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Weighted generalized weak type inequalities for modified Hardy operators

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Abstract

We consider the operator $T_g f(x)=g(x)\int_0^x f$, where g is a positive nonincreasing function, and characterize the pairs of positive measurable functions (u,v) such that the generalized weak type inequality

$$\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\{x\in(0,\infty);|T_g f(x)|>\lambda\}} u\right)\le\Phi_1^{-1}\left(\int_0^\infty \Phi_1(K|f|)v\right)$$

holds, where either Φ_1 is a N-function and Φ_2 is a positive increasing function such that $\Phi_1\circ\Phi_2^{-1}$ is countably subadditive or $\Phi_1(t)=t$ and Φ_2 is a positive increasing function whose inverse is countably subadditive.

Let g be a positive measurable function on $(0,\infty)$ and let T_g be the operator defined for locally integrable functions f on $(0,\infty)$ by

$$T_g f(x) = g(x)\int_0^x f(y)dy \quad (x\in(0,\infty)). \tag{1}$$

The characterization of the couples of weights (u,v) such that

$$\left(\int_{\{x\in(0,\infty);|T_g f(x)|>\lambda\}} \lambda^q u\right)^{1/q} \le C\left(\int_0^\infty |f|^p v\right)^{1/p} \tag{2}$$

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holds in the case $1 \leq p \leq q < \infty$ has been done in [6], where results due to Andersen and Muckhenhoupt [1], Sawyer [9] and Ferreyra [3] have been generalized and improved. If $q < p$ and g is a monotone function, the characterization of the couples of weights such that (2) holds has been done in [5].

In this note, we work with a nonincreasing g and characterize the couples of weights (u, v) such that the weighted generalized weak type inequality

$$\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\{\,x\in(0,\infty);|T_g f(x)|>\lambda\,\}} u\right) \leq \Phi_1^{-1}\left(\int_0^\infty \Phi_1(K|f|)v\right) \tag{3}$$

holds, where either Φ_1 is a N-function and Φ_2 is a positive increasing function such that $\Phi_1 \circ \Phi_2^{-1}$ is countably subadditive or $\Phi_1(t) = t$ and Φ_2 is a positive increasing function whose inverse is countably subadditive. It is clear that, under these conditions over Φ_1 and Φ_2, inequality (3) is a generalization of (2) in the case $1 \leq p \leq q$.

By a N-function we mean a continuous and convex function Φ defined on $[0, \infty)$ such that $\Phi(s) > 0$ if $s > 0$, $\frac{\Phi(s)}{s} \to 0$ when $s \to 0$ and $\frac{\Phi(s)}{s} \to \infty$ when $s \to \infty$. Every N-function Φ admits a representation of the form $\Phi(x) = \int_0^x \phi(t)dt$, where ϕ is nondecreasing, continuous by the right at every point and verifies $\phi(0) = 0$, $\phi(s) > 0$ if $s > 0$ and $\phi(s) \to \infty$ when $s \to \infty$. The function ϕ is called the density function of Φ. Given a N-function Φ, the function $\Psi : [0, \infty) \to R$ defined by $\Psi(t) = \sup_{s\geq 0}(st - \Phi(s))$ is also a N-function which is called the complementary function of Φ. Two complementary N-functions Φ and Ψ verify Young's inequality, which is a fundamental tool to prove our theorems: if $s, t \geq 0$, then $st \leq \Phi(s) + \Psi(t)$.

Inequality (3) has been studied by L. Qinsheng [8] in the case $g \equiv 1$ (the Hardy operator) and by S. Bloom and R. Kerman [2] for nondecreasing g. When g is nondecreasing, T_g is a monotone operator (see [2]) and the set $O_\lambda = \{x \in (0, \infty); T_g f(x) > \lambda\}$ is an interval. This is not true for nonincreasing g. The difficulties that appear in this case are solved by mean of methods already applied in [5] and [6], which are based on [4]. We also use the standard methods of N-functions ([7] and [2]).

The results and their proofs are the following ones:

Theorem 1

Let u, v be positive locally integrable functions on $(0, \infty)$. Let Φ_2 be a positive increasing function such that Φ_2^{-1} is countably subadditive. Then, the weak type inequality (3) *holds with $\Phi_1(t) = t$ if and only if*

$$\frac{\Phi_2^{-1}\left(\Phi_2(\lambda)\int_b^\beta u\right)}{\lambda}\left(\operatorname*{ess\,sup}_{x\in(0,b)} v^{-1}(x)\right) g(\beta-) \leq K \tag{4}$$

holds for every $\lambda > 0$ and every b, β with $0 < b < \beta$, where $g(\beta-) = \lim_{x\to\beta^-} g(x)$.

