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Asymptotic behaviour for a parabolic system with nonlinear boundary conditions

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Abstract

In this paper we obtain the blow-up rate for positive solutions of a system of two heat equations, $u_t=\Delta u$, $v_t=\Delta v$, in a bounded smooth domain Ω, with boundary conditions $\frac{\partial u}{\partial\eta}=v^p$, $\frac{\partial v}{\partial\eta}=u^q$. Under some assumptions on the initial data u_0, v_0 and p,q subcritical, we find that the behaviour of u and v is given by $\|u(\cdot,t)\|_\infty\sim(T-t)^{-\frac{p+1}{2(pq-1)}}$ and $\|v(\cdot,t)\|_\infty\sim(T-t)^{-\frac{q+1}{2(pq-1)}}$. As a corollary of the blow-up rate we obtain the localization of the blow-up set at the boundary of the domain. The main tool in the proof, is a nonexistence theorem for an elliptic system; we prove that the only nonnegative classical solution of the system $\Delta u=0$, $\Delta v=0$ in $\mathbb{R}^n_+$, with boundary conditions $\frac{\partial u}{\partial\eta}=v^p$, $\frac{\partial v}{\partial\eta}=u^q$ on $\partial\mathbb{R}^n_+$ is the trivial solution $u\equiv0$, $v\equiv0$, when $p\le\frac{n}{n-2}$, $q<\frac{n}{n-2}$ and $pq>1$.

1. Introduction

In this paper we obtain the blow-up rate for positive solutions of the following parabolic system

$$\begin{cases} u_t = \Delta u & \text{in } \Omega\times(0,T),\\ v_t = \Delta v & \text{in } \Omega\times(0,T),\end{cases} \tag{1.1}$$

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$$\begin{cases} \dfrac{\partial u}{\partial \eta} = v^p & \text{on} \quad \partial\Omega \times (0,T), \\ \dfrac{\partial v}{\partial \eta} = u^q & \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{1.2}$$

$$\begin{cases} u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ v(x,0) = v_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Parabolic reaction-diffusion problems or systems like (1.1)-(1.2) or of a more general form, allowing for example source terms or with different boundary conditions, appear in several branches of applied mathematics. They have been used to model, for example, chemical reactions, heat transfer or population dynamics and have been studied by several authors. See [18] and the references therein.

The question of whether the solution develops singularities in finite time has deserve a great deal of interest. In particular, for (1.1)-(1.3) it is well known (see [5], [20] and [21]) that if $pq > 1$ the solution (u, v) blows up in finite time, i.e. there exists a finite time T such that

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

We observe that both functions, u and v, go to infinity simultaneously at time T. In [1] the blow-up problem is considered for more general nonlinearities, in the equation and in the boundary conditions, in a general smooth domain Ω.

The question of how this blow-up phenomena happens is therefore a natural one and a lot of work has been done in that direction. In the case of a single equation (i.e. $p = q$ and $u_0 = v_0$ which imply $u = v$) we cite the work of [13] where they prove that the blow-up rate in that case was

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \sim (T-t)^{-1/(2(p-1))}.$$

For the blow-up rate of the system (1.1)-(1.3) , we refer to [5], [19] and [22] where the authors consider only the radial case.

Here we obtain the blow-up rate problem for (1.1)-(1.3) in a general bounded smooth domain, under suitable assumptions on the exponents p, q and on the initial datum (u_0, v_0). More precisely, throughout this paper we assume that $q \le p$ (for symmetry reasons, this is not a restriction). Also we assume that, if $n \ge 3$, $pq > 1$, $p \le \frac{n}{n-2}$, $q < \frac{n}{n-2}$ and, if $n = 2$, $pq > 1$. On the initial data we suppose that are positive, verify a compatibility condition and $\Delta u_0, \Delta v_0 \ge \alpha > 0$ in order to guarantee $u_t, v_t \ge 0$.

The main result of the paper is:

Theorem 1.1

Under the above assumptions on p, q, u_0 and v_0, there exists positive constants C, c such that

$$c \le \max_{\overline{\Omega}} u(\cdot,t)(T-t)^{(p+1)/(2(pq-1))} \le C \qquad (t \nearrow T),$$

$$c \le \max_{\overline{\Omega}} v(\cdot,t)(T-t)^{(q+1)/(2(pq-1))} \le C \qquad (t \nearrow T).$$

As a Corollary we obtain the localization of the blow-up set at the boundary of Ω.

Corollary 1.1

Let p, q, u_0 and v_0 be as in Theorem 1.1. Then if $\Omega' \subset\subset \Omega$ there exists a constant $C = C(dist(\Omega', \partial\Omega))$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega')} + \|v(\cdot,t)\|_{L^\infty(\Omega')} < C \qquad (t \in [0,T))$$

(i.e. the blow-up set is localized at $\partial\Omega$).

The proof is based on a "blow-up" type argument introduced by Gidas-Spruck [11] and that was adapted for the parabolic case by [13]. Here, we use these ideas to deal with our system.

After this "blow-up" technique is used, the proof relays on the following Liouville-type theorems for an elliptic system in the half space with nonlinear boundary conditions:

Theorem 1.2

Suppose $n \ge 3$, and $p \le \frac{n}{n-2}$, $q < \frac{n}{n-2}$ with $pq > 1$. Let (u,v) be a classical nonnegative solution of the following problem:

$$\begin{cases} \Delta u = 0 & in \quad \mathbb{R}^n_+, \\ \Delta v = 0 & in \quad \mathbb{R}^n_+, \end{cases} \tag{1.4}$$

with boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial \eta} = v^p & on \quad \partial\mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial \eta} = u^q & on \quad \partial\mathbb{R}^n_+, \end{cases} \tag{1.5}$$

then $u \equiv 0,\ v \equiv 0$.

Theorem 1.3

Let $n = 2$, and $p, q > 0$. Let (u, v) be a classical nonnegative solution of (1.4), (1.5) with u bounded, then $u \equiv 0,\ v \equiv 0$.

These theorems are of independent interest. In fact it have been used by the authors to prove an existence result for an elliptic system with a nonlinear boundary condition in a bounded domain [6].

The proof of Theorem 1.2 is based on the *Moving Plane Method*, introduced by Alexandroff and then used by several authors to study the symmetry properties of many elliptic equations [10], [4], [16], etc). In [14] the *Moving Plane Method* is used to study the single equation

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial n} = u^p & \text{on} \quad \partial\mathbb{R}^n_+. \end{cases}$$

It is proved there that the only classical solution is $u \equiv 0$ when p is subcritical ($p < \frac{n}{n-2}$) and greater than one.

The paper is organized as follows, in §2, we prove Theorem 1.1, in §3 the nonexistence results (Theorems 1.2 and 1.3) and we leave for the Appendix some uniform Schauder estimates needed in the proof of Theorem 1.1.

2. Blow-up rate for the system

To prove Theorem 1.1 we need a result that gives the asymptotic behavior for solutions of

$$\begin{cases} w_t = \Delta w & \text{in} \quad \Omega \times [0, T), \\ \dfrac{\partial w}{\partial \eta} (\geq) \leq \dfrac{k}{(T-t)^s} & \text{on} \quad \partial\Omega \times [0, T), \\ w(x, 0) = w_0(x) > 0 & \text{on} \quad \Omega, \end{cases} \tag{2.1}$$

where $s > 1/2$. We state this result as follows.

Lemma 2.1

Let w be a positive solution of (2.1) that blows-up at time T, then

$$(c \leq) \|w(\cdot, t)\|_\infty (T - t)^{s-1/2} \leq C \qquad (t \nearrow T).$$

Proof. It is enough to prove the Lemma for w such that $w_t \geq 0$, because, given w_0 we can choose an initial datum $\widetilde{w}_0$ such that $\Delta\widetilde{w}_0 > \delta > 0$ (this guarantees $\widetilde{w}_t \geq 0$) below or above w_0, then we obtain the result by a comparison argument.

