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Tensor products and p-induction of representations on Banach spaces

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ABSTRACT

In this paper we obtain L^p versions of the classical theorems of induced representations, namely, the inducing in stages theorem, the Kronecker product theorem, the Frobenius Reciprocity theorem and the subgroup theorem. In doing so we adopt the tensor product approach of Rieffel to inducing.

1. Introduction

The aim of the present paper is to carry over the theory of induced representations of locally compact groups on Hilbert spaces to more general Banach spaces. The cornerstone of this theory is the work of Mackey. Several generalizations have already been considered by various authors (*cf.* [1], [12], [22]). However these treatments do not give a complete and coherent account of the basic theorems of induced representations: the Inducing-in-stages Theorem, the Kronecker Product Theorem, the Frobenius Reciprocity Theorem and the Subgroup Theorem, in this context. This statement is slightly misleading; in fact, [12] does contain an inducing-in-stages theorem and [19], [22] contain Frobenius Reciprocity Theorems for "1-inducing". Our aim here is to investigate the problems involved in finding such theorems in the more general context of p-inducing, rather than the classical 2-inducing. We obtain versions of all of these theorems. To do this, we follow the philosophy of Rieffel in using tensor products as the mechanism for inducing. In doing this, we have as does Rieffel to impose restrictions which prevent us from obtaining an inducing in stages theorem as sharp as that of [12]. On the other hand, our version of the Frobenius Reciprocity Theorem is valid for $1 < p < \infty$ instead of $p = 1$ from [19], [22]. We also obtain a version of the subgroup theorem and of the Kronecker product theorem, neither of which are, to our knowledge, available in the literature.

It turns out that the extension of the basic theorems to this context relies heavily on properties of the Banach spaces involved and that a full theory requires the Banach spaces on which the groups are represented to be close to L^p-spaces. Accordingly we spend some time discussing the properties of these spaces in the next section of the paper, followed by the new definition of p-inducing as a tensor product in Section 3. In Section 4, we prove the inducing in stages theorem, the Kronecker product theorem and the Frobenius Reciprocity theorem. Finally we give a version of the subgroup theorem.

2. Preliminaries

All groups considered here will be locally compact and separable. All Banach spaces considered will be complex, separable and reflexive. In particular they have the Radon-Nikodym property. We will also assume that they have the approximation property. Let us also define what we mean by a representation of a group G on a Banach space X.

2.1 Representations of groups

DEFINITION. Let G be a group and X a Banach space. A *representation* π of G on X is a set $(\pi_g)_{g\in G}$ of linear mappings $\pi_g : X \mapsto X$ such that

1. $\pi_e = I$ and for all $g_1, g_2 \in G$, $\pi_{g_1 g_2} = \pi_{g_1}\pi_{g_2}$;
2. for every $g \in G$, π_g is continuous ;

3. for every $x \in X$ the map $\begin{array}{ccc} G & \mapsto & X \\ g & \mapsto & \pi_g x \end{array}$ is continuous (*i.e.* π is *strongly continuous*).

A representation π is said to be *uniformly bounded* if $\sup_{g \in G} \|\pi_g\| < \infty$, π is *isometric* if every π_g is an isometry.

Remark. Assume π is a uniformly bounded representation of a group G on a Banach space X. Define a new norm on X by

$$\|x\|_\pi = \sup_{g \in G} \|\pi_g x\|$$

then $\|.\|_\pi$ is equivalent to $\|.\|$ on X and π is an isometric representation of G on $(X, \|.\|_\pi)$.

In the sequel, every representation considered will be isometric.

EXAMPLE: Let $(\mathcal{M}, \mu)$ be a measured space and let G be a group of transformations of $\mathcal{M}$ ($\mathcal{M}$ is then called a G-*space*). Assume that G leaves μ invariant (*i.e.* $\mu(gM) = \mu(M)$ for every $g \in G$ and every measurable $M \subset \mathcal{M}$). Let $1 \leq p \leq \infty$ and define, for $g \in G$, $\pi_g : L^p(\mathcal{M}, \mu) \mapsto L^p(\mathcal{M}, \mu)$ by $\pi_g f(x) = f(g^{-1}x)$, then $(\pi_g)_{g \in G}$ is an isometric representation of G on $L^p(\mathcal{M}, \mu)$.

2.2 p-spaces

We first describe some results on Banach spaces and tensor products that we will need. They can all be found in [3], Ch. 23 and 25.6.

Let X be a Banach space, Ω a locally compact space and μ a Radon measure on Ω. We shall be considering the spaces $L^p(\mu), L^p(\mu, X)$, defined in the usual way.

Define $i_p(\mu) : L^p(\mu) \otimes X \mapsto L^p(\mu, X)$ by

$$f \otimes x \mapsto \big(t \mapsto f(t)x\big).$$

Then i_p produces on $L^p(\mu) \otimes X$ a norm Δ_p induced by the norm of $L^p(\mu, X)$. We denote by $L^p(\mu)\hat{\otimes}_{\Delta_p} X$ the completion of $L^p(\mu) \otimes X$ under this norm, so that $L^p(\mu)\hat{\otimes}_{\Delta_p} X \simeq L^p(\mu, X)$.

For X and Y two Banach spaces, we define two norms d_p and g_p on the tensor product $X \otimes Y$ as follows. For $y_1, \ldots, y_n \in Y$, $1 < p' < \infty$ define

$$\varepsilon_{p'}(y_1, \ldots, y_n) = \sup \left\{ \left(\sum_{i=1}^n |\psi(y_i)|^{p'} \right)^{1/p'} : \psi \in Y', \|\psi\| = 1 \right\}.$$

For $z \in X \otimes Y$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ let

$$d_p(z) = \inf \left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \varepsilon_{p'}(y_1, \ldots, y_n) \right\}$$

where the infimum is taken over all representations of z of the form $z = \sum_{i=1}^{n} x_i \otimes y_i$.

The norm $g_p(z)$ is defined by exchanging the roles of x_i and y_i in the above definition. We write $X \otimes_{d_p} Y$ (resp. $X \otimes_{g_p} Y$) for the completion of $X \otimes Y$ with respect to the norm d_p (resp. g_p).

This norms have been introduced independently by S. Chevet [2] and P. Saphar [21] in order to generalize the projective tensor product norm. If we identify $z \in X \otimes Y$ with an operator $T_z : X' \mapsto Y$ then, under this identification, operators corresponding to elements of $X \otimes_{d_p} Y$ will be called *right p-nuclear* and those corresponding to elements of $X \otimes_{g_p} Y$ will be called *left p-nuclear*. We write $N_p(X', Y)$ for the class of all right p-nuclear operators from X' to Y and $N^p(X', Y)$ for the class of all left p-nuclear operators. The following result from [8] (Corollary 1.6) tells us that the d_p tensor norm is the nearest ideal norm to Δ_p:

Theorem 2.1.1

Let X be a Banach space and $1 < p < \infty$. The following are equivalent:
1. X is isomorphic to a quotient of a subspace of an L_p space (a QSL_p space);
2. there exists an infinite dimensional $L_p(\mu)$ and an ideal norm α equivalent to Δ_p on $L^p(\mu) \otimes X$;
3. for every infinite dimensional $L_p(\mu)$ there exists an ideal norm α equivalent to Δ_p on $L^p(\mu) \otimes X$.
Moreover, the ideal norm α can be chosen to be the d_p norm.

Specialists of representation theory may be more familiar with p-spaces as defined by Herz [10]. We refrain from giving this definition, since it turns out that QSL_p spaces and p-spaces are the same. The following result follows at once from the preceding one and the observation that a p-space is a subspace of a quotient of an L^p space, by Proposition 0 of [10] and Theorem 2' of [13].

Theorem 2.1.2

Let X be a Banach space and $1 < p < \infty$. Then X is a QSL_p space if and only if it is a p-space.

The d_p and g_p tensor products are also of particular interest when X and Y are both L^p spaces. Indeed, if (Ω, μ) and (Ω', μ') are two measure spaces, we have

$$L^p(\Omega) \otimes_{d_p} L^p(\Omega') = L^p(\Omega) \otimes_{g_p} L^p(\Omega') \simeq L^p(\Omega \times \Omega') \simeq N_p\big(L^{p'}(\Omega), L^p(\Omega')\big). \quad (1)$$

It is then obvious from (1) that, if R, S, T are measure spaces, then

$$L^p(R) \otimes_{d_p} \big(L^p(S) \otimes_{d_p} L^p(T)\big) \simeq \big(L^p(R) \otimes_{d_p} L^p(S)\big) \otimes_{d_p} L^p(T). \quad (2)$$

In other words, if X, Y, Z are all L^p spaces, then

$$X \otimes_{d_p} Y \simeq Y \otimes_{d_p} X$$
$$X \otimes_{d_p} (Y \otimes_{d_p} Z) \simeq (X \otimes_{d_p} Y) \otimes_{d_p} Z.$$

We will now generalize these two identities to a larger class of Banach spaces.

