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An inverse problem for a general annular drum with positive smooth functions in the Robin boundary conditions

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Abstract

The asymptotic expansion of the trace of the heat kernel $\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j)$ as $t \to 0^+$ has been derived for a variety of domains, where $\{\lambda_j\}$ are the eigenvalues of the negative Laplace operator $-\Delta = -\sum_{i=1}^{2}(\frac{\partial}{\partial x^i})^2$ in the (x^1, x^2)-plane. The dependence of $\Theta(t)$ on the connectivity of domains and the boundary conditions is analyzed. Particular attention is given for a general annular drum in $\mathbb{R}^2$ together with Robin boundary conditions, where the coefficients in the boundary conditions are positive smooth functions. Some applications of an ideal gas enclosed in the general annular drum are given.

1. Introduction

Let Ω be a given arbitrary simply connected bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Consider the Robin problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega\,, \qquad \Big(\frac{\partial}{\partial n} + \gamma\Big) u = 0 \quad \text{on } \partial\Omega\,, \tag{1.1}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$, where the coefficient γ is a positive smooth function and $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Denote its eigenvalues counted according to multiplicity, by

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_j \le \ldots \to \infty \qquad \text{as } j \to \infty\,. \tag{1.2}$$

The basic problem is that of determining some geometric quantities associated with the bounded domain Ω from complete knowledge of the eigenvalues (1.2) using the asymptotic expansion of the trace of the heat kernel

$$\Theta(t)=\sum_{j=1}^{\infty}\exp(-t\lambda_j) \quad \text{as} \quad t\to 0^+. \tag{1.3}$$

The Robin problem (1.1) has been investigated by many authors (see for example, Pleijel [14], Kac [9], McKean and Singer [10], Stewartson and Waechter [19], Gottlieb [5] and Hsu [8]) in the following special cases:

(i) case 1. $\gamma=0$ (the Neumann problem)

$$\Theta(t)=\frac{|\Omega|}{4\pi t}+\frac{|\partial\Omega|}{8(\pi t)^{1/2}}+a_0+\frac{7}{256}\Big(\frac{t}{\pi}\Big)^{1/2}\int\limits_{\partial\Omega}K^2(z)\,dz+O(t) \quad \text{as}\, t\to 0^+, \tag{1.4}$$

(ii) case 2. $\gamma\to\infty$ (the Dirichlet problem)

$$\Theta(t)=\frac{|\Omega|}{4\pi t}-\frac{|\partial\Omega|}{8(\pi t)^{1/2}}+a_0+\frac{1}{256}\Big(\frac{t}{\pi}\Big)^{1/2}\int\limits_{\partial\Omega}K^2(z)\,dz+O(t) \quad \text{as}\, t\to 0^+. \tag{1.5}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the total length of $\partial\Omega$ and $K(z)$ is the curvature of $\partial\Omega$, where z is the arc length of the counterclockwise oriented boundary $\partial\Omega$. The constant term a_0 has geometric significance, e.g., if Ω is smooth and convex, then $a_0=\frac{1}{6}$ and if Ω is permitted to have a finite number of smooth convex holes "H", then $a_0=(1-H)/6$.

We merely note that aspects of the question of Kac [9], namely, can one hear the shape of a drum? have been discussed by Sleeman and Zayed [15] for the Robin problem (1.1) when γ is a positive constant, and by Zayed [25] when γ is a positive smooth function. Further, the Robin problem (1.1) has been investigated by Hsu [8] in the general situation where Ω is a compact Riemannian manifold of n-dimensions with smooth boundary, and has determined the first four terms of $\Theta(t)$ as $t\to 0^+$.

Thus, in the mathematical terms, Kac's question becomes: Does the boundary condition define the spectrum uniquely? The proof of Gordon et al [6] uses drums made by piecing together with few (identical) basic shapes, for examples, triangles. Gordon et al [6] proved that different shaped drums can posses identical spectra, that is they are "isospectral". The examples of Hajima Urakawa [20] and peter Buser [4] also show that one can not always hear the shape of a domain in $\mathbb{R}^n$, for $n\geq 3$. Also, Milnor [11] has constructed two non-congruent 16−dimensional tori whose Laplace Beltrami operators have precisely the same eigenvalues.

In the present paper we are not concerned with the discussions of the non-uniqueness of the inverse problems, but we are concerned with the determination of some geometric quantities of some general bounded domains from complete knowledge of its eigenvalues.