Let $\Gamma(x,t)$ be the fundamental solution of the heat equation, namely

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Now for $x \in \partial\Omega$, using Green's identity and the jump relation (see [7]) we have

$$\begin{aligned} \frac{1}{2}w(x,t) = & \int_\Omega \Gamma(x-y,t-z)w(y,z)\ dy \\ & + \int_z^t \int_{\partial\Omega} \frac{\partial w}{\partial\eta}(y,\tau)\,\Gamma(x-y,t-\tau)\ dS_y d\tau \\ & - \int_z^t \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\eta}(x-y,t-\tau)w(y,\tau)\ dS_y d\tau. \end{aligned} \tag{2.2}$$

Now we set $W(t) = \sup_\Omega w(\cdot,t)$. Since Ω is smooth, for instance $\partial\Omega \in C^{1+\alpha}$, Γ satisfies (see [7])

$$\left|\frac{\partial\Gamma}{\partial\eta}(x-y,t-\tau)\right| \leq \frac{C}{(t-\tau)^\mu |x-y|^{n+1-2\mu-\alpha}}$$

if $\frac{\partial w}{\partial\eta} \leq \frac{k}{(T-t)^s}$ by (2.2) we obtain, for $1-\alpha/2 < \mu < 1$

$$\frac{1}{2}\,W(t) \leq W(z) + C\int_z^t \frac{k}{(T-\tau)^s(t-\tau)^{1/2}}\ d\tau + CW(t)(T-z)^{1-\mu}.$$

We choose z such that $C(T-z)^{1-\mu} < 1/4$ then multiplying by $(T-t)^{s-1/2}$ we get

$$\begin{aligned} \frac{(T-t)^{s-1/2}}{4}\,W(t) \leq & (T-t)^{s-1/2}W(z) \\ & + C(T-t)^{s-1/2}\int_z^t \frac{k}{(T-\tau)^s(t-\tau)^{1/2}}\ d\tau. \end{aligned}$$

One can check that the right hand side of the last inequality is bounded uniformly in t as we wanted to prove.

For the other inequality, if $\frac{\partial w}{\partial\eta} \geq \frac{k}{(T-t)^s}$,

$$\frac{1}{2}W(t) \geq \int_z^t \int_{\partial\Omega} \frac{k}{(T-t)^s}\,\Gamma(x-y,t-\tau)\ dS_y d\tau - CW(t)(T-z)^{1-\mu}.$$

As before, we choose z such that $C(T-z)^{1-\mu} < 1/2$ then

$$\begin{aligned} W(t) &\geq \int_z^t \frac{k}{(T-t)^s} \left(\int_{\partial\Omega} \Gamma(x-y, t-\tau)\, dS_y \right) d\tau \\ &\geq c \int_z^t \frac{k}{(T-t)^s} \frac{1}{(t-\tau)^{1/2}}\, d\tau. \end{aligned}$$

As before, one can check that the right hand side multiplied by $(T-t)^{s-1/2}$, is bounded by below uniformly in t. This completes the proof. □

Now we state two results.

Lemma 2.2

Let z be a positive solution of

$$\begin{cases} z_t = \Delta z & \text{in} \quad \Omega \times [0, T), \\ \dfrac{\partial z}{\partial \eta} \leq z^\kappa & \text{on} \quad \partial\Omega \times [0, T), \\ z(x, 0) = z_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.3}$$

with $\kappa > 1$ and blow-up time T. Then there exists $c > 0$ such that

$$c \leq \max_\Omega z(\cdot, t)(T-t)^{1/(2(\kappa-1))}.$$

The proof can be found in [13].

The second result is a comparison between the pair of functions u and v^γ (with $\gamma = \frac{p+1}{q+1}$), where (u, v) is the solution of (1.1)-(1.3). This comparison result allows us to reduce the problem to a single equation and then apply Lemma 2.1. The proof of this Lemma can be found in [19] and [5].

Lemma 2.3

There exists a constant $C > 0$ such that

$$Cu \geq v^{(p+1)/(q+1)}$$

where (u, v) is a solution of (1.1)-(1.3).

Now we prove that the converse of Lemma 2.3 is, in some sense, true. In fact, we prove the following result (see [9] for a similar result for a semilinear system).

Lemma 2.4

Let

$$M(t) = \max_{\overline{\Omega}} u(\cdot, t), \qquad N(t) = \max_{\overline{\Omega}} v(\cdot, t). \tag{2.4}$$

There exists a constant $\delta > 0$ *such that*

$$\delta \max\big\{M^{q+1}(t), N^{p+1}(t)\big\} \le \min\big\{M^{q+1}(t), N^{p+1}(t)\big\}.$$

Proof. We argue by contradiction. Assume that there exists a sequence $t_n \to T$ such that

$$\max\big\{M^{q+1}(t_n), N^{p+1}(t_n)\big\} = M^{q+1}(t_n), \qquad M^{-(q+1)}(t_n)N^{p+1}(t_n) \to 0.$$

Let $x_n \in \partial\Omega$ be a point such that $u(x_n, t_n) = M(t_n)$. We define

$$\varphi_n(y, s) = \frac{1}{M(t_n)} u\big(\lambda_n R_n y + x_n, \lambda_n^2 s + t_n\big),$$

$$\psi_n(y, s) = \frac{1}{\lambda_n^{\frac{q+1}{1-pq}}} v\big(\lambda_n R_n y + x_n, \lambda_n^2 s + t_n\big).$$

Where R_n is an orthogonal transformation that maps the unit normal vector at x_n to $-e_1$. We choose $\lambda_n = M^{\frac{1-pq}{p+1}}(t_n)$. These functions φ_n, ψ_n satisfy $0 \le \varphi_n \le 1$, $\varphi_n(0,0) = 1$, $0 \le \psi_n \le \frac{N(t_n)}{M^{\frac{q+1}{p+1}}(t_n)} \to 0$ and

$$\begin{cases} (\varphi_n)_s = \Delta\varphi_n, & (\psi_n)_s = \Delta\psi_n, \\ \dfrac{\partial\varphi_n}{\partial\eta} = \psi_n^p, & \dfrac{\partial\psi_n}{\partial\eta} = \varphi_n^q, \end{cases}$$

in $\Omega_n \times I_n$ where $\Omega_n = \{y \mid \lambda_n R_n y + x_n \in \Omega\}$ and $I_n = (-\lambda_n^{-2} t_n, 0]$. We observe that $\lambda_n \to 0$ as $n \to \infty$. Hence Ω_n approaches to the half space $\mathbb{R}^N_+ = \{y_1 > 0\}$ and $I_n \to (-\infty, 0]$. The Schauder estimates allows us to pass to the limit as $n \to \infty$ (using a subsequence, if necessary) in the space $C^{2+\mu, 1+\mu/2}$ (see the appendix for the details) obtaining that $\varphi_n \to \varphi$, and $\psi_n \to \psi \equiv 0$. Hence we have $0 = \frac{\partial\psi}{\partial\eta}(0,0) = \varphi^p(0,0) = 1$, a contradiction. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We use a scaling argument similar to that of Lemma 2.4. With $M(t^*)$ and $N(t^*)$ given by (2.4) we define

$$\varphi_\lambda(y,s) = \frac{1}{M(t^*)} u(\lambda R y + x^*, \lambda^2 s + t^*),$$
$$\psi_\lambda(y,s) = \frac{1}{N(t^*)} v(\lambda R y + x^*, \lambda^2 s + t^*),$$

where $T/2 < t^* < T$ and $u(x^*,t^*) = \max_{\overline{\Omega}} u(\cdot,t^*)$ and $R = R(t^*)$ is as in Lemma 2.4.

These functions φ_λ, ψ_λ satisfy $0 \le \varphi_\lambda, \psi_\lambda \le 1$, $\varphi_\lambda(0,0) = 1$, $\frac{\partial \varphi_\lambda}{\partial s}, \frac{\partial \psi_\lambda}{\partial s} \ge 0$ and

$$\begin{cases} (\varphi_\lambda)_s = \Delta \varphi_\lambda, & (\psi_\lambda)_s = \Delta \psi_\lambda, \\ \dfrac{\partial \varphi_\lambda}{\partial \eta} = \lambda M^{-1} N^p (\psi_\lambda)^p, & \dfrac{\partial \psi_\lambda}{\partial \eta} = \lambda M^q N^{-1} (\varphi_\lambda)^q. \end{cases}$$

Now we choose $\lambda = \frac{N}{M^q}$ and observe that λ goes to zero as t^* goes to T because by Lemma 2.3, $\lambda = \frac{N}{M^q} \le c N^{1 - q\frac{p+1}{q+1}} \to 0$.

