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Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures

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Abstract

We prove that the Morrey space is contained in the space CMO^q of functions with bounded central mean oscillation, explaining in this way the dependence of CMO^q on q. We also define a natural extension of CMO^2, to prove the existence of a continuous bijection with central Carleson measures of order λ. This connection is further studied to extend and refine duality results involving tent spaces and Hardy spaces associated with Herz-type spaces. Finally, we prove continuity results on these spaces for general non-convolution singular integral operators, including pseudo-differential operators with symbols in the Hörmander class $S^m_{\rho,\delta}$, $\rho<1$, and linear commutators.

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1. Introduction

In his monumental papers, "Generalized Harmonic Analysis" [31, pp. 148, 159], and "Tauberian Theorems" [32, pp. 80, 91], Norbert Wiener looked for ways other than the O and o symbols to describe the behavior of a function at infinity. To this effect, he considered several alternatives, for instance

$$\frac{1}{T}\int_0^T |f(x)|^2\,dx \ \text{ is bounded for large } T\,,$$
$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^T |f(x)|^2\,dx = 0\,,$$
$$\frac{1}{T^{1-\alpha}}\int_0^T |f(x)|\,dx \ \text{ is bounded for some } \alpha\in(0,1) \text{ and every } T>0\,.$$

Wiener also observed that these conditions are related to appropriate weighted L^q spaces.

A. Beurling [5] extended these ideas looking for a more general setting in which the Wiener-Lévy Theorem and the Wiener Approximation Theorem still hold. So, Beurling defined a pair of dual Banach spaces, A^q and $B^{q'}$, $1/q+1/q'=1$. More precisely, A^q is a Banach algebra with respect to the convolution, expressible as a union of weighted L^q spaces. The space $B^{q'}$ is expressible as the intersection of weighted $L^{q'}$ spaces, or equivalently, it is the space of functions bounded in the central mean of order q'. C. Herz [18] further generalized the space A^q, into the space $A^{p,q}$, depending on a second parameter p.

H. Feichtinger [13] observed that the space B^q can be described by the condition

$$\|f\|_{B^q} = \sup_{k\geq 0}\left(2^{-kn/q}\,\|f\chi_k\|_q\right) < \infty \tag{1.1}$$

where χ_0 is the characteristic function of the unit ball $\{x\in\mathbb{R}^n : |x|\leq 1\}$, χ_k is the characteristic function of the annulus $\left\{x\in\mathbb{R}^n : 2^{k-1}<|x|\leq 2^k\right\}$, $k=1,2,3,...$ and $\|\cdot\|_q$ is the norm in L^q. Actually, this observation is a special case of much earlier results of J. Gilbert [16].

By duality, the space A^q, appropriately called now the Beurling algebra, can be described by the condition

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'}\,\|f\chi_k\|_q < \infty. \tag{1.2}$$

Chen and Lau [9] (see also García-Cuerva [14]) introduced an atomic space HA^q associated with the Beurling algebra A^q, and identified its dual as the space CMO^q defined by the condition

$$\sup_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f-f_R|^q\,dx\right)^{1/q}<\infty \tag{1.3}$$

where $B(0,R)=\{x\in\mathbb{R}^n:|x|<R\}$, $|B(0,R)|$ is the measure of $B(0,R)$ and $f_R=\frac{1}{|B(0,R)|}\int_{B(0,R)}f(x)dx$. It is clear that CMO^q contains both BMO and B^q. Chen and Lau [9] showed by means of a counterexample that CMO^q depends on q. This observation establishes a dramatic difference between CMO^q and BMO. On the other hand, both spaces are duals of atomic spaces. Furthermore, Chen and Lau [9] and Lu and Yang [25] showed that functions in CMO^2 are related to a notion of central Carleson measure in the same way that functions in BMO are related to classical Carleson measures. So, CMO^q exhibits both similarities and differences with respect to the space BMO.

In recent years, several authors have extended and studied these and other related spaces and have considered the action of various operators on them. We mention, for instance, the work of J. García-Cuerva [14], J. García-Cuerva and M. J. Herrero [15], S. Lu and D. Yang [24], [25], [26], [27], X. Li and D. Yang [23], J. Lakey [22], E. Hernández, G. Weiss and D. Yang [21], L. Grafakos, X. Li, and D. Yang [17].

The main purpose of our paper is to make precise and extend several properties pertaining to the spaces B^q, CMO^q, and HA^q.

In Section 3 we prove that the Morrey space is contained in B^q. This observation explains why CMO^q depends on q [9]. In fact, it was proved in [1] that the distribution function of functions in the Morrey space, in general, decays at infinity not better than $t^{-\alpha}$ for some appropriate exponent $\alpha>0$. Thus, from this point of view, we can say that B^q and BMO are roughly the bad part and the good part of CMO^q.

In Section 4 we investigate duality results at two levels. Downstairs, that is to say in $\mathbb{R}^n$, we consider Herz-type spaces $A_q^{p,\alpha}$ and their associated Hardy spaces $HA_q^{p,\alpha}$. In this context, we need to define the following extension of (1.3),

$$\sup_{R\geq 1}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,R)|^{1+2\lambda}}\int_{B(0,R)}|f(x)-c|^2\,dx\right)^{1/2}<\infty \tag{1.4}$$

for $0<\lambda<1/n$. When $\lambda\geq 1/n$ we will subtract higher degree polynomials, and when $\lambda<0$, we do not need any polynomial correction. We prove that (1.4) describes the dual $CMO^{2,\lambda}$ of $HA_2^{p,\alpha}$ for $0 < p \leq 1$, $\lambda = \alpha - 1/2$, and analogously for the homogeneous version using $\sup_{R>0}$. The case $p = 1$ provides the Banach space end point.

Upstairs, that is to say, in $\mathbb{R}_+^{n+1}$, we introduce central Carleson measures of type λ in the sense of Amar and Bonami [4]. We consider both the continuous and the discrete case and use classical techniques and wavelet decomposition techniques, respectively, to obtain in each case the correspondence between the measures and functions in $CMO^{2,\lambda}$. Our proofs apply in particular to obtain the classical correspondence between Carleson measures and BMO functions. We remark that the results in Section 4 apply to both the homogeneous and inhomogeneous cases of the spaces we consider. These results are related to the work of Hernández, Weiss, and Yang [21]. The connections will be made precise in Section 4.

In Section 5 we study the action of non-convolution singular integral operators on inhomogeneous spaces. The main observation is that the inhomogeneous nature of the space, allows for the continuous action of operators associated to kernels more singular than standard kernels in the sense of Coifman and Meyer [8]. The results we obtain extend and refine previous work of García-Cuerva [14] and Lu and Yang [24].

In Section 2, we collect preliminary definitions and results. The paper ends with a list of references.