The object of this paper is to discuss the following inverse problem: Suppose that Ω is a general annular drum in $\mathbb{R}^2$ consisting of a simply connected bounded inner domain Ω_1 with a smooth boundary $\partial\Omega_1$ and a simply connected bounded outer domain $\Omega_2 \supset \bar{\Omega}_1$ with a smooth boundary $\partial\Omega_2$. Suppose that the eigenvalues (1.2) are given for the Helmholtz equation

$$-\Delta u = \lambda u \qquad \text{in } \ \Omega \tag{1.6}$$

together with the Robin boundary conditions

$$\begin{aligned} \Big(\frac{\partial}{\partial n_1} + \gamma_1\Big)\, u = 0 \quad &\text{on } \ \partial\Omega_1, \\ \Big(\frac{\partial}{\partial n_2} + \gamma_2\Big)\, u = 0 \quad &\text{on } \ \partial\Omega_2, \end{aligned} \tag{1.7}$$

where $\frac{\partial}{\partial n_i}$ $(i = 1, 2)$ denote differentiations along the inward pointing normals to the boundaries $\partial\Omega_i$ $(i = 1, 2)$ respectively, while the coefficients γ_i $(i = 1, 2)$ are positive smooth functions defined on $\partial\Omega_i$ $(i = 1, 2)$ respectively.

The basic problem is to determine some geometric quantities associated with the general annular drum Ω from complete knowledge of its eigenvalues (1.2) using the asymptotic expansion of $\Theta(t)$ for small positive t.

Note that the problem (1.6)-(1.7) has been discussed by Zayed [21, 24] in the case where γ_i $(i = 1, 2)$ are positive constants.

2. Statement of results

Before we state our main results, we remind the reader that x and y are two-dimensional variables, while dx, dy (or dy_i and dz_i, $i = 1, 2$) denote the two (or one)-dimensional Euclidean area elements, as indicated in Section 2 and 3.

Theorem

Suppose that the boundaries $\partial\Omega_i$ $(i = 1, 2)$ of the general annular drum Ω are given locally by the equations $x^n = y^n(z_i)$ $(n = 1, 2)$ in which z_i $(i = 1, 2)$ are the arc lengths of the counter clockwise-oriented boundaries $\partial\Omega_i$ $(i = 1, 2)$ and $y^n(z_i) \in C^\infty(\partial\Omega_i)$. Let L_i be the lengths of $\partial\Omega_i$ $(i = 1, 2)$ and let $K_i(z_i)$ $(i = 1, 2)$ be their curvatures respectively. Then, the result of our main problem (1.6)-(1.7) can be summarized in the following form:

$$\Theta(t) = \frac{a_1}{t} + \frac{a_2}{t^{1/2}} + a_3 + a_4 t^{1/2} + O(t) \quad \text{as } t \to 0, \tag{2.1}$$

where

$$a_1 = \frac{|\Omega|}{4\pi},$$

$$a_2 = \Big(\sum_{i=1}^{2} L_i\Big) \,/\, 8\pi^{1/2},$$

$$a_3 = \frac{1}{2\pi}\Big[\int\limits_{\partial\Omega_1} \gamma_1(z_1)\, dz_1 - \int\limits_{\partial\Omega_2} \gamma_2(z_2)\, dz_2\Big],$$

$$a_4 = \frac{7}{256\,\pi^{1/2}} \sum_{i=1}^{2} \int\limits_{\partial\Omega_i} \Big[K_i^2(z_i) - \frac{32}{7}\left\{\gamma_i(z_i)K_i(z_i) - 2\gamma_i^2(z_i)\right\}\Big]\, dz_i.$$

From this theorem, we note that the formula (2.1) is in agreement with Zayed's result [21, 24] when γ_i $(i = 1, 2)$ are positive constants.

With reference to the formula (1.4) and Zayed [24], the asymptotic expansion (2.1) may be interpreted as follows:

(i) Ω is a general annular drum in $\mathbb{R}^2$ and we have the Robin boundary conditions (1.7) where γ_i $(i = 1, 2)$ are positive smooth functions.

(ii) For the first four terms, Ω is a general annular drum in $\mathbb{R}^2$ with $H = 1 - \frac{3}{\pi}\,[\int_{\partial\Omega_1} \gamma_1(z_1)\, dz_1 - \int_{\partial\Omega_2} \gamma_2(z_2)\, dz_2]$ holes, and has area $|\Omega|$ and the inner component of the boundary has length L_1 and curvature $[\,K_1^2(z_1) - \frac{32}{7}\{\gamma_1(z_1)K_1(z_1) - 2\gamma_1^2(z_1)\}]^{1/2}$ together with the Neumann boundary condition, while the outer component of the boundary has length L_2 and curvature $[\,K_2^2(z_2) - \frac{32}{7}\{\gamma_2(z_2)K_2(z_2) - 2\gamma_2^2(z_2)\}]^{1/2}$ together with the Neumann boundary condition, provided "H" is a positive integer.