Definition. Let $\lambda > 1$ and $1 < p < \infty$. We will say that a Banach space X is an $\mathcal{L}^g_{p\lambda}$ space if there exists a projection P of norm $\|P\| \le \lambda$ from an L^p-space onto X.
X is called an $\mathcal{L}^g_p$ space if it is an $\mathcal{L}^g_{p\lambda}$ for some λ.

It turns out that this spaces have a local characterization close to the $\mathcal{L}_p$ spaces investigated by Lindenstrauss and Pelczyński [14].

Proposition 2.1.3 (cf. [3])

A Banach space X is an $\mathcal{L}^g_{p\lambda}$ if and only if, for every $\varepsilon > 0$, and every finite dimensional subspace M of X, there exists operators $R : M \mapsto \ell_p^m$ and $S : \ell_p^m \mapsto X$ that factors the inclusion map $I_M^X = SR$ and such that $\|S\|\|R\| \le \lambda + \varepsilon$.

These spaces have a few nice properties:

Proposition 2.1.4 (cf. [3])

For $1 < p < \infty$:
1) If X is an $\mathcal{L}^g_p$ space, then it has the Radon Nikodym property and the bounded approximation property;
2) X is an $\mathcal{L}^g_{p\lambda}$ space if and only if X' is an $\mathcal{L}^g_{p'\lambda}$ space;
3) if X is an $\mathcal{L}^g_p$ space then either it is an $\mathcal{L}_p$ space or it is isomorphic to a Hilbert space;
4) if X and Y are $\mathcal{L}^g_p$ then $X \otimes_{d_p} Y$ is an $\mathcal{L}^g_p$ space.

Proposition 2.1.5 (cf. [3])

Let $1 < p < \infty$. The following propositions are equivalent:
1) X is isomorphic to a quotient of an L^p space;
2) $L^p \otimes_{d_p} X \simeq L^p \otimes_{g_p} X = X \otimes_{d_p} L^p$.
In particular, this is true for complemented subspaces of L^p spaces i.e. $\mathcal{L}_p^g$ spaces.

Since an $\mathcal{L}_p^g$ space X is a (complemented) subspace of an L^p space, we have, by Proposition 2.1.1,

$$L^p(\mu) \otimes_{d_p} X \simeq L^p(\mu) \otimes_{\Delta_p} X = L^p(\mu, X).$$

Using local techniques we can derive from (2) and Proposition 2.1.4 (1) and (5), that if X, Y, Z are $\mathcal{L}_p^g$ spaces then

$$X \otimes_{d_p} (Y \otimes_{d_p} Z) \simeq (X \otimes_{d_p} Y) \otimes_{d_p} Z. \tag{3}$$

This identity has an operator counterpart:

Lemma 2.1.6

Let $1 < p < \infty$ and R be a measure space and let X and Y be $\mathcal{L}_p^g$ spaces. Then

$$N_{p'}\big(L^p(R) \otimes_{d_p} X, Y'\big) \simeq N_{p'}\Big(X, N_{p'}(L^p(R), Y')\Big) \tag{4}$$

where the operator $T : L^p(R) \otimes_{d_p} X \mapsto Y'$ is identified with the operator $\tilde{T} : X \mapsto N_{p'}(L^p(R), Y')$ via $\tilde{T}(\varphi)(\psi) = T(\psi \otimes \varphi)$.

Proof. Equation (3) can be read, using the identification of tensor products and operators as:

$$\begin{aligned}
N_{p'}\big(L^p(R) \otimes_{d_p} X, Y'\big) &= N_{p'}\big(L^p(R, X), Y'\big) = L^p(R, X)' \otimes_{d_{p'}} Y' \\
&= L^{p'}(R, X') \otimes_{d_{p'}} Y' = \big(L^{p'}(R) \otimes_{d_{p'}} X'\big) \otimes_{d_{p'}} Y' \\
&= \big(X' \otimes_{d_{p'}} L^{p'}(R)\big) \otimes_{d_{p'}} Y' = X' \otimes_{d_{p'}} \big(L^{p'}(R) \otimes_{d_{p'}} Y'\big) \\
&= N_{p'}(X, L^{p'}(R) \otimes_{d_{p'}} Y') = N_{p'}\Big(X, N_{p'}(L^p(R), Y')\Big). \ \square
\end{aligned}$$

Remark. For a fixed p $(1 < p < \infty)$, $\mathcal{L}_p^g$ is a rather large class of Banach spaces. In particular, it contains the L_p spaces, the Hilbert spaces and the Hardy spaces H_p.

2.3 p-induction

The concept of p-induction has been defined in various places, eg. [5], [12], and [22]. Here we will follow Anker [1]. To fix notation we repeat the definitions of that paper. Let G be a separable locally compact group and H be a closed subgroup. Let $1 \leq p < \infty$. Let ν_G (resp. ν_H) denote the (left) Haar measure on G (resp. H). Denote by Δ_G (resp. Δ_H) the modular function of G (resp. H), and let $\delta(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$.

Let q be a continuous positive function defined on G that satisfies the covariance condition $q(xh) = q(x)\delta(h)$ for all $x \in G, h \in H$. We write μ for the *quasi-invariant measure*[1] on G/H that is associated to q by

$$\int_{G/H} \left[\int_H \frac{f(xh)}{q(xh)} d\nu_H(h) \right] d\mu(xH) = \int_G f(x) d\nu_G(x)$$

for all $f \in \mathcal{C}_c(G)$. The fact that such a measure exists can be found in [16].

Let β be a *Bruhat function* for the pair $H \subset G$, that is, a non-negative continuous function on G that satisfies

1) supp $\beta \cap CH$ is compact for every compact set C in G;

2) $\int_H \beta(xh) d\nu_H(h) = 1$ for every $x \in G$.

(For details, see for instance [6] Chapter 5 or [18] Chapter 8.)

Let π be a strongly continuous isometric representation of the subgroup H in a Banach space X. For $1 \leq p < \infty$, we denote by $L^p(G, H, \pi)$ the space of functions $f : G \mapsto X$ that satisfy the following conditions:

1) for every $\xi \in X^*$, $x \mapsto < f(x), \xi >$ is measurable;

2) for every $x \in G, h \in H$,

$$f(xh) = \delta(h)^{1/p} \pi_h^{-1} f(x).$$

This condition is called the *covariance condition*. Note that it implies that $\frac{\|f(x)\|^p}{q(x)}$ is constant on the cosets xH. Thus, the following condition makes sense

3)

$$\|f\|_p = \left[\int_{G|_H} \frac{\|f(x)\|^p}{q(x)} d\mu(xH) \right]^{1/p} = \left[\int_G \|f(x)\|^p \beta(x) d\nu_G(x) \right]^{1/p} < \infty.$$

[1] Recall that a measure μ on a G-space $\mathcal{M}$ is quasi-invariant if for every $g \in G$, and every measurable $M \subset \mathcal{M}$, $\mu(gM) = 0$ if and only if $\mu(M) = 0$.

This space is the completion for the norm $\|f\|_p$ of the space $\mathcal{C}_c^p(G;H;\pi)$ of all continuous functions $f : G \mapsto X$ with compact support that satisfy the covariance condition.

We recall also *Mackey's Mapping* $f \mapsto M_p f$ from $\mathcal{C}_c(G,X)$ (the space of all continuous functions $G \mapsto X$ with compact support) to $\mathcal{C}_c^p(G,H,\pi)$ defined by the integral

$$M_p f(x) = \int_H \frac{1}{\delta(h)^{1/p}} \pi_h f(xh) d\nu_H(h).$$

The p-induced representation $Ind_H^G(p,\pi)$ then operates on $L^p(G;H;\pi)$ by left translation: for $g \in G$

$$\big(Ind_H^G(p,\pi)_g f\big)(x) = f(g^{-1}x).$$

The first result on p-induction follows as in the L^2 case and is given in detail in [12].

Theorem 2.2.1 (Induction In Stages)

Let G be a locally compact group, K a closed subgroup of G and H a closed subgroup of K. Let π be a representation of H in a Banach space X. Then the representations $Ind_K^G(p, Ind_H^K(p,\pi))$ and $Ind_H^G(p,\pi)$ are equivalent.

2.4 Modules

We recall a few properties of Banach modules over groups and Banach algebras. The reader is referred to [19] for basic definitions. For a locally compact group G, every Banach G-module V becomes a Banach $L^1(G)$-module under the action

$$f.v = \int_G f(g) g.v d\nu_G(g) \qquad f \in L^1(G), v \in V.$$

Notation. If V and W are two G-modules (thus $L^1(G)$-modules) and if α is a tensor norm, let K (resp. K_1) be the closed subspace of $V \otimes_\alpha W$ spanned by elements of the form $g.v \otimes w - v \otimes g.w$ with $v \in V, w \in W, g \in G$ (resp. spanned by elements of the form $f.v \otimes w - v \otimes f.w$ with $v \in V, w \in W, f \in L^1(G)$). Define then $V \otimes_G^\alpha W = (V \otimes_\alpha W)\,|_K$ and $V \otimes_{L^1(G)}^\alpha W = (V \otimes_\alpha W)\,|_{K_1}$.

