when p is close to 2, while $2n_p$ is better for $p \to \infty$ or $p \to 1$.

There is another example where essentially the same approach we used for S leads to a sharp result. Furthermore, a shortcut in the proof is available! It is the case of the "gap Hilbert transform" H_r defined, for any $r > 0$, by

$$(H_r f)(x) = -i \int_{\mathbb{R}} \left(\chi_{(r,\infty)}(\xi) - \chi_{(-\infty,-r)}(\xi)\right) \hat{f}(\xi) e^{2\pi ix\xi}\ d\xi. \tag{35}$$

We have the following

Theorem 3

The (p, p) norm of H_r, the multiplier operator defined in (35)*, coincides with n_p, norm of the Hilbert transform H.*

Proof. We observe first that H_r maps real-valued functions into real-valued functions. One way to check this exploits the fact that the Fourier transform of a real-valued function f can be written in the form $g_0 + ig_1$ with g_0 even and real, and with g_1 odd and real. This implies that $\big(\chi_{(r,\infty)}(\xi) - \chi_{(-\infty,-r)}(\xi)\big)\, \hat{f}(\xi) = \gamma_1(\xi) + i\gamma_0(\xi)$ with γ_0 even and real, and with γ_1 odd and real. The integral (35) reduces to the real-valued expression

$$\begin{aligned} -i \int_{\mathbb{R}} \big(\gamma_1(\xi) + i\gamma_0(\xi)\big)\ \big(\cos(2\pi x\xi) + i\ \sin(2\pi x\xi)\big)\ d\xi \\ = \int_{\mathbb{R}} \big(\gamma_0(\xi)\ \cos(2\pi x\xi) + \gamma_1(\xi)\ \sin(2\pi x\xi)\big)\ d\xi. \end{aligned}$$

Because of the theorem in [7] that we already mentioned in the introduction, we can restrict our attention to real-valued $L^p(\mathbb{R})$ functions f . We claim that

$$H_r f = \mathrm{Re}\,(M_r\ HM_{-r}\ f), \tag{36}$$

where M_r is the operator defined in (4). In fact we have

$$\begin{aligned} (M_r\ HM_{-r}\ f)\ (x) &= -i \int_{\mathbb{R}} \mathrm{sgn}\,(\xi - r)\ \hat{f}(\xi)\ e^{2\pi i x\xi}\ d\xi \\ &= -i \int_{\mathbb{R}} \big(\chi_{(r,\infty)}(\xi) - \chi_{(-\infty,-r)}(\xi) - \chi_{[-r,r]}(\xi)\big)\hat{f}(\xi)\ e^{2\pi i x\xi}\ d\xi. \end{aligned}$$

Therefore, denoting by S_r the segment multiplier associated to $[-r, r]$, and remembering that it maps real-valued functions into real-valued functions, we obtain

$$\mathrm{Re}\,(M_r\ HM_{-r}\ f)(x) = \mathrm{Re}\,\big((H_r f)(x) + i(S_r f)(x)\big) = (H_r f)(x),$$

which is (36).

A proof of the theorem could now be constructed following the same strategy we used for the proof of Theorem 1. More simply we observe that by dilation and Fatou's lemma the norm of H_r must be at least equal to n_p. On the other hand, because of (36), the same estimate we did in (5) readily shows that this norm is also less than or equal to n_p. $\square$

In higher dimensions it would be natural to try extending Theorem 2 to more general multipliers, associated to polyhedral sets. In particular, when $n = 2$, it would be very interesting to know the norms of multipliers associated to regular polygons, or at least a sharp lower estimate of the rate of growth of these norms with the number of sides. Some remarkable work in this direction can be found in [1].

Added in proof. After this paper was submitted and accepted for publication J. Lang showed us a preprint of his, in collaboration with R. Kerman, where they prove that there are Orlicz spaces in which the norm of the segment multiplier is bounded while the norm of the Hilbert transform is unbounded. Notice the contrast with the classic L^p case, where we have just shown that the (p, p) norms of both operators actually coincide for all $1 < p < \infty$.

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