We close this section with the following question: Can one construct two non-congruent general annular drums (with Dirichlet or Neumann or Robin boundary conditions) whose Laplace operators have precisely the same eigenvalues? This is an open problem, which has been left for the interested readers.

3. Construction of the results

With reference to Kac [9], Hsu [8] and Zayed [25], it is easily seen that $\Theta(t)$ associated with the main problem (1.6)-(1.7) is given by

$$\Theta(t) = \iint_{\Omega} G_{\gamma}(t, x, x)\, dx \tag{3.1}$$

where the heat kernel $G_{\gamma}(t, x, y)$ is defined on $(0, \infty) \times \bar{\Omega} \times \bar{\Omega}$, which satisfies the following: For fixed $x \in \bar{\Omega}$, and γ is a function of γ_i $(i = 1, 2)$, $G_{\gamma}(t, x, y)$ satisfies the heat equation in t, y:

$$\left(\frac{\partial}{\partial t} - \Delta_y\right) G_{\gamma}(t, x, y) = 0, \tag{3.2}$$

and the Robin boundary conditions

$$\left[\frac{\partial}{\partial n_{iy}} + \gamma_i(y)\right] G_{\gamma}(t, x, y) = 0 \quad \text{on } \partial\Omega_i \ (i = 1, 2), \tag{3.3}$$

and the initial condition

$$\lim_{t \to 0^+} G_{\gamma}(t, x, y) = \delta(x - y), \tag{3.4}$$

where $\delta(x - y)$ is the Dirac delta function located at the source point $x = y$.

Note that in (3.2), (3.3) the subscript "y" means that the derivatives are taken in y-variable. By the super position principle of the heat equation, we write

$$G_{\gamma}(t, x, y) = G_N(t, x, y) + \kappa_{\gamma}(t, x, y), \tag{3.5}$$

where $G_N(t, x, y)$ is the Neumann heat kernel on Ω which satisfies (3.2) and the Neumann boundary conditions $\frac{\partial}{\partial n_{iy}} G_N(t, x, y) = 0$ on $\partial\Omega_i$ $(i = 1, 2)$ and the initial condition (3.4), while $\kappa_{\gamma}(t, x, y)$ satisfies (3.2) and the boundary conditions

$$\frac{\partial}{\partial n_{iy}} \kappa_{\gamma}(t, x, y) = -\gamma_i(y)\, G_{\gamma}(t, x, y) \quad \text{on } \partial\Omega_i \ (i = 1, 2), \tag{3.6}$$

and the initial condition

$$\lim_{t \to 0^+} \kappa_{\gamma}(t, x, y) = 0\,. \tag{3.7}$$

Now, the solution $\kappa_\gamma(t,\, x,\, y)$ which satisfies (3.2), (3.6) and (3.7), (see for example [8], [17, 18], [25]) can be written in the form

$$\kappa_\gamma(t,\, x,\, y) = \int_0^t ds \int_{\partial\Omega_1} G_N(t-s,\, x,\, z_1)\, \gamma_1(z_1)\, G_\gamma(s,\, z_1,\, y)\, dz_1 - \int_0^t ds \int_{\partial\Omega_2} G_N(t-s,\, x,\, z_2)\, \gamma_2(z_2)\, G_\gamma(s,\, z_2,\, y)\, dz_2. \tag{3.8}$$

From (3.5) and (3.8) we have the following integral equation:

$$G_\gamma(t,\, x,\, y) = G_N(t,\, x, y) + \int_0^t ds \int_{\partial\Omega_1} G_N(t-s,\, x,\, z_1)\gamma_1(z_1) G_\gamma(s,\, z_1,\, y) dz_1 - \int_0^t ds \int_{\partial\Omega_2} G_N(t-s,\, x,\, z_2)\, \gamma_2(z_2)\, G_\gamma(s,\, z_2,\, y)\, dz_2. \tag{3.9}$$

On applying the iteration method (see for example [1, 2], [12], [22, 23]) to the integral equation (3.9) we obtain an infinite convergent series

$$G_\gamma(t,\, x,\, y) = \sum_{m=0}^{\infty} (-1)^m F_m(t,\, x,\, y), \tag{3.10}$$

where

$$F_0(t,\, x,\, y) = G_N(t,\, x,\, y) \tag{3.11}$$

and

$$F_m(t,\, x,\, y) = \int_0^t ds \int_{\partial\Omega_2} G_N(t-s,\, x,\, z_2)\, \gamma_2(z_2)\, F_{m-1}(s,\, z_2,\, y)\, dz_2 - \int_0^t ds \int_{\partial\Omega_1} G_N(t-s,\, x,\, z_1)\, \gamma_1(z_1)\, F_{m-1}(s,\, z_1,\, y)\, dz_1 \tag{3.12}$$

where $m = 1,\, 2,\, 3, \ldots$

We will often use the following simple estimate for the Neumann heat kernel (see for example [1, 2], [3], [8]): There exists positive constants t_0, c_1 such that for all $t < t_0$, $x,\, y \in \bar{\Omega} \times \bar{\Omega}$, we get

$$G_N(t,\, x, y) \le c_1 t^{-1} \exp\Big\{ - \frac{|x-y|^2}{c_1 t} \Big\}. \tag{3.13}$$

