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Some multiplier theorems on the sphere

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Abstract

The n-dimensional sphere, Σ_n, can be seen as the quotient between the group of rotations of $\mathbb{R}^{n+1}$ and the subgroup of all the rotations that fix one point. Using representation theory, one can see that any operator on $L^p(\Sigma_n)$ that commutes with the action of the group of rotations (called multiplier) may be associated with a sequence of complex numbers. We prove that, if a certain "discrete derivative" of a given sequence represents a bounded multiplier on $L^p(\Sigma_1)$, then the given sequence represents a bounded multiplier on $L^p(\Sigma_n)$. As a corollary of this, we obtain the multidimensional version of the Marcinkiewicz theorem on multipliers. An associated problem related to expansions in ultraspherical polynomials is also studied

1. Introduction

The results in this paper arise from an effort to apply the "transference method" of Coifman and Weiss [5] to operators acting on the spaces $L^p(\Sigma_n)$, $1 \le p < \infty$, Σ_n being the n-dimensional sphere in $\mathbb{R}^{n+1}$. Let us begin with a formal, brief description of this method. Suppose G is a locally compact group and T is a representation of G acting on an L^p-space of functions defined on some measure space $(\mathcal{M}, \mu)$. Suppose we have a bounded convolution operator $\varphi \to k * \varphi$ on $L^p(G, \lambda)$ (say, we are dealing with left Haar measure λ and the convolution is left-invariant). The representation T can be used to transfer this convolution to an operator H_k on $L^p(\mathcal{M})$ by letting

$$(H_k f)(x) \stackrel{\text{def}}{=} \int_G k(u)(T_{u^{-1}} f)(x)\, d\lambda(u).$$

Minkowski's integral inequality, then, gives us

$$\|H_k f\|_{L^p(\mathcal{M})} \le \|k\|_{L^1(G)} c \|f\|_{L^p(\mathcal{M})} \tag{1.1}$$

where c is $\sup_{u \in G} \|T_u\|$, $\|T_u\|$ being the norm of the operator T_u acting on $L^p(\mathcal{M})$. Inequality (1.1) is far from best possible in practically all examples of this situation, even if $c < \infty$. Ideally, we would like the norm $\|H_k\|$ of the operator H_k to be, essentially, the norm $N_p(k)$ of the convolution operator $\varphi \to k * \varphi$. When G is amenable this is indeed the case (see [5]):

$$\|H_k\| \le c^2 N_p(k). \tag{1.2}$$

Since it is often true that the ratio $\|k\|_{L^1(G)}/N_p(k)$ is arbitrarily large, we see that (1.2) is a much stronger result than (1.1).

As we stated at the beginning, we show that such results are true when subgroups of $G = SO(n+1)$ act (in the obvious way) on $L^p(\Sigma_n)$. This enables us to obtain the boundedness of a class of "zonal operators" from known 1-dimensional (Fourier series) convolution estimates. In fact, this program was initiated by the first named author in his Ph.D. thesis and some parts of it appeared in a research announcement [7]. In this thesis, as well as in the announcement, extensions of these results were begun (the "transferred" results were associated with series involving certain special functions; in particular, the development of functions in terms of ultraspherical polynomials that include the spaces $L^p(\Sigma_n)$ as special cases). Each of us has gone further in analogous directions. In order to present these advances, however, we need a complete presentation of the original work. Since this has been formulated by both of us, we decided to present this in this paper. We will also describe other directions, but only in the last section, when all the necessary definitions and notation will be available to us, and plan to write them up in different articles in the near future. Thus, to be more explicit, this work concerns itself mainly with the developments in terms of spherical harmonics (the $L^p(\Sigma_n)$ case) and the natural extensions involving ultraspherical polynomials.

2. Preliminaries and notation

The following facts can be found in [11] and [5].

Let n be an integer greater than 1, Σ_n the unit sphere in $\mathbb{R}^{n+1}$,

$$\Sigma_n = \left\{ x \in \mathbb{R}^{n+1} \, : |x| = 1 \right\}$$

and $e = (1, 0, \ldots, 0)$ the "east pole" of Σ_n. Let $G = SO(n+1)$ be the group of orthogonal transformations of $\mathbb{R}^{n+1}$ with determinant 1; that is, the group of rotations of $\mathbb{R}^{n+1}$. The subgroup K of G composed by all the rotations that fix the east pole e,

$$K = \left\{ u \in G \, : ue = e \right\}$$

is isomorphic to $SO(n)$.

One can easily see that the sphere Σ_n is diffeomorphic to the homogeneous space G/K (simply identify a point $x \in \Sigma_n$ with the coset of all the rotations that take e to x).

Definition 2.1. A function $\mathbf{F}$ on G is said to be right (left) invariant (with respect to K) if $\mathbf{F}(uk) = \mathbf{F}(u)$ $(\mathbf{F}(ku) = \mathbf{F}(u))$ for all $u \in G$, $k \in K$. If $\mathbf{F}$ is both left and right invariant, it is called biinvariant. A zonal function on the sphere is a function which is constant on the parallels $L_\theta = \{x \in \Sigma_n \, : x \cdot e = cos\theta \}$.

Thanks to the diffeomorphism between G/K and Σ_n we can identify right invariant functions on G with functions on Σ_n. More precisely, if $\mathbf{F}$ is a right invariant function on G, we can associate with it a function $\mathbf{F}^1$ on Σ_n by the relation $\mathbf{F}^1(x) = \mathbf{F}(u)$ whenever $ue = x \in \Sigma_n$. Conversely, any function f on Σ_n determines a right invariant function on G, $f^\sharp$, given by $f^\sharp(u) = f(ue)$. It is clear that f is zonal if and only if $f^\sharp$ is biinvariant. Also if f is zonal, we can associate to it a function f^0 defined on $[-1, 1]$ by the relation $f^0(\cos\theta) = f(x)$ whenever $x \cdot e = \cos\theta$. Conversely, each function f^0 on $[-1, 1]$ defines a zonal function by the last equality.

Proposition 2.2

If $f \in L^1(\Sigma_n)$ then $f^\sharp \in L^1(G)$ and

$$\int_{\Sigma_n} f(x)\,dx = \omega_n \int_G f^\sharp(u)\,du \tag{2.1}$$

where dx is the element of Lebesgue surface area on Σ_n, $\omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the surface area of Σ_n and du is the normalized Haar measure of G.

Consider the subgroup A of G consisting of all the matrices $\alpha(\theta) \in G$ having the form

$$\alpha(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \dots & 0 & 0 \\ \sin\theta & \cos\theta & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{2.2}$$

It is well known that each element of G can be written in the form $k\alpha k'$ with $k, k' \in K$ and $\alpha \in A$. Thus G admits a Cartan decomposition

$$G = KAK.$$

Proposition 2.3

If $\mathbf{F} \in L^1(G)$ then

$$\int_G \mathbf{F}(u)\,du = c_n \int_K \int_K \int_{-\pi}^{\pi} \mathbf{F}(k\alpha(\theta)k')|\sin\theta|^{n-1}\,d\theta\,dk\,dk' \tag{2.3}$$

where dk is the normalized Haar measure of K and $c_n^{-1} = \int_{-\pi}^{\pi} |\sin\theta|^{n-1}\,d\theta = \frac{2\omega_n}{\omega_{n-1}}$.

Suppose $\mathbf{F}$ is a biinvariant function, then for all $u \in G$ there are $k, k' \in K$ such that $kuk' = \alpha(\theta)$ for some θ and

$$\begin{aligned} \mathbf{F}(u) &= \mathbf{F}(kuk') = \mathbf{F}(\alpha(\theta)) = \mathbf{F}^1(\alpha(\theta)e) \\ &= (\mathbf{F}^1)^0((\alpha(\theta)e \cdot e) = (\mathbf{F}^1)^0(\cos\theta). \end{aligned}$$

Definition 2.4. The restriction to Σ_n of a homogeneous harmonic polynomial of degree l in $n+1$ variables is called a spherical harmonic of degree l. The space of these functions will be denoted by $\mathcal{H}_l(\Sigma_n)$. A finite sum of elements of $\bigcup_{l=0}^{\infty} \mathcal{H}_l(\Sigma_n)$ is called a generalized trigonometric polynomial and the space of these functions will be denoted by $\mathcal{P}(\Sigma_n)$.

The typical generalized trigonometric polynomial has the form $\sum_{l=0}^{k} f_l$, where $f_l \in \mathcal{H}_l(\Sigma_n)$, $l = 0, \dots, k$.

Proposition 2.5

The space $\mathcal{H}_l(\Sigma_n)$ has finite dimension

$$a_l = \binom{n+l}{l} - \binom{n+l-2}{l-2}.$$

Furthermore $L^2(\Sigma_n) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l(\Sigma_n)$ (direct sum) and $\mathcal{P}(\Sigma_n)$ is dense in $L^p(\Sigma_n)$, whenever $1 \leq p < \infty$.

Definition 2.6. A zonal multiplier on Σ_n is a linear operator from $\mathcal{P}(\Sigma_n)$ to $\mathcal{C}(\Sigma_n)$ that commutes with rotations.

The following theorem gives us a characterization of zonal multipliers, and explains why the term "multiplier" is used in this last definition.

Theorem 2.7

Let T be a zonal multiplier on Σ_n. For each $l \geq 0$, the elements f_l of $\mathcal{H}_l(\Sigma_n)$ are proper vectors of T corresponding to the same proper value, m_l, of T.

In other words, T is a zonal multiplier if and only if there is a sequence of complex numbers $\{m_l\}_{l=0}^{\infty}$ (called the *Fourier transform* of T) such that, for every generalized trigonometric polynomial $\sum_{l=0}^{k} f_l$, T has the form

$$T\left(\sum_{l=0}^{k} f_l\right) = \sum_{l=0}^{k} m_l f_l\,.$$

Let $1 \leq p < \infty$ and suppose that for every $f \in \mathcal{P}(\Sigma_n)$, we have

$$\|Tf\|_p \leq A_p \|f\|_p$$

where A_p is a constant independent of f, then we say that T is a bounded zonal multiplier on $L^p(\Sigma_n)$, since then T admits a unique extension to a bounded operator on $L^p(\Sigma_n)$.

Basic examples of such operators are provided by "convolutions":

Definition 2.8. Suppose $\mathbf{G}$, $\mathbf{F} \in L^1(G)$, the convolution in G between $\mathbf{G}$ and $\mathbf{F}$ is a function $\mathbf{G} * \mathbf{F}$ defined by

$$\mathbf{G} * \mathbf{F}(u) \stackrel{\text{def}}{=} \int_G \mathbf{G}(v^{-1}u)\mathbf{F}(v)\, dv.$$

If $g, f \in L^1(\Sigma_n)$, the convolution in Σ_n between g and f is a function $g * f$ defined by

$$\begin{aligned} g * f(x) &\stackrel{\text{def}}{=} \omega_n (g^\sharp * f^\sharp)^1(x) \\ &= \omega_n \int_G g(v^{-1}x) f(ve)\, dv \end{aligned} \tag{2.4}$$

where $g^\sharp * f^\sharp$ denotes the convolution in the group G previously defined.

Note that the definition is consistent since $g^\sharp * f^\sharp$ is right invariant. Also, if g is zonal,

$$g * f(x) = \int_{\Sigma_n} g^0(x \cdot y) f(y)\, dy. \tag{2.5}$$

It is easy to see that convolution itself provides us with a basic example of zonal multipliers bounded on $L^p(\Sigma_n)$. Indeed, let $g \in L^1(\Sigma_n)$. Since, for $1 \le p < \infty$, we have $\|g * f\|_p \le \|g\|_1 \|f\|_p$, the operator

$$\begin{aligned} T_g : \mathcal{P}(\Sigma_n) &\longrightarrow \mathcal{C}(\Sigma_n) \\ f &\longmapsto g * f \end{aligned}$$

is linear, commutes with rotations and is bounded on $L^p(\Sigma_n)$. A convolution kernel that will be used often is the Poisson kernel

$$P_r(x) = \frac{1 - r^2}{\omega_n |rx - e|^{n+1}}, \quad r \in [0, 1). \tag{2.6}$$

This defines a bounded zonal multiplier on $L^p(\Sigma_n)$ given by $T_r(f) = P_r * f$ which has L^p-norm bounded by $\|P_r\|_1 = 1$. Its Fourier transform is $\{r^l\}_{l=0}^{\infty}$.

$\mathcal{H}_l(\Sigma_n)$ is a finite dimensional Hilbert space (with inner product induced by $L^2(\Sigma_n)$), composed of continuous functions. Therefore, if $y \in \Sigma_n$, $\mathcal{F}_y p = p(y)$ defines a bounded linear functional on $\mathcal{H}_l(\Sigma_n)$. Thus, there is a function $Z_y^{l,n} \in \mathcal{H}_l(\Sigma_n)$, called the *zonal spherical harmonic of degree l with pole y*, that represents this functional:

$$\mathcal{F}_y p = p(y) = \int_{\Sigma_n} p(x) \overline{Z_y^{l,n}(x)}\, dx, \quad \forall\, p \in \mathcal{H}_l(\Sigma_n).$$

Proposition 2.9

(i) *$Z_e^{l,n}$ is real valued and zonal.*
(ii) $Z_y^{l,n}(x) = Z_{uy}^{l,n}(ux)$ *for all* $x, y \in \Sigma_n$, $u \in G$.
(iii) *For all* $f \in L^2(\Sigma_n)$, $Z_e^{l,n} * f = P_l f$ *(projection onto* $\mathcal{H}_l(\Sigma_n)$*).*
(iv) *For all* $x \in \Sigma_n$, $|Z_e^{l,n}(x)| \le |Z_e^{l,n}(e)| = \dim(\mathcal{H}_l(\Sigma_n))/\omega_n \stackrel{\text{def}}{=} A_n(l)$ *and* $A_n(l)$ *has polynomial growth of degree* $n - 1$.

Proof. (i), (ii) and the first part of (iv) can be found in [11]. As for (iii), assume $\{Y_j\}_{j=1}^{a_l}$ is an orthonormal basis for $\mathcal{H}_l(\Sigma_n)$. Then

$$\begin{aligned}
Z_e^{l,n} * f(x) &= \omega_n \int_G Z_e^{l,n}(v^{-1}x) f(ve)\, dv = \omega_n \int_G Z_{ve}^{l,n}(x) f(ve)\, dv \\
&= \omega_n \int_G \left[\sum_{j=1}^{a_l} (Z_{ve}^{l,n}, Y_j) Y_j(x) \right] f(ve)\, dv \\
&= \omega_n \sum_{j=1}^{a_l} \left[\int_G f(ve) \overline{Y_j(ve)}\, dv \right] Y_j(x) \\
&= \sum_{j=1}^{a_l} \left[\int_{\Sigma_n} f(y) \overline{Y_j(y)}\, dy \right] Y_j(x) = \sum_{j=1}^{a_l} (f, Y_j) Y_j(x) = (P_l f)(x),
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Sigma_n)$. Observe that

$$\begin{aligned}
A_n(l) = \frac{\dim(\mathcal{H}_l(\Sigma_n))}{\omega_n} &= \frac{1}{\omega_n} \left[\binom{n+l}{l} - \binom{n+l-2}{l-2} \right] = \frac{n+2l-1}{\omega_n l} \binom{n+l-2}{l-1} \\
&= \frac{n+2l-1}{\omega_n l} \, \frac{(n+l-2)(n+l-3)\dots(l+1)l}{(n-1)!} \\
&= \frac{1}{\omega_n (n-1)!} \, \frac{n+2l-1}{l} (l^{n-1} + \dots)
\end{aligned}$$

which proves (iv). $\square$

We make a final observation. All we have said so far holds for $n = 1$ as well, with some important exceptions. First of all note that, if $G = SO(2)$, then $K = \{\text{Identity}\}$ and therefore $SO(2)$ and Σ_1 are canonically identified by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longleftrightarrow (\cos\theta, \sin\theta).$$

With this notation, the Haar measure of G is $\frac{d\theta}{2\pi}$. Keeping this in mind, it is easy to check that all definitions and propositions 2.1 through 2.5, 2.8 and 2.9 hold for $n = 1$. In this case $\mathcal{H}_l(\Sigma_1) = \text{Span}\{e^{il\theta}, e^{-il\theta}\}$ for $l \geq 1$, $\mathcal{H}_0(\Sigma_1) = \{\text{constant functions}\}$, $\mathcal{P}(\Sigma_1)$ is the space of classical trigonometric polynomials and $Z^{l,1}_{(\cos\phi, \sin\phi)}((\cos\theta, \sin\theta)) = \frac{\cos l(\phi+\theta)}{\pi}$. As for the differences with the n-dimensional case, first of all note that all functions on $SO(2)$ are biinvariant (with respect to K), whereas the zonal functions on Σ_1 are only those for which $f((\cos\theta, \sin\theta)) = f((\cos\theta, -\sin\theta))$ (in other words, even functions, when considered as functions of

θ, $-\pi \leq \theta < \pi$). If $n = 1$, Theorem 2.7 is no longer true, essentially because the spaces $\mathcal{H}_l(\Sigma_n)$ are invariant but not minimal invariant under the representation of $SO(2)$ given by

$$S_{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} f(\cos\varphi, \sin\varphi) \stackrel{\text{def}}{=} f\big(\cos(\varphi+\theta), \sin(\varphi+\theta)\big).$$

Indeed, $SO(2)$ being abelian, the minimal invariant spaces of the representation S are 1-dimensional, namely $\mathrm{Span}\{e^{il\theta}\}$, for all $l \in \mathbb{Z}$. This implies that any operator of the form

$$T\left(a_0 + \sum_{l=1}^{k} a_l e^{il\theta} + b_l e^{-il\theta}\right) = m_0 a_0 + \sum_{l=1}^{k} m_l a_l e^{il\theta} + \mu_l b_l e^{-il\theta}$$

commutes with rotations. We will then call zonal multiplier on Σ_1 any operator from $\mathcal{P}(\Sigma_1)$ to $\mathcal{C}(\Sigma_1)$ of the above form, with the additional condition $m_l = \mu_l$ for all $l \geq 1$.