We define $K_\lambda = \lambda M^{-1} N^p$ and observe that, by Lemmas 2.3 and 2.4, $0 < c \le K_\lambda \le C < +\infty$ as t^* goes to T.

We claim that there exists a constant C such that for every λ small

$$\frac{\partial \psi_\lambda}{\partial s}(0,0) \ge C.$$

To prove this claim, suppose not. Then there exists a sequence $\lambda_j \to 0$ such that

$$\frac{\partial \psi_{\lambda_j}}{\partial s}(0,0) \to 0.$$

As φ_{λ_j} and ψ_{λ_j} are uniformly bounded in $C^{2+\gamma,1+\gamma/2}$ (see the appendix for the details) we obtain a pair of positive functions φ, ψ such that $\varphi_{\lambda_j} \to \varphi$, $\psi_{\lambda_j} \to \psi$, $K_{\lambda_j} \to K_0 \ne 0$ and verify $0 \le \varphi, \psi \le 1$, $\varphi(0,0) = 1$, $\frac{\partial \varphi}{\partial s}, \frac{\partial \psi}{\partial s} \ge 0$ and

$$\begin{cases} \varphi_s = \Delta \varphi, & \psi_s = \Delta \psi, \\ \dfrac{\partial \varphi}{\partial \eta} = K_0 \psi^p, & \dfrac{\partial \psi}{\partial \eta} = \varphi^q, \end{cases}$$

in $\mathbb{R}^N_+ \times (-\infty, 0]$. We set $w = \psi_s$ and as w satisfies the heat equation, a boundary condition of the type $\frac{\partial w}{\partial \eta} \ge 0$ and $w(0,0) = 0$, then by Hopf's lemma we obtain that $w \equiv 0$, that is ψ does not depend on s.

Let $z = \varphi_s$, z is positive and satisfies the heat equation with a boundary condition of the form $\frac{\partial z}{\partial \eta} \ge 0$.

On the other hand we have that $0 = \frac{\partial w}{\partial \eta} = q\varphi^{q-1}z$, but φ^{q-1} is not zero at the boundary of the domain $\mathbb{R}^N_+ \times (-\infty, 0]$ (if it is zero at a point in the boundary it has a minimum there and then by Hopf's lemma it has to be zero everywhere, a contradiction), then z is zero on the boundary of $\mathbb{R}^N_+ \times (-\infty, 0]$ and using again Hopf's lemma $z = 0$ in all the domain. This proves that φ and ψ are independent of s and by Theorems 1.2 and 1.3, we obtain a contradiction as $K_0 \neq 0$.

So we have proved that

$$\frac{\partial \psi_\lambda}{\partial s}(0,0) \geq C$$

in terms of v, that is $\frac{\lambda^2 v_t}{N} \geq C$. As N is Lipschitz continuous, this implies

$$N^{1-2(p+1)/(q+1)q} N' \geq C.$$

Let $r = 1 - 2\frac{p+1}{q+1}q < -1$, now we integrate between t and T and obtain

$$C(T-t) \leq \int_t^T N^r(t) N'(t)\, dt \leq \int_{N(t)}^{+\infty} s^r\, ds = \frac{C}{N(t)^{-1-r}}.$$

Finally

$$N(t) \leq \frac{C}{(T-t)^{(q+1)/(2(pq-1))}}.$$

Using this bound for v, u verifies the heat equation and $\frac{\partial u}{\partial \eta} = v^p \leq \frac{C}{(T-t)^{\frac{p(q+1)}{2(pq-1)}}}$. Then by Lemma 2.1 we obtain

$$M(t) \leq \frac{C}{(T-t)^{(p+1)/(2(pq-1))}}.$$

Let us prove the reverse inequalities in order to finish the proof of Theorem 1.1. Now we begin by u. Using Lemma 2.3, u satisfies

$$\begin{cases} u_t = \Delta u, \\ \dfrac{\partial u}{\partial \eta} = v^p \leq C u^{p\gamma} \end{cases}$$

where $p\gamma = \frac{p(q+1)}{p+1} > 1$, then Lemma 2.2 tells us that,

$$M(t) \geq \frac{c}{(T-t)^{1/(2(p\gamma-1))}} = \frac{c}{(T-t)^{(p+1)/(2(pq-1))}}.$$

By the previous bound, v satisfies the heat equation and $\frac{\partial v}{\partial \eta} = u^q \geq \frac{C}{(T-t)^s}$, in this case $s = \frac{q(p+1)}{2(pq-1)} > \frac{1}{2}$ and by Lemma 2.1, v satisfies

$$N(t) \geq \frac{c}{(T-t)^{(q+1)/(2(pq-1))}}$$

so we have finished the proof of Theorem 1.1. □

We observe that with this blow-up rate we can localize the blow-up set at the boundary of the domain.

Proof of Corollary 1.1. We just observe that we fall into the hypothesis of Theorem 4.1 of [13]. □

3. Nonexistence results

Throughout this section, to apply the *Moving plane method* we use the following notation, for $\lambda \in R$ let

$$\Sigma_\lambda = \big\{(x_1, ..., x_n); x_1 > 0,\ x_n < \lambda\big\}, \qquad T_\lambda = \big\{(x_1, ..., x_n); x_1 \geq 0,\ x_n = \lambda\big\},$$

$$\widetilde{\Sigma}_\lambda = \overline{\Sigma_\lambda} - \{(0, ..., 0, 2\lambda)\}, \qquad B^+_\mu(y_0) = B_\mu(y_0) \cap \{x_1 > 0\}.$$

Let (u, v) be a positive solution of (1.4)-(1.5) and $\alpha_1 = -\frac{p+1}{pq-1}$, $\alpha_2 = -\frac{q+1}{pq-1}$ (we observe that, as $pq > 1$, α_1 and α_2 are negatives). Then define

$$\overline{u}(x) = \mu^{-\alpha_1} u(\mu x), \qquad \overline{v}(x) = \mu^{-\alpha_2} v(\mu x).$$

As u, v satisfy (1.4)-(1.5), $\overline{u}$, $\overline{v}$ verify

$$\begin{cases} \Delta\overline{u}(x) = 0, & \Delta\overline{v}(x) = 0, \\ \dfrac{\partial\overline{u}}{\partial\eta} = \overline{v}^p, & \dfrac{\partial\overline{v}}{\partial\eta} = \overline{u}^q. \end{cases} \tag{3.1}$$

By (3.1), if $\overline{u} \equiv 0$, then $\overline{v} \equiv 0$, then we can suppose that $u \not\equiv 0$, $v \not\equiv 0$. Now we observe that if $\mu < 1$

$$\begin{aligned} \sup_{x\in B^+_1(0)} \overline{u}(x) &\leq \mu^{-\alpha_1} \sup_{x\in B^+_\mu(0)} u(x) \leq C\mu^{-\alpha_1}, \\ \sup_{x\in B^+_1(0)} \overline{v}(x) &\leq \mu^{-\alpha_2} \sup_{x\in B^+_\mu(0)} v(x) \leq C\mu^{-\alpha_2}. \end{aligned} \tag{3.2}$$

Also

$$\begin{aligned} \inf_{x\in B^+_1(0)} \overline{u}(x) &\geq \mu^{-\alpha_1} \inf_{x\in B^+_\mu(0)} u(x) \geq c\mu^{-\alpha_1}, \\ \inf_{x\in B^+_1(0)} \overline{v}(x) &\geq \mu^{-\alpha_2} \inf_{x\in B^+_\mu(0)} v(x) \geq c\mu^{-\alpha_2}. \end{aligned} \tag{3.3}$$