Lemma 3.1

We have that

$$\sum_{m=3}^{\infty} \iint_{\Omega} |F_m(t,\, x,\, x)|\, dx = O(t) \qquad \text{as } \ t \to 0^+. \tag{3.14}$$

Proof. With reference to the article [10], we use the convolution property of the Gaussian kernel to verify by induction that $F_m(t,\, x,\, y)$ has an estimate of the form

$$|F_m(t,\, x,\, y)| \leq c_2\, c_3^m \Big[\Gamma\big(\frac{m+1}{2}\big)\Big]^{-1} t^{(m-2)/2} \, \exp\Big\{ -\frac{|x-y|^2}{c_1 t}\Big\}\,, \tag{3.15}$$

where c_2 and c_3 are positive constants and Γ is the gamma function. Summing over m and using (3.10) we see that there exists positive constant $c_4,\ t_0$ such that for all $t < t_0$ and $x,\, y \in \bar{\Omega} \times \bar{\Omega}$, we get

$$G_{\gamma}(t,\, x, y) \leq c_4 t^{-1} \exp\Big\{ -\frac{|x-y|^2}{c_1 t}\Big\}. \tag{3.16}$$

Let $g_{\gamma m} = (-1)^m F_m(t,\, x,\, y),\ g_{\gamma_1 m} = (-1)^{m+1} F_{\gamma_1 m}(t,\, x,\, y)\,,\ g_{\gamma_2 m} = (-1)^m F_{\gamma_2 m}(t,\, x,\, y)$ with the boundary functions $\gamma_i = -\|\gamma_i\|_{\infty}$ (i =1,2) where $F_m = F_{\gamma_2 m} - F_{\gamma_1 m}$. Then by the recursive relation (3.12) of $g_{\gamma m}(t,\, x,\, y) = g_{\gamma_2 m}(t,\, x,\, y) + g_{\gamma_1 m}(t,\, x,\, y)$ we get

$$\begin{aligned}
\sum_{m=2}^{\infty} g_{\gamma m}(t,\, x,\, y) \leq \|\gamma_2\|_{\infty}^2 & \int_0^t ds_1 \int_{\partial\Omega_2} G_N(t-s_1,\, x,\, z_2')\, dz_2' \\
& \times \int_0^{s_1} ds_2 \int_{\partial\Omega_2} G_N\big(s_1-s_2,\, z_2',\, z_2''\big) \sum_{m=0}^{\infty} g_{\gamma_2 m}\big(s_2,\, z_2'',\, y\big)\, dz_2'' \\
& + \|\gamma_1\|_{\infty}^2 \int_0^t ds_1 \int_{\partial\Omega_1} G_N\big(t-s_1,\, x,\, z_1'\big)\, dz_1' \\
& \times \int_0^{s_1} ds_2 \int_{\partial\Omega_1} G_N\big(s_1-s_2,\, z_1',\, z_1''\big) \sum_{m=0}^{\infty} g_{\gamma_1 m}\big(s_2,\, z_1'',\, y\big)\, dz_1''.
\end{aligned} \tag{3.17}$$

But it is clear from (3.10) and (3.12) that

$$\sum_{m=0}^{\infty} g_{\gamma_i\, m}(s,\, z,\, y)\ \leq G_{-\|\gamma_i\|_{\infty}}(s,\, z,\, y), \qquad (i = 1, 2).$$

Hence the right-hand side of (3.17) can be estimated by the Gaussian type upper bounds (3.13) and (3.16) of the heat kernels $G_N(t,\, x,\, y)$ and $G_{-\|\gamma_i\|_\infty}(t,\, x,\, y),\ (i = 1,2)$ and we get

$$\sum_{m=2}^{\infty} g_{\gamma m}(t,\, x, y) \leq c_5 \exp\Big\{ -\frac{|x-y|^2}{c_6 t} \Big\}. \tag{3.18}$$

where c_5 and c_6 are positive constants. From the recursive relation of $g_{\gamma m}$ again, we have