The notation used in this paper is standard. It may be useful to point out that given $1 \leq q \leq \infty$, q' will denote the conjugate exponent, $1/q + 1/q' = 1$. The symbols C_0^∞, $\mathcal{S}'$, $\mathcal{D}'$, L^q, L^q_{loc}, l^p, etc. will indicate the usual spaces of distributions, sequences, or functions defined on $\mathbb{R}^n$, with complex values. Moreover, $\|f\|_q$ will denote the L^q norm of the function f.

When a particular setting is needed, we will write $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, $L^q_{\text{loc}}\left(\mathbb{R}_+^{n+1}\right)$, $L^q(\mu)$, $\|f\|_{L^q\left(\mathbb{R}_+^{n+1}\right)}$, as needed. With $|A|$ we will denote the Lebesgue measure of a measurable set A, and $B(z, R)$ will be a ball centered at z with radius R. With χ_B we will denote the characteristic function of a set B. As usual, the letter C will indicate an absolute constant, probably different at different occurrences. Other notations will be introduced at the appropriate time.

2. Preliminary definitions and results

Definition 2.1. Given $\lambda < 1/n$, $1 < q < \infty$ we define the space $CMO^{q,\lambda}$ by the condition

$$\|f\|_{CMO^{q,\lambda}} = \sup_{R \geq 1} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_R|^q \, dx \right)^{1/q} < \infty \tag{2.1}$$

where $f_R = \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x) dx$.

$CMO^{q,\lambda}$ becomes a Banach space if we identify functions that differ in a constant.

It is easy to see that (2.1) is equivalent to the condition

$$\sup_{R\geq 1}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)-c|^q\,dx\right)^{1/q}<\infty\,.$$

DEFINITION 2.2. Given $\lambda\in\mathbb{R}$, $1<q<\infty$, we define the space $B^{q,\lambda}$ by the condition

$$\|f\|_{B^{q,\lambda}}=\sup_{R\geq 1}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q\,dx\right)^{1/q}<\infty\,. \tag{2.2}$$

$B^{q,\lambda}$ is a Banach space continuously included in $CMO^{q,\lambda}$.

When $\lambda=0$ we obtain the spaces CMO^q and B^q defined in the introduction.

One can also consider the homogeneous versions of (2.1) and (2.2) by taking the supremum over $R>0$.

In Section 4 we will extend Definition 2.1 to $\lambda\geq 1/n$.

It is clear that $B^{q,\lambda}$ reduces to zero when $\lambda<-1/q$ and that $B^{q,-1/q}$ is L^q.

Lemma 2.3

The space $CMO^{q,\lambda}$ reduces to the constant functions when $\lambda<-1/q$ and it coincides with L^q modulo constants, when $\lambda=-1/q$.

Proof. Given $f\in CMO^{q,\lambda}$, $\lambda<-1/q$, we have

$$\lim_{R\to\infty}\int_{B(0,R)}|f(x)-f_R|^q\,dx=0\,.$$

In particular, taking an increasing sequence $\{R_k\}$ such that $R_k\to\infty$ as $k\to\infty$, we can write

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}|f(x)-f_{R_k}|^q\,\chi_{B(0,R_k)}dx=0\,.$$

Thus, there exists a subsequence $\left\{\left|f-f_{R_{k_j}}\right|^q\chi_{B(0,R_{k_j})}\right\}$ that converges to zero *a.e.* as $j\to\infty$. Since $\left\{\chi_{B(0,R_{k_j})}\right\}$ converges pointwise to 1 as $j\to\infty$, we conclude that $f(x)-f_{R_{k_j}}\to 0$ *a.e.* as $j\to\infty$. Hence, f is constant *a.e.*

If $\lambda=-1/q$, the functions in $CMO^{q,-1/q}$ satisfy the condition

$$\sup_{R\geq 1}\inf_{c\in\mathbb{C}}\left(\int_{B(0,R)}|f(x)-c|^q\,dx\right)^{1/q}<\infty\,.$$

Let $B_k = B(0,k)$, $k = 1, 2, \ldots$. The uniform convexity of the L^q-norm jointly with a compactness argument, (see [7, p. 142]), imply that for each k there exists a unique constant $c_k \in \mathbb{C}$ such that

$$\inf_{c\in\mathbb{C}} \left(\int_{B(0,k)} |f(x) - c|^q \, dx \right)^{1/q} = \left(\int_{B(0,k)} |f(x) - c_k|^q \, dx \right)^{1/q} .$$

Now, given $l = 1, 2, \ldots$, we have

$$\begin{aligned} |B_k|^{1/q} |c_k - c_{k+l}| &\leq \left(\int_{B_k} |f(x) - c_k|^q \, dx \right)^{1/q} + \left(\int_{B_{k+l}} |f(x) - c_{k+l}|^q \, dx \right)^{1/q} \\ &\leq 2 \, \|f\|_{CMO^{q,-1/q}} . \end{aligned}$$

Thus, the sequence $\{c_k\}$ is a Cauchy sequence in $\mathbb{C}$. Let $c = \lim_{k\to\infty} c_k$. Then,

$$|B_k|^{1/q} |c_k - c| \leq 2 \, \|f\|_{CMO^{q,-1/q}} .$$

Moreover,

$$\begin{aligned} \left(\int_{B_k} |f(x) - c|^q \, dx \right)^{1/q} &\leq \left(\int_{B_k} |f(x) - c_k|^q \, dx \right)^{1/q} + |B_k|^{1/q} |c_k - c| \\ &\leq 3 \, \|f\|_{CMO^{q,-1/q}} . \end{aligned}$$

Hence, the function $f - c \in L^q$. This completes the proof of the lemma. □

Let us point out that the spaces $B^{q,\lambda}$ and $CMO^{q,\lambda}$ are different for different values of $\lambda \geq -1/q$. In fact, this is clear when $\lambda = -1/q$. On the other hand, if we fix $\lambda > -1/q$, we can consider the function f defined on $\mathbb{R}$ as

$$f(x) = \sum_{k=0}^{\infty} 2^{k(1+q\lambda)/q} \, 2^{-k/2q} \chi_{A_k}(x) \operatorname{sgn}(x),$$

where $A_k = \left\{ x \in \mathbb{R} : 2^k \leq |x| < 2^k + 2^{k/2} \right\}$.

It is not difficult to show that $f \in CMO^{q,\lambda}$ but $f \notin CMO^{q,\mu}$ for every $\mu < \lambda$. This example also shows that $B^{q,\lambda}$ is strictly larger than L^q when $\lambda > -1/q$.

In the spirit of the equivalent norm obtained by Feichtinger [13] for the space B^q, we can prove that given $\lambda > -1/q$, the space $B^{q,\lambda}$ can be described by the condition

$$\sup_{k\geq 0} 2^{-nk/q(1+q\lambda)} \, \|f\chi_k\|_q < \infty. \tag{2.3}$$

In fact, (2.3) gives an equivalent norm in $B^{q,\lambda}$.