3. The main theorem

Let $n \geq 1$ and $N = \left[\frac{n}{2}\right]$, the biggest integer less than or equal to $\frac{n}{2}$. Let J be the operator on the space of sequences defined by

$$(Jm)_l = \begin{cases} l m_l - (l-1) m_{l-1} & \text{if } l \geq 1 \\ 0 & \text{if } l = 0\,. \end{cases}$$

Define $J^N = J \circ J \circ \ldots \circ J$ (Ntimes).

The following is the main result of this paper.

Theorem 3.1

Let $n \geq 1$, $N = \left[\frac{n}{2}\right]$, $1 < p < \infty$, $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers. If $J^N m$ defines a bounded zonal multiplier on $L^p(\Sigma_1)$, then m defines a bounded zonal multiplier on $L^p(\Sigma_n)$.

The proof of Theorem 3.1 will follow from a series of results contained in this and the next two sections. More precisely, in this section we will show how to deduce the boundedness of a zonal multiplier on $L^p(\Sigma_n)$ from the boundedness of three associated zonal multipliers on $L^p(\Sigma_{n-2})$ (Theorem 3.9). If n is an odd integer, by applying this step N times, we arrive at the Σ_1 case, which is essentially our goal. But if n is even, after $N-1$ steps we arrive at the Σ_2 case. In Section 4, we study how to deduce the boundedness of a zonal multiplier on $L^p(\Sigma_2)$ from the boundedness of a zonal multiplier on $L^p(\Sigma_1)$ (Theorem 4.9). Finally, in Section 5, we put together all these results and use some facts about multipliers on $L^p(\Sigma_1)$ to polish our theorem and get it in the final, simple version that we just stated (Theorem 3.1).

Theorem 3.2

Let $n \geq 2$, $1 \leq p < \infty$, $k \in L^1(\Sigma_n)$ be zonal. Assume that $h^0(\cos\theta) = k^0(\cos\theta)|\sin\theta|$ defines a zonal function h on Σ_{n-1} which defines a bounded convolution operator on $L^p(\Sigma_{n-1})$ with norm A_p. Then k defines a bounded convolution operator on $L^p(\Sigma_n)$, with norm less than or equal to $\frac{\omega_{n-1}}{\omega_{n-2}}A_p$ (assume $\omega_0 = 2$).

Proof. Let $f \in L^p(\Sigma_n)$. Then

$$\begin{aligned}
\|k * f\|_p &= \left[\omega_n \int_G \left|\omega_n \int_G k^\sharp(u) f^\sharp(vu^{-1})\, du\right|^p dv\right]^{1/p} \\
&= \omega_n^{1+1/p} c_n \left[\int_G \left|\int_K \int_K \int_{-\pi}^{\pi} k^\sharp(\alpha(\theta)) f^\sharp(v\gamma\alpha(-\theta)\gamma') |\sin\theta|^{n-1} d\theta\, d\gamma'\, d\gamma\right|^p dv\right]^{1/p} \\
&= \frac{\omega_n^{1/p}\omega_{n-1}}{2} \left[\int_G \left|\int_K \int_{-\pi}^{\pi} h^0(\cos\theta) f^\sharp(v\gamma\alpha(-\theta)) |\sin\theta|^{n-2} d\theta\, d\gamma\right|^p dv\right]^{1/p}.
\end{aligned}$$

Let $H = \{u \in G \mid ue = e, u\tilde{e} = \tilde{e}\} \subset K$, where $\tilde{e} = (0, \ldots, 0, 1)$ is the "north pole" of Σ_n. Obviously, $H \cong SO(n-2)$. Since $d\gamma$ is the Haar measure of K, we can replace γ with $\gamma\beta$, for any $\beta \in H$, and integrate in $d\beta$ over H. Thus the last expression equals

$$\frac{\omega_n^{1/p}\omega_{n-1}}{2} \left[\int_G \left|\int_H \int_K \int_{-\pi}^{\pi} h^0(\cos\theta) f^\sharp(v\gamma\beta\alpha(-\theta)) |\sin\theta|^{n-2} d\theta\, d\gamma\, d\beta\right|^p dv\right]^{1/p}.$$

Applying Fubini's theorem to the integrals in H and K, and then Minkowski's integral inequality we see that the last expression is majorized by

$$\begin{aligned}
&\frac{\omega_n^{1/p}\omega_{n-1}}{2} \int_K \left[\int_G \left|\int_H \int_{-\pi}^{\pi} h^0(\cos\theta) f^\sharp(v\gamma\beta\alpha(-\theta)) |\sin\theta|^{n-2} d\theta\, d\beta\right|^p dv\right]^{1/p} d\gamma \\
&= \frac{\omega_n^{1/p}\omega_{n-1}}{2} \left[\int_G \left|\int_H \int_H \int_{-\pi}^{\pi} h^0(\cos\theta) f^\sharp(v\beta\alpha(-\theta)\beta') |\sin\theta|^{n-2} d\theta\, d\beta\, d\beta'\right|^p dv\right]^{1/p}.
\end{aligned}$$

Now observe that $Q \stackrel{\text{def}}{=} \{u \in G \mid u\tilde{e} = \tilde{e}\}$ is isomorphic to $SO(n)$ and, therefore, has the following Cartan decomposition

$$Q = HAH.$$

Thus, by Proposition 2.3 applied to Q, the last expression equals

$$\begin{aligned} &\frac{\omega_n^{1/p}\omega_{n-1}}{2}\left[\int_G \left|\frac{1}{c_{n-1}}\int_Q h^\sharp(\eta) f^\sharp(v\eta^{-1})\,d\eta\right|^p dv\right]^{1/p} \\ &= \frac{\omega_n^{1/p}\omega_{n-1}}{2c_{n-1}}\left[\int_G\int_Q \left|\int_Q h^\sharp(\eta) f^\sharp(v\eta'\eta^{-1})\,d\eta\right|^p d\eta'\,dv\right]^{1/p}. \end{aligned} \tag{3.1}$$

The innermost integral is the convolution (in Q) of $h^\sharp$ with $(f^\sharp)_v$, where $(f^\sharp)_v$ is defined on Q by $(f^\sharp)_v(\eta) \stackrel{\text{def}}{=} f^\sharp(v\eta)$. If we define f_v on $\Sigma_{n-1} = \{x \in \Sigma_n \mid x_{n+1} = 0\} = Q/H$ by $f_v(x) \stackrel{\text{def}}{=} f(vx)$, the corresponding right invariant (with respect to H) function on Q will be precisely $(f^\sharp)_v$, since

$$(f^\sharp)_v(\eta) = f^\sharp(v\eta) = f(v\eta e) = f_v(\eta e).$$

Thus, denoting by $\star$ the convolution in Σ_{n-1},

$$\begin{aligned} \int_Q \left|\int_Q h^\sharp(\eta)(f^\sharp)_v(\eta'\eta^{-1})\,d\eta\right|^p d\eta' &= \int_Q \left|\frac{h \star f_v(\eta' e)}{\omega_{n-1}}\right|^p d\eta' \\ &= \frac{1}{\omega_{n-1}^{1+p}}\int_{\Sigma_{n-1}} |h \star f_v(x)|^p dx \\ &\le \frac{1}{\omega_{n-1}^{1+p}} A_p^p \int_{\Sigma_{n-1}} |f_v(x)|^p dx \\ &= \frac{1}{\omega_{n-1}^{p}} A_p^p \int_Q |f^\sharp(v\eta)|^p d\eta. \end{aligned}$$

We can therefore conclude that

$$\begin{aligned} \|k * f\|_p &\le \frac{\omega_n^{1/p}}{2c_{n-1}} A_p \left[\int_G\int_Q |f^\sharp(v\eta)|^p d\eta\,dv\right]^{1/p} = \frac{\omega_n^{1/p}\omega_{n-1}}{\omega_{n-2}} A_p \left[\int_G |f^\sharp(v)|^p dv\right]^{1/p} \\ &= \frac{\omega_{n-1}}{\omega_{n-2}} A_p \left[\int_{\Sigma_n} |f(x)|^p dx\right]^{1/p} = \frac{\omega_{n-1}}{\omega_{n-2}} A_p \|f\|_p. \ \Box \end{aligned}$$

Observation. The last part of the proof of Theorem 3.2 is really a transference result. Indeed the proof could have finish as follows:

Consider the representation of Q acting on $L^p(G)$ given by

$$T_\eta \mathbf{F}(v) = \mathbf{F}(v\eta).$$

Notice that $\|T_\eta\| = 1$, for all $\eta \in Q$. Suppose $\mathbf{F} = f^\sharp$ for some $f \in L^p(\Sigma_n)$; then for all $v \in G$, the function on Q

$$\eta \to f^\sharp(v\eta)$$

is H-right invariant. We can thus apply the transference theorem we discussed in the introduction, obtaining

$$\int_G \left| \int_Q h^\sharp(\eta) f^\sharp(v\eta^{-1})\, d\eta \right|^p dv \le N_p^p(h^\sharp)\|f^\sharp\|^p_{L^p(G)} = \frac{A_p^p}{\omega_{n-1}^p \omega_n} \|f\|^p_{L^p(\Sigma_n)}.$$

Applying this inequality to (3.1) we have the desired result.
In this particular context, we decided to present a proof that could be more easily followed by the reader non familiar with transference.

Theorem 3.3

Let $n \ge 3$, $1 \le p < \infty$, $\{m_l\}_{l=0}^\infty$ be a bounded sequence of complex numbers and define T on $\mathcal{P}(\Sigma_n)$ by

$$Tf(x) = \lim_{r\to 1^-} (k_r * f)(x)$$

where $k_r(x) \overset{\text{def}}{=} \sum_{l=0}^\infty r^l m_l Z_e^{l,n}(x)$, for $0 \le r < 1$. Let S_r be the convolution operator on $L^p(\Sigma_{n-2})$ given by the zonal kernel $k_r^0(\cos\theta)|\sin\theta|^2$. If the operators S_r have uniformly bounded (by a constant A_p) norms as $r \to 1^-$, then T is bounded on $L^p(\Sigma_n)$, with norm less than or equal to $\frac{\omega_{n-1}}{\omega_{n-3}} A_p$.

Observation. The series defining k_r converges uniformly, by Proposition 2.9 (iv). If f is a generalized trigonometric polynomial, $f = \sum_{l=0}^j f_l$, $f_l \in \mathcal{H}_l(\Sigma_n)$, then

$$\begin{aligned} Tf(x) &= \lim_{r\to 1^-} (k_r * f)(x) = \lim_{r\to 1^-} \sum_{l=0}^\infty r^l m_l Z_e^{l,n} * f(x) \\ &= \lim_{r\to 1^-} \sum_{l=0}^j r^l m_l f_l(x) = \sum_{l=0}^j m_l f_l(x); \end{aligned}$$

thus, T is the zonal multiplier associated with the sequence $\{m_l\}_{l=0}^\infty$.

Proof. Applying Theorem 3.2 twice, first to the kernel $k_r^0(\cos\theta)|\sin\theta|$ and then to $k_r^0(\cos\theta)$, we see that $k_r^0(\cos\theta)$ defines a convolution kernel on Σ_n, bounded on $L^p(\Sigma_n)$ by the constant $\frac{\omega_{n-1}}{\omega_{n-2}}\frac{\omega_{n-2}}{\omega_{n-3}}A_p = \frac{\omega_{n-1}}{\omega_{n-3}}A_p$, for all $0 \le r < 1$. Now observe that, for $f \in \mathcal{P}(\Sigma_n)$,

$$\begin{aligned}\|Tf\|_p &= \left[\int_{\Sigma_n} \lim_{r\to 1^-} |(k_r * f)(x)|^p dx\right]^{1/p} \\ &\le \liminf_{r\to 1^-} \left[\int_{\Sigma_n} |(k_r * f)(x)|^p dx\right]^{1/p} \le \frac{\omega_{n-1}}{\omega_{n-3}} A_p \|f\|_p . \ \square\end{aligned}$$

Theorem 3.3 allows us to conclude that a certain sequence m defines a zonal multiplier bounded on $L^p(\Sigma_n)$ if some family of convolution operators is uniformly bounded in $L^p(\Sigma_{n-2})$ $(n \ge 3)$. The next step expresses these operators in terms of their Fourier transform.

DEFINITION 3.4. Let $\lambda > 0$. If we write

$$\left(1 - 2rt + r^2\right)^{-\lambda} = \sum_{l=0}^{\infty} C_l^\lambda(t) r^l$$

where $|r| < 1$, $|t| \le 1$, then the coefficient $C_l^\lambda(t)$ is called the Gegenbauer (or ultraspherical) polynomial of degree l associated with λ.

Proposition 3.5

Let $l \ge 0$ be an integer. C_l^λ is a polynomial of degree l and the following identities hold.

(i) *For $\lambda = \frac{n-1}{2}$, $n \ge 2$ integer,*

$$Z_e^{l,n}(x) = \frac{l+\lambda}{\omega_n \lambda} C_l^\lambda(x \cdot e) .$$

(ii) *For any $\lambda > 1$,*

$$\sin^2\theta C_l^\lambda(\cos\theta) = \frac{(l+2\lambda-2)(l+2\lambda-1)C_l^{\lambda-1}(\cos\theta) - (l+1)(l+2)C_{l+2}^{\lambda-1}(\cos\theta)}{4(\lambda-1)(\lambda+l)} .$$

(iii) $\sin^2\theta C_l^1(\cos\theta) = \sin\theta\sin(l+1)\theta = \frac{1}{2}[\cos l\theta - \cos(l+2)\theta]$.

Proof. The proof of (i) can be found on [11], as for (ii) and (iii) see [13], page 83. □

In what follows we will put $\lambda = \frac{n-1}{2}$.

Lemma 3.6

Let $n \geq 3$ and let $\{m_l\}_{l=0}^{\infty}$ be a sequence of complex numbers with exponential decay. Then

$$\sum_{l=0}^{\infty} m_l \sin^2\theta \left(Z_e^{l,n}\right)^0 (\cos\theta) = \frac{\omega_{n-2}}{4\lambda\omega_n} \sum_{l=0}^{\infty} \left[(\mathcal{D}m)_l + (T_\lambda m)_l\right] \left(Z_e^{l,n-2}\right)^0 (\cos\theta)$$

where $\mathcal{D}m$ and $T_\lambda m$ are sequences defined by

$$(\mathcal{D}m)_l = l m_l - (l-2) m_{l-2}$$
$$(T_\lambda m)_l = \frac{\lambda(\lambda-1)}{l+\lambda-1}(m_l - m_{l-2}) + (3\lambda - 2)m_l + (\lambda-2)m_{l-2}$$

for all $l \geq 0$ (define $m_{-1} = m_{-2} = 0$).