Let ε_1, ε_2 be the following numbers which are positive by the maximum principle,

$$\varepsilon_1 = \min_{|x|=1, x_n \geq 0} \overline{u}(x) > 0, \qquad \varepsilon_2 = \min_{|x|=1, x_n \geq 0} \overline{v}(x) > 0.$$

Next we observe that if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then by a comparison argument,

$$\begin{cases} \overline{u}(x) \geq \dfrac{\varepsilon}{|x|^{n-2}} & |x| \geq 1 \quad x_n > 0, \\ \overline{v}(x) \geq \dfrac{\varepsilon}{|x|^{n-2}}. \end{cases} \tag{3.4}$$

Now we use the Kelvin's inversion to define

$$\varphi(x) = \frac{\overline{u}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}}, \qquad \psi(x) = \frac{\overline{v}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}}.$$

As $\overline{u}, \overline{v}$ satisfy (3.1), these functions φ, ψ satisfy

$$\begin{cases} \Delta\varphi(x) = 0, & \Delta\psi(x) = 0, \\ \dfrac{\partial\varphi}{\partial\eta}(x) = \dfrac{\psi^p(x)}{|x|^{n-(n-2)p}}, & \dfrac{\partial\psi}{\partial\eta}(x) = \dfrac{\varphi^q(x)}{|x|^{n-(n-2)q}}. \end{cases}$$

As a consequence of (3.4), we obtain

$$\psi(x) = \frac{\overline{v}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}} \geq \varepsilon, \quad \varphi(x) = \frac{\overline{u}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}} \geq \varepsilon, \quad \text{ in } |x| \leq 1 \quad x_n > 0,$$

Also, by (3.2)

$$\begin{aligned} \varphi(x) &= \frac{\overline{u}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}} \leq \frac{\sup_{y \in B_1^+(0)} \overline{u}(y)}{|x|^{n-2}} \leq \frac{C\mu^{-\alpha_1}}{|x|^{n-2}} \quad \text{if} \quad |x| \geq 1, \quad x_n > 0, \\ \psi(x) &= \frac{\overline{v}\left(\frac{x}{|x|^2}\right)}{|x|^{n-2}} \leq \frac{\sup_{y \in B_1^+(0)} \overline{v}(y)}{|x|^{n-2}} \leq \frac{C\mu^{-\alpha_2}}{|x|^{n-2}} \quad \text{if} \quad |x| \geq 1, \quad x_n > 0. \end{aligned} \tag{3.5}$$

In order to prove symmetry properties of φ and ψ, we set

$$\Phi_\lambda(x) = \varphi_\lambda(x) - \varphi(x), \qquad \Psi_\lambda(x) = \psi_\lambda(x) - \psi(x),$$

where for $\lambda < 0$ we define

$$\begin{aligned} \varphi_\lambda(x_1, ..., x_n) &= \varphi(x_1, ..., x_{n-1}, 2\lambda - x_n) = \varphi(x_\lambda), \\ \psi_\lambda(x_1, ..., x_n) &= \psi(x_1, ..., x_{n-1}, 2\lambda - x_n) = \psi(x_\lambda). \end{aligned}$$

Now we can begin the *moving plane method*.

Lemma 3.1

If $-\lambda$ is big enough, then

$$\Phi_\lambda, \Psi_\lambda \geq 0 \qquad in \quad \widetilde{\Sigma}_\lambda.$$

Proof. Let us start by defining the following functions:

$$\overline{\Phi}_\lambda(x) = |z|^\beta \Phi_\lambda(x), \qquad \overline{\Psi}_\lambda(x) = |z|^\beta \Psi_\lambda(x),$$

where $z = x + e_1 = x + (1, 0, ..., 0)$. This functions satisfy

$$\begin{aligned} & -\Delta\overline{\Phi}_\lambda + \frac{2\beta}{|z|^2}\, z\cdot\nabla\overline{\Phi}_\lambda + \frac{\beta(n-2-\beta)}{|z|^2}\overline{\Phi}_\lambda = 0, \\ & -\Delta\overline{\Psi}_\lambda + \frac{2\beta}{|z|^2}\, z\cdot\nabla\overline{\Psi}_\lambda + \frac{\beta(n-2-\beta)}{|z|^2}\overline{\Psi}_\lambda = 0. \end{aligned} \qquad \text{in} \quad \Sigma_\lambda$$

We choose $\beta = \frac{n-2}{2}$ so that the coefficient of order zero in both equations is nonnegative.

At the boundary, this functions verify

$$\begin{aligned} -\frac{\partial\overline{\Phi}_\lambda}{\partial x_1}\,|_{x_1=0} &= -\left(\frac{\partial|z|^\beta}{\partial x_1}\Phi_\lambda(x) + |z|^\beta\frac{\partial\Phi_\lambda}{\partial x_1}(x)\right)|_{x_1=0} \\ &= -\left(\frac{\beta}{|z|^2}\overline{\Phi}_\lambda + |z|^\beta\frac{\partial}{\partial x_1}(\varphi_\lambda(x) - \varphi(x))\right)|_{x_1=0} \\ &= -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda + |z|^\beta\left(\frac{1}{|x_\lambda|^{n-(n-2)p}}\psi_\lambda^p - \frac{1}{|x|^{n-(n-2)p}}\psi^p\right). \end{aligned}$$

Now, as $|x_\lambda| \leq |x|$ in $\overline{\Sigma_\lambda}$, $(\lambda < 0)$, by the mean value theorem,

$$\begin{aligned} &\left(\frac{1}{|x_\lambda|^{n-(n-2)p}}\psi_\lambda^p - \frac{1}{|x|^{n-(n-2)p}}\psi^p\right) \\ &\geq \frac{1}{|x|^{n-(n-2)p}}\left(\psi_\lambda^p - \psi^p\right) = \frac{1}{|x|^{n-(n-2)p}}\left(p\xi^{p-1}\Psi_\lambda\right) \end{aligned}$$

where ξ lies between ψ_λ and ψ. Then

$$-\frac{\partial\overline{\Phi}_\lambda}{\partial x_1}\,|_{x_1=0} \geq -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda + \overline{\Psi}_\lambda\frac{1}{|x|^{n-(n-2)p}}p\xi^{p-1}. \tag{3.6}$$

Analogously

$$-\frac{\partial \overline{\Psi}_\lambda}{\partial x_1}\ |_{x_1=0} \geq -\frac{\beta}{|z|^2}\overline{\Psi}_\lambda + \overline{\Phi}_\lambda \frac{1}{|x|^{n-(n-2)q}} q\zeta^{q-1} \tag{3.7}$$

where ζ lies between φ_λ and φ.

Now suppose that the statement of the lemma is false, that is,

$$\inf_{x\in\widetilde{\Sigma}_\lambda} \overline{\Phi}_\lambda = -\delta < 0.$$

We have

$$|\overline{\Phi}_\lambda(x)| = |z|^\beta |\varphi_\lambda(x) - \varphi(x)| \leq |z|^\beta \big(|\varphi_\lambda(x)| + |\varphi(x)|\big)$$
$$\leq \left(\frac{C\mu^{-\alpha_1}}{|x_\lambda|^{n-2}} + \frac{C\mu^{-\alpha_1}}{|x|^{n-2}}\right)|z|^\beta \leq \frac{C\mu^{-\alpha_1}}{|x|^{(n-2)/2}}, \quad \text{if } |x| \text{ is big enough.}$$

Analogously

$$|\overline{\Psi}_\lambda(x)| \leq \frac{C\mu^{-\alpha_2}}{|x|^{(n-2)/2}}.$$

Now, near the point $(0,...,0,2\lambda)$ (more precisely, for $|x-(0,...,0,2\lambda)| \leq 1$), we have

$$\overline{\Phi}_\lambda(x) \geq |z|^\beta\big(\varepsilon - \varphi(x)\big) \geq |z|^\beta \left(\varepsilon - \frac{C\mu^{-\alpha_1}}{|x|^{n-2}}\right)$$
$$\geq |z|^\beta \left(\varepsilon - \frac{C\mu^{-\alpha_1}}{|\lambda|^{n-2}}\right) > 0, \qquad \text{if } -\lambda \text{ is big enough.}$$

In a similar way we obtain, for $|x-(0,...,0,2\lambda)| \leq 1$, $\overline{\Psi}_\lambda(x) > 0$. Then the infimum must be located in $x_0 \in \overline{\Sigma}_\lambda \backslash B_1(0,...,0,2\lambda)$.