However, when $\lambda = -1/q$, the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k/q} \chi_{C_k}(x)\,,$$

$C_0 = [-1,1]$, $C_k = \left\{x \in \mathbb{R} : 2^{k-1} < |x| \leq 2^k\right\}$, $k = 1, 2, ...$ satisfies condition (2.3), but $f \notin L^q(\mathbb{R})$.

Finally, when $\lambda < -1/q$, condition (2.3) defines a non-zero space strictly included in L^q. In fact, we can consider the function

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{-k/q}}{k^{2/q}\,(1+q\lambda)^{2/q}} \chi_{C_k}(x)\,.$$

This function belongs to $L^q\,(\mathbb{R})$, but it does not satisfy condition (2.3).

On the other hand, the non-zero function

$$f(x) = \sum_{k=0}^{\infty} 2^{k\lambda} \chi_{C_k}(x)$$

belongs to $L^q\,(\mathbb{R})$ and satisfies (2.3).

It should be also pointed out that the spaces $B^{q,\lambda}$ are isomorphic for different values of $\lambda \geq -1/q$. This is because of the surjective isometry between $B^{q,\lambda}$ and $\bigoplus_{k=1}^{\infty} L^q(C_k)$ given by the map

$$f \longrightarrow \left(2^{-nk/q(1+q\lambda)}\, \|f\chi_k\|_q\right)_{k=0}^{\infty}\,.$$

Proposition 2.4

Let $\theta < -n(\lambda q + 1)$. Then, given $f \in CMO^{q,\lambda}$ we have

$$\int_{\mathbb{R}^n} |f(x) - f_0|^q\,(1+|x|)^{\theta}\,dx < \infty\,,$$

where $f_0 = \frac{1}{|B(0,1)|}\int_{B(0,1)} f(x)dx$.

Proof. For $k = 0, 1, 2, ...$ let

$$B_k = B\left(0, 2^k\right) \text{ and } f_k = \frac{1}{|B_k|} \int_{B_k} f(x) dx\,.$$

We can write

$$\begin{aligned}
\left(\frac{1}{|B_k|} \int_{B_k} |f - f_0|^q \, dx\right)^{1/q} &\le \left(\frac{1}{|B_k|} \int_{B_k} |f - f_k|^q \, dx\right)^{1/q} + |f_0 - f_k| \\
&\le C 2^{nk\lambda} \|f\|_{CMO^{q,\lambda}} + \sum_{j=0}^{k-1} \left(\frac{1}{|B_j|} \int_{B_j} |f_{j+1} - f_j|^q \, dx\right)^{1/q} \\
&\le C 2^{nk\lambda} \|f\|_{CMO^{q,\lambda}} \\
&\quad + C \sum_{j=0}^{k-1} \left(\frac{1}{|B_{j+1}|} \int_{B_{j+1}} |f - f_{j+1}|^q \, dx\right)^{1/q} \\
&\quad + C \sum_{j=0}^{k-1} \left(\frac{1}{|B_j|} \int_{B_j} |f - f_j|^q \, dx\right)^{1/q} .
\end{aligned}$$

Thus,

$$\int_{B_k} |f(x) - f_0|^q \, dx \le \begin{cases} C(k+1)^q 2^{nk} \|f\|_{CMO^q} & \text{if } \lambda = 0 \\ C 2^{n(\lambda q + 1)k} \|f\|_{CMO^{q,\lambda}} & \text{if } \lambda \neq 0 \end{cases} .$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x) - f_0|^q (1 + |x|)^\theta \, dx &= \int_{B_0} |f(x) - f_0|^q (1 + |x|)^\theta \, dx \\
&\quad + \sum_{k=1}^{\infty} \int_{B_k \setminus B_{k-1}} |f(x) - f_0|^q (1 + |x|)^\theta \, dx \\
&\le C \int_{B_0} |f(x) - f_0|^q \, dx + C \sum_{k=1}^{\infty} 2^{k\theta} \int_{B_k} |f(x) - f_0|^q \, dx \\
&\le C \|f\|_{CMO^{q,\lambda}} .
\end{aligned}$$

This completes the proof of the proposition. □

The following result is an immediate consequence of Proposition 2.4:

Corollary 2.5

If $\theta < \min\{-n, -n(1+q\lambda)\}$, the space $CMO^{q,\lambda}$ is continuously included in $L^q\left((1+|x|)^\theta\, dx\right)$.

Lemma 2.6

Given $\lambda \geq -1/q$, and $\theta \geq -n(1+q\lambda)$, the space $L^q\left((1+|x|)^\theta\, dx\right)$ is continuously included in $B^{q,\lambda}$.

Proof. Given $f \in L^q\left((1+|x|)^\theta\, dx\right)$, $B = B(0,R)$, $R \geq 1$, we have

$$\frac{1}{|B|^{1+q\lambda}} \int_B |f(x)|^q\, dx \leq C \int_B |f(x)|^q \frac{(1+|x|)^\theta}{R^{n(1+q\lambda)}(1+|x|)^\theta} dx\,. \tag{2.4}$$

When $\theta \geq 0$, we can estimate (2.4) as

$$C \int_{\mathbb{R}^n} |f(x)|^q (1+|x|)^\theta\, dx\,.$$

When $-n(1+q\lambda) \leq \theta < 0$, we estimate (2.4) as

$$C \int_B |f(x)|^q \frac{(1+|x|)^\theta}{R^{n(1+q\lambda)+\theta}} dx \leq C \int_{\mathbb{R}^n} |f(x)|^q (1+|x|)^\theta\,. \ \square$$

Remark 1. The results above make more precise and extend inclusion results proved for the one-dimensional space CMO^q by Chen and Lau [9].

If we use Corollary 2.5 with $\lambda = 0$, we obtain in particular the known continuous inclusion of BMO into $L^q\left((1+|x|)^\theta\, dx\right)$ for any $\theta < -n$, $1 < q < \infty$.

We finish this preliminary section observing that much of what we said about the spaces $B^{q,\lambda}$, $CMO^{q,\lambda}$ remains true for their homogeneous versions. In particular, it is important to stress for future use that Lemma 2.3, Proposition 2.4, Corollary 2.5, and Lemma 2.6 remain true in the homogeneous setting.

3. Morrey spaces and central mean oscillation

We discussed in Section 2 properties of the space CMO^q that point to the similarities that exist between CMO^q and BMO.