Proof. First of all observe that both series converge uniformly. Then, assuming $n > 3$, by Proposition 3.5,

$$\begin{aligned}
\sum_{l=0}^{\infty} m_l \sin^2\theta \left(Z_e^{l,n}\right)^0 (\cos\theta) &= \frac{1}{\omega_n} \sum_{l=0}^{\infty} m_l \frac{l+\lambda}{\lambda} \sin^2\theta C_l^{\lambda}(\cos\theta) \\
&= \frac{1}{\omega_n} \sum_{l=0}^{\infty} \frac{1}{4(\lambda-1)\lambda} \\
&\quad \times \left[(l+2\lambda-2)(l+2\lambda-1) m_l C_l^{\lambda-1}(\cos\theta) - (l+1)(l+2) m_l C_{l+2}^{\lambda-1}(\cos\theta)\right] \\
&= \frac{\omega_{n-2}}{4\lambda\omega_n} \sum_{l=0}^{\infty} \frac{(l+2\lambda-2)(l+2\lambda-1)}{l+\lambda-1} m_l \left(Z_e^{l,n-2}\right)^0 (\cos\theta) + \\
&\quad - \frac{(l+1)(l+2)}{l+\lambda+1} m_l \left(Z_e^{l+2,n-2}\right)^0 (\cos\theta) \\
&= \frac{\omega_{n-2}}{4\lambda\omega_n} \sum_{l=0}^{\infty} \left[\frac{(l+2\lambda-2)(l+2\lambda-1)}{l+\lambda-1} m_l - \frac{l(l-1)}{l+\lambda-1} m_{l-2}\right] \left(Z_e^{l,n-2}\right)^0 (\cos\theta)
\end{aligned}$$

and the lemma is proved, once we observe that

$$\begin{aligned}
\frac{(l+2\lambda-2)(l+2\lambda-1)}{l+\lambda-1} &= l + \frac{\lambda(\lambda-1)}{l+\lambda-1} + 3\lambda - 2 \\
\frac{l(l-1)}{l+\lambda-1} &= l - 2 + \frac{\lambda(\lambda-1)}{l+\lambda-1} + 2 - \lambda\,.
\end{aligned}$$

The case $n = 3$ follows in a similar way; we only need to remember that

$$Z_e^{l,1}(\cos\theta) = \frac{\cos l\theta}{\pi}\,.\square$$

Lemma 3.7

Let $n \geq 3$, $1 \leq p < \infty$.

(i) The sequence $\left\{\frac{\lambda-1}{l+\lambda-1}\right\}_{l=0}^{\infty}$ defines a bounded multiplier Q_λ on $L^p(\Sigma_{n-2})$.

(ii) $\left\|\frac{\partial}{\partial r}\left[\left(r-\frac{1}{r}\right)P_r\right]\right\|_{L^1(\Sigma_{n-2})}$ are uniformly bounded as $r \to 1^-$.

Proof. (i) If $n = 3$, that is if $\lambda = 1$, there is nothing to prove. Assume $\lambda > 1$. Recall that

$$\begin{aligned} h_r^0(\cos\theta) &\stackrel{\text{def}}{=} \frac{1}{(1-2r\cos\theta+r^2)^{\lambda-1}} = \sum_{l=0}^{\infty} C_l^{\lambda-1}(\cos\theta)r^l \\ &= \sum_{l=0}^{\infty} r^l\,\frac{\omega_{n-2}(\lambda-1)}{l+\lambda-1}\left(Z_e^{l,n-2}\right)^0(\cos\theta). \end{aligned}$$

Thus h_r is a zonal convolution kernel with Fourier transform

$$\left\{\omega_{n-2}r^l\,\frac{\lambda-1}{l+\lambda-1}\right\}_{l=0}^{\infty}.$$

It is easy to see that, for all $r \in [0,1)$,

$$\|h_r\|_{L^1(\Sigma_{n-2})} = \frac{\omega_{n-3}}{2}\int_{-\pi}^{\pi} h_r^0(\cos\theta)|\sin\theta|^{n-3}d\theta \leq \pi\omega_{n-3}.$$

If $f \in \mathcal{P}(\Sigma_{n-2})$ has the form $f = \sum_{l=0}^{k} f_l$, with $f_l \in \mathcal{H}_l(\Sigma_{n-2})$, then

$$\begin{aligned} \|Q_\lambda f\|_{L^p(\Sigma_{n-2})} &= \left\|\sum_{l=0}^{k}\frac{\lambda-1}{l+\lambda-1}f_l\right\|_{L^p(\Sigma_{n-2})} \\ &= \frac{1}{\omega_{n-2}}\left\|\lim_{r\to1^-}\sum_{l=0}^{k}\omega_{n-2}r^l\,\frac{\lambda-1}{l+\lambda-1}f_l\right\|_{L^p(\Sigma_{n-2})} \\ &\leq \frac{1}{\omega_{n-2}}\liminf_{r\to1^-}\|h_r * f\|_{L^p(\Sigma_{n-2})} \\ &\leq \frac{\pi\omega_{n-3}}{\omega_{n-2}}\|f\|_{L^p(\Sigma_{n-2})}. \end{aligned}$$

(ii) $\frac{\partial}{\partial r}\left[\left(r-\frac{1}{r}\right)P_r\right]$ is zonal and

$$\begin{aligned}
&\left[\frac{\partial}{\partial r}\left[\left(r-\frac{1}{r}\right)P_r\right]\right]^0(\cos\theta)=\frac{\partial}{\partial r}\left[\left(r-\frac{1}{r}\right)P_r^0(\cos\theta)\right]\\
&=\frac{\partial}{\partial r}\left[\left(r-\frac{1}{r}\right)\frac{1-r^2}{\omega_{n-2}(r^2-2r\cos\theta+1)^{(n-1)/2}}\right]\\
&=-P_r^0(\cos\theta)\frac{r\cos\theta[(7-n)r^2+1+n]+[(n-4)r^4-(n+3)r^2-1]}{r^2(r^2-2r\cos\theta+1)}\,.
\end{aligned}$$

It is enough to check that the second factor in the above product is uniformly bounded in absolute value. Put $t=\cos\theta$, $t\in[-1,1]$, and define

$$f_r(t)\stackrel{\text{def}}{=}\frac{rt[(7-n)r^2+1+n]+[(n-4)r^4-(n+3)r^2-1]}{r^2(r^2-2rt+1)}\,;$$

f_r is differentiable in $[-1,1]$ for all $r\in(0,1)$ and, for those values of r and t,

$$f_r'(t)=\frac{n-1}{r}\frac{(r^2-1)^2}{(r^2-2rt+1)^2}>0\,.$$

Thus f_r achieves its minimum at -1 and its maximum at 1. But

$$\begin{aligned}
f_r(-1)&=\frac{(n-4)r^2+(1-n)r-1}{r^2}\\
f_r(1)&=\frac{(n-4)r^2+(n-1)r-1}{r^2}\,.
\end{aligned}$$

This proves that the absolute maximum of f_r is bounded as $r\to 1^-$. □

Definition 3.8. For any integer $k>0$, we will denote by τ_k the operator on the space of sequences given by

$$(\tau_k m)_l=\begin{cases} m_{l-k} & \text{if } l\geq k\\ 0 & \text{if } 0\leq l<k\,.\end{cases}$$

Theorem 3.9

Let $n\geq 3$, $1\leq p<\infty$. Let $m=\{m_l\}_{l=0}^\infty$ be a bounded sequence of complex numbers and let M be the zonal multiplier on Σ_n defined by m. If $\mathcal{D}m$, m, $\tau_2 m$ define bounded zonal multipliers on $L^p(\Sigma_{n-2})$, respectively M_1, M_2, M_3, then M is bounded on $L^p(\Sigma_n)$.

Proof. According to Theorem 3.3, it suffices to show that the convolution operators S_r with zonal kernels

$$\sin^2\theta k_r^0(\cos\theta) = \sum_{l=0}^{\infty} r^l m_l \sin^2\theta \left(Z_e^{l,n}\right)^0(\cos\theta)$$

are uniformly bounded on $L^p(\Sigma_{n-2})$ as $r \to 1^-$. Observe that the sequence $\mathcal{R}_r m \overset{\text{def}}{=} \{r^l m_l\}_{l=0}^{\infty}$ has exponential decay (if $0 < r < 1$). We can therefore apply Lemma 3.6 to the sequence $\mathcal{R}_r m$ and obtain

$$\sum_{l=0}^{\infty} r^l m_l \sin^2\theta \left(Z_e^{l,n}\right)^0(\cos\theta) = \frac{\omega_{n-2}}{4\lambda\omega_n}\sum_{l=0}^{\infty}\left[(\mathcal{D}\mathcal{R}_r m)_l + (T_\lambda \mathcal{R}_r m)_l\right]\left(Z_e^{l,n-2}\right)^0(\cos\theta)\,.$$

We must, therefore, prove that the sequences $\mathcal{D}\mathcal{R}_r m + T_\lambda\mathcal{R}_r m$ define uniformly bounded multipliers on $L^p(\Sigma_{n-2})$. Observe that

$$\begin{aligned}
(\mathcal{D}\mathcal{R}_r m)_l &= r^l(\mathcal{D}m)_l + \left[(l+1)r^l - (l-1)r^{l-2}\right]m_{l-2} - \left[3r^l - r^{l-2}\right]m_{l-2} \\
(T_\lambda\mathcal{R}_r m)_l &= \frac{\lambda(\lambda-1)}{l+\lambda-1}\left(r^l m_l - r^{l-2}m_{l-2}\right) + (3\lambda-2)r^l m_l + (\lambda-2)r^{l-2}m_{l-2}\,.
\end{aligned}$$

Thus, assuming $f \in \mathcal{P}(\Sigma_{n-2})$, we have

$$\begin{aligned}
&\left\|\sum_{l=0}^{\infty}(\mathcal{D}\mathcal{R}_r m)_l\, Z_e^{l,n-2} * f\right\|_{L^p(\Sigma_{n-2})} \\
&\le \left\|\sum_{l=0}^{\infty} r^l(\mathcal{D}m)_l Z_e^{l,n-2} * f\right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\|\sum_{l=0}^{\infty}\left[(l+1)r^l - (l-1)r^{(l-2)}\right](\tau_2 m)_l Z_e^{l,n-2} * f\right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\|\sum_{l=0}^{\infty}\left(3r^l - r^{l-2}\right)(\tau_2 m)_l Z_e^{l,n-2} * f\right\|_{L^p(\Sigma_{n-2})} \\
&\le \|P_r * M_1 f\|_{L^p(\Sigma_{n-2})} + \left\|\frac{\partial}{\partial r}\left[\left(r - \frac{1}{r}\right)P_r\right] * M_3 f\right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\|\left(3 - \frac{1}{r^2}\right)P_r * M_3 f\right\|_{L^p(\Sigma_{n-2})} \\
&\le \|M_1 f\|_{L^p(\Sigma_{n-2})} + C_n\|M_3 f\|_{L^p(\Sigma_{n-2})} + 3\|M_3 f\|_{L^p(\Sigma_{n-2})}\,.
\end{aligned}$$

Similarly

$$
\begin{aligned}
&\left\| \sum_{l=0}^{\infty} (T_\lambda \mathcal{R}_r m)_l Z_e^{l,n-2} * f \right\|_{L^p(\Sigma_{n-2})} \\
&\leq \left\| \sum_{l=0}^{\infty} \frac{\lambda(\lambda-1)}{l+\lambda-1} r^l m_l Z_e^{l,n-2} * f \right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\| \sum_{l=0}^{\infty} \frac{\lambda(\lambda-1)}{l+\lambda-1} r^{l-2} m_{l-2} Z_e^{l,n-2} * f \right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\| \sum_{l=0}^{\infty} (3\lambda-2) r^l m_l Z_e^{l,n-2} * f \right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \left\| \sum_{l=0}^{\infty} (\lambda-2) r^{l-2} m_{l-2} Z_e^{l,n-2} * f \right\|_{L^p(\Sigma_{n-2})} \\
&= \| \lambda P_r * Q_\lambda M_2 f \|_{L^p(\Sigma_{n-2})} + \left\| \lambda \frac{1}{r^2} P_r * Q_\lambda M_3 f \right\|_{L^p(\Sigma_{n-2})} \\
&\quad + \| (3\lambda-2) P_r * M_2 f \|_{L^p(\Sigma_{n-2})} + \left\| (\lambda-2) \frac{1}{r^2} P_r * M_3 f \right\|_{L^p(\Sigma_{n-2})} \\
&\leq \lambda \| Q_\lambda M_2 f \|_{L^p(\Sigma_{n-2})} + \lambda \frac{1}{r^2} \| Q_\lambda M_3 f \|_{L^p(\Sigma_{n-2})} \\
&\quad + (3\lambda-2) \| M_2 f \|_{L^p(\Sigma_{n-2})} + |\lambda-2| \frac{1}{r^2} \| M_3 f \|_{L^p(\Sigma_{n-2})}. \square
\end{aligned}
$$

Suppose n is an odd integer. We can apply Theorem 3.9 to a sequence m on Σ_n, then to the sequences $\mathcal{D}m$, m, $\tau_2 m$ on Σ_{n-2}, and so on, so that, eventually, we obtain that the boundedness on $L^p(\Sigma_1)$ of a certain (finite) family of multipliers will imply the boundedness of m on $L^p(\Sigma_n)$. This is the basic idea in the proof of Theorem 3.1. If n is even, though, by reducing the dimension of the sphere two units at each step, we arrive at the Σ_2 case. The next section, as we anticipated at the beginning, will be devoted precisely to the answer to the question: the boundedness of which multipliers (if any) on $L^p(\Sigma_1)$ imply the boundedness of a given multiplier on $L^p(\Sigma_2)$?

4. The 2-dimensional case

We begin this section with some facts about $SU(2)$ and its relation to the sphere Σ_2. For the details, see [16], [12] Chapters 1 and 2, [3], page 105 and [5].

The group $SU(2)$ (special unitary group of degree 2) consists of all 2×2 matrices u satisfying

$$u^*u = I \quad \text{and} \quad \det u = 1.$$

The space M af all matrices of the form

$$X = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$$

with x_1, x_2, $x_3 \in \mathbb{R}$, is clearly isomorphic to $\mathbb{R}^3$.

DEFINITION 4.1. For any $u \in SU(2)$, define $\Phi(u)$ as the linear map on M given by

$$\Phi(u)X = uXu^*.$$

One can check that $uXu^* \in M$, so that $\Phi(u)$ maps into M, and that $\Phi(uv) = \Phi(u)\Phi(v)$, so that Φ is a homomorphism. Note that, defining

$$a(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad b(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

we have $a(\theta)$, $b(\varphi) \in SU(2)$. $\Phi(a(\theta))$ and $\Phi(b(\varphi))$ are then linear maps on M and their matrix representation with respect to the basis of M

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}$$

is given by

$$\Phi(a(\theta)) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad \Phi(b(\varphi)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

It is well known that each $u \in SU(2)$ can be written as $u = b(\varphi_1)a(\theta)b(\varphi_2)$, and that each $\eta \in SO(3)$ can be written as $\eta = \Phi(b(\varphi_1))\Phi(a(\theta))\Phi(b(\varphi_2))$ (φ_1, θ, φ_2 are the so-called Euler angles). Thus $\Phi(SU(2)) = SO(3)$ and, as one can check, $\ker\Phi = \{I, -I\}$. We can define an action

$$\begin{aligned} SU(2) \times \Sigma_2 &\longrightarrow \Sigma_2 \\ (u, x) &\longmapsto \Phi(u)x. \end{aligned}$$

Proposition 4.2

Let $e = (1, 0, 0)$ be the "east pole" of Σ_2, $B = \{b(\varphi) \mid \varphi \in [0, 4\pi]\}$, a subgroup of $SU(2)$. Then Σ_2 is diffeomorphic to the homogeneous space $SU(2)/B$.

Proof. It is enough to show that

$$B = \{u \in SU(2) \mid \Phi(u)e = e\},$$

and this follows easily from the fact that

$$\Phi(b(\varphi)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \quad \square$$

Notation. For any $f \in L^1(\Sigma_2)$, we will define $\tilde{f}$ on $SU(2)$ by

$$\tilde{f}(u) \overset{\text{def}}{=} f\big(\Phi(u)e\big).$$

Note that $\tilde{f}$ is right invariant with respect to B.