Proof. Suppose that condition (4) holds. Let f be a non negative measurable function supported on a bounded interval $(0, A)$. Let $x_0 = A$ and, given x_k, let x_{k+1} be the unique real number such that $\int_0^{x_k} f = 2\int_0^{x_{k+1}} f$. The sequence $\{x_k\}$ is decreasing and has limit 0. Moreover,

$$\int_0^{x_k} f = 4\int_{x_{k+2}}^{x_{k+1}} f \tag{5}$$

for every k. Let $\lambda > 0$, $k \in N$ and $E_k = \{\, x \in (x_{k+1}, x_k); T_g f(x) > \lambda \,\}$. Let $\beta_k = \sup E_k$. If $x \in E_k$, then

$$\lambda < g(x)\int_0^{x} f < g(x)\int_0^{x_k} f. \tag{6}$$

Since (6) holds for every $x \in E_k$ and g is nonincreasing, we have

$$\lambda \le g(\beta_k-)\int_0^{x_k} f. \tag{7}$$

Then, by (7), (5) and (4) we obtain

$$\begin{aligned}
&\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{E_k} u\right) \le \Phi_2^{-1}\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k} u\right)\\
&\le \frac{4}{\lambda} g(\beta_k-)\left(\int_{x_{k+2}}^{x_{k+1}} f\right)\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k} u\right)\\
&\le \frac{4}{\lambda} g(\beta_k-)\left(\operatorname{ess}\sup_{x\in(x_{k+2},x_{k+1})} v^{-1}(x)\right)\left(\int_{x_{k+2}}^{x_{k+1}} fv\right)\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k} u\right)\\
&\le K\int_{x_{k+2}}^{x_{k+1}} fv
\end{aligned} \tag{8}$$

for every k. Summing up in k, the subadditivity of Φ_2^{-1} gives the weak type inequality (3).

Conversely, let $\lambda > 0$ and let β and b be real numbers with $0 < b < \beta$. Let $\varepsilon > 0$ and let F be a measurable subset of $(0, b)$ with positive measure such that $v(x) \le \varepsilon + \operatorname{ess\,inf}\{v(t); t \in (0, b)\}$ for every $x \in F$. Since we can assume $g(\beta-) > 0$, there exists $\eta > 0$ such that $\eta|F|g(\beta-) = (1+\varepsilon)\lambda$. Let $f = \eta\chi_F$. Then, if $x \in [b, \beta)$, $T_g(f)(x) = g(x)\eta|F| \ge \eta|F|g(\beta-) = (1+\varepsilon)\lambda > \lambda$. Therefore, $[b, \beta) \subset \{x; T_g(f)(x) > \lambda\}$ and inequality (3) gives

$$\Phi_2^{-1}\left(\Phi_2(\lambda)\int_b^{\beta} u\right) \le K\int_F \eta v \le K\eta|F|\left(\varepsilon + \operatorname{ess\,inf}\{v(t); t \in (0, b)\}\right). \tag{9}$$

Multiplying by $g(\beta-)$, we obtain

$$\Phi_2^{-1}\left(\Phi_2(\lambda)\int_b^\beta u\right)g(\beta-)\le(1+\varepsilon)K\lambda\left(\varepsilon+\operatorname{ess\,inf}\{v(t);t\in(0,b)\}\right). \tag{10}$$

Since inequality (10) holds for all $\varepsilon>0$, we are done. □

Theorem 2

Let u, v be positive locally integrable functions on $(0,\infty)$. Let Φ_1 be a N-function and let Φ_2 be a positive increasing function such that $\Phi_1\circ\Phi_2^{-1}$ is countably subadditive. Let Ψ_1 be the complementary N-function of Φ_1. Then the weak type inequality (3) *implies that the inequality*

$$\int_0^b\Psi_1\left(\frac{g(\beta-)(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_b^\beta u\right)}{K\lambda v}\right)v\le(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_b^\beta u\right) \tag{11}$$

holds for every $\lambda>0$ and every b, β with $0<b<\beta$. Conversely, condition (11) *with constant K implies the weak type inequality* (3) *with constant $8K$.*

