Collect. Math. **51**, 3 (2000), 277–284

On strong M-bases in Banach spaces with PRI

Deba P. Sinha

Department of Mathematics, Dyal Singh College, (University of Delhi),
Lodi Road, New Delhi 110003, INDIA
E-mail: dpsinha@nda.vsnl.net.in

Received January 10, 2000. Revised April 14, 2000

Abstract

If every member of a class $\mathcal{P}$ of Banach spaces has a projectional resolution of the identity such that certain subspaces arising out of this resolution are also in the class $\mathcal{P}$, then it is proved that every Banach space in $\mathcal{P}$ has a strong M-basis. Consequently, every weakly countably determined space, the dual of every Asplund space, every Banach space with an M-basis such that the dual unit ball is weak* angelic and every $\mathcal{C}(K)$ space for a Valdivia compact set K, has a strong M-basis.

1. Introduction

It is well known that every separable Banach space has a complete total biorthogonal system, in particular, a Markuševič basis (in short, M-basis). The problem regarding the existence of a basis in every separable Banach space remained open until an example of a separable Banach space without a basis was given by Enflo [6]. Meanwhile, a type of M-basis called the strong M-basis was defined by Ruckle [10]. Every basis of a Banach space is a strong M-basis while the converse is not necessarily true [11, Example 7.1]. Following Enflo's counterexample, Davis and Singer [4, Problem 2] raised the question whether every separable Banach space admits a strong M-basis. A partial negative solution to this problem was given by the author [12] for strong M-bases with a little extra condition. Very recently

Terenzi [14] has finally solved the problem by proving that every separable Banach space admits a bounded norming strong M-basis.

For non-separable Banach spaces in general, Reif [8, 9] proved that every weakly compactly generated Banach space, and every subspace there of, have M-bases. Fabian and Godefroy [7] have shown the existence of M-bases in duals of Asplund spaces. On the other hand Alexandrov and Plichko [1] proved that there exists a non-separable Banach space E with an M-basis and containing a copy of ℓ_∞ such that E does not admit any strong M-bases. It is now natural to ask whether every non-separable weakly compactly generated Banach space or the dual of an Asplund space has a strong M-bases. What we prove in this note answers this question positively. In fact, some recent progress in the theory of the projectional resolutions of the identity (in short, PRI) enables us to conclude much more.

Deville, Godefroy and Zizler [5] in their book, have considered the following hypothesis : *Assume that every member E of a class $\mathcal{P}$ of Banach spaces admits a PRI $(p_\alpha)_{\omega\leq\alpha\leq\mu}$ with* dens $E =$ card μ *such that* $p_\alpha(E)$*(respectively* $(p_{\alpha+1}-p_\alpha)(E)$*) belongs to $\mathcal{P}$ for every $\alpha \in [\omega,\mu)$.* Besides the class of separable Banach spaces many well known classes of non-separable Banach spaces also satisfy this hypothesis. Recently, it has been shown in [13] that every Banach space in a class satisfying this hypothesis has the Gelfand-Phillips property. In this note we prove that every Banach space in a class satisfying the hypothesis has a strong M-basis. As a consequence, we obtain strong M-bases in weakly countably determined (in particular, weakly compactly generated) Banach spaces, in the duals of Asplund spaces, in Banach spaces whose duals equipped with the weak* topology belong to a certain class Σ and in $\mathcal{C}(K)$ spaces for Valdivia compact sets K. As an immediate corollary we get the already known result [5, Theorem 7.1.8] that every Banach space in a class satisfying this hypothesis admits an equivalent locally uniformly rotund norm.

Very recently, Alexandrov and Plichko [2] have shown that every non separable Banach space with a norming M-basis has a strong M-basis and that the converse may not be true. Thus it follows that the result in this note can not be improved in general to get a complete extension of the theorem of Terenzi [14] to a class of non-separable Banach spaces satisfying the hypothesis.