We need a definition from Rieffel:

DEFINITION 2.3.1. Let A be a Banach algebra and let V be a Banach G-module. We say that V is *essential* if the space $\{a.v \ : \ a \in A, \ v \in V\}$ is dense in V.

Then, following Rieffel ([19] Theorem 4.14) every Banach G-module is an essential $L^1(G)$ module and

$$V \otimes_G^{d_p} W = V \otimes_{L^1(G)}^{d_p} W.$$

The remaining of this section is taken from [17].

Proposition 2.3.2

Let G be a compact group and let V and W be two Banach G-modules. Then $V \otimes_G^{d_p} W$ is isometrically isomorphic to the 1-complemented linear subspace $(V \otimes^{d_p} W)^G$ consisting of those z in $V \otimes^{d_p} W$ for which $g \otimes e(z) = e \otimes g(z)$ for all $g \in G$ (e the unit element of G), that is,

$$V \otimes_G^{d_p} W = (V \otimes^{d_p} W)^G$$

isometrically isomorphic. Moreover, the projection from $V \otimes_{d_p} W$ onto $(V \otimes_{d_p} W)^G$ is given by

$$P(v \otimes w) = \int_G g^{-1}.v \otimes g.w d\nu_G(g).$$

We will also need the following version of Proposition 2.4 in [17]:

Proposition 2.3.3

Let G be a compact group and let V and W be two Banach G-modules, V being a reflexive Banach space with the approximation property. Denote by $N_p^G(V, W)$ the set of all right p-nuclear operators T such that for every $g \in G$ and every $v \in V$, $T(g.v) = g.Tv$, then

$$N_p^G(V, W) = V' \otimes_G^{d_p} W.$$

From (3) and (4) we then immediately obtain the two following identities:

Lemma 2.3.4

Let $1 < p < \infty$. Let H, K be compact groups, let R be a measure space, and let V, W be $\mathcal{L}_p^g$ spaces such that V is an H-module, W is a K-module and $L^p(R)$ is an $H - K$-bimodule, then

$$\left(L^p(R) \otimes_{d_p}^H V\right) \otimes_{d_p}^K W = L^p(R) \otimes_{d_p}^H (V \otimes_{d_p}^K W) \tag{5}$$

and

$$N_{p'}^K\left(L^p(R) \otimes_{d_p}^H V, W'\right) = N_{p'}^H\Big(V, N_{p'}^K\left(L^p(R), W'\right)\Big). \tag{6}$$

2.5 Rieffel's 1-induction

We summarize here Chapter 10 of [19].

Grothendieck [9] has shown that $L^1(G)\hat{\otimes}_\pi V$ can be naturally and isometrically identified with $L^1(G, V)$ through the mapping $f \otimes v \mapsto \big(x \mapsto f(x)v\big)$. We will not distinguish between $L^1(G)\hat{\otimes}_\pi V$ and $L^1(G, V)$. For $f \in L^1(G)$ and $s \in H$, let $(f_s)(x) = \Delta_G(s^{-1})f(xs^{-1})$ $(x \in G)$ and let $\tilde{K}$ be the closed subspace of $L^1(G)\hat{\otimes}_\pi V$ spanned by the elements of the form $f_s \otimes v - f \otimes \pi_s v$ $(s \in H, f \in L^1(G)$ and $v \in V)$. We define $L^1(G)\hat{\otimes}_\pi^H V = L^1(G)\hat{\otimes}_\pi V|_{\tilde{K}}$.

Mackey's transform defined in Section 2.2 will allow us to identify the spaces $L^1(G; H; \pi)$ and $L^1(G)\hat{\otimes}_\pi^H V$. This is Theorem 10.4 of [19]:

Theorem 2.4

For $g \in L^1(G,V)$, recall that Mg has been defined on G by

$$Mg(x) = \int_H \frac{1}{\delta(h)}\, \pi_h g(xh)\, d\nu_H(h).$$

Then Mg is defined almost everywhere, $Mg \in L^1(G;H;\pi)$, and M is a G-module homomorphism from $L^1(G,V)$ to $L^1(G;H;\pi)$. Moreover, the kernel of M is exactly $\tilde{K}$ and the norm in $L^1(G;H;\pi)$ can be regarded as the quotient norm in $L^1(G,V)|_K$. Thus $L^1(G;H;\pi)$ is isometrically G-module isomorphic to $L^1(G)\hat{\otimes}_\pi^H V$.

We shall extend this result to the case $p \geq 1$.

3. p-induction using tensor products

In this section we will show that the Mackey mapping allows us to define $L^p(G;H;\pi)$ as a tensor product. The proves will be adapted from [19], [20].

Let $1 < p < \infty$. We will now assume that V is a reflexive Banach space. In particular V has the Radon-Nikodým property.

Let G be a locally compact group, and let H be a *compact* subgroup of G. Note that since H is compact, $\Delta_H = 1$. Let β be a Bruhat function of the pair $H \subset G$.

Let q be the function on G defined by

$$q(x) = \int_H \beta(xs)\Delta_G(s) d\nu_H(s).$$

Then q satisfies, for all $x \in G$ and all $h \in H$,

$$q(xh) = \frac{1}{\Delta_G(h)}\, q(x) = \delta(h)\, q(x).$$

Let μ be the *quasi-invariant measure* on G/H associated with q, defined in the following way:

$$\int_{G|H} \left[\int_H \frac{f(xh)}{q(xh)}\, d\nu_H(h) \right] d\mu(xH) = \int_G f(x)\, d\nu_G(x) \tag{7}$$

for every continuous compactly supported function $f : G \mapsto \mathbb{C}$. The existence of such a measure has been established in various places, eg. [19] Proposition 10.1.

Let π be a representation of H on the Banach space V. This space being reflexive, we can define the coadjoint representation π^* of H on V^* by letting $\pi_h^* = (\pi_{h^{-1}})^*$.

Remember that we defined the Mackey map $f \mapsto M_p f$ from $\mathcal{C}_c(G, B)$ to $\mathcal{C}_c^p(G, H, \pi)$ by

$$M_p f(x) = \int_H \frac{1}{\delta(h)^{1/p}} \pi_h f(xh) d\nu_H(h) = \int_H \Delta_G(s)^{1/p} \pi_s f(xs) d\nu_H(s).$$

We want to show that this defines a continuous projection

$$M_p : L^p(G, V) \mapsto L^p(G; H; \pi).$$

Let $f \in L^p(G, V)$. We show that $M_p f \in L^p(G; H; \pi)$. Observe first that $M_p f$ is defined a.e., is measurable and satisfies the covariance condition. The argument is strictly similar to [19] pp. 484–486 and will not be reproduced here.

We now show that M_p is continuous (with norm ≤ 1).

$$\begin{aligned}
\|M_p f\|^p &= \int_{G|_H} \frac{\|M_p f(x)\|_V^p}{q(x)} d\mu(xH) \\
&= \int_{G|_H} \frac{1}{q(x)} \left\| \int_H \Delta_G(h)^{1/p} \pi_h f(xh) d\nu_H(h) \right\|_V^p d\mu(xH) \\
&\leq \int_{G|_H} \frac{1}{q(x)} \int_H \Delta_G(h)^{1/p} \|\pi_h f(xh)\|_V^p d\nu_H(h) d\mu(xH).
\end{aligned}$$

But $\|\pi_h f(xh)\|_V = \|f(xh)\|_V$. Thus, by disintegration of measures (i.e. the definition of μ), we obtain $\|M_p f\|^p \leq \|f\|^p$.

Next, we identify the kernel of M_p. First, define the following representation of G on $L^p(G)$:

$$\rho_t f(x) = \Delta_G(t)^{1/p} f(xt)$$

and note that for $f \in L^p(G), v \in V, t \in H$

$$\begin{aligned}
M_p\big(f(.)\pi_t v\big)(x) &= \int_H \Delta_G(s)^{1/p} f(xs) \pi_s \pi_t v d\nu_H(s) \\
&= \int_H \Delta_G(st^{-1})^{1/p} f(xst^{-1}) \pi_s v d\nu_H(s) \\
&= \int_H \Delta_G(s)^{1/p} \pi_s \big(\Delta_G(t^{-1})^{1/p} f(xst^{-1}) v\big) d\nu_H(s) \\
&= M_p\big(\rho_{t^{-1}} f(.) v\big)(x).
\end{aligned}$$

Now, let K be the closed linear span of all the elements of the form $x \mapsto f(x)\pi_t v - \rho_{t^{-1}} f(x)v$ with $f \in L^p(G), v \in V$ and $t \in H$. By linearity and continuity of M_p we see that $\ker M_p \supset K$. It is now possible to adapt the proof of Rieffel [20] for Hilbert spaces (i.e. QSL_2) to yield $\ker M_p = K$ for reflexive Banach spaces.

First consider $L^p(G, V)$ and $L^p(G; H; \pi)$ as G-modules where the action of G is defined by left translation, i.e. $g.f(x) = f(g^{-1}x)$. It is then clear, as in [19], that M_p is a G-module homomorphism, that is, $M_p(g.f) = g.M_p f$.