Proposition 4.3

If $f \in L^1(\Sigma_2)$, then $\tilde{f} \in L^1(SU(2))$ and

$$\int_{\Sigma_2} f(x)\,dx = 4\pi \int_{SU(2)} \tilde{f}(u)\,du\,.$$

As usual, dx denotes the element of Lebesgue surface area and du is the normalized Haar measure of $SU(2)$.

Proof.

$$\begin{aligned} \int_{\Sigma_2} f(x)\,dx &= \int_{SU(2)} \int_{\Sigma_2} f(\Phi(u)x)\,dx\,du = \int_{\Sigma_2} \int_{SU(2)} f(\Phi(u)x)\,du\,dx \\ &= \int_{\Sigma_2} \int_{SU(2)} f(\Phi(u)e)\,du\,dx = 4\pi \int_{SU(2)} \tilde{f}(u)\,du\,. \ \square \end{aligned}$$

Let L be a non-negative integer and $\mathcal{P}^L$ the vector space of homogeneous polynomials of degree L in the complex variables $z_1,\, z_2$. The $L+1$ polynomials $e_j(z) = e_j(z_1,\, z_2) \overset{\text{def}}{=} \sqrt{\binom{L}{j}} z_1^j z_2^{L-j}$, $j = 0, \dots, L$, form a basis for this space and $\mathcal{P}^L$ can be considered to be a Hilbert space if we impose the condition that $\{e_j\}_{j=0}^L$ be an orthonormal basis. Consider the following mapping

$$\begin{aligned} S^L \quad : \quad SU(2) &\longrightarrow U(\mathcal{P}^L) \quad \text{(unitary operators on } \mathcal{P}^L) \\ u &\longmapsto S^L_u \end{aligned}$$

where $S^L_u p(z) \overset{\text{def}}{=} p(u'z) \qquad (u' = \text{transpose of } u).$

Lemma 4.4

S^L is an irreducible unitary representation of $SU(2)$ and any irreducible representation of $SU(2)$ is equivalent to one of the S^L.

See [12], pages 48 and 58 for the proof.

Let $[S_u^L]_{\{e_j\}}$ be the matrix representation of the unitary operator S_u^L with respect to the basis $\{e_j\}$, and let its entries be $t_{k,j}^L(u)$, $0 \le k, j \le L$. We know (see [16]) that if $u \in SU(2)$, then

$$u = -i \begin{pmatrix} -\overline{X}_2 & X_1 \\ \overline{X}_1 & X_2 \end{pmatrix}$$

with X_1, $X_2 \in \mathbb{C}$ and $|X_1|^2 + |X_2|^2 = 1$.

Proposition 4.5

With the above notation, we have

$$\begin{aligned} t_{k,j}^L(u) = \frac{1}{2\pi} \frac{\sqrt{k!(L-k)!}}{\sqrt{j!(L-j)!}} (-i)^L \\ \times \int_{-\pi}^{\pi} \big(-\overline{X}_2 e^{i\theta} + \overline{X}_1 e^{-i\theta} \big)^j \big(X_1 e^{i\theta} + X_2 e^{-i\theta} \big)^{L-j} e^{-i(2k-L)\theta} d\theta. \end{aligned}$$

Proof. Observe that

$$S_u^L e_j(z) = \sum_{k=0}^{L} t_{k,j}^L(u) e_k(z).$$

The left hand side of the above equality is

$$\begin{aligned} e_j(u'z) &= e_j \left(-i \begin{pmatrix} -\overline{X}_2 & \overline{X}_1 \\ X_1 & X_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) = e_j \begin{pmatrix} i\overline{X}_2 z_1 - i\overline{X}_1 z_2 \\ -iX_1 z_1 - iX_2 z_2 \end{pmatrix} \\ &= \frac{\sqrt{L!}}{\sqrt{j!(L-j)!}} (-i)^L (-\overline{X}_2 z_1 + \overline{X}_1 z_2)^j (X_1 z_1 + X_2 z_2)^{L-j}. \end{aligned}$$

On the other hand,

$$\sum_{k=0}^{L} t_{k,j}^L(u) e_k(z) = \sum_{k=0}^{L} t_{k,j}^L(u) \frac{\sqrt{L!}}{\sqrt{k!(L-k)!}} z_1^k z_2^{L-k}.$$

If we restrict ourselves to the values $z = (z_1, z_2) = (e^{i\theta}, e^{-i\theta})$, we have

$$(-i)^L \frac{(-\overline{X}_2 e^{i\theta} + \overline{X}_1 e^{-i\theta})^j (X_1 e^{i\theta} + X_2 e^{-i\theta})^{L-j}}{\sqrt{j!(L-j)!}} = \sum_{k=0}^{L} t^L_{k,j}(u) \frac{e^{i(2k-L)\theta}}{\sqrt{k!(L-k)!}}.$$

The desired result now follows once we integrate both sides against $e^{-i(2k_0-L)\theta} d\theta$ over the interval $[-\pi, \pi]$. □

Proposition 4.6

Suppose L is even, $L = 2l$. Using the notation after Proposition 4.2, we have $t^L_{k,l}(u) = \widetilde{Y}^l_k(u)$, where

$$Y^l_k(y_1, y_2, y_3) \overset{\text{def}}{=} \frac{(-1)^l}{2\pi} \frac{\sqrt{k!(2l-k)!}}{l!} \int_{-\pi}^{\pi} \big(-y_1 + iy_2 \sin 2\theta + iy_3 \cos 2\theta \big)^l e^{-2i(k-l)\theta} d\theta$$

is defined on Σ_2. Furthermore, $\{Y^l_k\}_{k=0}^{2l}$ is an orthogonal basis of $\mathcal{H}_l(\Sigma_2)$, the space of spherical harmonics of degree l, and $\|Y^l_k\|_2^2 = \frac{4\pi}{2l+1}$.

Proof. We have to show that $t^L_{k,l}(u) = Y^l_k(\Phi(u)e)$. Note that $e = (1, 0, 0)$ are the coordinates, with respect to the above mentioned basis of M, of the element

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus,

$$\begin{aligned}\Phi(u)e &= u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u^* \\ &= \begin{pmatrix} |X_2|^2 - |X_1|^2 & -2X_1\overline{X}_2 \\ -2\overline{X}_1 X_2 & |X_1|^2 - |X_2|^2 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} y_1 & y_2 + iy_3 \\ y_2 - iy_3 & -y_1 \end{pmatrix}.\end{aligned}$$

By Proposition 4.5 we have that $t^L_{k,l}(u)$ equals

$$\begin{aligned}&\frac{\sqrt{k!(L-k)!}}{2\pi l!}(-i)^L \int_{-\pi}^{\pi} \big(-X_1\overline{X}_2 e^{i2\theta} + \overline{X}_1 X_2 e^{-i2\theta} + |X_1|^2 - |X_2|^2 \big)^l e^{-i(2k-L)\theta} d\theta \\ &= \frac{\sqrt{k!(L-k)!}}{2\pi l!}(-1)^l \int_{-\pi}^{\pi} \Big(\frac{y_2 + iy_3}{2} e^{i2\theta} - \frac{y_2 - iy_3}{2} e^{-i2\theta} - y_1 \Big)^l e^{-i(2k-L)\theta} d\theta \\ &= \frac{(-1)^l}{2\pi} \frac{\sqrt{k!(2l-k)!}}{l!} \int_{-\pi}^{\pi} \big(-y_1 + iy_2 \sin 2\theta + iy_3 \cos 2\theta \big)^l e^{-2i(k-l)\theta} d\theta.\end{aligned}$$

By the Peter-Weyl Theorem, $\{t^L_{k,l}\}_{k=0}^{2l}$ are mutually orthogonal in $L^2(SU(2))$ each having norm $\frac{1}{\sqrt{2l+1}}$. Thus, by Proposition 4.3, $\{Y^l_k\}_{k=0}^{2l}$ are mutually orthogonal in $L^2(\Sigma_2)$, and

$$\|Y^l_k\|_2^2 = \frac{4\pi}{2l+1}.$$

Finally, we have to check that $Y^l_k \in \mathcal{H}_l(\Sigma_2)$. Since $\dim \mathcal{H}_l(\Sigma_2) = 2l+1$, this will prove the proposition. The definition of $Y^l_k(y)$ makes sense if we assume $y \in \mathbb{R}^3$ instead of just Σ_2. Clearly, this extension is a homogeneous polynomial of degree l. So, we only have to check that Y^l_k is harmonic. By the mean value theorem for harmonic functions, it is enough to show that $G^l_\theta(y) \stackrel{\text{def}}{=} (-y_1 + iy_2 \sin 2\theta + iy_3 \cos 2\theta)^l$ is harmonic for all θ. Consider the following rotation of $\mathbb{R}^3$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$$

and define F^l by $F^l(x_1, x_2, x_3) \stackrel{\text{def}}{=} (-x_1 + ix_3)^l$. Then

$$G^l_\theta(y) = F^l(R_\theta y).$$

Since F^l is trivially harmonic (by inspection), G^l_θ is also harmonic. $\square$

We now recall some facts about harmonic analysis on $SU(2)$ (see [16]). If we define $\mathcal{H}_L \stackrel{\text{def}}{=} \operatorname{span}\{t^L_{k,j}\}_{0\le k,j\le L}$, by the Peter-Weyl Theorem, $L^2(SU(2)) = \bigoplus_{L=0}^\infty \mathcal{H}_L$. Also, the characters of $SU(2)$

$$\chi_L(u) \stackrel{\text{def}}{=} \operatorname{tr} S^L_u = \sum_{j=0}^{L} t^L_{j,j}(u)$$

are, as all characters of compact groups, central functions (i.e. $\chi_L(vuv^{-1}) = \chi_L(u)$ for all $u, v \in SU(2)$). It is well known that, for all $u \in SU(2)$, there is a $v \in SU(2)$ and a $\varphi \in [0, 2\pi]$ such that $u = vb(2\varphi)v^{-1}$. Thus

$$\chi_L(u) = \chi_L(b(2\varphi)) \stackrel{\text{def}}{=} \chi^0_L(\varphi), \quad \varphi \in [0, 2\pi]$$

($e^{i\varphi}$, $e^{-i\varphi}$ are the eigenvalues of u). We have

$$\chi^0_L(\varphi) = \frac{\sin(L+1)\varphi}{\sin\varphi}, \quad L = 0, 1, \dots$$

and for all $f \in L^2(SU(2))$

$$f = \sum_{L=0}^{\infty} (L+1)\chi_L * f$$

where $(L+1)\chi_L * f$ equals the projection of f onto $\mathcal{H}_L$.

Theorem 4.7

Let $1 \le p < \infty$, $m = \{m_L\}_{L=0}^{\infty}$ a bounded sequence of complex numbers and M_r the convolution operators on $SU(2)$ given by the central kernels k_r, where

$$k_r^0(\theta) = \sum_{L=0}^{\infty} (L+1) m_L r^L \chi_L^0(\theta).$$

If $h_r^0(\theta) \stackrel{\text{def}}{=} \sin^2\theta k_r^0(\theta)$ define uniformly bounded (by some constant A_p) convolution operators on $L^p(\Sigma_1)$ as $r \to 1^-$, then M_r are uniformly bounded on $L^p(SU(2))$, with norms bounded above by $\frac{A_p}{\pi}$.

Proof. (cfr. [16], page 216). Recall the following formula for the Haar measure on $SU(2)$

$$\int_{SU(2)} f(u)\, du = \frac{1}{\pi} \int_{SU(2)} \int_{-\pi}^{\pi} f(vb(\theta)v^{-1}) \sin^2\theta \, d\theta \, dv.$$

Thus:

$$\begin{aligned}\|k_r * f\|_{L^p(SU(2))} &= \left\| \int_{SU(2)} k_r(v) f(\cdot\, v^{-1})\, dv \right\|_{L^p(SU(2))} \\ &= \frac{1}{\pi} \left[\int_{SU(2)} \left| \int_{SU(2)} \int_{-\pi}^{\pi} h_r^0(\theta) f(uvb(-\theta)v^{-1})\, d\theta dv \right|^p du \right]^{1/p}.\end{aligned}$$

By Minkowski's integral inequality, we see that the last expression is dominated by

$$\frac{1}{\pi} \int_{SU(2)} \left[\int_{SU(2)} \left| \int_{-\pi}^{\pi} h_r^0(\theta) f(uvb(-\theta)v^{-1})\, d\theta \right|^p du \right]^{1/p} dv.$$

Because of the right invariance of Haar measure, the integral within the parentheses is unchanged if we multiply u on the right by $vb(\varphi)v^{-1}$. Doing so and averaging over all $\varphi \in [-\pi, \pi]$ we obtain that the last expression equals

$$\frac{1}{\pi} \int_{SU(2)} \left[\int_{SU(2)} \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} h_r^0(\theta) f\big(uvb(\varphi - \theta)v^{-1}\big)\, d\theta \right|^p d\varphi \right\} du \right]^{1/p} dv.$$

The expression in curly brackets is the pth power of the L^p norm of the convolution of h_r^0 with the function g having values $g(\varphi) = f(uvb(\varphi)v^{-1})$. Thus, the last expression is no larger than

$$\begin{aligned}
&\frac{1}{\pi}\int_{SU(2)}\left[\int_{SU(2)}\frac{A_p^p}{2\pi}\int_{-\pi}^{\pi}\left|f(uvb(\varphi)v^{-1})\right|^p d\varphi\, du\right]^{1/p} dv \\
&= \frac{A_p}{\pi}\int_{SU(2)}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{SU(2)}\left|f(uvb(\varphi)v^{-1})\right|^p du\, d\varphi\right]^{1/p} dv \\
&= \frac{A_p}{\pi}\left[\int_{SU(2)}|f(u)|^p du\right]^{1/p}. \ \square
\end{aligned}$$

DEFINITION 4.8. For any sequence $m = \{m_l\}_{l=0}^{\infty}$ define the associated sequences $\mathcal{Z}m$ and $\mathcal{W}m$ by

$$(\mathcal{Z}m)_l \stackrel{\text{def}}{=} \begin{cases} m_{l/2} & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd} \end{cases}$$

and

$$(\mathcal{W}m)_l \stackrel{\text{def}}{=} \begin{cases} (l+1)m_{l/2} - (l-1)m_{l/2-1} & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

The following theorem answers the question we posed at the end of the last section.

Theorem 4.9

Let $1 \leq p < \infty$ and $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers. Suppose $\tau_2\mathcal{Z}m$ and $\mathcal{W}m$ define bounded zonal multipliers on $L^p(\Sigma_1)$. Then m is a bounded zonal multiplier on $L^p(\Sigma_2)$.

Proof. The proof can be divided into four steps.

Step 1. Once again, for any $r \in (0,1)$ and for any sequence $a = \{a_L\}_{L=0}^{\infty}$, define the sequence $\mathcal{R}_r a \stackrel{\text{def}}{=} \{r^L a_L\}_{L=0}^{\infty}$. We will prove that $\mathcal{D}\mathcal{R}_r\mathcal{Z}m + T_1\mathcal{R}_r\mathcal{Z}m$ are uniformly bounded multipliers on $L^p(\Sigma_1)$ as $r \to 1^-$ ($\mathcal{D}$ and T_1 are defined in Lemma 3.6). First of all note that

$$\begin{aligned}
&(\mathcal{D}\mathcal{R}_r\mathcal{Z}m)_L + (T_1\mathcal{R}_r\mathcal{Z}m)_L \\
&\qquad = (\mathcal{R}_r\mathcal{W}m)_L + [(L+1)r^L + (L-1)r^{L-2}](\tau_2\mathcal{Z}m)_L - 2(\mathcal{R}_r\tau_2\mathcal{Z}m)_L.
\end{aligned}$$

Suppose $f \in \mathcal{P}(\Sigma_1)$; that is, $f = \sum_{L=0}^{k} f_L$ with $f_L \in \mathcal{H}_L(\Sigma_1)$. Then

$$\begin{aligned}\left\|\sum_{L=0}^{k}(\mathcal{R}_r\mathcal{W}m)_L f_L\right\|_{L^p(\Sigma_1)} &= \left\|P_r * \sum_{L=0}^{k}(\mathcal{W}m)_L f_L\right\|_{L^p(\Sigma_1)}\\ &\le \left\|\sum_{L=0}^{k}(\mathcal{W}m)_L f_L\right\|_{L^p(\Sigma_1)};\end{aligned}$$

$$\begin{aligned}&\left\|\sum_{L=0}^{k}[(L+1]r^L + (L-1)r^{L-2}](\tau_2\mathcal{Z}m)_L f_L\right\|_{L^p(\Sigma_1)}\\ &\qquad = \left\|\frac{\partial}{\partial r}\left(rP_r - \frac{1}{r}P_r\right) * \sum_{L=0}^{k}(\tau_2\mathcal{Z}m)_L f_L\right\|_{L^p(\Sigma_1)}\\ &\qquad \le C\left\|\sum_{L=0}^{k}(\tau_2\mathcal{Z}m)_L f_L\right\|_{L^p(\Sigma_1)};\end{aligned}$$

$$\begin{aligned}\left\|\sum_{L=0}^{k}(\mathcal{R}_r\mathcal{W}m)_L f_L\right\|_{L^p(\Sigma_1)} &= 2\left\|P_r * \sum_{L=0}^{k}(\tau_2\mathcal{Z}m)_L f_L\right\|_{L^p(\Sigma_1)}\\ &\le 2\left\|\sum_{L=0}^{k}(\tau_2\mathcal{Z}m)_L f_L\right\|_{L^p(\Sigma_1)}\end{aligned}$$

(the second to last inequality follows from Lemma 3.7 (ii)). This proves Step 1.