However, there are also some striking differences. The classical John-Nirenberg inequality shows that functions in BMO are locally exponentially integrable. This implies that given $1 \leq q < \infty$, the functions in BMO can be described by means of the condition

$$\sup_B \inf_{c\in\mathbb{C}} \left(\frac{1}{|B|} \int_B |f(x) - c|^q \, dx \right)^{1/q} < \infty \, .$$

On the contrary, the space CMO^q depends on q. This was observed by Chen and Lau in [9], in the one-dimensional case. Indeed, given $1 < q_1 < q_2 < \infty$, they constructed a function in CMO^{q_1} that is at a positive distance of CMO^{q_2}. Although Chen and Lau worked with the inhomogeneous version of CMO^q, a minor modification of their function will serve as a counterexample in the homogeneous case as well.

The dependence of CMO^q on q in the inhomogeneous case can be explained better in terms of Morrey spaces.

It is difficult to pinpoint a definition of these spaces that is generally accepted. Several versions appear in the literature. We will choose the original definition used by Campanato [6, p. 67].

Definition 3.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with diameter ρ_0 and let $1 < q < \infty$, $-1/q < \lambda < 0$. A function f, locally in $L^q(\Omega)$, belongs to the Morrey space $L^{q,\lambda}$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)\cap\Omega} |f(x)|^q \, dx \right)^{1/q} \leq C \, |B(x_0,\rho)|^\lambda \tag{3.1}$$

for every $x_0 \in \overline{\Omega}$, $0 < \rho \leq c\rho_0$, where $c = c(n) \geq 1$.

We take as a norm the infimum over all the constants $C > 0$ such that (3.1) holds. Then, the space $L^{q,\lambda}$ becomes a Banach space.

Let us recall that the distribution function δ_g of a measurable function g is defined for each $t > 0$ as $|\{x : |g(x)| > t\}|$.

Proposition 3.2 [1]

Given $1 < q < \infty$, let $g : (0,1) \longrightarrow (0,\infty)$ be such that $g \in L^q(0,1)$, g is increasing, and $g(t) \to \infty$ as $t \to 0^+$. Then we can find $T > 0$, a cube $Q_0 = (0,L) \times ... \times (0,L)$ for some $0 < L \leq 1$, and a function f in the Morrey space $L^{q,\lambda}(Q_0)$ such that $\delta_f(t) \geq C\delta_g(t)$ for some $C > 0$, $t \geq T$.

Remark 2. It was observed in [1] that the function f actually satisfies the following sharper version of (3.1):

$$\int_{B(x_0,\rho)\cap Q_0} |f(x)|^q \, dx \leq C \, |B(x_0,\rho)\cap Q_0|^{1+\lambda q} \, .$$

We can now prove the main result of this section.

Proposition 3.3

Given $1 < q_1 < q_2 < \infty$, there exists a function $F \in B^{q_1}$ such that $F \notin L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Given $1/q_2 < \alpha < 1/q_1$, we use Proposition 3.2 with $q = q_1$ and $g(t) = t^{-\alpha}$. Then, we can construct a function $f \in L^{q_1,\lambda}(Q_0)$, such that $f \notin L^{q_2}(Q_0)$.

We consider now the function $F(x)$ defined on $\mathbb{R}^n$ as

$$F(x) = \begin{cases} f(x) & \text{if } \; x \in Q_0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $F \notin L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n)$. However, we will show that $F \in B^{q_1}$.

In fact, given $B = B(0,R)$, with $R \geq 1$, we have

$$\left(\frac{1}{|B|}\int_B |F(x)|^{q_1} \, dx\right)^{1/q_1} = \left(\frac{1}{|B|}\int_{B\cap Q_0} |f(x)|^{q_1} \, dx\right)^{1/q_1} .$$

If $R \geq \sqrt{n}$, then $Q_0 \subset B$ and we can write

$$\begin{aligned} \left(\frac{1}{|B|}\int_{B\cap Q_0} |f(x)|^{q_1} \, dx\right)^{1/q_1} &\leq \left(\frac{1}{|Q_0|}\int_{Q_0} |f(x)|^{q_1} \, dx\right)^{1/q_1} \\ &\leq C \, |Q_0|^{\lambda} \, . \end{aligned}$$

If $R < \sqrt{n}$, then we can use (3.1) with $C(n) = \sqrt{n}$ to obtain

$$\left(\frac{1}{|B|}\int_{B\cap Q_0} |f(x)|^{q_1} \, dx\right)^{1/q_1} \leq C \, |B|^{\lambda} \leq C \, .$$

Thus, $f \in B^q$. This completes the proof of the proposition. □

As a consequence of Proposition 3.3, we can conclude that BMO is strictly included in CMO^q. Furthermore, BMO is strictly included in $\bigcap_{q>1} CMO^q$.

In fact, if we consider in $\mathbb{R}$ the function

$$f(x) = \sum_{j=1}^{\infty} j\chi_{A_j}(x)\mathrm{sgn}(x) + \chi_{\{|x|\geq 1\}}(x)\mathrm{sgn}(x)\,,$$

where $A_j = \left\{x \in \mathbb{R} : 4^{-j-1} < |x| \leq 4^j\right\}$, $j = 1, 2, ...$, then it is not difficult to show that $f \notin BMO$ but $f \in \bigcap_{q>1} CMO^q$.

Thus, there exists an interval I such that the distribution function of $f(x)$-f_I, $x \in I$, does not have exponential decay at infinity.

Finally, let us point out that while $\ln|x| \in BMO$, $\ln|x| \notin \bigcup_{q\geq 1} B^q$.

Remark 3. The proof of Proposition 3.3 shows that in general, the extension operator $f \longrightarrow F$ is well defined and continuous from $L^{q,\lambda}(\Omega)$ into B^q. Campanato has proved in [7], p. 167 that the space $L^{q,\lambda}(\Omega)$ can also be defined using a condition on the mean oscillation. Namely,

$$\inf_{c\in\mathbb{C}} \left(\frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)\cap\Omega} |f(x) - c|^q \, dx \right)^{1/q} \leq C\, |B(x_0,\rho)|^{\lambda} \,. \tag{3.2}$$

More precisely, (3.1) implies (3.2) for any Ω. The converse holds under a mild condition on Ω, namely, there exists a constant $A > 0$ such that

$$|B(x_0,\rho) \cap \Omega| \geq A\rho^n \tag{3.3}$$

for every $x_0 \in \overline{\Omega}$, $0 < \rho \leq c\rho_0$, where $c = c(n) \geq 1$. Condition (3.3) is satisfied if Ω has the cone property. This is a surprising result, certainly not applicable to BMO, B^q, or CMO^q.

4. λ-central Carleson measures and λ-central mean oscillation

4.1. The spaces $A^{p,\alpha}$ and $HA^{p,\alpha}$.

The most general families of Herz-type spaces on $\mathbb{R}^n$ were investigated by Hernández, Weiss, and Yang [21].

For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and weights ω_1, ω_2, they defined homogeneous spaces $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ via the condition

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left(\sum_{k\in\mathbb{Z}} [\omega_1(Q_k)]^{\alpha p} \|f\chi_k\|_{L^q(\omega_2)}^p\right)^{1/p} < \infty\,,$$

where Q_k denotes the cube centered at 0 with sidelength 2^k and χ_k is the characteristic function of Q_k. Hernández, Weiss, and Yang also considered inhomogeneous versions of the space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$.