$$\begin{aligned}
\iint\limits_{\Omega} g_{\gamma m}(t,\, x,\, x)\, dx &= \|\gamma_2\|_\infty^2 \int_0^t (t-u)\, du \int\limits_{\partial\Omega_2} dy_2 \\
&\quad \times \int\limits_{\partial\Omega_2} G_N(t-u,\, y_2,\, z_2) g_{(m-2)\gamma_2}(u,\, z_2,\, y_2)\, dz_2 \\
&\quad + \|\gamma_1\|_\infty^2 \int_0^t (t-u)\, du \int\limits_{\partial\Omega_1} dy_1 \\
&\quad \times \int\limits_{\partial\Omega_1} G_N(t-u,\, y_1,\, z_1) g_{(m-2)\gamma_1}(u,\, z_1,\, y_1)\, dz_1 \\
&\leq \|\gamma_2\|_\infty^2\ t \int_0^t du \int\limits_{\partial\Omega_2} dy_2 \\
&\quad \times \int\limits_{\partial\Omega_2} G_N(t-u,\, y_2,\, z_2) g_{(m-1)\gamma_2}(u,\, z_2,\, y_2)\, dz_2 \\
&\quad + \|\gamma_1\|_\infty^2\ t \int_0^t du \int\limits_{\partial\Omega_1} dy_1 \\
&\quad \times \int\limits_{\partial\Omega_1} G_N(t-u,\, y_1,\, z_1) g_{(m-1)\gamma_1}(u,\, z_1,\, y_1)\, dz_1 \\
&= \|\gamma_2\|_\infty\ t \int\limits_{\partial\Omega_2} g_{(m-1)\gamma_2}(t, y_2,\, y_2)\, dy_2 \\
&\quad + \|\gamma_1\|_\infty\ t \int\limits_{\partial\Omega_1} g_{(m-1)\gamma_1}(t, y_1,\, y_1)\, dy_1.
\end{aligned} \tag{3.19}$$

Summing over m from 3 to infinity and using (3.18) we obtain (3.14) from the inequality $|F_m| \leq g_{\gamma m}$. This proves Lemma 3.1.

From (3.1), (3.10), (3.11) and (3.14) we deduce for $t \to 0^+$ that

$$\Theta(t) = \Theta_N(t) - \iint_\Omega F_1(t,\, x,\, x)\, dx + \iint_\Omega F_2(t,\, x,\, x)\, dx + O(t), \tag{3.20}$$

where

$$\Theta_N(t) = \iint_\Omega G_N(t,\, x,\, x)\, dx\,. \tag{3.21}$$

On the other hand, the asymptotic expansion of $\Theta_N(t)$ as $t \to 0^+$ for the Neumann problems ($\gamma_1 = \gamma_2 = 0$) is well known (see, for example [5], [23], [24]) and is given by

$$\Theta_N(t) = \frac{|\Omega|}{4\pi t} + \frac{(L_1 + L_2)}{8(\pi t)^{1/2}} + \frac{7}{256}\left(\frac{t}{\pi}\right)^{1/2} \sum_{i=1}^{2} \int_{\partial\Omega_i} K_i^2(z_i)\, dz_i + O(t) \quad \text{as } \ t \to 0, \tag{3.22}$$

where the constant term a_0 disappears in (3.22) because the general annular drum Ω has only one hole (i.e. $H = 1$). $\square$

The problem now is to calculate the integrals of $F_i(t, x, x)$, $(i = 1, 2)$ over the general annular drum Ω as follows:

Lemma 3.2

If Ω is a general annular drum in $\mathbb{R}^2$, then we have

$$\iint_\Omega F_1(t, x, x)\, dx = \frac{1}{2\pi}\left[\int_{\partial\Omega_2} \gamma_2(z_2)\, dz_2 - \int_{\partial\Omega_1} \gamma_1(z_1)\, dz_1\right] + \frac{1}{8\pi}\left(\frac{t}{\pi}\right)^{1/2} \sum_{i=1}^{2} \int_{\partial\Omega_i} \gamma_i(z_i)\, K_i(z_i)\, dz_i + O(t), \quad \text{as } t \to 0^+\,. \tag{3.23}$$

Proof. The definition of $F_1(t, x, x)$ and the Chapman-Kolmogorov equation of the heat kernel (see [8], [13]) imply

$$\iint_\Omega F_1(t, x, x)\, dx = t\left\{\int_{\partial\Omega_2} G_N(t, z_2, z_2)\gamma_2(z_2)\, dz_2 - \int_{\partial\Omega_1} G_N(t, z_1, z_1)\gamma_1(z_1)\, dz_1\right\}. \tag{3.24}$$