Step 2. We will prove that the central kernels in $SU(2)$ given by

$$k_r^0(\theta) \stackrel{\text{def}}{=} \sum_{L=0}^{\infty} r^L(L+1)(\mathcal{Z}m)_L\chi_L^0(\theta)$$

define uniformly bounded convolution operators on $L^p(SU(2))$ as $r \to 1^-$. Consider the following equalities (where we define $(\mathcal{Z}m)_{-1} = (\mathcal{Z}m)_{-2} = 0$):

$$\sum_{L=0}^{\infty} r^L(L+1)(\mathcal{Z}m)_L \sin^2\theta\chi_L^0(\theta) = \sum_{L=0}^{\infty} r^L(L+1)(\mathcal{Z}m)_L \sin\theta\sin((L+1)\theta)$$
$$= \frac{\pi}{2}\sum_{L=0}^{\infty} r^L(L+1)(\mathcal{Z}m)_L \frac{\cos L\theta - \cos((L+2)\theta)}{\pi} = \frac{\pi}{2}\sum_{L=0}^{\infty}$$
$$\times \left[Lr^L(\mathcal{Z}m)_L + r^L(\mathcal{Z}m)_L - (L-2)r^{L-2}(\mathcal{Z}m)_{L-2} - r^{L-2}(\mathcal{Z}m)_{L-2}\right] \frac{\cos L\theta}{\pi}$$
$$= \frac{\pi}{2}\sum_{L=0}^{\infty}[(\mathcal{D}\mathcal{R}_r\mathcal{Z}m)_L + (T_1\mathcal{R}_r\mathcal{Z}m)_L]\left(Z_{e,1}^L\right)^0(\cos\theta).$$

By Step 1, these are uniformly bounded convolution kernels on $L^p(\Sigma_1)$; thus, by Theorem 4.7, k_r are uniformly bounded convolution kernels on $L^p(SU(2))$.

Step 3. The multipliers M_r associated with the sequences $\mathcal{R}_r m$ are uniformly bounded on $L^p(\Sigma_2)$ as $r \to 1^-$. Indeed, let f be a trigonometric polynomial on Σ_2; that is, in the notation of Proposition 4.6,

$$f = \sum_{l=0}^{j}\sum_{k=0}^{2l}\left(f, \frac{\sqrt{2l+1}}{2\sqrt{\pi}}Y_k^l\right)\frac{\sqrt{2l+1}}{2\sqrt{\pi}}Y_k^l.$$

Then

$$\tilde{f}(u) = \sum_{l=0}^{j}\sum_{k=0}^{2l}\left(f, \frac{\sqrt{2l+1}}{2\sqrt{\pi}}Y_k^l\right)\frac{\sqrt{2l+1}}{2\sqrt{\pi}}t_{k,l}^{2l}(u).$$

In particular, denoting by P_L the projection of $L^2(SU(2))$ onto $\mathcal{H}_L$,

$$k_{\sqrt{r}} * \tilde{f}(u) = \sum_{L=0}^{\infty} r^{L/2}(L+1)(\mathcal{Z}m)_L\chi_L * \tilde{f}(u) = \sum_{L=0}^{\infty} r^{L/2}(\mathcal{Z}m)_L P_L\tilde{f}(u)$$
$$= \sum_{l=0}^{\infty} r^l(\mathcal{Z}m)_{2l}P_{2l}\tilde{f}(u) + \sum_{l=0}^{\infty} r^{l+1/2}(\mathcal{Z}m)_{2l+1}P_{2l+1}\tilde{f}(u)$$
$$= \sum_{l=0}^{j} r^l(\mathcal{Z}m)_{2l}\sum_{k=0}^{2l}\frac{2l+1}{4\pi}\left(f, Y_k^l\right)t_{k,l}^{2l}(u)$$
$$= \left[\sum_{l=0}^{j} r^l m_l \sum_{k=0}^{2l}\frac{2l+1}{4\pi}\left(f, Y_k^l\right)Y_k^l\right]^{\sim}(u) = [M_r f]^{\sim}(u).$$

Thus,

$$\begin{aligned}\|M_r f\|^p_{L^p(\Sigma_2)} &= 4\pi \|(M_r f)^{\sim}\|^p_{L^p(SU(2))} = 4\pi \|k_{\sqrt{r}} * \tilde{f}\|^p_{L^p(SU(2))} \\ &\le 4\pi C \|\tilde{f}\|^p_{L^p(SU(2))} = C\|f\|^p_{L^p(\Sigma_2)}.\end{aligned}$$

Step 4. M, the multiplier given by the sequence m, is bounded on $L^p(\Sigma_2)$. Indeed, let $f = \sum_{l=0}^k f_l$ be a trigonometric polynomial ($f_l \in \mathcal{H}_l(\Sigma_2)$); then

$$\begin{aligned}\|Mf\|^p_{L^p(\Sigma_2)} &= \int_{\Sigma_2} \left|\sum_{l=0}^k m_l f_l(x)\right|^p dx = \int_{\Sigma_2} \lim_{r\to 1^-} \left|\sum_{l=0}^k r^l m_l f_l(x)\right|^p dx \\ &\le \liminf_{r\to 1^-} \int_{\Sigma_2} \left|\sum_{l=0}^k r^l m_l f_l(x)\right|^p dx \le C \int_{\Sigma_2} |f(x)|^p dx = C\|f\|^p_{L^p(\Sigma_2)}. \ \square\end{aligned}$$

5. Conclusion of the proof

In order to conclude the proof of Theorem 3.1, we need to establish some facts about zonal multipliers on $L^p(\Sigma_1)$. First of all, recall that we can identify functions on Σ_1 with 2π-periodic functions on the real line,

$$f_0(\theta) \stackrel{\text{def}}{=} f(\cos\theta, \sin\theta)$$

and, with this notation,

$$\int_{\Sigma_1} f(x)\, dx = \int_{-\pi}^{\pi} f_0(\theta)\, d\theta.$$

¿From now on, we shall identify f and f_0.

Lemma 5.1

Let $1 \le p < \infty$, $m = \{m_l\}_{l=0}^\infty$ be a bounded sequence of complex numbers. If m defines a bounded zonal multiplier on $L^p(\Sigma_1)$, then $\mathcal{Z}m$ defines a bounded zonal multiplier on $L^p(\Sigma_1)$.

Proof. (The Authors owe the following simple proof to Dr. Gustavo Garrigós.) Call $\Pi(\Sigma_1)$ the space of π-periodic functions in $L^p(\Sigma_1)$. $\Pi(\Sigma_1)$ is a Banach space with the norm induced by $L^p(\Sigma_1)$. Consider the following operators from $L^p(\Sigma_1)$ to $\Pi(\Sigma_1)$:

$$\begin{aligned} Tf(\theta) &\stackrel{\text{def}}{=} \frac{1}{2}\left(f(\theta)+f(\theta+\pi)\right) \\ \delta f(\theta) &\stackrel{\text{def}}{=} f(2\theta). \end{aligned}$$

T, δ and δ^{-1} are bounded linear operators (where defined). Denoting by M_m and $M_{\mathcal{Z}m}$ the zonal multipliers associated with m and $\mathcal{Z}m$ respectively, we have

$$\delta M_m \delta^{-1} T = M_{\mathcal{Z}m}.$$

Indeed, let $f \in \mathcal{P}(\Sigma_1)$, $f(\theta) = a_0 + \sum_{l=1}^{k}\left(a_l e^{il\theta} + b_l e^{-il\theta}\right)$, then

$$\begin{aligned} \delta M_m \delta^{-1} T f(\theta) &= \delta M_m \delta^{-1}\left(a_0 + \sum_{j=1}^{\left[\frac{k}{2}\right]} a_{2j} e^{i2j\theta} + b_{2j} e^{-i2j\theta}\right) \\ &= \delta M_m\left(a_0 + \sum_{j=1}^{\left[\frac{k}{2}\right]} a_{2j} e^{ij\theta} + b_{2j} e^{-ij\theta}\right) \\ &= \delta\left(m_0 a_0 + \sum_{j=1}^{\left[\frac{k}{2}\right]} m_j\left(a_{2j} e^{ij\theta} + b_{2j} e^{-ij\theta}\right)\right) \\ &= m_0 a_0 + \sum_{j=1}^{\left[\frac{k}{2}\right]} m_j\left(a_{2j} e^{i2j\theta} + b_{2j} e^{-i2j\theta}\right) = M_{\mathcal{Z}m} f(\theta). \ \square \end{aligned}$$

Lemma 5.2

Let $1 < p < \infty$ and $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers. If m is a bounded zonal multiplier on $L^p(\Sigma_1)$, then **(i)** *$\tau_h m$ and* **(ii)** *$\mathcal{A}m \stackrel{\text{def}}{=} \left\{\frac{1}{l+1}\sum_{j=0}^{l} m_j\right\}_{l=0}^{\infty}$ are bounded on $L^p(\Sigma_1)$. Also, if* **(iii)** *$\mathcal{D}m$ or* **(iv)** *Jm are bounded on $L^p(\Sigma_1)$, then m is bounded on $L^p(\Sigma_1)$.*

Proof. **(i)** Consider the following operators, bounded on $L^p(\Sigma_1)$ (here $f(\theta) = a_0 + \sum_{l=1}^{k}\left(a_l e^{il\theta} + b_l e^{-il\theta}\right) \in \mathcal{P}(\Sigma_1)$):

$$B_{\pm 1}f(\theta) \stackrel{\text{def}}{=} e^{\pm i\theta}f(\theta)$$
$$M_m f(\theta) \stackrel{\text{def}}{=} m_0 a_0 + \sum_{l=1}^{k} m_l \left(a_l e^{il\theta} + b_l e^{-il\theta}\right)$$
$$Hf(\theta) \stackrel{\text{def}}{=} \tilde{f}(\theta) \quad \text{(the conjugate function, see [9])}$$
$$\tilde{b}f(\theta) \stackrel{\text{def}}{=} -\frac{1}{2}a_0 + \frac{1}{2}\big(f(\theta) + iHf(\theta)\big) = \sum_{l=1}^{k} a_l e^{il\theta}$$
$$\tilde{\tilde{b}}f(\theta) \stackrel{\text{def}}{=} -\frac{1}{2}a_0 + \frac{1}{2}\big(f(\theta) - iHf(\theta)\big) = \sum_{l=1}^{k} b_l e^{-il\theta}.$$

We will show that the operator

$$\tilde{b}B_1 M_m B_{-1} + \tilde{\tilde{b}}B_{-1}M_m B_1$$

has $\tau_1 m$ as its Fourier transform. This will prove that $\tau_1 m$ defines a bounded multiplier on $L^p(\Sigma_1)$. Part **(i)** will then follow immediately.

$$\begin{aligned}
\tilde{b}B_1 M_m B_{-1}f(\theta) &= \tilde{b}B_1 M_m \left(a_0 e^{-i\theta} + \sum_{l=1}^{k} a_l e^{i(l-1)\theta} + b_l e^{-i(l+1)\theta}\right) \\
&= \tilde{b}B_1 \left(m_1 a_0 e^{-i\theta} + \sum_{l=1}^{k} m_{l-1} a_l e^{i(l-1)\theta} + m_{l+1} b_l e^{-i(l+1)\theta}\right) \\
&= \tilde{b}\left(m_1 a_0 + \sum_{l=1}^{k} m_{l-1} a_l e^{il\theta} + m_{l+1} b_l e^{-il\theta}\right) = \sum_{l=1}^{k} m_{l-1} a_l e^{il\theta}.
\end{aligned}$$

Similarly

$$\tilde{\tilde{b}}B_{-1}M_m B_1 f(\theta) = \sum_{l=1}^{k} m_{l-1} b_l e^{-il\theta}.$$

(ii) This is a trivial application of the Marcinkiewicz Theorem on multipliers (see [10]).

(iii) Without loss of generality, we can assume $m_0 = 0$ (if not, recall that the sequence $\widetilde{m} = \{m_0, 0, \ldots, 0, \ldots\}$ defines a bounded multiplier on $L^p(\Sigma_1)$, namely convolution with the constant function $g = m_0$, that $\mathcal{D}m = \mathcal{D}(m - \widetilde{m})$ and that $m = \widetilde{m} + (m - \widetilde{m})$). Applying Lemma 5.1 to the identity map, we see that the sequence $\mu = \{\mu_l\}_{l=0}^{\infty} = \{1,0,1,0,\ldots\}$ defines a bounded multiplier on $L^p(\Sigma_1)$ and, by part **(ii)**, the same is true for

$$\gamma = \{\gamma_l\}_{l=0}^{\infty} \stackrel{\text{def}}{=} \left\{ \frac{1}{l+1} \sum_{j=0}^{l} \mu_j (\mathcal{D}m)_j \right\}_{l=0}^{\infty}$$

and

$$\mu\gamma \stackrel{\text{def}}{=} \{\mu_l \gamma_l\}_{l=0}^{\infty}.$$

Note that

$$\mu_l \gamma_l = \begin{cases} \frac{l}{l+1} m_l & l \text{ even} \\ 0 & l \text{ odd.} \end{cases}$$

The sequence $\xi = \{\xi_l\}_{l=0}^{\infty}$ with

$$\xi_l \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } l = 0 \\ \frac{l+1}{l} & \text{if } l > 0 \end{cases}$$

is a bounded multiplier on $L^p(\Sigma_1)$. Thus the sequence

$$\xi_l \mu_l \gamma_l = \begin{cases} m_l & l \text{ even} \\ 0 & l \text{ odd} \end{cases}$$

is a bounded multiplier on $L^p(\Sigma_1)$. A similar argument shows that the sequence given by

$$\begin{cases} 0 & l \text{ even} \\ m_l & l \text{ odd} \end{cases}$$

defines a bounded multiplier on $L^p(\Sigma_1)$.

The proof of **(iv)** is similar to (but simpler than) that of part **(iii)**. □

Lemma 5.3

Let $m = \{m_l\}_{l=0}^{\infty}$ be a sequence of complex numbers and j a positive integer. Then:

(i) $\mathcal{D}\tau_{2j}m = \tau_{2j}\mathcal{D}m + 2j\tau_{2j}m - 2j\tau_{2j+2}m.$
(ii) $J\tau_j m = \tau_j Jm + j\tau_j m - j\tau_{j+1}m.$
(iii) $\mathcal{Z}\tau_{2j}m = \tau_{4j}\mathcal{Z}m.$
(iv) $\mathcal{Z}\mathcal{D}m = \frac{1}{2}\mathcal{D}\mathcal{Z}m + \frac{1}{2}\tau_2\mathcal{D}\mathcal{Z}m.$
(v) $\mathcal{D}m = Jm + \tau_1 Jm.$
(vi) $\mathcal{D}\mathcal{Z}m = 2\mathcal{Z}Jm.$

All these equalities are easily verified.

Lemma 5.4

Let $1 < p < \infty$, $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers and N a positive integer. Suppose $\mathcal{D}^N \mathcal{Z} m$ is a bounded multiplier on $L^p(\Sigma_1)$. Then $\mathcal{D}^k \mathcal{Z} \mathcal{D}^j m$ is a bounded multiplier on $L^p(\Sigma_1)$, for $k \geq 0$, $j \geq 0$, $k + j \leq N$.