By the maximum principle, $x_0 \notin \mathrm{int}(\widetilde{\Sigma}_\lambda)$ and $x_0 \notin \mathrm{T}_\lambda$ because $\overline{\Phi}_\lambda \equiv 0$ in T_λ, then x_0 must be in $\{(x_1,...,x_n)/x_1 = 0\}$.

If $\overline{\Psi}_\lambda(x_0) \geq 0$ we are done because by (3.6) the normal derivative of $\overline{\Phi}_\lambda$ must be positive at x_0 a fact that contradicts Hopf's Lemma.

If not, $\psi_\lambda(x_0) < \psi(x_0)$ and then $\inf \overline{\Psi}_\lambda(x) = \overline{\Psi}_\lambda(x_1) < 0$, and by an analogous argument, $\varphi_\lambda(x_1) < \varphi(x_1)$.

Then we have, by (3.5)

$$\xi(x_0) \leq \frac{C\mu^{-\alpha_2}}{|x_0|^{n-2}}, \qquad \zeta(x_1) \leq \frac{C\mu^{-\alpha_1}}{|x_1|^{n-2}}. \tag{3.8}$$

By Hopf's Lemma, we can suppose that the normal derivative of $\overline{\Phi}_\lambda$ is negative at x_0, that is, using (3.8)

$$\begin{aligned} 0 > -\frac{\partial \overline{\Phi}_\lambda}{\partial x_1} \,|_{x=x_0} &\geq -\frac{\beta}{|z|^2}\overline{\Phi}_\lambda(x_0) + \overline{\Psi}_\lambda(x_0)\frac{1}{|x_0|^{n-(n-2)p}}p\xi^{p-1} \\ &\geq -\frac{\beta}{1+|x_0|^2}\overline{\Phi}_\lambda(x_0) + \overline{\Psi}_\lambda(x_0)\frac{1}{|x_0|^2}pC\mu^{-\alpha_2(p-1)}. \end{aligned}$$

Then, we have

$$\frac{\beta}{1+|x_0|^2}\delta < -\frac{p}{|x_0|^2}C\mu^{-\alpha_2(p-1)}\overline{\Psi}_\lambda(x_0).$$

Replacing in (3.7) we get

$$\begin{aligned} -\frac{\partial \overline{\Psi}_\lambda}{\partial x_1} \,|_{x=x_1} &\geq -\frac{\beta}{1+|x_1|^2}\overline{\Psi}_\lambda(x_0) - \frac{q}{|x_1|^2}C\mu^{-\alpha_1(q-1)}\delta \\ &\geq \frac{\beta^2}{1+|x_1|^2}\delta\frac{|x_0|^2}{1+|x_0|^2}\frac{1}{pC\mu^{-\alpha_2(p-1)}} - \frac{q}{|x_1|^2}\delta C\mu^{-\alpha_1(q-1)} \\ &\geq \left[\frac{\beta^2}{pC\mu^{-\alpha_2(p-1)}} - qC\mu^{-\alpha_1(q-1)}\right]\frac{\delta}{|x_1|^2}. \end{aligned} \tag{3.9}$$

We observe that, as $pq > 1$, if we choose μ small enough, we get that the last term is positive which is a contradiction, and the Lemma is proved. □

Let us now start to move the plane.

Lemma 3.2

If $\lambda_0 = \sup\{\lambda < 0 : \Phi_\gamma, \Psi_\gamma \geq 0 \text{ in } \widetilde{\Sigma}_\gamma \ \forall\ \gamma < \lambda\}$ *then*

$$\lambda_0 = 0.$$

Proof. Suppose that $\lambda_0 < 0$. By continuity, we have

$$\Phi_{\lambda_0}, \Psi_{\lambda_0} \geq 0 \qquad \text{in} \quad \widetilde{\Sigma}_{\lambda_0}.$$

In the boundary $\{x_1 = 0\} \cap \overline{\Sigma}_{\lambda_0}$, by (3.6) and (3.7) this functions verify

$$\frac{\partial \Phi_{\lambda_0}}{\partial \eta} = \frac{\psi_\lambda^p}{|x_\lambda|^{n-(n-2)p}} - \frac{\psi^p}{|x|^{n-(n-2)p}} \geq \frac{p}{|x|^{n-p(n-2)}}\xi^{p-1}\Psi_{\lambda_0} \geq 0, \tag{3.10}$$

$$\frac{\partial \Psi_{\lambda_0}}{\partial \eta} = \frac{\varphi_\lambda^q}{|x_\lambda|^{n-(n-2)q}} - \frac{\varphi^q}{|x|^{n-(n-2)q}} \geq \frac{q}{|x|^{n-q(n-2)}}\zeta^{q-1}\Phi_{\lambda_0} \geq 0.$$

Now, by (3.10) (as $n-p(n-2)\geq 0$, $n-q(n-2)>0$ and $\lambda_0<0$), $\Phi_{\lambda_0},\Psi_{\lambda_0}\not\equiv 0$ in $\widetilde{\Sigma}_{\lambda_0}$, then, by the maximum principle, we have

$$\Phi_{\lambda_0},\Psi_{\lambda_0}>0 \qquad \text{in } \overline{\Sigma}_{\lambda_0}-\big\{T_{\lambda_0}\cup\{(0,...,0,2\lambda_0)\}\big\}. \tag{3.11}$$

Now, let us define the following numbers, which by (3.11) are positive

$$\begin{aligned}
\delta_1 &= \inf\Big\{\Phi_{\lambda_0}: x_1>0,\ |x-(0,...,0,2\lambda_0)|=\frac{|\lambda_0|}{2}\Big\},\\
\delta_2 &= \inf\Big\{\Psi_{\lambda_0}: x_1>0,\ |x-(0,...,0,2\lambda_0)|=\frac{|\lambda_0|}{2}\Big\},\\
\delta &= \min\big\{\delta_1,\delta_2\big\}.
\end{aligned}$$

The point $(0,...,0,2\lambda_0)$ might be a singularity point for Φ_{λ_0} and Ψ_{λ_0}, to control this fact, we define h_ε to be the solution of the following problem:

$$\begin{cases}
\Delta h_\varepsilon = 0 & \text{in } \ \varepsilon<|x-(0,...,0,2\lambda_0)|<\dfrac{1}{2}|\lambda_0|,\ x_1>0,\\
h_\varepsilon=\delta & \text{on } \ |x-(0,...,0,2\lambda_0)|=\dfrac{1}{2}|\lambda_0|,\ x_1\geq 0,\\
h_\varepsilon=0 & \text{on } \ |x-(0,...,0,2\lambda_0)|=\varepsilon,\ x_1\geq 0,\\
\dfrac{\partial h_\varepsilon}{\partial\eta}=0 & \text{on } \ \varepsilon<|x-(0,...,0,2\lambda_0)|<\frac{1}{2}|\lambda_0|,\ x_1=0.
\end{cases}$$

By the maximum principle, we have

$$\Phi_{\lambda_0},\Psi_{\lambda_0}\geq h_\varepsilon \ \text{ in } \ \varepsilon\leq|x-(0,...,0,2\lambda_0)|\leq\frac{1}{2}|\lambda_0|,\ |x_1|\geq 0.$$

Now, let $\varepsilon\to 0$, and as $\lim_{\varepsilon\to 0^+}h_\varepsilon(x)\equiv\delta$, we obtain

$$\Phi_{\lambda_0},\Psi_{\lambda_0}\geq\delta \ \text{ in } \ 0<|x-(0,...,0,2\lambda_0)|\leq\frac{1}{2}|\lambda_0|,\ |x_1|\geq 0.$$

As, in $\widetilde{\Sigma}_{\lambda_0}$ $\overline{\Phi}_{\lambda_0}\geq\Phi_{\lambda_0}$, $\overline{\Psi}_{\lambda_0}\geq\Psi_{\lambda_0}$, we obtain

$$\lim_{\lambda\searrow\lambda_0}\inf_{\substack{|x-(0,...,0,2\lambda_0)|\leq|\lambda_0|/2\\ x_1\geq 0}}\overline{\Phi}_\lambda\geq\inf_{\substack{|x-(0,...,0,2\lambda_0)|\leq|\lambda_0|/2\\ x_1\geq 0}}\Phi_{\lambda_0}\geq\delta$$

and an analogous inequality holds for $\overline{\Psi}_\lambda$.