We will restrict our attention to the unweighted homogeneous version, $\omega_1 = \omega_2 = 1$, which can then be described by the condition

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k\in\mathbb{Z}} 2^{nk\alpha p} \|f\chi_k\|_{L^q}^p\right)^{1/p} < \infty\,.$$

To simplify the notation, we will drop the dot. So, the unweighted homogeneous Herz-type space on $\mathbb{R}^n$ will be denoted $K_q^{\alpha,p}$. We will follow the same convention in the rest of the section.

Rather than presenting results in the utmost generality possible, we will restrict our attention to a few special cases, focusing on specific parameters to try to bring issues surrounding unresolved containment, isomorphism, and mapping properties to the forefront. In one direction, we shall work with centered analogues of H^1 and BMO, while in somewhat different directions we will consider centered analogues of H^p, $p < 1$. We will work with the homogeneous version of the spaces, the inhomogeneous case being similar.

The atomic space analogue of H^1, ([9], [14]), is based on the Beurling algebra $A^q = K_q^{1/q',1}$, where q' is the conjugate exponent of q. To define this atomic space analogue of H^1 it is necessary to give an appropriate notion of atom.

Definition 4.1. A function a defined on $\mathbb{R}^n$ is called a central $(1, q)$-atom if the following conditions hold:

1. a has support in a ball B centered at 0 with radius $R > 0$.
2. a has average 0.
3. $\|a\|_q \le |B|^{-1+\ 1/q}$.

It turns out ([9], [14]), that for $1 < q < \infty$, the following statements are equivalent for a real function f on $\mathbb{R}^n$:

a) $f \in A^q$ and $R_j f \in A^q$ for $j = 1, 2, ...$, where R_j are the Riesz transforms.

b) $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the a_j are central $(1, q)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

Moreover, $\|f\|_{A^q} + \sum_{j=1}^{n} \|R_j f\|_{A^q}$ and $\inf_{f=\sum \lambda_j a_j} \sum |\lambda_j|$ define equivalent norms on the Hardy space HA^q defined by either of the equivalent conditions **a)** or **b)**.

The atomic characterization of HA^q also allows to prove the duality result, (see [9], [14])

$$(HA^q)^* = CMO^{q'}$$

in the following sense:

Given $g \in CMO^{q'}$ the functional L_g defined on compactly supported functions $f \in HA^q$ by

$$L_g(f) = \int_{\mathbb{R}^n} f(x)g(x)dx$$

extends in a unique way to a continuous linear functional $L_g \in (HA^q)^*$ with norm $\|L_g\| \leq C \|g\|_{CMO^{q'}}$. Conversely, given $L \in (HA^q)^*$ there is a unique, modulo constants, $g \in CMO^{q'}$ such that $L = L_g$ and $\|g\|_{CMO^{q'}} \leq C \|L\|$.

The space HA^q is a localized version of the Hardy space H^1.

In the notation of [21], García-Cuerva and Herrero considered in [15] the spaces $HK_q^{(1/p-1/q),p}$, as localized versions of H^p, $p < 1$. The fact that H^p is the union of all the translates of $HK_q^{(1/p-1/q),p}$ can be seen from the point of view of atomic decompositions of each, (see [21], [15]). But there is a third possibility, which is to vary the parameter α in $K_q^{\alpha,p}$. To illustrate the effect that this has, we shall stick to the case $q = 2$, although interesting results are obtainable for $p \leq q$. Then we consider $A^{p,\alpha} = K_2^{\alpha,p}$.

When $p \leq 1$, the condition

$$\|f\|_{A^{p,\alpha}}^p = \sum_{k \in \mathbb{Z}} 2^{nk\alpha p} \|f\chi_k\|_{L^2}^p < \infty$$

defines a quasi-norm in $A^{p,\alpha}$.

The space $A^{p,\alpha}$ increases with the parameter p when α is fixed because l^p is increasing. The inhomogeneous version is decreasing with the parameter α.

One can define the corresponding Hardy space $HA^{p,\alpha}$ by the condition that $\|G(f)\|_{A^{p,\alpha}} < \infty$ where $G(f)$ is the grand maximal function of f, (see [24]).

For α fixed, $HA^{1,\alpha}$ will be the largest space of interest, in the sense that the Hardy space differs from the Herz-Beurling space.

In general, $HA^{p,\alpha} = A^{p,\alpha}$ when $\alpha < 1/2$, but $HA^{p,\alpha}$ is a proper subspace of $A^{p,\alpha}$ for $\alpha \geq 1/2$ ([21]).

The space $HA^{p,\alpha}$, $0 < p \leq 1$ and $\alpha > -1/2$, possesses an atomic decomposition, and one can take this as the definition of $HA^{p,\alpha}$ in this case, (see [24]). To make this precise:

Definition 4.2. A function $a(x)$ is called a *central type* $(2,\alpha)$-*atom* provided

(i) supp $(a) \subset B(0,R)$ for some $R>0$.

(ii) $\|a\|_{L^2} \leq |B(0,R)|^{-\alpha}$.

(iii) $\int a(x)P(x)dx = 0$ for any polynomial P of degree at most $[n(\alpha-1/2)]$.

The following is a special case of [21], Theorem 1.2. See also [24]:

Theorem 4.3

If $0<p\leq 1$ *and* $\alpha \geq 1/2$, *then* $f \in HA^{p,\alpha}$ *if and only if* $f(x)=\sum_{k=0}^{\infty}\lambda_k a_k$ *with convergence in* $\mathcal{S}'$, *where each* a_k *is a central type* $(2,\alpha)$*-atom.*

Furthermore, $\|f\|_{HA^{p,\alpha}}$ *is equivalent to* $\inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p}$, *where the infimum is taken over all such representations.*

This theorem shows that the parameter α is the important one from the point of view of Hardy space theory. For fixed $p<1$, $\alpha = 1/p-1/2$ is a critical exponent in the sense that $HA^{p,\alpha} \subset H^p$ and H^p is the union of all the translates of $HA^{p,\alpha}$. In fact, a central type $(2,\alpha)$-atom is also an H^p atom then.

In the Appendix we shall prove the special case $\alpha \geq 1/p-1/2$ of the atomic decomposition from the point of view of discrete tent spaces. This was also done in [21]. However, we define the tent space atoms in a slightly different way, which facilitates the proof of duality for the tent spaces. This will be made clear in Subsection 4.3.

The duals of certain $HA^{p,\alpha}$ spaces were considered in [15], but not in [21]. However, the space $HK_q^{p,\alpha}$ with weights, originally defined as the class of functions whose grand maximal functions belong to $K_q^{p,\alpha}$, was completely characterized in terms of Littlewood-Paley functions or, what is equivalent, in terms of wavelets, in [21].