Proof. First note that, by Lemma 5.2 (iii), $\mathcal{Z}m$, $\mathcal{D}\mathcal{Z}m$, $\mathcal{D}^2\mathcal{Z}m, \ldots, \mathcal{D}^N\mathcal{Z}m$ are all bounded multipliers on $L^p(\Sigma_1)$. The lemma is therefore true for $j = 0$. Applying an induction argument on j, assume that the lemma is true for $j-1$, $k = 0, 1, \ldots, N - j + 1$, and let us prove it for j, $k = 0, 1, \ldots, N - j$, $(j > 0)$. Applying Lemma 5.3 (iv), we obtain

$$\mathcal{D}^k \mathcal{Z} \mathcal{D}^j m = \frac{1}{2}\mathcal{D}^{k+1}\mathcal{Z}\mathcal{D}^{j-1}m + \frac{1}{2}\mathcal{D}^k \tau_2 \mathcal{D}\mathcal{Z}\mathcal{D}^{j-1}m.$$

The first term is a bounded multiplier on $L^p(\Sigma_1)$ by the induction hypothesis. As for the second, note the following: for all positive integers t, for all non-negative integers h and s and for all sequences $M = \{M_l\}_{l=0}^{\infty}$, we have (applying Lemma 5.3 (i))

$$\mathcal{D}^t \tau_{2h} \mathcal{D}^s M = \mathcal{D}^{t-1}\tau_{2h}\mathcal{D}^{s+1}M + 2h\mathcal{D}^{t-1}\tau_{2h}\mathcal{D}^s M - 2h\mathcal{D}^{t-1}\tau_{2h+2}\mathcal{D}^s M \tag{5.1}$$

The last equality implies that a sufficient condition for $\mathcal{D}^k\tau_2\mathcal{D}\mathcal{Z}\mathcal{D}^{j-1}m$ to be a bounded multiplier on $L^p(\Sigma_1)$ is that

$$\mathcal{D}^{k-1}\tau_{2h_1}\mathcal{D}^{s_1}\mathcal{Z}\mathcal{D}^{j-1}m \tag{5.2}$$

be bounded multipliers on $L^p(\Sigma_1)$ for all $h_1 \geq 1$ and for $s_1 = 1$, 2. Using (5.1) again, (5.2) are bounded multipliers on $L^p(\Sigma_1)$ if

$$\mathcal{D}^{k-2}\tau_{2h_2}\mathcal{D}^{s_2}\mathcal{Z}\mathcal{D}^{j-1}m$$

are bounded multipliers on $L^p(\Sigma_1)$ for all $h_2 \geq 1$ and for $s_2 = 1$, 2, 3. Continuing this reasoning, we obtain that $\mathcal{D}^k\tau_2\mathcal{D}\mathcal{Z}\mathcal{D}^{j-1}m$ is a bounded multiplier on $L^p(\Sigma_1)$ if $\tau_{2h_k}\mathcal{D}^{s_k}\mathcal{Z}\mathcal{D}^{j-1}m$ are bounded multipliers on $L^p(\Sigma_1)$, for all $h_k \geq 1$, for $s_k = 1, 2, \ldots, k+1$. Lemma 5.2 (i) implies that it is enough to require that $\mathcal{D}^{s_k}\mathcal{Z}\mathcal{D}^{j-1}m$ be bounded multipliers for $s_k = 1, 2, \ldots, k+1$, which is true by the induction hypothesis. □

Lemma 5.5

Let $1 < p < \infty$, $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers and N a positive integer. Suppose $J^N m$ is a bounded multiplier on $L^p(\Sigma_1)$. Then $J^j\mathcal{D}^k m$ is a bounded multiplier on $L^p(\Sigma_1)$, for $k \geq 0$, $j \geq 0$, $j + k \leq N$.

Proof. First note that Lemma 5.2 (iv) implies that m, Jm, $J^2m, \ldots, J^Nm$ are bounded multipliers on $L^p(\Sigma_1)$. Let $h = j + k$ and let us make an induction on h:
$h = 0$ implies $j = k = 0$. Thus $J^j\mathcal{D}^km = m$, which is bounded on $L^p(\Sigma_1)$.
$h = 1$ implies either $j = 1$ and $k = 0$ (in which case we obtain Jm, which is bounded on $L^p(\Sigma_1)$) or $j = 0$ and $k = 1$. In the latter case we get $\mathcal{D}m$ which, by Lemma 5.3 (v), equals $Jm + \tau_1 Jm$ and this is a bounded multiplier on $L^p(\Sigma_1)$ (use Lemma 5.2 (i) again).
Suppose now the lemma is true for all j's and k's such that $j + k < h$. We claim that the induction hypothesis and the fact that $J^j\mathcal{D}^km$ is a bounded multiplier on $L^p(\Sigma_1)$, $j + k = h$, $j > 0$, imply that $J^{j-1}\mathcal{D}^{k+1}m$ is a bounded multiplier on $L^p(\Sigma_1)$. This claim, together with the fact that J^hm is a bounded multiplier on $L^p(\Sigma_1)$, proves the lemma. Let us prove the claim. By Lemma 5.3 (v),

$$J^{j-1}\mathcal{D}^{k+1}m = J^j\mathcal{D}^km + J^{j-1}\tau_1 J\mathcal{D}^km.$$

$J^j\mathcal{D}^km$ is a bounded multiplier by the hypothesis of the claim. By Lemma 5.3 (ii), the boundedness of $J^{j-1}\tau_1 J\mathcal{D}^km$ on $L^p(\Sigma_1)$ is implied by the boundedness of

$$J^{j-2}\tau_1 J^2\mathcal{D}^km, \quad J^{j-2}\tau_1 J\mathcal{D}^km, \quad J^{j-2}\tau_2 J\mathcal{D}^km.$$

Being generous, we seek for the boundedness of

$$J^{j-2}\tau_{r_2} J^{s_2}\mathcal{D}^km, \quad \forall\, r_2 \geq 1,\ s_2 = 1, 2$$

which, again using Lemma 5.3 (ii), is implied by the boundedness of

$$J^{j-3}\tau_{r_3} J^{s_3}\mathcal{D}^km, \quad \forall\, r_3 \geq 1,\ s_3 = 1, 2, 3.$$

Proceeding similarly, all we need is the boundedness on $L^p(\Sigma_1)$ of

$$\tau_{r_j} J^{s_j}\mathcal{D}^km, \quad \forall\, r_j \geq 1,\ s_j = 1, 2, 3, \ldots, j.$$

By Lemma 5.2 (i), we only need the boundedness of

$$J^{s_j}\mathcal{D}^km, \quad \text{for } s_j = 1, 2, 3, \ldots, j.$$

which follows from the induction hypothesis and the hypothesis of the claim. □

The following is a corollary of Theorem 4.9.

Theorem 5.6

Let $1 < p < \infty$ and $m = \{m_l\}_{l=0}^{\infty}$ a sequence of complex numbers. Suppose $\mathcal{D}\mathcal{Z}m$ is a bounded multiplier on $L^p(\Sigma_1)$. Then m is a bounded multiplier on $L^p(\Sigma_2)$.

Proof. This follows from Lemma 5.2 (i) and (iii) and the easily verified equality

$$\mathcal{W}m = \mathcal{D}\mathcal{Z}m + \mathcal{Z}m - \tau_2\mathcal{Z}m. \square$$

DEFINITION 5.7. Let $m = \{m_l\}_{l=0}^{\infty}$ be a bounded sequence of complex numbers. For any integer $j \geq 0$ we define the family of sequences $\mathcal{A}_j$ by

$$\mathcal{A}_j \stackrel{\text{def}}{=} \left\{\tau_{2h}\mathcal{D}^{j-k}m,\, 0 \leq h \leq k \leq j\right\}.$$

We can now state and prove a preliminary version of Theorem 3.1.

Theorem 5.8

Let $1 < p < \infty$, n a positive integer, $N = \left[\frac{n}{2}\right]$ and $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers.

(i) Assume n is odd. If $\mathcal{D}^N m$ is a bounded multiplier on $L^p(\Sigma_1)$, then m is a bounded multiplier on $L^p(\Sigma_n)$.

(ii) Assume n is even. If $\mathcal{D}^N\mathcal{Z}m$ is a bounded multiplier on $L^p(\Sigma_1)$, then m is a bounded multiplier on $L^p(\Sigma_n)$.

Proof. The case $n = 1$ is trivial, and the case $n = 2$ coincides with Theorem 5.6. Suppose $n \geq 3$. We claim that, for any integer j, $1 \leq j \leq \frac{n-1}{2}$, if the sequences in $\mathcal{A}_j$ are bounded multipliers on $L^p(\Sigma_{n-2j})$, then the sequences in $\mathcal{A}_{j-1}$ are bounded multipliers on $L^p(\Sigma_{n-2j+2})$. Note that

$$\mathcal{A}_{j-1} = \left\{\tau_{2h}\mathcal{D}^{j-k}m,\, 0 \leq h < k \leq j\right\}$$

and, for $j = 1$, this claim coincides with Theorem 3.9.

By Theorem 3.9, we have to show that, for any sequence $\mu \in \mathcal{A}_{j-1}$, the sequences $\mathcal{D}\mu$, $\tau_2\mu$, μ are bounded multipliers on $L^p(\Sigma_{n-2j})$. Note that $\tau_2\mu$ and μ belong to $\mathcal{A}_j$, so we only have to worry about sequences of the type $\mathcal{D}\mu$, with $\mu \in \mathcal{A}_{j-1}$. But, by Lemma 5.3 (i),

$$\mathcal{D}\mu = \mathcal{D}\tau_{2h}\mathcal{D}^{j-k}m = \tau_{2h}\mathcal{D}^{j-k+1}m + 2h\tau_{2h}\mathcal{D}^{j-k}m - 2h\tau_{2h+2}\mathcal{D}^{j-k}m$$

which is a bounded multiplier on $L^p(\Sigma_{n-2j})$ since the three sequences

$$\tau_{2h}\mathcal{D}^{j-(k-1)}m, \quad \tau_{2h}\mathcal{D}^{j-k}m, \quad \tau_{2(h+1)}\mathcal{D}^{j-k}m$$

belong to $\mathcal{A}_j$, for $0 \leq h < k \leq j$. This proves the claim.

(i) Assume n is odd. Applying the above claim N times, we see that (starting with $j = 1$) $\mathcal{A}_0 = \{m\}$ is a bounded multiplier on $L^p(\Sigma_{n-2+2}) = L^p(\Sigma_n)$ if (finishing with $j = N$) the sequences in $\mathcal{A}_N$ are bounded multipliers on $L^p(\Sigma_{n-2N}) = L^p(\Sigma_1)$. By Lemma 5.2 (i) and (iii), it is sufficient to assume that $\mathcal{D}^N m$ be a bounded multiplier on $L^p(\Sigma_1)$.

(ii) Assume n is even. Applying the claim $N-1$ times, we obtain that m is a bounded multiplier on $L^p(\Sigma_n)$ if the sequences in $\mathcal{A}_{N-1}$ are bounded multipliers on $L^p(\Sigma_2)$. Thus, the typical sequence that we have to prove to be bounded on $L^p(\Sigma_2)$ is

$$\tau_{2h}\mathcal{D}^{N-1-k}m, \quad 0 \le h \le k \le N-1.$$

By Theorem 5.6, this happens if

$$\mathcal{D}\mathcal{Z}\tau_{2h}\mathcal{D}^{N-1-k}m, \quad 0 \le h \le k \le N-1$$

are bounded multipliers on $L^p(\Sigma_1)$. But, by Lemma 5.3 (i) and (iii),

$$\begin{aligned}\mathcal{D}\mathcal{Z}\tau_{2h}\mathcal{D}^{N-1-k}m = \mathcal{D}\tau_{4h}\mathcal{Z}\mathcal{D}^{N-1-k}m &= \tau_{4h}\mathcal{D}\mathcal{Z}\mathcal{D}^{N-1-k}m \\ &+ 4h\tau_{4h}\mathcal{Z}\mathcal{D}^{N-1-k}m - 4h\tau_{4h+2}\mathcal{Z}\mathcal{D}^{N-1-k}m.\end{aligned}$$

By Lemma 5.2 (i) and (iii),

$$\mathcal{D}\mathcal{Z}\tau_{2h}\mathcal{D}^{N-1-k}m, \quad 0 \le h \le k \le N-1$$

is then a bounded multiplier on $L^p(\Sigma_1)$ if

$$\mathcal{D}\mathcal{Z}\mathcal{D}^{N-1-k}m, \quad 0 \le k \le N-1$$

are bounded multipliers on $L^p(\Sigma_1)$, which, by Lemma 5.4, is true if $\mathcal{D}^N\mathcal{Z}m$ is bounded on $L^p(\Sigma_1)$. □

Proof of Theorem 3.1. Assume first n is odd. Since $J^N m$ is a bounded multiplier on $L^p(\Sigma_1)$, Lemma 5.5 implies that $\mathcal{D}^N m$ is a bounded multiplier on $L^p(\Sigma_1)$. Theorem 5.8 implies that m is a bounded multiplier on $L^p(\Sigma_n)$.

Suppose now that n is even. Just as in the proof of Theorem 5.8, all we need to show is that

$$\mathcal{D}\mathcal{Z}\mathcal{D}^{N-1-k}m, \quad 0 \le k \le N-1$$

be bounded multipliers on $L^p(\Sigma_1)$. But, by Lemma 5.3 (vi),

$$\mathcal{D}\mathcal{Z}\mathcal{D}^{N-1-k}m = 2\mathcal{Z}J\mathcal{D}^{N-1-k}m$$

and $\mathcal{Z}J\mathcal{D}^{N-1-k}m$ is a bounded multiplier on $L^p(\Sigma_1)$ if (Lemma 5.1)

$$J\mathcal{D}^{N-1-k}m$$

is a bounded multiplier on $L^p(\Sigma_1)$, for any $0 \le k \le N-1$. The conclusion now follows from the hypothesis of the theorem and Lemma 5.5. □

6. A theorem by Bonami and Clerc

In their paper [2], Bonami and Clerc proved an extension to the n-dimensional sphere of the Marcinkiewicz theorem on multipliers (see [10]). Here we show how their theorem can be obtained by "transferring" the 1-dimensional result to the n-dimensional sphere by means of Theorem 3.1.

DEFINITION 6.1. Let $m = \{m_l\}_{l=0}^{\infty}$ be a bounded sequence of complex numbers. The sequence Δm is defined by

$$(\Delta m)_l \stackrel{\text{def}}{=} m_l - m_{l-1}$$

where, as usual, we assume $m_{-1} = 0$.

Lemma 6.2

Let $N \geq 1$, $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers. Then $J^N m$ is a linear combination of sequences of the type

$$\{l^h(\Delta^s m)_l\}_{l=0}^{\infty},$$

with $0 \leq h \leq s \leq N$.

Proof. By induction. We have $(Jm)_l = l(\Delta m)_l - (\Delta m)_l + m_l$ and the lemma is true for $N = 1$. Suppose the lemma is true for $N - 1$. Then

$$\begin{aligned}(J^N m)_l &= l(J^{N-1}m)_l - (l-1)(J^{N-1}m)_{l-1} \\ &= \sum_{0\leq h\leq s\leq N-1} a_{h,s}\left[l^{h+1}(\Delta^s m)_l - (l-1)^{h+1}(\Delta^s m)_{l-1}\right] \\ &= \sum_{0\leq h\leq s\leq N-1} a_{h,s}\left[(l-1)^{h+1}(\Delta^{s+1}m)_l + (l^{h+1} - (l-1)^{h+1})(\Delta^s m)_l\right].\end{aligned}$$

Since $l^{h+1} - (l-1)^{h+1}$ is a polynomial of degree h, the lemma is true for N. $\square$

Lemma 6.3

Let $N \geq 1$, $m = \{m_l\}_{l=0}^{\infty}$ a bounded sequence of complex numbers. Then $\Delta J^N m$ is a linear combination of sequences of the type

$$\{l^h(\Delta^{s+1} m)_l\}_{l=0}^{\infty},$$

with $0 \leq h \leq s \leq N$.

Proof. Applying the previous lemma, we have

$$\begin{aligned}(\Delta J^N m)_l &= (J^N m)_l - (J^N m)_{l-1} \\ &= \sum_{0\le h\le s\le N} a_{h,s}\left[l^h(\Delta^s m)_l - (l-1)^h(\Delta^s m)_{l-1}\right] \\ &= \sum_{0\le h\le s\le N} a_{h,s}\left[(l-1)^h(\Delta^{s+1} m)_l + (l^h-(l-1)^h)(\Delta^s m)_l\right].\end{aligned}$$

Since $l^h - (l-1)^h$ has degree $h-1$, the lemma is proved. □

Theorem 6.4 (Bonami and Clerc.)