By the definition of λ_0, there exists a sequence (λ_k), $\lambda_k\searrow\lambda_0$ such that

$$\inf_{x\in\widetilde{\Sigma}_{\lambda_k}}\overline{\Phi}_{\lambda_k}(x)<0 \qquad \text{or} \qquad \inf_{x\in\widetilde{\Sigma}_{\lambda_k}}\overline{\Psi}_{\lambda_k}(x)<0.$$

Let us suppose that

$$\inf_{x\in\widetilde{\Sigma}_{\lambda_k}} \overline{\Phi}_{\lambda_k}(x) < 0. \tag{3.12}$$

Clearly, $\lim_{|x|\to\infty} \overline{\Phi}_{\lambda_k}(x) = 0$, then the infimum (3.12) must be located in some point $x^k \in \overline{\Sigma}_{\lambda_k} - B_{\frac{|\lambda_0|}{2}}(0,...,0,2\lambda_0)$ if $|\lambda_k - \lambda_0|$ is small enough.

Now, x^k cannot be an interior point by the equation that satisfies $\overline{\Phi}_{\lambda_k}$, and as $\overline{\Phi}_{\lambda_k} \equiv 0$ in T_{λ_k}, thus x^k must be located on the lateral wall

$$\Big\{x/x_1 = 0,\ x_n < \lambda_k,\ |x-(0,...,0,2\lambda_0)| \ge \frac{|\lambda_0|}{2}\Big\}.$$

Then the tangential derivative $\frac{\partial \overline{\Phi}_{\lambda_k}}{\partial x_n}(x^k) = 0$. Now, as $\overline{\Phi}_{\lambda_k}$, $\overline{\Psi}_{\lambda_k}$ verify (3.6) and (3.7), the infimum of $\overline{\Psi}_{\lambda_k}$ must also be less than 0, and by analogous considerations must be located in the lateral wall too.

By the boundary conditions (3.6), (3.7) and by (3.9) we have that $\overline{\Phi}_{\lambda_k}$ cannot take a negative minimum at a point on the boundary $\{x_1 = 0\} \cap \{|x| > 1\}$, then we must have $|x^k| \le 1$. Therefore we can assume (via a subsequence) that $\lim_{k\to\infty} x^k = x_0$.

Then we have

$$\overline{\Phi}_{\lambda_0}(x_0) = 0, \qquad \frac{\partial \overline{\Phi}_{\lambda_0}}{\partial x_n} = 0, \qquad x_0 \in T_{\lambda_0} \cap \{x_1 = 0\} \tag{3.13}$$

and, as a consequence of (3.13), we get

$$\frac{\partial \Phi_{\lambda_0}}{\partial x_n}(x_0) = 0. \tag{3.14}$$

Let g be the solution of the following elliptic problem

$$\begin{cases} \Delta g = 0 & \text{in } \{3/2\lambda_0 < x_n < \lambda_0,\ x_1^2 + \cdots + x_{n-1}^2 < 1\}, \\ g(x) = 0 & \text{on } \{x_n = \lambda_0\} \cap \{x_1^2 + \cdots + x_{n-1}^2 \le 1\}, \\ g(x) = 0 & \text{on } \{x_1^2 + \cdots + x_{n-1}^2 = 1\} \cap \{3/2\lambda_0 \le x_n \le \lambda_0\}, \\ g(x) = \eta & \text{on } \{x_n = 3/2\lambda_0\} \cap \{x_1^2 + \cdots + x_{n-1}^2 \le 1\}, \end{cases}$$

where $\eta = \inf\{\Phi_{\lambda_0}(x) :\ x_n = 3/2\lambda_0,\ x_1^2 + \cdots + x_{n-1}^2 \le 1\} > 0$. By construction, we have

$$\Phi_{\lambda_0} \ge g.$$

Now, as g is symmetric respect to $\{x_1 = 0\}$, we have

$$\frac{\partial g}{\partial \eta}(x) = -\frac{\partial g}{\partial x_1}(x) = 0 \quad \text{on} \quad \{x_1 = 0\}$$

and as $\Phi_{\lambda_0}(x_0) = g(x_0) = 0$,

$$\frac{\partial \Phi_{\lambda_0}}{\partial x_n}(x_0) \leq \frac{\partial g}{\partial x_n}(x_0).$$

But, by Hopf's Lemma, $\frac{\partial g}{\partial x_n}(x_0)$ must be negative which is a contradiction to (3.14) and proves our claim. □

End of the proof of Theorem 1.2. From the last Lemma we have that

$$\varphi(x_1, ..., -x_n) \geq \varphi(x_1, ..., x_n), \qquad x_n < 0.$$

As the same is valid for $x_n > 0$ we obtain that φ is symmetric with respect to the x_n axis.

The same argument shows that φ is symmetric with respect to every direction perpendicular to x_1, and hence

$$\varphi(x) = q\big(x_1, |(x_2, ..., x_n)|\big).$$

We conclude that u and v depends also of x_1 and $|(x_2, ..., x_n)|$. As the origin is arbitrary we obtain that u and v are functions of x_1 only and we can easily see that this is not possible unless $u \equiv v \equiv 0$. □

Proof of Theorem 1.3. As before, if $u \equiv 0$, then $v \equiv 0$, then we can suppose that u and v are not identically zero. By the maximum principle, we have

$$c = \inf_{|x|=2R;\ x_1 \geq 0} v(x) > 0$$

and by hypothesis $\|u\|_{L^\infty} \leq L$.

We now construct the auxiliary function

$$\psi(x) = c\frac{(2R)^\varepsilon}{|x|^\varepsilon}.$$

A direct calculation shows that

$$\begin{cases} -\Delta\psi < 0 & \text{for} \quad x \neq 0 \text{ since } n = 2 \text{ and } \varepsilon > 0, \\ \dfrac{\partial \psi}{\partial \eta} = 0 \leq \dfrac{\partial v}{\partial \eta} & \text{on} \quad \{x_1 = 0\}, \\ \psi(x) = c \leq v(x) & \text{on} \quad \{x_1 = 2R\} \cap \{x_1 \geq 0\}, \end{cases}$$

$$\lim_{M\to\infty} \inf_{|x|>M} \big(v(x) - \psi(x)\big) \geq 0.$$

It follows from the maximum principle that

$$v(x) \geq \psi(x), \qquad \text{for} \quad |x| \geq 2R, \ x_1 \geq 0.$$

Now, letting $\varepsilon \to 0^+$, we obtain

$$v(x) \geq c, \qquad \text{for} \quad |x| \geq 2R, \ x_1 \geq 0.$$

Next, let $K > 2R$ be a large positive number and take a smooth cut-off function $\zeta(x)$ such that

$$\begin{aligned} &\zeta(x) \equiv 0 && \text{on} \ \ \{|x| \leq K\} \cup \{|x| \geq 4K\}, \\ &\zeta(x) \equiv 1 && \text{on} \ \ \{2K \leq |x| \leq 3K\}, \\ &0 \leq \zeta(x) \leq 1, \quad |\nabla \zeta(x)| \leq \frac{C}{K}. \end{aligned}$$

Multiplying the equation $\Delta u = 0$ by $u^{-1}\zeta^2$ and integrating by parts, we obtain

$$\begin{aligned} \int_{\{x_1=0\}} \frac{\zeta^2}{u} v^p dS + \int\!\!\int_{\{x_1>0\}} \zeta^2 \frac{|\nabla u|^2}{u^2} dx &= \int\!\!\int_{\{x_1>0\}} 2\zeta \nabla \zeta \frac{\nabla u}{u} dx \\ &\leq \int\!\!\int_{\{x_1>0\}} |\nabla \zeta|^2 dx + \int\!\!\int_{\{x_1>0\}} \zeta^2 \frac{|\nabla u|^2}{u^2} dx. \end{aligned}$$

It follows that

$$\int_{\{x_1=0\}} \frac{\zeta^2}{u} v^p dS \leq \int\!\!\int_{\{x_1>0\}} |\nabla \zeta|^2 dx,$$

which implies that

$$\frac{c^p}{L} K \leq \int_{2K}^{3K} \frac{v^p}{u}(0, x_2) dx_2 \leq \frac{C^2}{K^2} |B_{4K}(0)| \leq \frac{C}{K^2} K^2 \leq C.$$

This is a contradiction if K is large enough. $\square$

A. Appendix

In this Appendix we prove the uniform bounds needed in the proof of Theorem 1.1. The main difficulty comes from the fact that q can be less than one, so one of the nonlinearities needs not be Lipschitz.