Assume now that $\ker M_p \neq K$. Then, there exists a φ such that $M_p\varphi = 0$ but $\varphi \notin K$. By the Hahn-Banach theorem and the Radon-Nikodým property of V (V is reflexive), we can find a functional

$$Q \in K^\perp \subset \big(L^p(G, V)\big)' = L^{p'}(G, V')$$

such that $< Q, \varphi > \neq 0$. Since $L^p(G, V)$ is a G-module, it is an essential $L^1(G)$-module. Therefore, there exists an $i \in L^1(G)$ such that $< Q, i\varphi > \neq 0$, and if we use a continuous compactly supported approximation of unity we can even assume that i is continuous and compactly supported. Thus $< iQ, \varphi > = < Q, i\varphi > \neq 0$.

Now, K is G invariant, and hence so is $K^\perp$, so that $K^\perp$ is invariant under convolution by continuous compactly supported functions, from which it follows that, for all $\psi \in K$, $< iQ, \psi > = 0$.

By [11] Theorem 20.6, since iQ is a convolution involving a continuous compactly supported function, iQ is a continuous function F. Arguing as in [20], page 168, it follows from $F = iQ \in K^\perp$ that

$$F(xh) = \frac{1}{\Delta_G(h)^{1/p'}} \pi_h^*\big(F(x)\big) \quad \text{for all } \ h \in H, x \in G.$$

Note too that

$$\begin{aligned} < F(xh), \varphi(xh) > &= \frac{1}{\Delta_G(h)^{1/p'}} < \pi_h^*\big(F(x)\big), \varphi(xh) > \\ &= \frac{1}{\Delta_G(h)^{1/p'}} < F(x), \pi_h\big(\varphi(xh)\big) > \ . \end{aligned}$$

Now, by disintegration of measures (7),

$$
\begin{aligned}
< Q, \varphi > &= \int_G < F(x), \varphi(x) > d\nu_G(x) \\
&= \int_{G|_H} \left[\int_H \frac{< F(xh), \varphi(xh) >}{q(xh)} \, d\nu_H(h) \right] d\mu(xH) \\
&= \int_{G|_H} \left[\int_H \frac{1}{\Delta_G(h)^{1/p'}} \, \frac{< F(x), h\big(\varphi(xh)\big) >}{\delta(h) q(x)} \, d\nu_H(h) \right] d\mu(xH) \\
&= \int_{G|_H} \frac{1}{q(x)} < F(x), \int_H \frac{1}{\Delta_G(h)^{1/p'}} \, \frac{h\big(\varphi(xh)\big)}{\delta(h)} \, d\nu_H(h) > d\mu(xH) \\
&= \int_{G|_H} \frac{1}{q(x)} < F(x), M_p\varphi(x) > d\mu(xH) = 0
\end{aligned}
$$

since $M_p\varphi = 0$. This contradicts the assumption $< Q, \varphi > \neq 0$ and the kernel of M_p is exactly K.

Note that the proof of [19], Lemma 10.9 carries over to yield that M_p is surjective and that the norm on $L^p(G; H; \pi)$ is the quotient norm of $L^p(G, V)/K \simeq \big(L^p(G) \otimes_{\Delta_p} V\big)/K$. We leave the details to the reader.

We summarize the preceding discussion in the following theorem.

Theorem 3.1

Let $1 < p < \infty$. Let G be a locally compact group and H a compact subgroup of G. Let V be a reflexive Banach space and let π be a representation of H on V, for which V is an H-module. Let K be the closed linear subspace of $L^p(G, V)$ spanned by the elements of the form $x \mapsto f(x)\pi_t v - \big(\rho_{t^{-1}} f\big)(x)v$ with $f \in L^p(G), v \in V$ and $t \in H$. Identifying $L^p(G, V)$ and $L^p(G) \otimes_{\Delta_p} V$, we also regard K as being spanned by elements of the form $x \mapsto f(x) \otimes \pi_t v - \rho_{t^{-1}} f(x) \otimes v$ and write $L^p(G) \otimes^H_{\Delta_p} V$ for $\big(L^p(G) \otimes_{\Delta_p} V\big)/K$. Moreover, if for $f \in L^p(G, V)$ we define $M_p f$ by

$$
M_p f(x) = \int_H \frac{1}{\delta(h)^{1/p}} \, \pi_h f(xh) \, d\nu_H(h) = \int_H \Delta_G(h)^{1/p} \pi_h \, f(xh) \, d\nu_H(h),
$$

then M_p is a G-module homeomorphism from $L^p(G, V)$ onto $L^p(G; H; \pi)$. The kernel of M_p is exactly K and the norm of $L^p(G; H; \pi)$ is the quotient norm. Consequently, $L^p(G; H; \pi)$ is isometrically G-module homeomorphic to $L^p(G) \otimes^H_{\Delta_p} V$.

If V is a QSL_p space, then $L^p(G; H; \pi)$ is in fact isometrically G-module homeomorphic to $L^p(G) \otimes^H_{d_p} V$.

4. Applications to classical theorems on induction

The previous theorem allows us to define p-induction via tensor products. We now use that point of view to prove the results about induction in stages and a Kronecker product theorem. Finally we also obtain a new Frobenius reciprocity theorem.

At this stage, we will need various restrictions on the spaces on which we represent our groups. They will be QSL_p spaces or $\mathcal{L}_p^g$ spaces.

Let $1 < p < \infty$, let G be a locally compact group, H a compact subgroup of G, and V a QSL_p space. Fix a representation π of H on V so that V can be seen as an H-module.

We have seen in Section 3 that the p-induced representation of π can be identified with $L^p(G) \otimes_{d_p}^H V$. We write ${}^{G,p}V = L^p(G) \otimes_{d_p}^H V$ and call this the *p-induced module*.

We are now in a position to prove an inducing-in-stages theorem, the Kronecker product theorem and a new Frobenius reciprocity theorem. But first, we will need the following technical result:

Theorem 4.1.1

Let $1 < p < \infty$ and let V be a QSL_p space. Assume that G is a compact group. Assume also that V is a G-module and let us consider $L^{p'}(G)$ as a $G-G$-bimodule, then $N_p^G\big(L^{p'}(G), V\big) \simeq V$ as G-module. The identification is given by

$$v \in V \mapsto T_v(f) = \int_G f(x)x.v d\nu_G(x).$$

Equivalently, $L^p(G) \otimes_{d_p}^G V \simeq V$.

Proof. Let $v \in V$ and define for $f \in L^{p'}(G)$

$$T_v(f) = \int_G f(x)x.v d\nu_G(x).$$

As G is compact, $x \mapsto x.v \in \mathcal{C}(G, V) \subset L^p(G, V)$ thus $T_v \in N_p\big(L^{p'}(G), V\big)$.

Further, if $g \in G$ and $f \in L^{p'}(G)$

$$\begin{aligned} T_v(g.f) &= \int_G (g.f)(x)x.v d\nu_G(x) = \int_G f(g^{-1}x)x.v d\nu_G(x) \\ &= \int_G f(x)g.(x.v) d\nu_G(x) = g.\int_G f(x)x.v d\nu_G(x) = g.T_v(f). \end{aligned}$$

Thus $T_v \in N_p^G\big(L^{p'}(G), V\big)$. In the same way,

$$\begin{aligned}T_{g.v}(f) &= \int_G f(x)x.(g.v)d\nu_G(x) = \int_G f(xg^{-1})x.vd\nu_G(x)\\ &= \int_G (f.g)(x)x.vd\nu_G(x) = (T_v.g)(f).\end{aligned}$$

Moreover

$$\|T_v\|^p = \int_G \|x.v\|^p d\nu_G(x) = \int_G \|v\|^p d\nu_G(x) = \|v\|^p$$

as the action of G has been assumed to be isometric and G is compact. Thus $v \mapsto T_v$ is an isometric G-module homomorphism from V to $N_p^G\big(L^{p'}(G), V\big)$.

We just have to prove that $v \mapsto T_v$ is onto to complete the proof. For $T \in N_p^G\big(L^{p'}(G), V\big)$, we want to find $v \in V$ such that $T = T_v$. As $N_p\big(L^{p'}(G), V\big) \simeq L^p(G, V)$ (V is a QSL_p space), there exists $F \in L^p(G, V)$ such that, for all $f \in L^{p'}(G)$

$$T(f) = \int_G f(x)\, F(x)\, d\nu_G(x).$$

But, as T is a G-module homomorphism, for all $g \in G$ and all $f \in L^{p'}(G)$,

$$\int_G f(g^{-1}x)\, F(x)\, d\nu_G(x) = g.\int_G f(x)F(x)\, d\nu_G(x),$$

that is,

$$\int_G f(x)\, F(gx)\, d\nu_G(x) = \int_G f(x)g.F(x)d\nu_G(x).$$

Therefore, for all $g \in G$, $F(gx) = g.\big(F(x)\big)$, x a.e. It is easy, however, to see that $F(gx) - g.F(x)$ is measurable in (x, g) and by Fubini's theorem

$$Q = \{(x, g) : F(gx) \neq g.F(x)\}$$

is of measure zero. By Fubini's theorem again, except for x in a set of measure zero, $F(gx) = gF(x)$, g almost everywhere. Let x_0 be any x from this set and let $v = x_0^{-1}\big(F(x_0)\big)$. Then we have, for almost all x,

$$F(x) = F\big((xx_0^{-1})x_0\big) = (xx_0^{-1}).F(x_0) = x.\big[x_0^{-1}.F(x_0)\big] = x.v$$

in other words, $F(x) = x.v$ almost everywhere, and $T = T_v$. $\square$

Before we go on, we indicate what happens if H is not compact.