Let $1 < p < \infty$, $n \ge 1$, $N = \left[\frac{n}{2}\right]$. Suppose

(i) $m = \{m_l\}_{l=0}^{\infty}$ *is a bounded sequence of complex numbers.*

(ii) $\sup_{j\ge 0} 2^{jN} \sum_{l=2^j}^{2^{j+1}} \left|(\Delta^{N+1} m)_l\right| < \infty,$

then m is a bounded multiplier on $L^p(\Sigma_n)$.

Proof. Note that, for $n = 1$, we have exactly the Marcinkiewicz Theorem for multipliers. Assume $n > 1$. Our goal is to show that (i) and (ii) imply that

(iii) $J^N m$ is bounded, and

(iv) $\sup_{j\ge 0} \sum_{l=2^j}^{2^{j+1}} \left|(\Delta J^N m)_l\right| < \infty.$

By the Marcinkiewicz Theorem on multipliers, this implies that $J^N m$ is a bounded multiplier on $L^p(\Sigma_1)$ and, by Theorem 3.1, that m is a bounded multiplier on $L^p(\Sigma_n)$.

Define $\mu_k = (\Delta^N m)_k$. The sequence $\{\mu_k\}_{k=0}^{\infty}$ is Cauchy. Indeed, if $k > j$ and $2^r \le j < 2^{r+1}$, then

$$\begin{aligned}|\mu_k - \mu_j| &= \left|\sum_{l=j+1}^{k} (\Delta^{N+1} m)_l\right| \le \sum_{t\ge r}\sum_{l=2^t}^{2^{t+1}} |(\Delta^{N+1} m)_l| \\ &= \sum_{t\ge r} \frac{1}{2^{tN}}\left[2^{tN}\sum_{l=2^t}^{2^{t+1}} |(\Delta^{N+1} m)_l|\right] \\ &\le C\sum_{t\ge r}\frac{1}{2^{tN}} = C'\frac{1}{2^{Nr}} \to 0, \qquad \text{as}\, r\to\infty.\end{aligned} \tag{6.1}$$

Let $b = \lim_{k\to\infty} \mu_k$. Then

$$b = \lim_{k\to\infty}\frac{1}{k+1}\sum_{j=0}^{k}\mu_j = \lim_{k\to\infty}\frac{1}{k+1}\sum_{j=0}^{k}(\Delta^N m)_j = \lim_{k\to\infty}\frac{1}{k+1}(\Delta^{N-1} m)_k = 0,$$

since (i) implies $\Delta^{N-1}m$ is a bounded sequence. Taking limits as $k \to \infty$ in (6.1), we have

$$|\mu_j| \le C' \frac{1}{2^{Nr}}, \quad \text{for } 2^r \le j < 2^{r+1},$$

and, therefore,

$$|j^N \mu_j| \le C' 2^N, \quad \forall j \ge 0. \tag{6.2}$$

We claim that for any $s \le N$ there exists a constant $D_s > 0$ such that for any $l \ge 0$, $|l^s(\Delta^s m)_l| \le D_s$. Indeed, by (6.2) we know the claim is true for $s = N$. By induction, suppose it is true for $s+1$ and let us prove it for s. Let $j < k$, $2^r \le j < 2^{r+1}$, then

$$\begin{aligned} |(\Delta^s m)_k - (\Delta^s m)_j| &\le \sum_{l=j+1}^{k} |(\Delta^{s+1} m)_l| \le \sum_{t \ge r} \sum_{l=2^t}^{2^{t+1}-1} |(\Delta^{s+1} m)_l| \\ &\le \sum_{t \ge r} \sum_{l=2^t}^{2^{t+1}-1} \frac{D_{s+1}}{l^{s+1}} \le D'_s \frac{1}{2^{sr}}. \end{aligned} \tag{6.3}$$

Thus, $\Delta^s m$ is a Cauchy sequence. Let b be its limit; then

$$b = \lim_{k\to\infty} \frac{1}{k+1} \sum_{j=0}^{k} (\Delta^s m)_j = \lim_{k\to\infty} \frac{1}{k+1} (\Delta^{s-1} m)_k = 0.$$

Taking limits as $k \to \infty$ in (6.3), we obtain

$$|(\Delta^s m)_j| \le D'_s \frac{1}{2^{sr}}, \quad \text{for } 2^r \le j < 2^{r+1}$$

and, therefore,

$$|j^s (\Delta^s m)_j| \le D'_s 2^s, \quad \forall j \ge 0,$$

which proves the claim.

We can now use the claim and Lemma 6.2 to prove that $J^N m$ is bounded:

$$|(J^N m)_l| \le B \sum_{0 \le h \le s \le N} |l^h (\Delta^s m)_l| \le B' \sum_{0 \le h \le s \le N} |l^{h-s}| \le B'' < \infty.$$

Note that, for $0 \le s \le N-1$,

$$\sup_{j \ge 0} 2^{js} \sum_{l=2^j}^{2^{j+1}} |(\Delta^{s+1} m)_l| \le \sup_{j \ge 0} 2^{js} \sum_{l=2^j}^{2^{j+1}} \frac{D_{s+1}}{l^{s+1}} \le \sup_{j \ge 0} D_{s+1} \left(1 + \frac{1}{2^j}\right) < \infty$$

(the above inequality is also true for $s = N$, by hypothesis). Thus, using Lemma 6.3,

$$\begin{aligned}
\sup_{j\geq 0}\sum_{l=2^j}^{2^{j+1}} |(\Delta J^N m)_l| &\leq C \sup_{j\geq 0}\sum_{l=2^j}^{2^{j+1}} \sum_{0\leq h\leq s\leq N} |l^h(\Delta^{s+1}m)_l| \\
&\leq C \sup_{j\geq 0} \sum_{0\leq h\leq s\leq N} 2^{s(j+1)} \sum_{l=2^j}^{2^{j+1}} |(\Delta^{s+1}m)_l| \\
&\leq 2^N C \sum_{0\leq h\leq s\leq N} \left[\sup_{j\geq 0} 2^{sj} \sum_{l=2^j}^{2^{j+1}} |(\Delta^{s+1}m)_l|\right] < \ infty. \ \square
\end{aligned}$$

7. Ultraspherical series

Let λ be a non-negative real number and, for all $1 \leq p < \infty$, define

$$L^p_\lambda = \left\{ f : [0, \pi] \to \mathbb{C} : \int_0^\pi |f(x)|^p d\eta_\lambda(x) < \infty \right\}$$

where $d\eta_\lambda(x) = (\sin x)^{2\lambda} dx$. Define also the L^p_λ-norm of f by

$$\|f\|_{p,\lambda} \stackrel{\text{def}}{=} \left\{ \int_0^\pi |f(x)|^p d\eta_\lambda(x) \right\}^{1/p}.$$

If $\lambda > 0$, the functions $\{C^\lambda_l(\cos x)\}_{l=0}^\infty$ (recall C^λ_l are the ultraspherical polynomials. See Definition 3.4.) form an orthogonal basis for the space L^2_λ (see [13]); define

$$\begin{aligned}
R^\lambda_l(x) &\stackrel{\text{def}}{=} \frac{C^\lambda_l(\cos x)}{C^\lambda_l(1)} \quad \text{for } \lambda > 0, \\
R^0_l(x) &\stackrel{\text{def}}{=} \cos(lx).
\end{aligned}$$

Note that $\{R^0_l\}_{l=0}^\infty$ is an orthogonal basis for L^2_0. For any $\lambda \geq 0$ define the ultraspherical series of any $f \in L^p_\lambda$ as the formal sum

$$f(x) \sim \sum_{l=0}^\infty c_l \hat{f}(l) R^\lambda_l(x)$$

where $\hat{f}(l)$ is the Fourier coefficient of f given by

$$\hat{f}(l) \stackrel{\text{def}}{=} \int_0^\pi f(x)R_l^\lambda(x)\,d\eta_\lambda(x)$$

and $c_l^{-1} = \|R_l^\lambda\|_{2,\lambda}^2$. For any $\lambda \geq 0$ we can define a convolution $\overset{\lambda}{*}$ on the regular complex measures on $[0,\pi]$ that makes the couple $([0,\pi], \overset{\lambda}{*})$ into a hypergroup (see [1], [8], [14]). Here we will only be interested in convolutions between functions and, therefore, shall present only the definitions that are pertinent to this particular case. For any $f \in \mathcal{C}([0,\pi])$, define the λ-translation of f by x evaluated at y ($x, y \in [0,\pi]$) by

$$\begin{aligned} T_x^\lambda f(y) &\stackrel{\text{def}}{=} \gamma_\lambda \int_0^\pi f(\arccos(\cos x\cos y + \sin x \sin y \cos\theta))(\sin\theta)^{2\lambda-1}d\theta \quad \text{if} \quad \lambda > 0,\\ T_x^0 f(y) &\stackrel{\text{def}}{=} \frac{1}{2} f(\arccos(\cos x \cos y + \sin x \sin y))\\ &\quad + \frac{1}{2} f(\arccos(\cos x\cos y - \sin x \sin y)) \end{aligned}$$

where $\gamma_\lambda^{-1} \stackrel{\text{def}}{=} \int_0^\pi (\sin\theta)^{2\lambda-1}d\theta = \frac{\sqrt{\pi}\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}$. These translation operators satisfy the following property (see, for example, [14], page 8): for all $x \in [0,\pi]$ and $f \in \mathcal{C}([0,\pi])$, we have

$$\|T_x^\lambda f\|_{p,\lambda} \leq \|f\|_{p,\lambda}.$$

Notice that the density of $\mathcal{C}([0,\pi])$ in L_λ^p, $1 \leq p < \infty$, and the above mentioned property, allow us to extend the definition of λ-translation to the spaces L_λ^p, $1 \leq p < \infty$.

The convolution $g \overset{\lambda}{*} f$ of two functions g and f in L_λ^1 is defined by

$$g \overset{\lambda}{*} f(x) \stackrel{\text{def}}{=} \int_0^\pi g(y)T_x^\lambda f(y)\,d\eta_\lambda(y).$$

Using Fubini's theorem, the fact that $T_x^\lambda f(y) = T_y^\lambda f(x)$ and the inequality

$$\|T_y f\|_{1,\lambda} \leq \|f\|_{1,\lambda},$$

we see that $g \overset{\lambda}{*} f$ is a well defined function in L_λ^1.

This convolution is commutative, associative and

$$\begin{aligned} \|g \overset{\lambda}{*} f\|_{p,\lambda} &\leq \|g\|_{1,\lambda}\|f\|_{p,\lambda}, \qquad && 1 \leq p < \infty\\ (g \overset{\lambda}{*} f)\widehat{\ }(l) &= \hat{g}(l)\hat{f}(l), && l = 0,\ldots,\infty. \end{aligned}$$

We shall now motivate the "transference type" theorem that will be proved later. For any zonal function $f \in L^1(\Sigma_n)$, $n \geq 2$, define f_0 on $[0, \pi]$ by $f_0(\theta) = f^0(\cos\theta) = f(\alpha(\theta)e)$ (see §2). Let f and g be zonal functions in $L^1(\Sigma_n)$, $n \geq 2$. It is easy to check that $f * g$ is zonal as well. Furthermore, keeping in mind the formulas from §2, we have

$$\begin{aligned}(f*g)_0(\theta) &= \int_{\Sigma_n} f^0(\alpha(\theta)e\cdot y)g(y)\,dy = \omega_n \int_{SO(n+1)} f^0(\alpha(\theta)e\cdot ue)g(ue)\,du \\ &= \omega_n c_n \int_K \int_K \int_{-\pi}^{\pi} f^0(\alpha(\theta)e\cdot k\alpha(\varphi)k'e)g(k\alpha(\varphi)k'e)|\sin\varphi|^{n-1}d\varphi\,dk\,dk' \\ &= \frac{\omega_{n-1}}{2}\int_K \int_{-\pi}^{\pi} f^0(k\alpha(\theta)e\cdot\alpha(\varphi)e)g_0(\varphi)|\sin\varphi|^{n-1}d\varphi\,dk.\end{aligned}$$

Since K is isomorphic to $SO(n)$, we have the following Cartan decomposition:

$$K = LBL,$$

where $L = \{u \in SO(n+1) : ue = e \text{ and } u(0,1,0,\dots,0) = (0,1,0,\dots,0)\}$ and B is the subgroup of K consisting of all the matrices $\beta(\psi)$ of the form

$$\beta(\psi) \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \cos\psi & -\sin\psi & 0 & \dots & 0 \\ 0 & \sin\psi & \cos\psi & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Thus, the last integral equals

$$\begin{aligned}&= \frac{c_{n-1}\omega_{n-1}}{2}\int_L\int_L\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \\ &\qquad \times f^0(h\beta(\psi)h'\alpha(\theta)e\cdot\alpha(\varphi)e)g_0(\varphi)|\sin\varphi|^{n-1}|\sin\psi|^{n-2}d\varphi\,d\psi\,dh\,dh' \\ &= \frac{\omega_{n-2}}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^0(\beta(\psi)\alpha(\theta)e\cdot\alpha(\varphi)e)g_0(\varphi)|\sin\psi|^{n-2}|\sin\varphi|^{n-1}d\psi\,d\varphi \\ &= \frac{\omega_{n-2}}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^0(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos\psi)g_0(\varphi)|\sin\psi|^{n-2}|\sin\varphi|^{n-1}d\psi\,d\varphi \\ &= \omega_{n-2}\int_0^{\pi}\int_0^{\pi} \\ &\qquad \times f_0(\arccos(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos\psi))|\sin\psi|^{n-2}d\psi g_0(\varphi)|\sin\varphi|^{n-1}d\varphi \\ &= \frac{\omega_{n-2}}{\gamma_{(n-1)/2}}\int_0^{\pi} g_0(\varphi)T_\theta^{(n-1)/2}f_0(\varphi)\,d\eta_{(n-1)/2}(\varphi) = \omega_{n-1}\left(f_0 \overset{(n-1)/2}{*} g_0\right)(\theta).\end{aligned}$$

Note that $\|f\|^p_{L^p(\Sigma_n)} = \omega_{n-1}\|f_0\|^p_{p,(n-1)/2}$ and, if we define $M_{p,n}(h)$ as the smallest constant such that, for all zonal $\phi \in L^p(\Sigma_n)$ we have

$$\|h * \phi\|_{L^p(\Sigma_n)} \leq M_{p,n}(h)\|\phi\|_{L^p(\Sigma_n)}$$

and $N_{p,(n-1)/2}(h_0)$ as the smallest constant such that, for all $\phi_0 \in L^p_{(n-1)/2}$ we have

$$\|h_0 \overset{(n-1)/2}{*} \phi_0\|_{p,(n-1)/2} \leq N_{p,(n-1)/2}(h_0)\|\phi_0\|_{p,(n-1)/2},$$

we obtain the equality

$$\omega_{n-1}N_{p,(n-1)/2}(h_0) = M_{p,n}(h).$$

Similar equalities hold for $n = 1$, namely, if f and g are zonal functions in $L^p(\Sigma_1)$, then (recall that $\omega_0 = 2$)

$$\begin{aligned} (f * g)_0 &= 2(f_0 \overset{0}{*} g_0), \\ \|f\|^p_{L^p(\Sigma_1)} &= 2\|f_0\|^p_{p,0}, \\ M_{p,1}(h) &= 2N_{p,0}(h_0). \end{aligned}$$

We can now restate Theorem 3.2 in this new setting as follows.

Theorem 7.1

Let $\lambda = \frac{n-1}{2}$, with $n \geq 2$ an integer, and suppose $1 \leq p < \infty$. Let $k_0 \in L^1_\lambda$ and define h_0 by $h_0(\theta) = k_0(\theta)\sin\theta$. Suppose h_0, as a convolution operator on $L^p_{\lambda-1/2}$ has norm $N_{p,\lambda-1/2}(h_0)$. Then k_0, as a convolution operator on L^p_λ, has norm less than or equal to $N_{p,\lambda-1/2}(h_0)$.