Let Ω be a bounded domain with boundary $\partial\Omega \in C^{2+\alpha}$, $\Omega_\lambda = \{y \in \mathbb{R}^n : \lambda R y + x^* \in \Omega\}$ and $\varphi_\lambda,\ \psi_\lambda$ the solutions of

$$\begin{cases} \dfrac{\partial \varphi_\lambda}{\partial s} = \Delta\varphi_\lambda & \text{in } \ \Omega_\lambda \times \Big[-\dfrac{T}{2\lambda^2}, 0\Big], \\ \dfrac{\partial \psi_\lambda}{\partial s} = \Delta\psi_\lambda & \text{in } \ \Omega_\lambda \times \Big[-\dfrac{T}{2\lambda^2}, 0\Big], \end{cases} \tag{A.1}$$

with the following boundary conditions

$$\begin{cases} \dfrac{\partial \varphi_\lambda}{\partial \eta} = K_\lambda \psi_\lambda^p & \text{in } \ \partial\Omega_\lambda \times \Big[-\dfrac{T}{2\lambda^2}, 0\Big], \\ \dfrac{\partial \psi_\lambda}{\partial \eta} = \varphi_\lambda^q & \text{in } \ \partial\Omega_\lambda \times \Big[-\dfrac{T}{2\lambda^2}, 0\Big]. \end{cases} \tag{A.2}$$

These functions φ_λ and ψ_λ also verify

$$0 \le \varphi_\lambda(y,s); \psi_\lambda(y,s) \le 1, \qquad \frac{\partial \varphi_\lambda}{\partial s}(y,s);\ \frac{\partial \psi_\lambda}{\partial s}(y,s) \ge 0, \tag{A.3}$$

$$\varphi_\lambda(0,0) = 1. \tag{A.4}$$

Let $D_K = \Omega_\lambda \cap \{|y| < K\} \times (-K^2, 0)$. For each point $(y,s) \in \mathbb{R}^n_+ \times (-\infty, 0]$, there exists a cylinder $D_{2R}(y,s) \subset \mathbb{R}^n_+ \times (-\infty, 0]$. Therefore, following the argument of [3] we obtain a countable number of cylinders $\{D_{2R_i}\}_{i\in N}$ such that $D_{2R_i} \subset \mathbb{R}^n_+ \times (-\infty, 0]$ and $\{D_{R_i}\}_{i\in N}$ covers $\mathbb{R}^n_+ \times (-\infty, 0]$ where D_{R_i} is the cylinder with its top having the same center as the top of the cylinder D_{2R_i}, but with half the radius.

Since Ω_λ approaches $\mathbb{R}^n_+$ as $\lambda \to 0^+$ (see [3]), the families $\{\varphi_\lambda\}$ and $\{\psi_\lambda\}$ will be defined on each cylinder if λ is small enough. Therefore, by (A.1), (A.3) and the Schauder interior estimates, we obtain that

$$\|\varphi_\lambda\|_{C^{2+\alpha,1+\alpha/2}}(D_{R_i}) \le C\|\varphi_\lambda\|_{L^\infty(D_{2R_i})} \le C,$$
$$\|\psi_\lambda\|_{C^{2+\alpha,1+\alpha/2}}(D_{R_i}) \le C\|\psi_\lambda\|_{L^\infty(D_{2R_i})} \le C,$$

for each i (see [7]), where the constant C is independent of λ.

Since the sets $\{\varphi_\lambda\}, \{\psi_\lambda\}$ forms bounded sets in $C^{2+\alpha,1+\alpha/2}(D_{R_i})$, we obtain that $\{\varphi_\lambda\}, \{\psi_\lambda\}$ are precompact in $C^{2+\beta,1+\beta/2}(D_{R_i})$ for $0 < \beta < \alpha$ (see [12]). Therefore, by the diagonal method, we form a sequence $\lambda_j \to 0^+$ such that

$$\varphi_{\lambda_j} \to \varphi \qquad \text{and} \qquad \psi_{\lambda_j} \to \psi \tag{A.5}$$

in $C^{2+\beta,1+\beta/2}(D_{R_i})$ for each i.

Now, let us obtain some boundary estimates for φ_λ and ψ_λ. Let $C > 0$ such that $K_\lambda \le C\ \forall\lambda$, then we have

$$\Big\| \frac{\partial \varphi_\lambda}{\partial \eta} \Big\|_{L^\infty(\partial D_{2K} \cap \partial\Omega_\lambda)} \le C, \qquad \Big\| \frac{\partial \psi_\lambda}{\partial \eta} \Big\|_{L^\infty(\partial D_{2K} \cap \partial\Omega_\lambda)} \le 1$$

therefore, from [15], we obtain

$$\|\varphi_\lambda\|_{C^{\alpha,\alpha/2}(\overline{D_K})}; \|\psi_\lambda\|_{C^{\alpha,\alpha/2}(\overline{D_K})} \le C_K.$$

Also, if $B = \partial D_{2K} \cap \partial\Omega_\lambda$

$$\begin{aligned}
\Big\| \frac{\partial \psi_\lambda}{\partial \eta} \Big\|_{C^{\gamma,\gamma/2}(B)} &= \|K_\lambda \varphi_\lambda^q\|_{C^{\gamma,\gamma/2}(B)} \le C\|\varphi_\lambda^q\|_{C^{\gamma,\gamma/2}(B)} \\
&\le C\left(\|\varphi_\lambda^q\|_{L^\infty(B)} + [\varphi_\lambda^q]_{C^{\gamma,\gamma/2}(B)}\right) \\
&\le C\left(1 + \sup_{(y_i,s)\in B; y_1\ne y_2} \frac{|\varphi_\lambda^q(y_1,s) - \varphi_\lambda^q(y_2,s)|}{|y_1-y_2|^\gamma}\right. \\
&\qquad \left. + \sup_{(y,s_i)\in B; s_1\ne s_2} \frac{|\varphi_\lambda^q(y,s_1) - \varphi_\lambda^q(y,s_2)|}{|s_1-s_2|^{\gamma/2}}\right).
\end{aligned}$$

If $q \ge 1$, from the mean value theorem, we get

$$\frac{|\varphi_\lambda^q(y_1,s) - \varphi_\lambda^q(y_2,s)|}{|y_1-y_2|^\gamma} = q|\xi|^{q-1} \frac{|\varphi_\lambda(y_1,s) - \varphi_\lambda(y_2,s)|}{|y_1-y_2|^\gamma}$$

where ξ is an intermediate value between $\varphi_\lambda(y_1,s)$ and $\varphi_\lambda(y_2,s)$, then we obtain

$$\frac{|\varphi_\lambda^q(y_1,s) - \varphi_\lambda^q(y_2,s)|}{|y_1-y_2|^\gamma} \le q \frac{|\varphi_\lambda(y_1,s) - \varphi_\lambda(y_2,s)|}{|y_1-y_2|^\gamma}.$$