Theorem 4.1.2

Let $1<p<\infty$ and let V be a QSL^p space. Let G be a locally compact group and let H be a closed non-compact subgroup. Let π be a representation of H on V, making V into a H-module. Then

$$N_p^H\big(L^{p'}(G),V\big)=0 \quad \text{and} \quad L^p(G)\otimes_H^{d_p} V=0.$$

Proof. Let $T\in N_p^H\big(L^{p'}(G),V\big)$. Then, as in the end of the proof of Theorem 4.1, there exists $F\in L^p(G,V)$ such that $T=T_F$ and then $F(sx)=s.F(x)$ for all $s\in H$ and almost all x. But then F is of constant norm on cosets of H, and so is integrable if and only if it is identically zero. The second assertion is just the standard identification between the two spaces under consideration. □

We can now give a new proof of the theorem of Inducing-In-Stages. This proof is simpler then the proof given in [12], but we need some restrictive hypothesis on the subgroup H and on the Banach space V.

Theorem 4.1.3 (Inducing-In-Stages)

Let $1<p<\infty$ and let V be an $\mathcal{L}_p^g$ space. Let G be a locally compact group, K a compact subgroup of G and H a closed subgroup of K. Let π be a representation of H on V allowing us to consider V as an H-module. Then

$$^{G,p}\big(^{K,p}V\big)\simeq {}^{G,p}V.$$

Proof. Using the definition, the associativity of the d_p tensor product ,*i.e.* (5), and Theorem 4.1, it is immediate that

$$\begin{aligned} ^{G,p}\big(^{K,p}V\big) &= L^p(G)\otimes_{d_p}^K\big(L^p(K)\otimes_{d_p}^H V\big)\simeq\big(L^p(G)\otimes_{d_p}^K L^p(K)\big)\otimes_{d_p}^H V\\ &\simeq\big(L^p(G)\otimes_{d_p}^K L^p(K)\big)\otimes_{d_p}^H V\simeq L^p(G)\otimes_{d_p}^H V={}^{G,p}V. \ \square\end{aligned}$$

We now define the p-Kronecker product of two representations. Let H and K be two locally compact groups and V and W be two Banach spaces. Fix π to be a representation of H on V and γ to be a representation of K on W. We define the p-Kronecker product of π and γ as the representation of $H\times K$ on $V\otimes_{d_p}W$ defined by

$$\pi\times\gamma_{(h,k)}v\otimes w=\pi_h v\otimes\gamma_k w.$$

The next theorem asserts that taking p-Kronecker products and p-inducing are two commutative operations. This theorem is new to our knowledge.

Theorem 4.1.4 (p-Kronecker Product)

Let $1 < p < \infty$ and let V_1, V_2 be $\mathcal{L}^g_p$ spaces. Let G_1, G_2 be two locally compact groups, let H_1 be a compact subgroup of G_1 and H_2 a compact subgroup of G_2 and let π_i $(i = 1, 2)$ be representations of H_i on V_i. Then

$$^{G_1 \times G_2, p}(V_1 \otimes_{d_p} V_2) \simeq {}^{G_1,p}V_1 \otimes_{d_p} {}^{G_2,p}V_2.$$

Proof. Using properties of the d_p tensor product, we have

$$\begin{aligned} ^{G_1 \times G_2, p}(V_1 \otimes_{d_p} V_2) &= L^p(G_1 \times G_2) \otimes^{H_1 \times H_2}_{d_p} (V_1 \otimes_{d_p} V_2) \\ &\simeq \big(L^p(G_1) \otimes_{d_p} L^p(G_2)\big) \otimes^{H_1 \times H_2}_{d_p} (V_1 \otimes_{d_p} V_2) \\ &\simeq \big(L^p(G_1) \otimes^{H}_{d_p} V_1\big) \otimes_{d_p} \big(L^p(G_2) \otimes^{H}_{d_p} V_2\big) \\ &\simeq {}^{G_1,p}V_1 \otimes_{d_p} {}^{G_2,p}V_2. \ \square \end{aligned}$$

For W a G-module and H a subgroup of G, we write W_H for W seen as a H-module. We will now prove the following version of the Frobenius Reciprocity Theorem.

Theorem 4.1.5 (Frobenius Reciprocity)

Let $1 < p < \infty$ and let V be an $\mathcal{L}^g_p$ space and W an $\mathcal{L}^g_{p'}$ space. Let G be a compact group and H be a closed subgroup of G. Let π be a representation of H on V, making V an H-module, and let γ be a representation of G on W making W a G-module, so that W is also an H-module W_H. Then

$$N^G_{p'}({}^{G,p}V, W) \simeq N^H_{p'}(V, W_H).$$

and

$$N^G_{p}(W, {}^{G,p}V) \simeq N^H_{p}(W_H, V).$$

Proof. By definition

$$N^G_{p'}({}^{G,p}V, W) = N^G_{p'}\big(L^p(G) \otimes^H_{d_p} V, W\big) \simeq N^G_{p'}\big(V \otimes^H_{d_p} L^p(G), W\big)$$

and by Theorem 2.3.4, $N^G_{p'}\big(V \otimes^H_{d_p} L^p(G), W\big) \simeq N^H_{p'}\Big(V, N^G_{p'}\big(L^p(G), W\big)\Big)$. But, according to Theorem 4.1, $N^G_{p'}\big(L^p(G), W\big) \simeq W$, so that,

$$N^G_{p'}({}^{G,p}V, W) \simeq N^H_{p'}(V, W_H).$$

The other identity is obtained in a similar way. $\square$

5. The Subgroup Theorem

We shall now generalize Mackey's subgroup theorem ([16] Theorem 12.1) to the context of p-inducing. For technical reasons, we will restrict to the case when the group is *unimodular*, one subgroup considered is compact and the other one is also unimodular.

We will make extensive use of regularly related subgroups and their measure theoretic properties as may be found in [16] Section 11. For sake of completeness, we will now recall those that we shall use.

Let μ be a finite measure on a set X and suppose there is an equivalence relation R given on X. For $x \in X$, let $r(x) \in X/R$ be the equivalence class of x. The equivalence relation is said to be *measurable* if there exists a countable family $E_1, E_2, \dots$ of subsets of X/R such that $r^{-1}(E_i)$ is measurable for each i and such that each point in X/R is the intersection of the E_i's which contain it.

Let G be a locally compact group and let G_1 and G_2 be two subgroups of G. We say that G_1 and G_2 are *regularly related* if there exists a sequence $E_0, E_1, E_2, \dots$ of measurable subsets of G each of which is a union of $G_1 : G_2$ double cosets such that E_0 has Haar measure zero and each double coset not in E_0 is the intersection of the E_i's which contain it. Hence G_1 and G_2 are regularly related if and only if the orbits of $X = G/G_1$ under the action of G_2, outside a certain set of measure zero, form the equivalence classes of a measurable equivalence relation. In other words, there is a measurable cross-section ψ of the set $\mathcal{D}$ of all $G_1 : G_2$ double cosets in G *i.e.* $\psi : \mathcal{D} \mapsto G$ measurable. The following lemma ([16] Lemma 11.1) states that a measure μ defined on X may be decomposed as an integral over X/R of measures μ_y concentrated on the equivalence classes.

Lemma 5.1

Let $\tilde{\mu}$ be the measure in X/R such that a subset E of X/R is measurable if and only if $r^{-1}(E)$ is μ measurable and that $\tilde{\mu}(E) = \mu\big(r^{-1}(E)\big)$. Then for each y in X/R there exists a finite Borel measure μ_y on X such that $\mu_y\big(X \setminus r^{-1}(\{y\})\big) = 0$ and

$$\int_{X/R} f(y) \int_{r^{-1}(y)} g(x)\, d\mu_y(x)\, d\tilde{\mu}(y) = \int_X f\big(r(x)\big)\, g(x)\, d\mu\,(x),$$

whenever f is in $L^1(X/R, \tilde{\mu})$ and g is bounded and measurable on X.

Lemma 5.2

Let X be a G-space, and assume that the measure μ on X is quasi-invariant. Then, in the decomposition of μ in the previous lemma, almost all of the μ_y's are also quasi-invariant under the action of G.

Notation. In what follows, G will be a locally compact group, G_1 a compact subgroup of G and G_2 a closed subgroup of G. We will also assume that G and G_2 are *unimodular*. We will further assume that G_1 and G_2 are regularly related.