Proof. Suppose that condition (11) holds. Let f be a non negative measurable function supported on a bounded interval $(0,A)$. Let $\{x_k\}$, $\{E_k\}$ and β_k be defined as in the proof of Theorem 1, so that (5), (6) and (7) hold. Then, (7), (5), Young's inequality and condition (11) yield

$$\begin{aligned}
&2(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k}u\right)\\
&\le(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k}u\right)\frac{1}{\lambda}g(\beta_k-)\int_{x_{k+2}}^{x_{k+1}}8f\\
&=\int_{x_{k+2}}^{x_{k+1}}8Kf\frac{(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k}u\right)g(\beta_k-)}{K\lambda v}v\\
&\le\int_{x_{k+2}}^{x_{k+1}}\Phi_1(8Kf)v+\int_{x_{k+2}}^{x_{k+1}}\Psi_1\left(\frac{(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k}u\right)g(\beta_k-)}{K\lambda v}\right)v\\
&\le\int_{x_{k+2}}^{x_{k+1}}\Phi_1(8Kf)v+(\Phi_1\circ\Phi_2^{-1})\left(\Phi_2(\lambda)\int_{x_{k+1}}^{\beta_k}u\right).
\end{aligned} \tag{12}$$

The above inequality is equivalent to

$$(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_{x_{k+1}}^{\beta_k} u \right) \le \int_{x_{k+2}}^{x_{k+1}} \Phi_1(8Kf)v, \tag{13}$$

which implies

$$(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_{E_k} u \right) \le \int_{x_{k+2}}^{x_{k+1}} \Phi_1(8Kf)v. \tag{14}$$

Summing up in k, the subadditivity of $\Phi_1 \circ \Phi_2^{-1}$ gives the weak type inequality (3) with constant $8K$.

Suppose now that (3) holds. Let $\lambda > 0$ and let β and b be real numbers with $0 < b < \beta$. Let ρ be a positive number and let n be a natural number. Then, for every $\varepsilon > 0$,

$$\int_0^b \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon} dy \le b g(\beta-) \psi_1(n\varepsilon g(\beta-)), \tag{15}$$

where ψ_1 is the density function of Ψ_1, and, therefore, the integral is finite. The fact that the function $\frac{\Psi_1(t)}{t}$ increases taking all values from 0 to ∞, the continuity of the above integral as a function of ε and the fact that we can assume $g(\beta-) > 0$ imply that there exists $\varepsilon > 0$ such that

$$\int_0^b \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon} dy = (1+\rho)K\lambda. \tag{16}$$

Now, if f is the function defined on $(0, \infty)$ by

$$f(y) = \frac{1}{K} \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon g(\beta-)} \chi_{(0,b)}(y)$$

and $z \in [b, \beta)$, we have

$$\begin{aligned} T_g f(z) &= g(z) \int_0^b \frac{1}{K} \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon g(\beta-)} dy \\ &\ge \int_0^b \frac{1}{K} \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon} dy = (1+\rho)\lambda. \end{aligned} \tag{17}$$

Then, $[b, \beta) \subset \{ x \in (0, \infty); T_g f(x) > \lambda \}$ and the weak type inequality (3) together with the property $\Phi_1(\frac{\Psi_1(t)}{t}) \le \Psi_1(t)$ and (16) give

$$(18)\quad \begin{aligned} (\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_b^\beta u \right) &\le \int_0^b \Phi_1 \left(\Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{\varepsilon g(\beta-)} \right) v(y) dy \\ &\le \int_0^b \Psi_1 \left(\frac{\varepsilon g(\beta-)}{v(y) + \frac{1}{n}} \right) \left(v(y) + \frac{1}{n} \right) dy = (1+\rho) K \lambda \varepsilon. \end{aligned}$$

The fact that $S_{\Psi_1}(t) = \frac{\Psi_1(t)}{t}$ increases, (18) and (16) yield

$$(19)\quad \begin{aligned} &\int_0^b \Psi_1 \left(\frac{g(\beta-)(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_b^\beta u \right)}{(1+\rho) K \lambda \left(v(y) + \frac{1}{n} \right)} \right) \frac{v(y) + \frac{1}{n}}{(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_b^\beta u \right)} dy \\ &\le \int_0^b \Psi_1 \left(\frac{g(\beta-)\varepsilon}{v(y) + \frac{1}{n}} \right) \frac{v(y) + \frac{1}{n}}{(1+\rho) K \lambda \varepsilon} dy = 1. \end{aligned}$$

By the monotone convergence theorem, we obtain from (19) the inequality

$$(20)\quad \int_0^b \Psi_1 \left(\frac{g(\beta-)(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_b^\beta u \right)}{(1+\rho) K \lambda v(y)} \right) \frac{v(y)}{(\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_b^\beta u \right)} dy \le 1.$$

Since this inequality holds for all positive ρ, letting ρ tends to 0 we obtain (11) (again by monotone convergence). □

Final remark. It is worth noting that $\Phi_2(\lambda)$ can be replaced all over the paper by $h(\lambda)$, where h is an arbitrary positive function defined on $(0, \infty)$.

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