Hsu [8] has shown that the Neumann heat kernel $G_N(t, z_2, z_2)$ when $z_2 \in \partial\Omega_2$ is given by the following asymptotic formula:

$$G_N(t, z_2, z_2) = \frac{1}{2\pi t}\Big[1 + \frac{1}{4}(\pi t)^{1/2} K_2(z_2)\Big] + O(1) \quad \text{as } t \to 0^+. \tag{3.25}$$

Using methods similar to those obtained by Hsu, we deduce that the Neumann heat kernel $G_N(t, z_1, z_1)$ when $z_1 \in \partial\Omega_1$ is given by:

$$G_N(t, z_1, z_1) = \frac{1}{2\pi t}\Big[1 - \frac{1}{4}(\pi t)^{1/2} K_1(z_1)\Big] + O(1) \quad \text{as } t \to 0^+. \tag{3.26}$$

On inserting (3.25) and (3.26) into (3.24) we arrive at the proof of the Lemma 3.2. □

Lemma 3.3

If Ω is a general annular drum in $\mathbb{R}^2$, then we have

$$\iint\limits_{\Omega} F_2(t, x, x)\, dx = \frac{1}{4}\Big(\frac{t}{\pi}\Big)^{1/2} \sum_{i=1}^{2} \int\limits_{\partial\Omega_i} \gamma_i^2(z_i)\, dz_i + O(t) \quad \text{as } t \to 0^+. \tag{3.27}$$

Proof. From the definition of $F_2(t, x, x)$ and with the help of the expression of $F_1(t, x, x)$, we deduce that

$$\begin{aligned} \iint\limits_{\Omega} F_2(t, x, x)\, dx = \sum_{i=1}^{2} \int_0^t (t-u)\, du \int\limits_{\partial\Omega_i} \gamma_i^2(z_i)\, dz_i \\ \times \int\limits_{\partial\Omega_i} G_N(t-u, z_i, y_i)\, \gamma_i(y_i)\, G_N(u, y_i, z_i)\, dy_i\,. \end{aligned} \tag{3.28}$$

We replace $\gamma_i(y_i)$ in the above integral by $\gamma_i(z_i) + O(|z_i - y_i|)$ and split the integral into two integrals accordingly. On using the estimate (3.13) we deduce that

$$\begin{aligned} &\int\limits_{\partial\Omega_i} |z_i - y_i|\, G_N(t-u, y_i, z_i)\, G_N(u, z_i, y_i)\, dy_i \\ &\leq\; c_1[u(t-u)]^{-1} \int\limits_{R^1} |y_i| \exp\Big\{ -\frac{c_2 t\, |y_i|^2}{u(t-u)} \Big\}\, dy_i\,. \end{aligned} \tag{3.29}$$

Since the integral in the right-hand side of (3.29) is bounded by $c_7 t^{-1}$ where c_7 is a positive constant, we deduce as $t \to 0^+$ that

$$\iint\limits_{\Omega} F_2(t, x, x)\, dx = \sum_{i=1}^{2} \int\limits_{\partial\Omega_i} \gamma_i^2(z_i)\, g(t, z_i)\, dz_i + O(t), \tag{3.30}$$

where

$$g(t, z_i) = \int\limits_0^t (t-u)\, du \int\limits_{\partial\Omega_i} G_N(t-u,\, y_i,\, z_i)\, G_N(u, z_i, y_i)\, dy_i. \tag{3.31}$$

The right-hand side of (3.31) can be computed by taking the first term in the series expansion of the Neumann heat kernel

$$G_N(t-u,\, y_i,\, z_i) = 2q(t-u,\, y_i,\, z_i),\;\; G_N(u,\; z_i,\, y_i) = 2q(u,\; z_i,\, y_i)\,,$$

where

$$q(t,\, y_i,\, z_i) = (4\pi t)^{-1} \exp\Big\{ -\frac{|y_i - z_i|^2}{4t} \Big\}.$$

The explicit computation can be carried out with the help of a suitably chosen local coordinates system and the localization principle (see [8]). We leave the details of this computation to the interested reader and we content ourselves with the statement that the leading term $g(t, z_i)$ is equal to the same integral in the Euclidean plane, i.e.,

$$g(t, z_i) = \frac{1}{4\pi^2} \int\limits_0^t \frac{du}{u} \int\limits_{R^1} \exp\Big\{ -\frac{|z_i - y_i|^2}{4(t-u)} - \frac{|z_i - y_i|^2}{4u} \Big\}\, dy_i + O(t). \tag{3.32}$$

After some calculations, we deduced that

$$\begin{aligned} g(t, z_i) &= \frac{1}{2t^{1/2}\pi^{3/2}} \int\limits_0^t \Big(\frac{t-u}{u}\Big)^{1/2} du + O(t) \\ &= \frac{1}{4}\Big(\frac{t}{\pi}\Big)^{1/2} + O(t). \end{aligned} \tag{3.33}$$