In a similar way, we obtain

$$\frac{|\varphi_\lambda^q(y,s_1) - \varphi_\lambda^q(y,s_2)|}{|s_1-s_2|^{\gamma/2}} \le q \frac{|\varphi_\lambda(y,s_1) - \varphi_\lambda(y,s_2)|}{|s_1-s_2|^{\gamma/2}}.$$

Now, if $0 < q < 1$,

$$\begin{aligned}\frac{|\varphi_\lambda^q(y_1,s)-\varphi_\lambda^q(y_2,s)|}{|y_1-y_2|^\gamma} &= \frac{|\varphi_\lambda^q(y_1,s)-\varphi_\lambda^q(y_2,s)|}{|\varphi_\lambda(y_1,s)-\varphi_\lambda(y_2,s)|^q}\left(\frac{|\varphi_\lambda(y_1,s)-\varphi_\lambda(y_2,s)|}{|y_1-y_2|^{\gamma/q}}\right)^q \\ &\le \sup_{x,y\in(0,1)}\frac{|x^q-y^q|}{|x-y|^q}\left(\frac{|\varphi_\lambda(y_1,s)-\varphi_\lambda(y_2,s)|}{|y_1-y_2|^{\gamma/q}}\right)^q \\ &\le C\left(\frac{|\varphi_\lambda(y_1,s)-\varphi_\lambda(y_2,s)|}{|y_1-y_2|^{\gamma/q}}\right)^q.\end{aligned}$$

Then if we set $\gamma \le \min\{\alpha q;\alpha\}$, $\|\frac{\partial\psi_\lambda}{\partial\eta}\|_{C^{\gamma,\gamma/2}(B)} \le C_K$. Analogously, we get $\|\frac{\partial\varphi_\lambda}{\partial\eta}\|_{C^{\gamma,\gamma/2}(B)} \le C_K$, with $\gamma \le \min\{\alpha;\alpha q\}$ (observe that $p \ge q$). This implies (see [17]) that $\|\varphi_\lambda\|_{C^{1+\gamma,1/2+\gamma/2}(\overline{D_{K/2}})}, \|\psi_\lambda\|_{C^{1+\gamma,1/2+\gamma/2}(\overline{D_{K/2}})} \le C_K$, where the constant C_K is independent of λ.

Then, by the same argument as before, we can assume that the limit functions $\varphi,\psi \in C^{1+\beta,1/2+\beta/2}(\overline{\mathbb{R}^n_+}\times(-\infty,0]) \cap C^{2+\beta,1+\beta/2}(\mathbb{R}^n_+\times(-\infty,0])$ for $0<\beta<\gamma$. Also, we can assume that $K_{\lambda_j} \to K_0$.

By this estimates, we obtain that φ, ψ verify

$$\begin{cases}\dfrac{\partial\varphi}{\partial s} = \Delta\varphi & \text{in } \mathbb{R}^n_+\times(-\infty,0]\\ \dfrac{\partial\psi}{\partial s} = \Delta\psi & \text{in } \mathbb{R}^n_+\times(-\infty,0]\end{cases} \tag{A.6}$$

$$\begin{cases}\dfrac{\partial\varphi}{\partial\eta} = K_0\psi^p & \text{in } \{y_1=0\}\times(-\infty,0]\\ \dfrac{\partial\psi}{\partial\eta} = \varphi^q & \text{in } \{y_1=0\}\times(-\infty,0]\end{cases} \tag{A.7}$$

$$\varphi(0,0)=1, \qquad 0\le\varphi,\psi\le 1 \tag{A.8}$$

So by the regularity theory of parabolic PDEs [15], we find that $\psi,\varphi\in C^\infty$ for the y and s directions up to the boundary $\{y_1=0\}$.

By (A.3), (A.5) and the fact that the functions $\varphi_s(y,s),\psi_s(y,s)$ are continuous up to the boundary $\{y_1=0\}$, we get that

$$\varphi_s(y,s),\psi_s(y,s)\ge 0 \quad \text{for} \quad 0\le y_1<\infty,\ -\infty<s\le 0.$$

Now, by (A.8) and Hopf's lemma we obtain that for a fixed $K>0$ there exists $\delta_K>0$ such that $\varphi,\psi\ge\delta_K>0$ on $H_K\equiv\partial\mathbb{R}^n_+\cap\{|y|\le K\}\times[-K^2,0]$.

Therefore, by the use of this lower bound for φ, ψ and the fact that $\varphi_{\lambda_j} \to \varphi$; $\psi_{\lambda_j} \to \psi$ uniformly on H_K, we have that there exists $\epsilon_K > 0$ such that for sufficiently large j, $\varphi_{\lambda_j}, \psi_{\lambda_j} \geq \epsilon_K > 0$ on H_K.

We can use this fact to obtain more regularity on the boundary. We have that

$$\begin{aligned}
&\left[\frac{\partial \varphi_{\lambda_j}}{\partial \eta}\right]_{C^{1+\gamma,1/2+\gamma/2}(H_K)} = \left[\psi^p_{\lambda_j}\right]_{C^{1+\gamma,1/2+\gamma/2}(H_K)} \\
&= \sup_{|a|=1} \left[D^a_y(\psi^p_{\lambda_j})\right]_{C^\gamma_y(H_K)} + \left[\psi^p_{\lambda_j}\right]_{C_s^{1/2+\gamma/2}(H_K)} \\
&= \sup_{|a|=1} \left[p\psi^{p-1}_{\lambda_j} D^a_y(\psi_{\lambda_j})\right]_{C^\gamma_y(H_K)} + C_K \\
&\leq \sup_{|a|=1} \sup_{(y_i,s)\in H_K;\ y_1\neq y_2} \frac{|p\psi^{p-1}_{\lambda_j} D^a_y(\psi_{\lambda_j})(y_1,s) - p\psi^{p-1}_{\lambda_j} D^a_y(\psi_{\lambda_j})(y_2,s)|}{|y_1-y_2|^\gamma} + C_K \\
&\leq \sup_{|a|=1} \sup_{(y_i,s)\in H_K;\ y_1\neq y_2} |p\psi^{p-1}_{\lambda_j}(y_1,s)| \frac{|D^a_y(\psi_{\lambda_j}(y_1,s)) - D^a_y(\psi_{\lambda_j}(y_2,s))|}{|y_1-y_2|^\gamma} \\
&\quad + \sup_{|a|=1} \sup_{(y_i,s)\in H_K;\ y_1\neq y_2} |D^a_y(\psi_{\lambda_j}(y_2,s))| \frac{|p\psi^{p-1}_{\lambda_j}(y_1,s) - p\psi^{p-1}_{\lambda_j}(y_2,s)|}{|y_1-y_2|^\gamma} + C_K.
\end{aligned}$$

Now, by our previous estimates, the first term is bounded by a constant C_K, and because of the lower bound for φ_{λ_j}, ψ_{λ_j} and the mean value theorem, the second term is bounded by another constant. Therefore,

$$\left\|\frac{\partial \varphi_{\lambda_j}}{\partial \eta}\right\|_{C^{1+\gamma,1/2+\gamma/2}(H_K)} \leq C_K$$

and in a similar way

$$\left\|\frac{\partial \psi_{\lambda_j}}{\partial \eta}\right\|_{C^{1+\gamma,1/2+\gamma/2}(H_K)} \leq C_K.$$

This implies that

$$\|\varphi_{\lambda_j}\|_{C^{2+\gamma,1+\gamma/2}(H_{K/2})}; \|\varphi_{\lambda_j}\|_{C^{2+\gamma,1+\gamma/2}(H_{K/2})} \leq C_K,$$

where the constant C_K is independent of λ (see [12]).

So again, by compactness and if necessary by further refinement of the sequence, we obtain that

$$\|\varphi_{\lambda_j} - \varphi\|_{C^{2+\beta,1+\beta/2}(H_{K/2})} \to 0,$$
$$\|\psi_{\lambda_j} - \psi\|_{C^{2+\beta,1+\beta/2}(H_{K/2})} \to 0,$$

for $0 < \beta < \gamma$. $\square$

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