Let us assume, in what follows, that $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is an orthonormal wavelet family for $L^2(\mathbb{R}^n)$. Actually, such a basis requires 2^n-1 mother wavelets $\{\psi^{\varepsilon}\}_{\varepsilon\in E}$ where $E=\{0,1\}^n \setminus (0,0)$, (see [28]). Such wavelet mothers come from tensor products of unidimensional Daubechies wavelets. The crucial property is that such wavelets can be chosen to have compact support along with any specified number of derivatives and vanishing moments, which can then be transferred to the n-dimensional case. We will assume in the sequel that each mother wavelet is compactly supported and has continuous first derivatives.

For notational simplicity, there is no harm in pretending that only a single mother wavelet is needed. We can always sum estimates for individual $\{\psi_Q^{\varepsilon}\}_Q$ at the end.

Here is the wavelet characterization of $HK_q^{p,\alpha}$ in the special case ([21], Theorem 4.1) where $K_q^{p,\alpha} = A^{p,\alpha}$, $p\leq 1$ and $\alpha \geq 1/2$.

Theorem 4.4

Let $\alpha \geq 1/2$, $0 < p \leq 1$. Suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ with $f = \sum_{Q \in \mathcal{Q}} \langle f, \psi_Q \rangle \psi_Q$. Then the following statements are equivalent:

1. $\left(\sum_Q |\langle f, \psi_Q\rangle|^2 |\psi_Q(x)|^2\right)^{1/2} \in A^{p,\alpha}$.
2. *There is a constant $\eta > 0$ and a dyadic subcube R_0, $|R_0| > \eta$ such that, if $R(Q) = 2^j R_0 - k, Q = Q_{jk}$, then $\sigma(f)(x) = \left(\sum_Q |\langle f, \psi_Q\rangle|^2 \chi_{R(Q)}/|Q|\right)^{1/2}$ $\in A^{p\,,\,\alpha}$.*
3. $S(f)(x) = \left(\sum_Q |\langle f, \psi_Q\rangle|^2 \chi_Q/|Q|\right)^{1/2} \in A^{p,\alpha}$.
4. $f \in HA^{p,\alpha}$.

Furthermore, the $A^{p,\alpha}$ norms of the quantities appearing in (1)-(3) all define equivalent norms on $HA^{p,\alpha}$.

Once one has the atomic decomposition of $HA^{p,\alpha}$, it is a simple matter to identify the dual space. Actually, this has already been done in certain cases in [15]. Indeed, they also established the atomic decomposition from the classical point of view. However, the only $A^{p,\alpha}$ spaces considered therein correspond to the case $\alpha = 1/p - 1/2$, which are, after all, the critical spaces. Consider the space $CMO^{2,\lambda}$, (see Definition 2.1) for $\lambda = 1/p - 1$. Then,

Theorem 4.5 [15]

The space $CMO^{2,1/p-1}$ can be identified with the dual of $HA^{p,1/p-1/2}$.

The same space $CMO^{2,\lambda}$ introduced in Definition 2.1, provides a generalization of this theorem, as follows.

Theorem 4.6

Given $0 < p \leq 1$ and $\alpha \geq 0$, we have $CMO^{2,\lambda} = (HA^{p,\alpha})^*$, where $\lambda = \alpha - 1/2$.

Proof. First, we claim that any element g of $CMO^{2,\lambda}$ defines a continuous linear functional on $HA^{p,\alpha}$.

To see why, let $f = \sum_k \lambda_k a_k$ be an atomic decomposition of f, when $\alpha \geq 1/2$. Then

$$\begin{aligned}
\left|\int fg\right| &= \left|\int \left(\sum_k \lambda_k a_k\right) g\right| = \left|\sum_k \lambda_k \int a_k g\right| = \left|\sum_k \lambda_k \int_{B_k} a_k(g - p_k)\right| \\
&\leq C \sum_k |\lambda_k| \left(\int_{B_k} |a_k|^2\right)^{1/2} \left(\int_{B_k} |g - p_k|^2\right)^{1/2} \\
&\leq C \sum_k |\lambda_k| \left(\frac{1}{|B_k|^{2\alpha}} \int_{B_k} |g - p_k|^2\right)^{1/2}.
\end{aligned}$$

Taking the infimum over the polynomials p_k, we have

$$\left|\int fg\right| \le C\,\|g\|_{CMO^{2,\lambda}}\left(\sum_k |\lambda_k|^p\right)^{1/p}.$$

Or,

$$\left|\int fg\right| \le C\,\|g\|_{CMO^{2,\lambda}}\;\|f\|_{HA^{p,\alpha}}.$$

This proves that $(HA^{p,\alpha})^* \supset CMO^{2,\lambda}$.

We note that in the degenerate case, where $\alpha < 1/2$, there is a corresponding block decomposition of $HA^{p,\alpha} = A^{p,\alpha}$ and there is no need to knock off polynomials in the estimates above, (see [21]).

Next, if L defines a continuous linear functional on $HA^{p,\alpha}$, then for each ball B centered at the origin, L defines a continuous linear functional on the closed subspace $L^2_{[n\lambda]}(B)$ of $L^2(B)$ consisting of functions orthogonal to the space $\mathcal{P}([n\lambda])$ of polynomials of order at most $[n\lambda]$.

By the Riesz representation theorem, there exists a function $g^B \in L^2(B)$ with

$$\left\|g^B\right\|_{L^2(B)} \le C\,\|L\|\,|B|^{\lambda+\,1/2}$$

such that the functional L is represented on functions $f \in L^2_{[n\lambda]}(B)$ by

$$L\,(f) = \int_B f\,(x)\,g^B\,(x)\,dx\,.$$

This function g^B is not uniquely determined by L. If we add to g^B a polynomial of degree $\le [n\lambda]$, the above integral representation still holds.

Conversely, if L is represented as above by two functions, g_1^B and g_2^B, we claim that the difference $h^B = g_1^B - g_2^B$ has to be a polynomial of degree $\le [n\lambda]$.

In fact, the function h^B satisfies

$$\int_B f\,(x)\,h^B\,(x)\,dx = 0$$

for every $f \in L^2_{[n\lambda]}(B)$. Now, given $f \in L^2(B)$, there exists a unique polynomial $P^B\,(f) \in \mathcal{P}([n\lambda])$ that has over B the same moments of f up to the order $[n\lambda]$. Thus, $f - P^B\,(f)\,\chi_B \in L^2_{[n\lambda]}(B)$. Then,

$$\int_B \big(f\,(x) - P^B\,(f)\,(x)\big)\,h^B\,(x)\,dx = 0$$

Or,

$$\begin{aligned} 0 &= \int_B \big(f(x) - P^B(f)(x)\big)\, h^B(x)\, dx \\ &= \int_B \big(f(x) - P^B(f)(x)\big)\big(h^B(x) - P^B(h)(x)\big)\, dx \\ &= \int_B f(x)\big(h^B(x) - P^B(h)(x)\big)\, dx\,. \end{aligned}$$

Thus, the claim is proved.