2. Preliminaries

Let E be a Banach space. A system $(x_\alpha, f_\alpha)_{\alpha\in I}$, where $(x_\alpha)_{\alpha\in I} \subset E$ and $(f_\alpha)_{\alpha\in I} \subset E^*$, is said to be a *biorthogonal system* if $f_\alpha(x_\beta) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in I$, where $\delta_{\alpha\beta}$ is the Kronecker delta. A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha\in I}$ is said to be *complete* or fundamental in E if the linear span of $(x_\alpha)_{\alpha\in I}$ is dense in E, and *total* on E if $f_\alpha(x) = 0$ for all $\alpha \in I$ imply $x = 0$. The close linear subspace spanned by a subset A of a Banach space shall be denoted by [A].

A family $(x_\alpha)_{\alpha\in I}$ in E is said to be a *Markuševič basis* (in short, M-basis) if there is a family $(f_\alpha)_{\alpha\in I}$ in E^* such that $(x_\alpha, f_\alpha)_{\alpha\in I}$ is a biorthogonal system which is both complete in E and total on E. In this case the family $(f_\alpha)_{\alpha\in I}$ is unique and is said to be the associated family of functionals (in short, a.f.f.) to the M-basis $(x_\alpha)_{\alpha\in I}$.

An M-basis $(x_\alpha)_{\alpha\in I}$ of E with the a.f.f. $(f_\alpha)_{\alpha\in I}$ in E^* is said to be bounded if

$$\sup_{\alpha\in I} \|x_\alpha\| < \infty \quad \text{and} \quad \sup_{\alpha\in I} \|f_\alpha\| < \infty\,.$$

We denote by V the norm closed linear span of $(f_\alpha)_{\alpha\in I}$ in E^*. The M-basis $(x_\alpha)_{\alpha\in I}$ is said to be *norming* if the norm given by

$$\|x\|_v = \sup\big\{|f(x)| : f \in V\,,\ \|f\| \le 1\big\}$$

is equivalent to the original norm in E.

An M-basis $(x_\alpha)_{\alpha\in I}$ of E with the a.f.f. $(f_\alpha)_{\alpha\in I}$ in E^* is said to be *strong* if each $x \in E$ is in the closed linear span of $(f_\alpha(x)\, x_\alpha)_{\alpha\in I}$. Every Schauder basis of E is a norming strong M-basis while the converse is not usually true [11].

Let μ be the first ordinal of cardinality dens E. A long sequence $(p_\alpha)_{\omega\le\alpha\le\mu}$ of linear projection on E is said to be a projectional resolution of the identity (in short, PRI) of E if it satisfies the following:

(1.1) $\|p_\alpha\| = 1$ $\qquad(\omega \le \alpha \le \mu)$,

(1.2) $p_\alpha \cdot p_\beta = p_\beta \cdot p_\alpha = p_\beta$ $\qquad(\omega \le \beta \le \alpha \le \mu)$,

(1.3) $\operatorname{dens} p_\alpha(E) \le \operatorname{card} \alpha$ $\qquad(\omega \le \alpha \le \mu)$,

(1.4) $p_\alpha(E) = \overline{\cup\{p_{\beta+1}(E) : \omega \le \beta < \alpha\}}$ $\qquad(\omega \le \alpha \le \mu)$,

(1.5) $p_\mu$is the identity operator on E.

If a Banach space E has a PRI $(p_\alpha)_{\omega\le\alpha\le\mu}$ then for each $x \in E$ the map $\alpha \longrightarrow p_\alpha(x)$ from $[\omega, \mu]$ into E is continuous and if $\beta \le \mu$ is a limit ordinal then $p_\beta(x) = \lim\limits_{\alpha<\beta} p_\alpha(x)$ for each $x \in E$.

3. Main results

Amir and Lindenstrauss [3] introduced the concept of a PRI and constructed a PRI in a weakly compactly generated Banach space. Văsák [17] improved this result to show that every weakly countably determined Banach space has a PRI. Further, Fabian and Godefroy [7] proved the existence of a PRI in the dual of an Asplund space. Since the above mentioned properties of Banach spaces are hereditary, complimented subspaces of weakly countably determined spaces and that of dual of Asplund spaces have PRI. More examples of Banach spaces with similar properties given by Valdivia [15, 16] shall be discussed later. Now, we prove our main result.