Let $\mathcal{D}$ be the set of all $G_1 : G_2$ double cosets. For $x \in G$, we will note $s(x) = G_1 x G_2$ the $G_1 : G_2$ double coset to which x belongs. If ν is any finite measure on G with the same null sets as the Haar measure on G, we may define a measure ν_0 on $\mathcal{D}$ by setting $\nu_0(E) = \nu\big(s^{-1}(E)\big)$. Such a measure is called ([16] Section 12) an *admissible measure* on $\mathcal{D}$ (associated to ν).

Let $1 < p < \infty$ and let V be a QSL_p Banach space. Fix a representation π of G_1 on V, and consider V as a G_1 module. Let ${}^{G,p}V = L^p(G) \otimes_{d_p}^{G_1} V$ be the induced module. For $x \in G$ write $G_x = G_2 \cap (x^{-1} G_1 x)$ and denote π^x the representation of G_x on V defined by $\eta \mapsto \pi_{x\eta x^{-1}}$. We can consider V as a G_x-module (denoted by V^x) with the action defined by this representation. Furthermore, we define the module induced on G_2: ${}^{G_2,p}V^x = L^p(G_2) \otimes_{d_p}^{G_x} V^x$.

Lemma 5.3

${}^{G_2,p}V^x$ depends only (up to equivalence) on the coset $s(x) = G_1 x G_2$.

Proof. By definition ${}^{G_2,p}V^x = L^p(G_2)^x \otimes_{d_p}^{G_x} V^x$ where $L^p(G_2)^x = L^p(G_2)$ seen as a G_x-module with the action of $s \in G_x$ defined as $s.\varphi(t) = \varphi(s^{-1}t)$ and $V^x = V$ also seen as a G_x-module with the action of $s \in G_x$ defined by $s \bullet v = (xsx^{-1}).v$. Thus ${}^{G_2,p}V^x = \big(L^p(G_2)^x \otimes_{d_p} V^x\big)|_{K_x}$ with K_x the closed linear span of all

$$s.\varphi \otimes v - \varphi \otimes s \bullet v$$

such that $\varphi \in L^p(G_2), v \in V$ and $s \in G_x$.

We want to show that ${}^{G_2,p}V^x$ depends only on the double coset $s(x)$. In other words, we want to show that for all $g_1 \in G_1, g_2 \in G_2$,

$${}^{G_2,p}V^x \simeq {}^{G_2,p}V^{g_1 x g_2}.$$

It is enough to prove that $K_{g_1 x g_2} \simeq K_x$.

First, note that

$$G_{g_1 x g_2} = G_2 \cap (g_2^{-1} x^{-1} g_1^{-1} G_1 g_1 x g_2) = g_2^{-1}\big(G_2 \cap (x^{-1} G_1 x)\big) g_2.$$

Define the group isomorphism $a_{g_2} : G_x \mapsto G_{g_1 x g_2}$ by $a_{g_2} s = g_2^{-1} s g_2$. We can now regard $L^p(G_2)$ as a $G_{g_1 x g_2}$-module where the action is defined as $s \diamond \varphi = (g_2 s g_2^{-1}).\varphi$, and also regard V as a $G_{g_1 x g_2}$-module with action

$$s \diamond v = (x g_2 s g_2^{-1} x^{-1}).v.$$

By definition

$$\begin{aligned}K_{g_1 x g_2} &= \overline{span}\{s\diamond\varphi\otimes v - \varphi\otimes s\diamond v : \varphi\in L^p(G_2), v\in V, s\in G_{g_1xg_2}\}\\ &= \overline{span}\{a_{g_2}(s)\diamond\varphi\otimes v - \varphi\otimes a_{g_2}(s)\diamond v : \varphi\in L^p(G_2), v\in V, s\in G_x\}\\ &= \overline{span}\{s.\varphi\otimes v - \varphi\otimes s\bullet v : \varphi\in L^p(G_2), v\in V, s\in G_x\} = K_x\end{aligned}$$

which completes the proof. □

It now makes sense to write ${}^{G_2,p}V^x$ for x a $G_1 : G_2$ double coset. Recall that ${}^{G_2,p}V^x = L^p(G_2)\otimes_{d_p}^{G_x} V^x$ can be seen as a complemented subspace of $L^p(G_2)\otimes_{d_p} V$ via the projections

$$P_x(f\otimes v) = \int_{G_x} \rho_t f \pi_{x^{-1}tx} v d\nu_{G_x}(t)$$

where ν_{G_x} is a Haar measure on G_x. As V is QSL_p,

$$\big(L^p(G_2)\otimes_{d_p} V\big)^* = \big(L^p(G_2,V)\big)^* = L^{p'}(G_2,V^*) = L^{p'}(G_2)\otimes_{d_{p'}} V^*,$$

and $\big({}^{G_2,p}V^x\big)^*$ will be complemented in $L^{p'}(G_2)\otimes_{d_{p'}} V^*$ via P_x^*.

We will now show that the $P_x^*(g\otimes\xi) = \int_{G_x}\rho_{t^{-1}} g\pi^*_{x^{-1}tx}\xi d\nu_{G_x}(t)$ can be chosen to be "measurable". For this, we will need a few more definitions and lemmas.

Notation. Let G be a locally compact group. Let $\mathcal{X}(G)$ be the set of closed subsets of G and let $\mathcal{S}(G)$ be the set of all closed subgroups of G.

For K a compact subset of G and $U_1,\dots,U_n$ a finite family of open subsets of G, define

$$\mathcal{U}(K,U_1,\dots,U_n) = \{F\in\mathcal{X}(G) \ : \ F\cap K=\emptyset, \forall\, i=1\dots,n,\ F\cap U_i\neq\emptyset\}.$$

The compact open topology on $\mathcal{X}(G)$ is then the topology generated by the sets of the form

$$\mathcal{U}(K,U_1,\dots,U_n).$$

We will also call compact open topology on $\mathcal{S}(G)$ the induced topology. (*cf.* [4]).

Lemma 5.4

Let G be a locally compact group, G_1 a compact subgroup and G_2 a closed subgroup. Endow $\mathcal{S}(G)$ with the compact open topology. Then the mapping $\psi : G\mapsto\mathcal{S}(G)$ defined by $x\mapsto (x\,G_1\,x^{-1})\cap G_2$ is of the Baire first class, and is therefore measurable.

Proof. We will need two steps.

First step: Let $\mathcal{U}$ be a compact neighborhood of G_1, that is the closure of an open neighborhood of G_1 (in G) and let V be the closure of an open neighborhood of G_2. Then $\varphi : G \mapsto \mathcal{X}(G)$ defined by $x \mapsto x\mathcal{U}\, x^{-1} \cap V$ is continuous with respect of the topology of G and the compact open topology of $\mathcal{X}(G)$:

Let $x \in G$ and $x_n \in G$ be a sequence that converges to x, let K be a compact subset of G and $U_1, \ldots, U_k$ a finite family of open subset of G such that

$$x\mathcal{U}\, x^{-1} \cap V \cap K = \emptyset \text{ and for } i = 1, \ldots, k;\ x\mathcal{U}\, x^{-1} \cap V \cap U_i \neq \emptyset.$$

If there exists a subsequence of x_n, that for convenience we will still call x_n, such that $x_n\mathcal{U}\, x_n^{-1} \cap V \cap K \neq \emptyset$, then there exists a sequence $k_n \in \mathcal{U}$ such that $x_n\, k_n\, x_n^{-1} \in x_n\mathcal{U}\, x_n^{-1} \cap V \cap K$. $\mathcal{U}$ being compact, we can assume without loss of generality that k_n converges to $k \in \mathcal{U}$, but then $xkx^{-1} \in x\mathcal{U}\, x^{-1} \cap V \cap K$ contradicting the emptiness of that set. Thus, for n big enough, $x_n\mathcal{U}\, x_n^{-1} \cap V \cap K = \emptyset$.

As U_1 intersects $x\mathcal{U}\, x^{-1} \cap V$, U_1 intersects the interior $x\dot{\mathcal{U}}\, x^{-1} \cap \dot{V}$ of $x\mathcal{U}\, x^{-1} \cap V$. Let $k \in \dot{\mathcal{U}}$ be such that $xkx^{-1} \in x\dot{\mathcal{U}}\, x^{-1} \cap \dot{V} \cap U_1$. Then $x_n\, kx_n^{-1} \to xkx^{-1}$ thus is in $\overset{\circ}{V} \cap U_1$ for n big enough. Therefore, there exists N_1 such that, for $n \geq N_1$, $x_n\mathcal{U}\, x_n^{-1} \cap V \cap U_1 \neq \emptyset$. There exists then $N_2 \geq N_1$ such that for $n \geq N_2$, $x_n\mathcal{U}\, x_n^{-1} \cap V \cap U_2 \neq \emptyset$... thus, for n big enough and $i = 1, \ldots, k$ we get $x_n\mathcal{U}\, x_n^{-1} \cap V \cap U_i \neq \emptyset$.