We write $\mathbb{R}^n = \cup_j B_j$ where $\{B_j\}$ is an increasing family of balls, and we choose $g^{B_j} \in L^2(B_j)$ representing L such that $g^{B_{j+1}}$ extends g^{B_j}. Thus, we have found a function $g \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that the representation

$$L(f) = \int_{\mathbb{R}^n} f(x)\, g(x)\, dx$$

holds for every function $f \in L^2(\mathbb{R}^n)$ compactly supported and having vanishing moments up to order $[n\lambda]$.

Then,

$$\begin{aligned} &\left(\int_B |g(x) - P^B(g)(x)|^2 dx\right)^{1/2} \\ &= \sup_{\|f\|_2 \le 1} \left|\int_B \big(g(x) - P^B(g)(x)\big)\, f(x)\, dx\right| \\ &\le C\, \|L\|\, |B|^{1+\ 2\lambda}\,. \end{aligned}$$

Since the representation extends linearly to sums of atoms, it follows that $(HA^{p,\alpha})^* \subset CMO^{2,\lambda}$ and the theorem is proved. □

Corollary 4.7

If $p < 1$ and $\alpha \ge 0$ then $HA^{1,\alpha}$ is the containing Banach space of $HA^{p,\alpha}$.

This follows immediately from the facts that $HA^{p,\alpha} \subset HA^{1,\alpha}$ and that both spaces have the same dual.

4.2. The space $CMO^{2,\lambda}$ and λ-central Carleson measures

In this subsection we introduce the notion of a λ-central Carleson measure and of a λ-central Carleson sequence. The latter can be considered as a special case of the former where the measure is comprised of point masses. However, it will also be convenient to regard them simply as sequences indexed by the family $\mathcal{Q}$ of all the dyadic cubes in $\mathbb{R}^n$. As an example, we can consider sequences of wavelet coefficients. We shall state and prove our results in the homogeneous case. The corresponding results in the non-homogeneous case can be proved in the same way.

There are simple relationships between Carleson measures or Carleson sequences and the space $CMO^{2,\lambda}$. Indeed, as we shall show, the functions in $CMO^{2,\lambda}$ are exactly the *balayages* of certain λ-central Carleson measures or λ-central Carleson sequences.

Definition 4.8. A non-negative Borel measure μ on $\overline{\mathbb{R}^{n+1}_+}$ is called a λ-central Carleson measure if there exists a constant $C > 0$ such that for every $R > 0$

$$\mu\big(T(B(0,R))\big) \leq C\,|B\,(0,R)|^{1+2\lambda} \tag{4.1}$$

where $T\,(B\,(0,R))$ is the *cylindrical tent* $\Big\{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : |x| \leq R,\ 0 \leq t \leq R\Big\}$.

Equivalently, we can consider in Definition 4.8 the conical tent

$$\Big\{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : |x| \leq R - t,\ 0 \leq t \leq R\Big\}.$$

Definition 4.9. A sequence $\{b(Q) : Q \in \mathcal{Q}\}$ is called a λ-central Carleson sequence if there exists a constant $C > 0$ such that the following condition holds for every $R > 0$

$$\sum_{Q\subset B(0,R)} |b(Q)|^2 \leq C\,|B\,(0,R)|^{1+2\lambda}\,. \tag{4.2}$$

To show that Definition 4.9 is a natural discrete version of Definition 4.8, we note the following:

Suppose that μ is the measure given by $|F(x,t)|^2\,dxdt/t$, where $F(x,t) \in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+, dxdt/t)$. We can discretize F by replacing it by a constant equal to its average $b\,(Q)$, over each set of the form $Q_{jk} \times [2^{-j}, 2^{1-j})$, where Q_{jk} is the dyadic cube Q having lower vertex $(k_1/2^j, ..., k_n/2^j)$. Then, one has $\mu\,(T\,(B\,(0,R))) = \sum_{Q\subset B(0,R)} |b(Q)|^2$.

In particular, this heuristic argument is eminently justifiable when $F(x,t) = f * \psi_t$ where ψ is a wavelet.

The infimum of such constants C as in (4.1) or (4.2) is called the λ-Carleson constant for μ or b respectively. We remark again that any Carleson sequence can be regarded as a discrete Carleson measure with point masses at vertices of dyadic cubes. However, we prefer to think of them simply as sequences.

The case $\lambda = 0$ of Definition 4.8 appears in [25].

It is clear that only the zero measure can satisfy Definitions 4.8 or 4.9 when $\lambda < -1/2$.

Amar and Bonami [4] seem to have been the first to study Carleson measures of non-negative order systematically. In fact, given $\alpha \geq 0$, they introduced, in the homogeneous space setting, the space V^{α} of α-Carleson measures. In the Euclidean setting V^{α} consists of positive measures μ on $\overline{\mathbb{R}^{n+1}_{+}}$ such that for any open set $\Omega \subset \mathbb{R}^n$

$$\mu\left(T\left(\Omega\right)\right) \leq C|\Omega|^{\alpha}$$

where

$$T\left(\Omega\right) = \left\{(x,t) \in \overline{\mathbb{R}^{n+1}_{+}} : B(x,t) \subset \Omega\right\}.$$

Of course, when $\alpha \geq 0$, by countable additivity, it suffices to check the condition over open balls.

If we denote by CV^{λ} the space of λ-central Carleson measures, we have that $V^{1+2\lambda} \subset CV^{\lambda}$. But the CV^{λ} condition is much more lax than the $V^{1+2\lambda}$ condition. For instance, when $\lambda = 0$, V^1 is properly contained in CV^0. This assertion will become clear after we prove Theorem 4.10 below.

In this theorem, as in the remainder of this section, $\varphi \in \mathcal{S}$ will be a radial function with integral zero and support contained in $\{|x| \leq 1\}$. Moreover, φ will satisfy the normalization condition

$$\int_0^{\infty} |\widehat{\varphi}(t\xi)|^2 \, \frac{dt}{t} = 1 \text{ for } |\xi| \neq 0\,. \tag{4.3}$$

Everything that follows will hold with only minor modifications if we replace the pair $\varphi, \overline{\varphi}$ by a pair of radial functions φ and ψ supported in $\{|x| \leq 1\}$, having integral zero, and satisfying the normalization condition

$$\int_0^{\infty} \widehat{\varphi}(t\xi)\widehat{\psi}(t\xi) \, \frac{dt}{t} = 1 \text{ for } |\xi| \neq 0\,. \tag{4.4}$$

Theorem 4.10

Let $f \in L^2((1+|x|)^{-n-2}dx)$. Then, given $\lambda < 1/n$, the following statements are equivalent.