Theorem

Assume that every member E of a given class $\mathcal{P}$ of Banach spaces admits a PRI $(p_\alpha)_{\omega\le\alpha\le\mu}$ with $\operatorname{dens} E = \operatorname{card}\mu$ *such that $p_\alpha(E)$ (respectively, $(p_{\alpha+1}-p_\alpha)(E)$) belongs to $\mathcal{P}$ for each $\alpha\in[\omega,\mu)$. Then every E in $\mathcal{P}$ has a strong M-basis.*

Proof. We proceed by transfinite induction on dens E. If E is separable then by a result of Terenzi [14] E has a strong M-basis. Let E be non-separable, i.e. dens $E > \mathcal{N}_0$. We set up the first induction hypothesis that every Banach space F in the class $\mathcal{P}$ with dens $F <$ dens E has a strong M-basis. Let μ be the first ordinal with cardinality dens E. Since E is in the class $\mathcal{P}$ there is a long sequence $(p_\alpha)_{\omega\le\alpha\le\mu}$ of linear projections on E satisfying conditions (1.1) - (1.5) such that $p_\alpha(E)$ and $(p_{\alpha+1}-p_\alpha)(E)$ also have PRI for each $\omega\le\alpha<\mu$.

Let $E_\omega = p_\omega(E)$ and $E_{\alpha+1} = (p_{\alpha+1}-p_\alpha)(E)$ for $\omega\le\alpha<\mu$. Then, by (1.3) we have

$$\operatorname{dens} E_\omega = \operatorname{card}\omega = \mathcal{N}_0 < \operatorname{card}\mu = \operatorname{dens} E\,,$$
$$\operatorname{dens} E_{\alpha+1} \le \operatorname{card}(\alpha+1) < \operatorname{card}\mu = \operatorname{dens} E \qquad (\omega\le\alpha<\mu)\,.$$

Therefore, by the induction hypothesis E_ω has a strong M-basis $(x_i^\omega)_{i\in I_\omega}$ with the a.f.f. $(f_i^\omega)_{i\in I_\omega}$ in E_ω^* and $E_{\alpha+1}$ has a strong M-basis $(x_i^{\alpha+1})_{i\in I_{\alpha+1}}$ with the a.f.f. $(f_i^{\alpha+1})_{i\in I_{\alpha+1}}$ in $E_{\alpha+1}^*$ for each $\omega\le\alpha<\mu$, where card $I_\omega = \mathcal{N}_0$ and card $I_{\alpha+1} =$ dens $E_{\alpha+1}$ for $\omega\le\alpha<\mu$. Since p_ω is a projection on E_ω and $p_{\alpha+1}-p_\alpha$ is a projection on $E_{\alpha+1}$, by the definition, for each $\omega\le\alpha<\mu$ and $x\in E$, we have

$$p_\omega(x) \in \Big[\bigcup_{i\in I_\omega} (f_i^\omega\cdot p_\omega)(x)\, x_i^\omega\Big] \tag{1.6}$$

$$(p_{\alpha+1}-p_\alpha)(x) \in \Big[\bigcup_{i\in I_{\alpha+1}} \big(f_i^{\alpha+1}\cdot(p_{\alpha+1}-p_\alpha)\big)(x)\, x_i^{\alpha+1}\Big]\,. \tag{1.7}$$

Again, since $(x_i^\omega)_{i\in I_\omega}$ and $(x_i^{\alpha+1})_{i\in I_{\alpha+1}}$ for $\omega \le \alpha < \mu$ are M-bases, it follows by (1.1) that

$$\Big((x_i^\omega)_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (x_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\,,\big((g_i^\omega)_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (g_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\Big)$$

is a biorthogonal system in E, where $g_i^\omega = f_i^\omega \cdot p_\omega (i \in I_\omega)$ and $g_i^{\alpha+1} = f_i^{\alpha+1}(p_{\alpha+1} - p_\alpha)(i \in I_{\alpha+1}\,,\ \omega \le \alpha < \mu)$. We claim that for each $\omega \le \alpha \le \mu$ and $x \in E$

$$p_\alpha(x) \in \Big[\big(\bigcup_{i\in I_\omega} g_i^\omega(x)\, x_i^\omega\big) \cup \Big(\bigcup_{\omega\le\lambda<\alpha} \big(\bigcup_{i\in I_{\lambda+1}} g_i^{\lambda+1}(x)\,(x)_i^{\lambda+1}\big)\Big)\Big]\,. \tag{1.8}$$