Second step: Let $\mathcal{U}_n$ be a decreasing sequence of compact neighborhoods of G_1 such that $\bigcap \mathcal{U}_n = G_1$ and let V_n be a decreasing sequence of closed neighborhoods of G_2 such that $\bigcap V_n = G_2$. Let $\psi_n : G \mapsto \mathcal{X}(G)$ be defined by $\psi_n(x) = x\mathcal{U}_n x^{-1} \cap V_n$. According to the first step, ψ_n is continuous. Further, for each $x \in G$, $\psi_n(x) \to \psi(x)$ thus ψ is in Baire's first class:

Let $x \in G$, K be a compact subset of G and $U_1, \ldots, U_k$ a finite family of open subsets of G such that

$$xG_1x^{-1} \cap G_2 \cap K = \emptyset \text{ and for } i = 1, \ldots, k;\ x\, G_1\, x^{-1} \cap G_2 \cap U_i \neq \emptyset.$$

Then as $\mathcal{U}_n \supset G_1$ and $V_n \supset G_2$, for $i = 1, \ldots, k$

$$x\mathcal{U}_n\, x^{-1} \cap V_n \cap U_i \supset x\, G_1\, x^{-1} \cap G_2 \cap U_i \neq \emptyset.$$

Further $x\,\mathcal{U}_n x^{-1} \cap V_n \cap K$ is a decreasing sequence of compact sets whose intersection $x\, G_1\, x^{-1} \cap G_2 \cap K$ is empty, thus for n big enough, $x\,\mathcal{U}_n\, x^{-1} \cap V_n \cap K = \emptyset$, which concludes the proof of the convergence of $\psi_n\,(x)$ towards $\psi(x)$. □

DEFINITION. For each $K \in \mathcal{S}(G)$, let ν_K be a Haar measure on K. The map $K \mapsto \nu_K$ is said to be a continuous choice of Haar measures if, for every continuous compactly supported function f on G, the map $\mathcal{S}(G) \mapsto \mathbb{C}$ defined by

$$K \mapsto \int_K f(t)\, d\nu_K(t)$$

is continuous.

We will need the following lemma du to Fell (*cf.* [7]).

Lemma 5.5

Let f_0 be a non-negative continuous compactly supported function on G such that $f_0(e) > 0$ (e being the unit element of G). For each closed subgroup K of G let ν_K be the Haar measure on K such that $\int_K f_0(t)\, d\nu_K(t) = 1$. Then $K \mapsto \nu_K$ is a continuous choice of Haar measure.

Notation. In what follows, f_0 will be a fixed non-negative continuous compactly supported function on G such that $f_0(e) > 0$ and $K \mapsto \nu_K$ will denote the continuous choice of Haar measures associated to f_0.

Lemma 5.6

There exists $M > 0$ such that for every $x \in G$, $\nu_{G_x}(G_x) \leq M$.

Proof. Let $\varepsilon > 0$ and let U be a neighborhood of e such that $f_0(t) > \varepsilon > 0$ for $t \in U$.

For each $s \in G_1$, let U_s be a neighborhood of s such that $U_s^{-1} U_s \subset U$. As G_1 is compact, G_1 is covered by a finite subfamily $U_1, \ldots, U_n$ of the $\{U_s\}_{s \in G_1}$. Then, $xU_1x^{-1} \cap G_2, \ldots, x\, U_n\, x^{-1} \cap G_2$ is a cover of $x\, G_1\, x^{-1} \cap G_2$. Thus

$$\nu_{G_x}(G_x) = \int_{G_x} d\nu_{G_x} \leq \sum_{i=1}^{n} \int_{x\, U_i\, x^{-1} \cap G_2} d\nu_{G_x}.$$

Now choose a y_i in each U_i, and note that, for $t \in U$, $1 \leq \frac{1}{\varepsilon} f_0(t)$. Then if $s \in x\, U_i\, x^{-1}$, $y_i^{-1}\, x^{-1}\, sx \in U_i^{-1}\, U_i \subset U$ thus $1 \leq \frac{1}{\varepsilon}\, f_0(y_i^{-1}\, x^{-1}\, sx)$ and therefore

$$\nu_{G_x}(G_x) \leq \sum_{i=1}^{n} \frac{1}{\varepsilon} \int_{x\, U_i\, x^{-1} \cap G_2} f_0(y_i^{-1}\, x^{-1}\, sx)\, d\nu_{G_x} \leq \sum_{i=1}^{n} \frac{1}{\varepsilon} \int_{G_x} f_0(y_i^{-1}\, x^{-1}\, sx)\, d\nu_{G_x}$$

as $f_0 \geq 0$. But ν_{G_x} is a Haar measure of the compact (thus unimodular) group G_x so

$$\int_{G_x} f_0(y_i^{-1}\, x^{-1}\, sx)\, d\nu_{G_x} = \int_{G_x} f_0(s)\, d\nu_{G_x} = 1$$

(by the definition of ν_{G_x}). But then $\nu_{G_x}(G_x) \leq \frac{n}{\varepsilon}$. □

We are now able to prove the following

Proposition 5.7

For every $f \in L^p(G_2)$, $g \in L^{p'}(G_2)$, $v \in V$, $\xi \in V^*$,

$$x \mapsto < f \otimes v,\, P_x^*(g \otimes \xi) >= \int_{G_x} < f,\, \rho_{t^{-1}}\, g >< v, \pi^*_{x^{-1}tx}\, \xi > d\nu_{G_x}(t).$$

is measurable.

Proof. It is of course enough to prove that

$$x \mapsto \int_{G_x} < f,\, \rho_{t^{-1}} g > \pi^*_{x^{-1}\,tx}\; \xi d\nu_{G_x}\,(t)$$

is measurable.

Let $\varepsilon > 0$. As G_1 is compact and $t \mapsto \pi_t^* \xi$ is continuous, there exists a disjoint relatively compact cover $U_1, \ldots, U_n$ of G_1 and $t_1 \in U_1, \ldots t_n \in U_n$ such that for each $i = 1, \ldots, n$ and each $t \in U_i$, $\left\| \pi_t^*\, \xi - \pi_{t_i}^* \xi \right\| < \varepsilon$. Let χ_{U_i} be the characteristic function of U_i, then, the norm of

$$\Big\| \int_{G_x} < f,\, \rho_{t^{-1}}\, g > \pi^*_{x^{-1}tx} \xi d\nu_{G_x}(t) - \int_{G_x} < f, \rho_{t^{-1}} g > \sum_{i=1}^{n} \chi_{x^{-1}U_i x}(t) \pi^*_{x^{-1}t_i x} \xi d\nu_{G_x}(t) \Big\|$$

$$= \Big\| \sum_{i=1}^{n} \int_{xU_i x^{-1} \cap G_2} < f, \rho_{t^{-1}} g > (\pi^*_{x^{-1}tx}\xi - \pi^*_{x^{-1}t_i x}\xi) d\nu_{G_x}(t) \Big\|$$

$$\leq \sum_{i=1}^{n} \int_{xU_i x^{-1} \cap G_2} \|f\|\; \|g\|\; \left\| \pi^*_{x^{-1}tx}\xi - \pi^*_{x^{-1}t_i x}\xi \right\|\; d\nu_{G_x}(t)$$

$$\leq \varepsilon \|f\|\; \|g\|\; \nu_{G_x}(G_x) \leq \varepsilon \|f\|\; \|g\|\, M$$

by Lemma 5.6. It is thus enough to prove measurability for

$$x \mapsto \int_{G_x} < f,\, \rho_{t^{-1}}\, g > \chi_{x^{-1}\, U_i\, x}(t)\, d\nu_{G_x}\,(t) \pi^*_{x^{-1}\, t_i\, x}\, \xi.$$

Further, as $x \mapsto \pi^*_{x^{-1}t_i x}\, \xi$ is continuous, and as $\chi_{x^{-1}\, U_i\, x}(t) = \chi_{U_i}\,(xtx^{-1})$, we will just consider

$$x \mapsto \int_{G_x} < f,\, \rho_{t^{-1}}\, g > \chi_U\,(xtx^{-1})\, d\nu_{G_x}\,(t)$$

where U is a relatively compact measurable subset of G_1. Consider now a sequence φ_n of continuous compactly supported functions on G such that φ_n converges almost everywhere to χ_U and such that $0 \leq \varphi_n \leq 1$. Then, as for every $x \in G$,

$$\int_{G_x} < f, \rho_{t^{-1}}\, g > \varphi_n\,(xtx^{-1})\, d\nu_{G_x}\,(t) \to \int_{G_x} < f,\, \rho_{t^{-1}} g > \chi_U\,(xtx^{-1})\, d\nu_{G_x}\,(t)$$

we just need to consider

$$x \mapsto \int_{G_x} < f, \rho_{t^{-1}} g > \varphi(xtx^{-1}) d\nu_{G_x}(t)$$

where φ is a continuous compactly supported function on G. But, $K \mapsto \nu_K$ is a continuous choice of Haar measures, so

$$(K, x) \mapsto \int_K < f, \rho_{t^{-1}} g > \varphi(xtx^{-1}) d\nu_K(t)$$

is continuous, and as $x \mapsto G_x$ is measurable,

$$x \mapsto (G_x, x) \mapsto \int_{G_x} < f, \rho_{t^{-1}} g > \varphi(xtx^{-1}) d\nu_{G_x}(t)$$

is measurable. Finally x is in $\mathcal{D}$ and not in G. To overcome that difficulty, recall that G_1 and G_2 are assumed regularly related so that there exists a measurable cross-section ψ of $\mathcal{D}$ in G, thus we just have to compose the previous map and ψ. □