On inserting (3.33) into (3.30) we arrive at the proof of Lemma 2. Now, our main result (2.1) follows immediately from the formulae (3.20)-(3.22) and Lemmas 3.2 and 3.3. □

4. Some applications of the inverse problem for an ideal gas

Following Gutierrez and Yanez [7], we are interested in examining how the thermodynamic properties of an ideal gas are influenced by the geometry of its container. Thermodynamic properties of an ideal gas can be extracted from the partition function

$$Z(\beta) = \frac{z^N(\beta)}{N!}, \tag{4.1}$$

where N is the number of particles and $z(\beta)$ is given by

$$z(\beta) = \sum_j \exp(-\beta E_j), \tag{4.2}$$

where $\beta = (k_B T)^{-1}$, k_B is Boltzmann constant, and T is the absolute temperature. The energy level (eigenvalues) E_j of one particle are obtained from the stationary states $\Psi = u \exp\{-iEt/\hbar\}$ of the time-dependent Schrodinger equation

$$\frac{\hbar^2}{2M}\Delta\Psi + V(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{4.3}$$

with $V(x) = 0$, where M is the mass and $\hbar$ is the Planck constant. Thus, u satisfies the Helmholtz equation $-\Delta u = \lambda u$, with $\lambda = 2ME/\hbar^2$. Therefore, we deduce that the asymptotic expansion of the sum (1.3) of $\Theta(t)$ as $t \to 0^+$ which formally is the same as the one-particle function (4.2) of $z(\beta)$ as $\beta \to 0^+$. The purpose of this section is to use our main result (2.1) to derive a general expression for the corrections to the thermodynamic quantities, particularly, the energy of an ideal gas enclosed in the general annular drum Ω in $\mathbb{R}^2$.

Following the discussions of Section 2, we can obtain information about the shape of the general annular drum Ω, by studying the asymptotic expansion of the sum (4.2) when $\beta \to 0$ (i.e., $T \to \infty$, the ideal gas case). Noting that the eigenvalue problem of the Schrodinger equation is the same as the eigenvalue problem of the wave equation, we can use directly the asymptotic expansion (2.1) of $\Theta(t)$ replacing t by $(\frac{\hbar^2}{2M})\beta$.

Let us now consider the general partition function (4.1) in two-dimensions. On using the formula (2.1) for the Robin problem, equation (4.2) gives:

$$z(\beta) = \Big(\frac{2M}{\hbar^2}\Big)\frac{a_1}{\beta} + \Big(\frac{2M}{\hbar^2}\Big)^{1/2}\frac{a_2}{\beta^{1/2}} + a_3 + \Big(\frac{\hbar^2}{2M}\Big)^{1/2} a_4\beta^{1/2} + O(\beta) \quad \text{as} \quad \beta \to 0^+. \tag{4.4}$$

We set out to apply the formula (4.4) to the thermodynamic quantities such that the internal energy $U = -[\frac{\partial}{\partial\beta}\ln Z(\beta)]_{V,\,N}$, the pressure $P = \beta^{-1}[\frac{\partial}{\partial V}\ln Z(\beta)]_{T,\,N}$ and the specific heat $C = \left(\frac{\partial U}{\partial T}\right)_{N,\,V}$, among others (see [7]).

Thus, in the case of the internal energy, we get

$$U = -N\frac{\partial}{\partial\beta}\ln\left\{\left(\frac{2M}{\hbar^2}\right)\frac{a_1}{\beta} + \left(\frac{2M}{\hbar^2}\right)^{1/2}\frac{a_2}{\beta^{1/2}} + a_3 + \left(\frac{\hbar^2}{2M}\right)^{1/2} a_4\beta^{1/2} + O(\beta)\right\}.$$

Now, differentiating, expanding in powers of $\beta = (k_B T)^{-1}$ and using the definition of the thermal wave length $\Lambda(T) = (\frac{2\pi\hbar^2}{Mk_BT})^{1/2}$ we deduce, after some reduction, that the internal energy $U(T)$ has the asymptotic form:

$$\begin{aligned} U(T) = Nk_BT\Bigg\{1 &- \Big[\frac{a_2}{4a_1\sqrt{\pi}}\Big]\Lambda(T) + \frac{1}{8\pi}\Big[\Big(\frac{a_2}{a_1}\Big)^2 - \frac{2a_3}{a_1}\Big]\Lambda^2(T) \\ &- \frac{1}{16\pi^{3/2}}\Big[\Big(\frac{a_2}{a_1}\Big)^3 - \frac{3a_2a_3}{a_1^2} + \frac{3a_4}{a_1}\Big]\Lambda^3(T) + O[\Lambda^4(T)]\Bigg\} \quad \text{as } T \to \infty\,. \end{aligned} \tag{4.5}$$

Similar expressions hold for the pressure and the specific heat.