We shall use transfinite induction again on α to prove (1.8). It is enough to discuss the case when α is a limit ordinal. If $\alpha = \omega$, then (1.8) coincides with (1.6) and thus holds. Let $\omega < \alpha \le \mu$. We form a second induction hypothesis that (1.8) holds for all ordinals β with $\omega < \beta < \alpha$. Since α is a limit ordinal $\lim\limits_{\beta<\alpha} p_\beta(x) = p_\alpha(x)$ $(x \in E)$; it follows that

$$p_\alpha(x) \in \Big[\bigcup_{\omega<\beta<\alpha} \Big\{\big(\bigcup_{i\in I_\omega} g_i^\omega(x)\, x_i^\omega\big) \cup \big(\bigcup_{\omega\le\lambda<\beta} \big(\bigcup_{i\in I_{\lambda+1}} g_i^{\lambda+1}(x)\, x_i^{\lambda+1}\big)\big)\Big\}\Big]\ (x \in E)\,,$$

whence

$$p_\alpha(x) \in \Big[\big(\bigcup_{i\in I_\omega} g_i^\omega(x)\, x_i^\omega\big) \cup \big(\bigcup_{\omega\le\lambda\le\alpha} \big(\bigcup_{i\in I_{\lambda+1}} g_i^{\lambda+1}(x)\, x_i^{\lambda+1}\big)\big)\Big] \qquad (x \in E)\,.$$

This completes the proof of (1.8) and hence that of the second process of transfinite induction. In particular for $\alpha = \mu$, it follows by (1.5) and (1.8) that

$$x \in \Big[\big(\bigcup_{i\in I_\omega} g_i^\omega(x)\, x_i^\omega\big) \cup \big(\bigcup_{\omega\le\alpha<\mu} \big(\bigcup_{i\in I_{\alpha+1}} g_i^{\alpha+1}(x)\, x_i^{\alpha+1}\big)\big)\Big] \qquad (x \in E)\,. \tag{1.9}$$

Now, it follows immediately from (1.9) that the extended biorthogonal system

$$\Big((x_i^\omega)_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (x_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\,,\ (g_i^\omega)_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (g_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\Big)$$

is both complete in E and total over E. Hence,

$$\Big((x_i^\omega)_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (x_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\Big)$$

is an extended M-bases of E with the a.f.f.

$$\Big((g_i^{\omega})_{i\in I_\omega} \cup \big(\bigcup_{\omega\le\alpha<\mu} (g_i^{\alpha+1})_{i\in I_{\alpha+1}}\big)\Big),$$

which again by (1.9) is an strong M-bases. This completes the first process of transfinite induction on dens E and the proof as well. □

Some more examples of Banach spaces satisfying the hypothesis of the theorem are in order. For a set I, we denote by $\Sigma^{(I)}$ the topological subspace of $\mathbb{R}^I$ formed by the points $\{x_i : i \in I\}$ such that $\{i : x_i \neq 0\}$ is countable. A topological space X is a said to belong to the class Σ if there is a set I and a continuous one-to-one map from X into $\Sigma^{(I)}$. Further, a compact set K is said to be Valdivia compact if there is a set I and a subset K_0 of the compact space $[0,1]^I$ such that K is homeomorphic to K_0 and the subset of K_0 consisting of points $\{x_i : i \in I\}$ such that $\{i : x_i \neq 0\}$ is countable, is dense in K_0. Valdivia [15] has shown that a Banach space whose dual equipped with the weak* topology belongs to the class Σ, has a PRI and that the above mentioned property is also complimentably hereditary. It follows that such Banach spaces satisfy the hypothesis of the theorem. It has further been shown in [15, Corollary 2] that for a Banach space E with an M-basis, the dual E^* belongs to the class Σ if the dual unit ball is weak* angelic. Finally, it was observed in [16] that if K is a Valdivia compact set then $\mathcal{C}(K)$ satisfies the hypothesis of the theorem (also see [5, Corollary 7.1.10]).

The following corollary adds some new classes of Banach spaces to the list of those which are already known to have strong M-bases.

Corollary 1

Each of the following Banach spaces has a strong M-basis.