Proof. First of all, observe that the proof of Theorem 3.2 holds if we assume f to be zonal, and $M_{p,n-1}(h) = A_p$ to be the norm of the convolution operator associated with h acting on the subspace of $L^p(\Sigma_{n-1})$ of all p-integrable zonal functions on Σ_{n-1}. Thus, using Theorem 3.2 and the observations preceding the statement of Theorem 7.1,

$$\begin{aligned} \|k_0 \overset{\lambda}{*} f_0\|_{p,\lambda} &= \frac{1}{\omega_{n-1}^{1+1/p}}\|k * f\|_{L^p(\Sigma_n)} \\ &\leq \frac{M_{p,n-1}(h)}{\omega_{n-1}^{1+1/p}}\frac{\omega_{n-1}}{\omega_{n-2}}\|f\|_{L^p(\Sigma_n)} = N_{p,\lambda-1/2}(h_0)\|f_0\|_{p,\lambda}. \ \square \end{aligned}$$

Our goal is to generalize Theorem 7.1. More precisely, we shall prove that for any real $\lambda > 0$ and for any $\delta > 0$ such that $\lambda - \delta \geq 0$, the norm of the convolution operator on L^p_λ given by $k \in L^1_\lambda$ is bounded above by the norm of the convolution operator on $L^p_{\lambda-\delta}$ given by $h(x) = k(x)(\sin x)^{2\delta}$. Before we do this, let us present some definitions and lemmas.

DEFINITION 7.2. Let $\lambda \geq 0$. For any $f \in \mathcal{C}([0,\, \pi])$ and for any $b \in [0,\, \frac{\pi}{2}]$, define the function f_b on $[0, \pi]$ by

$$f_b(x) \stackrel{\text{def}}{=} f(\arccos(\cos b \cos x)).$$

For any $\delta > 0$ such that $\lambda - \delta \geq 0$, define the measures $m_{\lambda,\delta}$ and $\mu_{\lambda,\delta}$ on $[0, \frac{\pi}{2}]$ by

$$\begin{aligned} dm_{\lambda,\delta}(\psi) &\stackrel{\text{def}}{=} (\sin\psi)^{2(\lambda-\delta)}(\cos\psi)^{2\delta-1}d\psi \\ d\mu_{\lambda,\delta}(b) &\stackrel{\text{def}}{=} (\cos b)^{2\lambda-2\delta+1}(\sin b)^{2\delta-1}db \end{aligned}$$

and define the constant $d_{\lambda,\delta}$ by

$$d_{\lambda,\delta} \stackrel{\text{def}}{=} \frac{2\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}-\delta\right)\Gamma(\delta)} = \left(\int_0^1 (1-v^2)^{\lambda-\delta-1/2}v^{2\delta-1}dv\right)^{-1}.$$

Define, finally, the function $Q : [0, \pi] \times [0, \frac{\pi}{2}] \to [0, \pi] \times [0, \frac{\pi}{2}]$ by

$$Q(a, b) \stackrel{\text{def}}{=} \left(\arccos(\cos a \cos b),\, \arcsin\left(\frac{\sin a \cos b}{\sqrt{1-\cos^2 a \cos^2 b}}\right)\right).$$

Q will serve us in future in a change of variables.

Lemma 7.3

Let $\lambda > 0$, $\delta > 0$ and $\lambda - \delta \geq 0$. For any $f \in \mathcal{C}([0,\, \pi])$, we have

$$\|f\|^p_{p,\lambda} = d_{\lambda,\delta}\int_0^{\pi/2} \|f_b\|^p_{p,\lambda-\delta}d\mu_{\lambda,\delta}(b)\,.$$

Proof. The proof is a simple substitution:

$$\begin{aligned} &d_{\lambda,\delta}\int_0^{\pi/2} \|f_b\|^p_{p,\lambda-\delta}d\mu_{\lambda,\delta}(b) \\ =&d_{\lambda,\delta}\int_0^{\pi/2}\int_0^{\pi} |f(\arccos(\cos b\cos x))|^p(\sin x)^{2(\lambda-\delta)}dx(\cos b)^{2\lambda-2\delta+1}(\sin b)^{2\delta-1}db. \end{aligned}$$

Making the change of variable $\cos b \cos x = \cos t$ in the innermost integral, we obtain that the last expression equals

$$
\begin{aligned}
& d_{\lambda,\delta} \int_0^{\pi/2} \int_b^{\pi-b} |f(t)|^p (\cos^2 b - \cos^2 t)^{\lambda-\delta-1/2} (\sin b)^{2\delta-1} \cos b \sin t \, dt \, db \\
=& d_{\lambda,\delta} \int_0^{\pi} |f(t)|^p \left\{ \int_0^{\pi/2-|\pi/2-t|} (\cos^2 b - \cos^2 t)^{\lambda-\delta-1/2} (\sin b)^{2\delta-1} \cos b \, db \right\} \sin t \, dt \\
=& d_{\lambda,\delta} \int_0^{\pi} |f(t)|^p \left\{ \int_0^{\sin t} (\sin^2 t - u^2)^{\lambda-\delta-1/2} u^{2\delta-1} du \right\} \sin t \, dt \\
=& d_{\lambda,\delta} \int_0^{\pi} |f(t)|^p \left\{ \int_0^{1} (1 - v^2)^{\lambda-\delta-1/2} v^{2\delta-1} dv \right\} (\sin t)^{2\lambda} \, dt = \|f\|_{p,\lambda}^p . \ \square
\end{aligned}
$$

DEFINITION 7.4. For $\lambda > 0$, $\delta > 0$, $\lambda - \delta \geq 0$ and $\psi \in \left[0, \frac{\pi}{2}\right]$, define the $(\lambda - \delta, \psi)$-pseudo translations by x of a function $f \in \mathcal{C}([0, \pi])$, evaluated at y by

$$
T_x^{\lambda-\delta,\psi} f(y) = \gamma_{\lambda-\delta} \int_0^{\pi} f(\arccos(\cos x \cos y + \sin x \sin y \cos\theta \sin\psi))(\sin\theta)^{2\lambda-2\delta-1} d\theta
$$

if $\lambda - \delta > 0$, and

$$
\begin{aligned}
T_x^{0,\psi}(y) = & \frac{1}{2} f(\arccos(\cos x \cos y + \sin x \sin y \sin\psi)) \\
& + \frac{1}{2} f(\arccos(\cos x \cos y - \sin x \sin y \sin\psi)) ,
\end{aligned}
$$

where x, $y \in [0, \pi]$.

Lemma 7.5

For λ, $\delta > 0$, $\lambda - \delta \geq 0$, $f \in \mathcal{C}([0, \pi])$ and for all x, $y \in [0, \pi]$, we have

$$
T_x^{\lambda} f(y) = d_{\lambda,\delta} \int_0^{\pi/2} T_x^{\lambda-\delta,\psi} f(y) \, dm_{\lambda,\delta}(\psi) .
$$

Proof. We will prove the case $\lambda - \delta > 0$ (the case $\lambda - \delta = 0$ is similar).

$$
\begin{aligned}
& d_{\lambda,\delta} \int_0^{\pi/2} T_x^{\lambda-\delta,\psi} f(y) \, dm_{\lambda,\delta}(\psi) \\
& = d_{\lambda,\delta} \int_0^{\pi/2} \gamma_{\lambda-\delta} \int_0^{\pi} f(\arccos(\cos x \cos y + \sin x \sin y \cos\theta \sin\psi)) \\
& (\sin\theta)^{2\lambda-2\delta-1} d\theta (\sin\psi)^{2\lambda-2\delta} (\cos\psi)^{2\delta-1} d\psi.
\end{aligned}
$$

Making the substitution $\cos\theta\sin\psi = \cos\varphi$ in the innermost integral, and defining

$$\Phi(x, y, \varphi) \stackrel{\text{def}}{=} \arccos(\cos x \cos y + \sin x \sin y \cos\varphi)),$$

we get

$$\begin{aligned}
&= d_{\lambda,\delta}\gamma_{\lambda-\delta} \\
&\quad\times \int_0^{\pi/2} \int_{\pi/2-\psi}^{\pi/2+\psi} f(\Phi(x, y, \varphi))(\sin^2\psi - \cos^2\varphi)^{\lambda-\delta-1} \sin\varphi \, d\varphi (\cos\psi)^{2\delta-1} \sin\psi \, d\psi \\
&= d_{\lambda,\delta}\gamma_{\lambda-\delta} \int_0^{\pi} f(\Phi(x, y, \varphi)) \\
&\quad\times \left\{ \int_{|\pi/2-\varphi|}^{\pi/2} (\sin^2\psi - \cos^2\varphi)^{\lambda-\delta-1} (\cos\psi)^{2\delta-1} \sin\psi \, d\psi \right\} \sin\varphi \, d\varphi \\
&= d_{\lambda,\delta}\gamma_{\lambda-\delta} \int_0^{\pi} f(\Phi(x, y, \varphi)) \left\{ \int_0^{\sin\varphi} (\sin^2\varphi - u^2)^{\lambda-\delta-1} u^{2\delta-1} \, du \right\} \sin\varphi \, d\varphi \\
&= d_{\lambda,\delta}\gamma_{\lambda-\delta} \int_0^{\pi} f(\Phi(x, y, \varphi)) \left\{ \int_0^1 (1 - v^2)^{\lambda-\delta-1} v^{2\delta-1} \, dv \right\} (\sin\varphi)^{2\lambda-1} d\varphi \\
&= \frac{d_{\lambda,\delta}\gamma_{\lambda-\delta}}{d_{\lambda-1/2,\delta}} \int_0^{\pi} f(\Phi(x, y, \varphi))(\sin\varphi)^{2\lambda-1} d\varphi = \frac{d_{\lambda,\delta}\gamma_{\lambda-\delta}}{d_{\lambda-1/2,\delta}\gamma_\lambda} T_x^\lambda f(y) = T_x^\lambda f(y). \ \square
\end{aligned}$$

Lemma 7.6

Let $\lambda, \delta > 0$ and $\lambda - \delta \geq 0$. Let $G \in L^1\left([0, \pi] \times \left[0, \frac{\pi}{2}\right], \eta_\lambda \otimes m_{\lambda,\delta}\right)$. Then

$$\int_0^{\pi/2} \int_0^{\pi} G(x, \psi) \, d\eta_\lambda(x) \, dm_{\lambda,\delta}(\psi) = \int_0^{\pi/2} \int_0^{\pi} G(Q(a, b)) \, d\eta_{\lambda-\delta}(a) \, d\mu_{\lambda,\delta}(b)$$

and, for all a, $y \in [0, \pi]$, $b \in \left[0, \frac{\pi}{2}\right]$ and $f \in \mathcal{C}([0, \pi])$, we have

$$T_{Q_1(a,b)}^{\lambda-\delta, Q_2(a,b)} f(y) = T_a^{\lambda-\delta} f_b(y).$$

Proof. The first part is just a substitution. By the definition of Q, we have

$$\begin{cases} \cos x &= \cos b \cos a \\ \sin\psi \sin x &= \cos b \sin a. \end{cases}$$

This implies that

$$\sin^2 x \cos\psi \, dx \, d\psi = \sin b \cos b \, da \, db \, .$$

Thus

$$\int_0^{\pi/2}\int_0^{\pi} G(x,\psi)(\sin x)^{2\lambda}dx(\sin\psi)^{2\lambda-2\delta}(\cos\psi)^{2\delta-1}d\psi$$
$$=\int_0^{\pi/2}\int_0^{\pi} G(Q(a,b))(\sin a)^{2\lambda-2\delta}da(\cos b)^{2\lambda-2\delta+1}(\sin b)^{2\delta-1}db,$$

and this proves the first part. For the second part, it is enough to observe that, for any $\theta\in[0,\pi]$,

$$\arccos(\cos Q_1(a,b)\cos y+\sin Q_1(a,b)\sin y\cos\theta\sin Q_2(a,b))$$
$$=\arccos(\cos b(\cos a\cos y+\sin a\sin y\cos\theta)).\ \square$$

We can now state and prove a generalization of Theorem 7.1.

Theorem 7.7

Let $\lambda,\delta>0$ and $\lambda-\delta\geq 0$, $1\leq p<\infty$. Let $k\in L^1_\lambda$ and define h by $h(x)=k(x)(\sin x)^{2\delta}$, $x\in[0,\pi]$. Suppose h, as a convolution operator on $L^p_{\lambda-\delta}$, has norm $N_p(h)$. Then the convolution operator on L^p_λ given by k has norm less than or equal to $N_p(h)$.

Proof. Let $f\in\mathcal{C}([0,\pi])$. Then, applying Lemma 7.5,

$$\begin{aligned}k\overset{\lambda}{*}f(x)&=\int_0^{\pi}k(y)T_x^{\lambda}f(y)\,d\eta_\lambda(y)\\&=d_{\lambda,\delta}\int_0^{\pi}\int_0^{\pi/2}k(y)T_x^{\lambda-\delta,\psi}f(y)\,dm_{\lambda,\delta}(\psi)\,d\eta_\lambda(y)\\&=d_{\lambda,\delta}\int_0^{\pi/2}\int_0^{\pi}h(y)T_x^{\lambda-\delta,\psi}f(y)\,d\eta_{\lambda-\delta}(y)\,dm_{\lambda,\delta}(\psi).\end{aligned}$$

Thus, using Jensen's inequality and Lemmas 7.6 and 7.3,

$$
\begin{aligned}
&\left\|k \overset{\lambda}{*} f\right\|_{p,\lambda}^{p} = \int_0^\pi \left|k \overset{\lambda}{*} f(x)\right|^p d\eta_\lambda(x) \\
&= \int_0^\pi \left|\int_0^{\pi/2}\int_0^\pi h(y) T_x^{\lambda-\delta,\psi} f(y)\, d\eta_{\lambda-\delta}(y)\, d_{\lambda,\delta}\, dm_{\lambda,\delta}(\psi)\right|^p d\eta_\lambda(x) \\
&\le \int_0^\pi \left[\int_0^{\pi/2}\left|\int_0^\pi h(y) T_x^{\lambda-\delta,\psi} f(y)\, d\eta_{\lambda-\delta}(y)\right| d_{\lambda,\delta}\, dm_{\lambda,\delta}(\psi)\right]^p d\eta_\lambda(x) \\
&\le d_{\lambda,\delta} \int_0^{\pi/2}\int_0^\pi \left|\int_0^\pi h(y) T_x^{\lambda-\delta,\psi} f(y)\, d\eta_{\lambda-\delta}(y)\right|^p d\eta_\lambda(x)\, dm_{\lambda,\delta}(\psi) \\
&= d_{\lambda,\delta} \int_0^{\pi/2}\int_0^\pi \left|\int_0^\pi h(y) T_{Q_1(a,b)}^{\lambda-\delta,Q_2(a,b)} f(y)\, d\eta_{\lambda-\delta}(y)\right|^p d\eta_{\lambda-\delta}(a)\, d\mu_{\lambda,\delta}(b) \\
&= d_{\lambda,\delta} \int_0^{\pi/2}\int_0^\pi \left|\int_0^\pi h(y) T_a^{\lambda-\delta} f_b(y)\, d\eta_{\lambda-\delta}(y)\right|^p d\eta_{\lambda-\delta}(a)\, d\mu_{la,\delta}(b) \\
&= d_{\lambda,\delta} \int_0^{\pi/2} \left\|h \overset{\lambda-\delta}{*} f_b\right\|_{p,\lambda-\delta}^p d\mu_{\lambda,\delta}(b) \le d_{\lambda,\delta}\, (N_p(h))^p \int_0^{\pi/2} \|f_b\|_{p,\lambda-\delta}^p d\mu_{\lambda,\delta}(b) \\
&= (N_p(h))^p\, \|f\|_{p,\lambda}^p .
\end{aligned}
$$

The theorem now follows from the density of $\mathcal{C}([0,\pi])$ in L^p_λ. □

Let us observe that Theorem 7.7 is particularly interesting when $\delta = \lambda$. In this case, we can deduce estimates for a convolution operator on L^p_λ by studying a convolution operator on L^p_0, which coincides with studying a convolution operator on the space of zonal functions of $L^p(\Sigma_1)$.

We can now explain briefly what generalizations of these results we intend to prove in future articles. The first named author will show, in a subsequent paper with J. A. Tirao, an extension of Theorem 3.1 to compact two-point homogeneous spaces, whereas the second named author will prove an extension of Theorem 7.7 to more general hypergroups, including, as particular cases, the continuous Jacobi polynomial hypergroups (expansions in Jacobi polynomials), the Bessel-Kingman hypergroups, and the Jacobi hypergroups of non-compact type.

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