Note that Gutierrez and Yanez [7] have recently constructed a formula similar to (4.5) but for a simply connected bounded domain with Dirichlet boundary conditions by using the formula (1.5) of Section 1.

Our main result (4.5) shows, in principle, that an ideal gas could feel some aspects of the shape of the general annular drum $\Omega \subseteq \mathbb{R}^2$ because its thermodynamic quantities depend on some geometric properties of Ω. But, we note that an ideal gas, even with all terms in the expansion of the partition function completely known, is not able to discriminate between two different shapes. This theoretical result was experimentally verified in [16] by employing thin microwave cavities shaped in the form of two different domains known to be isospectral. Of course, with reference to the articles [6] and [16], there are domains where although different in shapes, the thermodynamic properties of an ideal gas will be exactly the same. From these discussions, we deduce that the answer of either Kac's question or Gutierrez and Yanez's question is negative.

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References

1. M. Berger, Eigenvalues of the Laplacian, *Proc. Sympos. Pure Math.* **16** (1970), 121–125.
2. M. Berger, Geometry of the spectrum I, *Proc. Sympos. Pure Math.* **27** (1975), 129–152.
3. L. Bers, F. John and M. Schechter, Partial Differential equations, *Amer. Math. Soc. Providence* (1974).
4. P. Buser, Isospectral Riemann surfaces, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 167–192.
5. H.P.W. Gottlieb, Eigenvalues of the Laplacian with Neumann boundary conditions, *J. Austral. Math. Soc. Ser. B* **26** (1985), 193–309.
6. C. Gordon, D.L. Webb and S. Wolpert, One can not hear the shape of a drum, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), 134–138.
7. G. Gutierrez and J.M. Yanez, Can an ideal gas feel the shape of its container?, *Amer. J. Phys.* **65** (1997), 739–743.
8. P. Hsu, On the Θ-function of a Riemannian manifold with boundary, *Trans. Amer. Math. Soc.* **333** (1992), 643–671.
9. M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* **73** (1966), 1–23.
10. H.P. McKean and I.M. Singer, Curvature and the eigenvalues of the Laplacian, *J. Differential Geom.* **1** (1967), 43–69.
11. J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, *Proc. Nat. Acad. Sci. U.S.A.* **51** (1964), 542.
12. S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, 1973.
13. V.G. Papanicolaou, Trace formulas and the behaviour of large eigenvalues, *SIAM J. Math. Anal.* **26** (1995), 218–237.
14. A. Pleijel, A study of certain Green's functions with applications in the theory of vibrating membranes, *Ark. Mat.* **2** (1954), 553–569.
15. B.D. Sleeman and E.M.E. Zayed, An inverse eigenvalue problem for a general convex domain, *J. Math. Anal. Appl.* **94** (1983), 78–95.
16. S. Sridhar and A. Kudrolli, Experiments on not hearing the shape of drums, *Phys. Rev. Lett.* **72** (1994), 2175–2178.
17. I. Stakgold, *Boundary value problems of Mathematical Physics*, **2**, Macmillan, New York, 1968.
18. I. Stakgold, *Green's functions and boundary value problems*, John Wiley & Sons, Inc. 1979.
19. K. Stewartson and R.T. Waechter, On hearing the shape of a drum: Further results, *Proc. Camb. Philos. Soc.* **69** (1971), 353–363.
20. H. Urakawa, Bounded domains which are isospectral but not congruent, *Ann. Sci. École Norm. Sup. (4)* **15** (1982), 441–456.
21. E.M.E. Zayed, Eigenvalues of the Laplacian for the third boundary value problem, *J. Austral. Math. Soc. Ser. B* **29** (1987), 79–87.
22. E.M.E. Zayed, Hearing the shape of a general convex domain, *J. Math. Anal. Appl.* **142** (1989), 170–187.
23. E.M.E. Zayed, Heat equation for an arbitrary doubly connected region in $\mathbb{R}^2$ with mixed boundary conditions, *Z. Angew. Math. Phys.* **40** (1989), 339–355.

24. E.M.E. Zayed, On hearing the shape of an arbitrary doubly connected region in $\mathbb{R}^2$, *J. Austral. Math. Soc. Ser. B* **31** (1990), 472–483.
25. E.M.E. Zayed, Short-time asymptotics of the heat kernel of the Laplacian of a bounded domain with Robin boundary conditions, *Houston J. Math.* **24** (1998), 377–385.