(a) A weakly countable determined Banach space.
(b) The dual of an Asplund space.
(c) A Banach space E such that the dual E^* with the weak topology belongs to the class Σ.*
(d) A Banach space with an M-basis whose dual unit ball is weak angelic.*
(e) A $\mathcal{C}(K)$ space where K is a Valdivia compact set.

Another immediate consequence of the theorem which follows from a result of Alexandrov and Plichko [1] is the following, which is already proved in [5, Theorem 7.1.8].

Corollary 2

Assume that every member E of a given class $\mathcal{P}$ of a Banach space admits a PRI $(p_\alpha)_{\omega\le\alpha\le\mu}$ with dens E = card μ *such that $p_\alpha(E)$ (respectively, $(p_{\alpha+1}-p_\alpha)(E)$) belongs to $\mathcal{P}$ for each $\alpha \in [\omega,\mu)$. Then every E in $\mathcal{P}$ has an equivalent locally uniformly rotund norm.*

Let us note that if Γ is any set with card $\Gamma >$ dens ℓ_∞, then $\ell_2(\Gamma) \oplus_2 \ell_\infty$ has a PRI, whereas it has no equivalent locally uniformly rotund norm (see [5, Remark 7.1.9]) and hence no strong M-basis (see [1, Theorem 1]). Thus, our theorem can not be proved for every Banach space with a PRI. Very recently, Alexandrov and Plichko [2] have shown that the space $\mathcal{C}[0,\omega_1]$ of continuous functionals on the ordinal interval $[0,\omega_1]$ does not have any norming M-basis. However, $\mathcal{C}[0,\omega_1]$ has a PRI and it satisfies the hypothesis of the theorem. It follows that though every Banach space satisfying the hypothesis in the theorem has a strong M-basis, it may not have a norming strong M-basis. Thus our theorem can not be improved to obtain Terenzi's result in its full strength in non separable Banach spaces satisfying our hypothesis.

Finally, the author is thankful to the referee for some valuable suggestions towards the improvement of the paper.

References

1. G.A. Alexandrov and A.N. Plichko, On the connection between strong *M*-bases and equivalent locally uniformly convex norms in Banach spaces, *C.R. Acad. Bulgare Sci.* **40** (1987), 15–16.
2. G.A. Alexandrov and A.N. Plichko, Connection between strong and norming Markushevich bases, preprint.
3. D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.* (2) **88** (1968), 35–46.
4. W.J. Davis and I. Singer, Boundedly complete *M*-bases and complemented subspaces in Banach spaces, *Trans. Amer. Math. Soc.* **175** (1973), 187–194.
5. R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces,* Longman Scientific and Technical, Harlow, 1993.
6. P. Enflo, A counterexample to the approximation problem in Banach spaces, *Acta Math.* **130** (1973), 309–317.
7. M. Fabian and G. Godefroy, The dual of every Asplund space admits a projectional resolution of the identity, *Studia Math.* **91** (1988), 141–151.
8. J. Reif, A note on Markuševič bases in weakly compactly generated Banach spaces, *Comment. Math. Univ. Carolin.* **15** (1974), 335–340.
9. J. Reif, Some remarks on subspaces of weakly compactly generated Banach spaces, *Comment. Math. Univ. Carolin.* **16** (1975), 787–793.
10. W.H. Ruckle, Representation and series summability of complete biorthogonal sequences, *Pacific J. Math.* **34** (1970), 511–528.
11. I. Singer, *Bases in Banach Spaces II,* Springer-Verlag, Berlin, 1981.
12. D.P. Sinha, Strong *M*-decompositions in separable Banach spaces, *J. Math. Anal. Appl.* **155** (1991), 338–344.

13. D.P. Sinha and K.K. Arora, On the Gelfand-Phillips property in Banach spaces with PRI, *Collect. Math.* **48** (1997), 347–354.
14. P. Terenzi, Every separable space has a bounded strong norming biorthogonal sequence, which is also a stenitz basis, *Studia Math.* **111** (1994), 207–222.
15. M. Valdivia, Resolutions of identity in certain Banach spaces, *Collect. Math.* **39** (1988), 127–140.
16. M. Valdivia, Projective resolution of identity in $\mathcal{C}(K)$ spaces, *Arch. Math.* **54** (1990), 493–498.
17. L. Vašák, On one generalization of weakly compactly generated Banach spaces, *Studia Math.* **70** (1981) 11–19.