Notation. Let μ_1 be the quasi-invariant measure on G/G_1 defined by

$$\int_{G/G_1} \left(\int_{G_1} f(st) d\nu_{G_1}(t) \right) d\mu_1(sG_1) = \int_G f(s) d\nu_G(s).$$

For $D \in \mathcal{D}$, let μ_D be the quasi-invariant measure on D obtained from μ_1 via Lemma 5.1 and 5.2:

$$\int_{\mathcal{D}} \int_D f(t) d\mu_D d\widetilde{\mu_1} = \int_{G/G_1} f(t) d\mu_1(t).$$

For $x \in G$, let μ_x be the measure on G_2/G_x defined by

$$\int_{G_2/G_x} \left(\int_{G_x} f(st) d\nu_{G_x}(t) \right) d\mu_x(sG_x) = \int_{G_2} f(s) d\nu_{G_2}(s).$$

Note that G_2 being unimodular, every quasi-invariant measure on G_2/G_x is proportional to μ_x. Thus, identifying G_2xG_1 with G_2/G_x we may assume that $\mu_x = \mu_{G_2xG_1}$.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a dense family of elements of $L^{p'}(G_2) \otimes_{d_{p'}} V^*$ of the form $g_n \otimes \xi_n$ where the g_n's are continuous compactly supported functions on G_2. Let $\psi_n(x) = P_x^*(\varphi_n)$. According to Proposition 5.7, $x \mapsto \psi_n(x)$ is weakly measurable. Further, for fixed x, $\{\psi_n(x)\}_{n\in\mathbb{N}}$ is dense in $({}^{G_2,p}V^x)^*$.

First let $\mathcal{B} = \prod_{x\in\mathcal{D}} {}^{G_2,p}V^x$, an element of $\mathcal{B}$ is thus a mapping $\varphi : x \mapsto \varphi(x)$ such that for every $x \in \mathcal{D}$, $\varphi(x) \in {}^{G_2,p}V^x$.

DEFINITION. Let $L^p(\mathcal{D},\mu,\mathcal{B})$ be the linear subset of $\mathcal{B}$ consisting of all φ such that

1) for every $n \in N$, $x \mapsto < \varphi(x), \psi_n(x) >$ is measurable, and
2) $\|\varphi\|_p = \left(\int_{\mathcal{D}} \|\varphi(x)\|^p_{G_2,pV^x} d\mu(x)\right)^{1/p} < \infty$.

We will of course identify two elements if they are equal almost everywhere. Then $L^p(\mathcal{D},\mu,\mathcal{B})$ is a Banach space and a G_2-module if we define the action of G_2 by $g_2\varphi : x \mapsto g_2\varphi(x)$.

Theorem 5.8

Under the above notations, ${}^{G,p}V_{G_2}$ is isometrically G_2-module homomorphic to $L^p(\mathcal{D},\mu,\mathcal{B})$.

Proof. Recall from Section 2 that we can identify ${}^{G_2,p}V^x$ as the set of all functions $f : G_2 \mapsto V$ such that

1) $x \mapsto < f(x), v' >$ is measurable for every $v' \in V^*$,
2) $f(sh) = \pi^{-1}_{xhx^{-1}} f(s)$ for all $s \in G_2, h \in G_x$,
3) $\|f\|^p_p = \int_{G_2|G_x} \|f(t)\|^p d\mu(tH) < \infty$.

Note that conditions (2) and (3) are simplified by the assumption that G_2 is unimodular.

We will take advantage of disintegration of measures (Lemma 5.1) to complete the proof. To do this we first need to write ${}^{G_2,p}V^x$ as a set of functions on the double coset G_2xG_1 instead of functions on G_2. This is done in the next lemma.

Lemma 5.9

Let $x \in G$ and define $\mathcal{E}^p_x$ to be the set of all $f : G_2xG_1 \mapsto V$ such that
1) $s \mapsto < f(s), v' >$ is measurable for all $v' \in V^*$,
2) $f(s\xi) = \pi^{-1}_\xi f(s)$ for all $\xi \in G_1$, $s \in G_2xG_1$,
3) $\int_{G_2|G_x} \|f(t)\|^p d\mu_x(t) < \infty$.

Then ${}^{G_2,p}V^x$ and $\mathcal{E}^p_x$ are G_2-module homomorphic and isometric.

Proof. Note first that π being isometric, the condition (2) implies that $\|f(t)\|^p$ is constant on G_x-cosets of G_2, thus condition (3) makes sense.

Let $f \in \mathcal{E}^p_x$ so that f is defined on G_2xG_1. We define $\tilde{f}(t) = f(tx)$ for $t \in G_2$. For all $v' \in V^*$, $t \mapsto < \tilde{f}(t), v' > = < f(tx), v' >$ is clearly measurable. Further, let $\eta \in G_x$ and let $\xi = x\eta x^{-1}$, then

$$\tilde{f}(t\eta) = \tilde{f}(tx\xi x^{-1}) = f(tx\xi) = \pi^{-1}_\xi f(tx) = \pi^{-1}_\xi \tilde{f}(t) = \pi^{-1}_{x^{-1}\eta x}\tilde{f}(t).$$

Now let $g \in {}^{G_2,p}V^x$ (seen as a function $G_2 \mapsto V$). Define a function f on G_2xG_1 by $f(tx\xi) = \pi_\xi^{-1} g(t)$ for $t \in G_2$ and $\xi \in G_1$.

Let us first check that f is unambiguously defined. Thus, assume that $t_1 x\xi_1 = t_2 x\xi_2$ with $t_1, t_2 \in G_2$ and $\xi_1, \xi_2 \in G_1$. Then $t_1 = t_2\, x\xi_2\, \xi_1^{-1}\, x^{-1}$ and $x\xi_2\, \xi_1^{-1}\, x^{-1} \in G_2 \cap (xG_1\, x^{-1}) = G_x$ thus

$$g(t_1) = \pi^{-1}_{(\xi_1\xi_2^{-1})^{-1}}\, g(t_2) = \pi^{-1}_{\xi_2\xi_1^{-1}}\, g(t_2) = \pi_{\xi_1}\pi^{-1}_{\xi_2}\, g(t_2)$$

thus $\pi^{-1}_{\xi_1}\, g(t_1) = \pi^{-1}_{\xi_2}\, g(t_2)$ and $f(t_1\, x\xi_1) = f(t_2\, x\xi_2)$ and f is unambiguously defined.

Fix $v' \in V^*$ and define for $(\xi, \eta) \in G_1 \times G_2$, $f_1(\xi, \eta) = \pi_\xi^{-1}\, g(\eta)$, then

$$< f_1(\xi, \eta), v' >=< g(\eta), (\pi_{\xi^{-1}})^* v' >$$

is a Borel function of $(\xi, \eta) \in G_1 \times G_2$. We can now finish the proof of the lemma in exactly the same way as the proof of the Lemma 6.1 of [16]. □

We have just established Lemma 5.9 for functions defined on G_2xG_1 double cosets in order to remain close to the proof of [16] Lemma 6.1. It is then obvious that a similar result is true for G_1xG_2.

Proof. (of the theorem) Recall from Section 2 that we can identify ${}^{G,p}V$ as the set of all functions $f : G \mapsto V$ such that

1) $s \mapsto < f(s), v' >$ is a Borel function for all $v' \in V^*$,

2) $F(s\xi) = \pi_\xi^{-1} f(s)$ for every $\xi \in G_1, s \in G$,

3) $\int_{G/G_1} \|f(t)\|^* d\mu_1(t) < \infty$.

We can now finish the proof of the theorem simply by using disintegration of measures as in [16]. Let $f \in {}^{G,p}V$ (seen as a function on G) then with Lemma 5.1,

$$\int_{D \in \mathcal{D}} \int_D \|f(t)\|^p d\mu_D\, d\widetilde{\mu_1}(D) = \int_{G/G_1} \|f\|^p d\mu_1 < \infty. \tag{8}$$

Thus, for almost all $D \in \mathcal{D}$,

$$\int_D \|f(t)\|^p d\mu_D < \infty.$$

Define then, for $D \in \mathcal{D}$, f_D to be the restriction of f to D. For almost all $D \in \mathcal{D}$, we then have that $f_D \in \mathcal{E}_x^p$ (where x is such that $D = G_1xG_2$) so that, by Lemma 5.3, we may assume that $f_D \in {}^{G_2,p}V^x$.

Equation (8) then asserts that ${}^{G,p}V$ is isometric to $L^p(\mathcal{D}, \mu, \mathcal{B})$. □

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