1. *$f \in CMO^{2,\lambda}$.*
2. *$|f * \varphi_t|^2 \, dxdt/t$ is a λ-central Carleson measure.*

Moreover, the norm $\|f\|_{CMO^{2,\lambda}}$ is equivalent to the λ-Carleson constant of the measure $|f * \varphi_t|^2 \, dxdt/t$.

Remark 4. 1. The case $\lambda = 0$ of Theorem 4.10 appears in [25] with a different proof. Our proof is elementary and it applies without change to balls centered at any point. In particular, it proves the connection between Carleson measures and BMO functions.

2. To prove that 2) implies 1) in the theorem above, we will use the following more general convergence result.

Lemma 4.11

Fix $\lambda < 1/n$. Let $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be a Lebesgue measurable function such that $|F(x,t)|^2\, dxdt/t$ is a λ-central Carleson measure. For $0 < a < b < \infty$ set

$$g_{ab}(x) = \int_a^b (F(\cdot,t) * \overline{\varphi_t})(x)\frac{dt}{t}\,. \tag{4.5}$$

Then $\{g_{ab}\}$ converges weakly to some function g in $CMO^{2,\lambda}$ as $a \to 0, b \to \infty$. Moreover, $\|g\|_{CMO^{2,\lambda}}$ can be estimated in terms of the λ-Carleson constant of $|F(x,t)|^2\, dxdt/t$.

Before proving this lemma, we shall use it to prove Theorem 4.10.

Proof of Theorem 4.10. First we shall prove that $d\mu = |f * \varphi_t|^2\, dxdt/t$ is a λ-central Carleson measure.

Let $R > 0$ and $B_R = B(0,R)$. Write $f = f_1 + f_2 + f_3$, where

$$f_1 = (f - f_{2R})\,\chi_{B_{2R}},\ f_2 = (f - f_{2R})\,\chi_{\mathbb{R}^n\setminus B_{2R}},\ f_3 = f_{2R}, \text{ for any } c \in \mathbb{C}\,.$$

Since $\int \varphi = 0$, we have that $f*\varphi_t = f_1*\varphi_t + f_2*\varphi_t$. Moreover, under the hypothesis $\operatorname{supp}(\varphi) \subset \{|x| \le 1\}$, we have $(f_2 * \varphi_t)(x) = 0$ for $(x,t) \in T(B)$.

Now, set $d\mu_1 = |f_1 * \varphi_t|^2\, dxdt/t$.

To estimate μ_1 we will use the square function s_φ and its basic property, namely, its L^2-continuity. Thus

$$\begin{aligned}\int_{B_R} \big(s_\varphi f_1(x)\big)^2 dx &\le C\,\|f_1\|_{L^2}^2 \\ &= C\int_{B_{2R}} |f(x) - f_{2R}|^2\, dx\,.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{B_R} \big(s_\varphi f_1(x)\big)^2 dx &= \int_{B_R}\int_0^\infty |f_1 * \varphi_t(x)|^2\,\frac{dt}{t}dx \\ &\ge \int_{T(B_R)} |f_1 * \varphi_t(x)|^2\, dx\frac{dt}{t} \\ &= \mu_1\big(T(B_R)\big)\,,\end{aligned}$$

Thus, we can conclude that μ is a λ-central Carleson measure. It is also clear from the proof that the λ-Carleson constant for μ is $O\left(\|f\|^2_{CMO^{2,\lambda}}\right)$.

We will now prove the converse. So, given $f \in L^2((1+|x|)^{-n-2})$, we consider (4.5) with $F(x,t) = f * \varphi_t$. That is to say

$$g_{ab}(x) = \int_a^b (f * \varphi_t * \overline{\varphi_t})(x)\frac{dt}{t}. \tag{4.6}$$

By Lemma 4.11, $\{g_{ab}\}$ converges weakly to some $g \in CMO^{2,\lambda}$ as $a \to 0, b \to \infty$. In particular, $\{g_{ab}\} \to g$ in $\mathcal{S}'$.

On the other hand, taking the $\mathcal{S}'$ Fourier transform of (4.6) gives

$$\widehat{g_{ab}}(\xi) = \widehat{f}(\xi)\int_a^b |\widehat{\varphi}(t\xi)|^2\frac{dt}{t} \to \widehat{f}(\xi) \text{ in } \mathcal{D}'(\mathbb{R}^n \setminus 0).$$

It follows that $\widehat{f}(\xi) = \widehat{g}(\xi)$ in $\mathbb{R}^n \setminus 0$. Therefore $f(x) = g(x) + P(x)$ for some polynomial P.

According to Corollary 2.5, $g \in L^2((1+|x|)^{-n-\ 2/n})$. Thus, $f - g \in L^2((1+|x|)^{-n-2})$. A polynomial P of degree k will belong to the space $L^2((1+|x|)^{-n-\ 2})$ if

$$2k + n - 1 - n - 2 < -1\,.$$

Or, $k < 1$. Thus, the polynomial P has degree zero. This proves that $f \in CMO^{2,\lambda}$.

Lemma 4.11 also guarantees the estimate on $\|f\|^2_{CMO^{2,\lambda}}$ in terms of the λ-Carleson constant for $|F(x,t)|^2\,dxdt/t$, thereby completing the proof of Theorem 4.10. □

Now we move on to the proof of Lemma 4.11. Though the following lemma is well-known, we shall include it for the sake of self-containment: it is the case $\lambda = -1/2$ of Lemma 4.11. It will exemplify some of the techniques used to prove Lemma 4.11.

Lemma 4.12

With φ as above, the mapping $f \mapsto F(x,t) = f * \varphi_t(x)$ is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{n+1}_+, dxdt/t)$.

Moreover, given $F(x,t) \in L^2(\mathbb{R}^{n+1}_+, dxdt/t)$, let $\{g_{ab}\}$ be as in 4.5. Then, there exists $f \in L^2(\mathbb{R}^n)$ such that $\{g_{ab}\}$ converges to f in $L^2(\mathbb{R}^n)$ as $a \to 0, b \to \infty$.

Proof. First, setting $F(x,t) = f * \varphi_t(x)$ gives

$$\|F\|^2_{L^2(\mathbb{R}^{n+1}_+, \frac{dxdt}{t})} = \int\limits_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \int\limits_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} d\xi = \|f\|^2_{L^2(\mathbb{R}^n)}$$

because of (4.3) and Plancherel's theorem. Therefore $f \mapsto f * \varphi_t(x)$ is an isometry.

To prove the second part of the lemma, first we note that

$$\widehat{g}_{ab}(z) = \int_a^b \widehat{F}(z,t)\widehat{\varphi}(tz)\frac{dt}{t}$$

with convergence in $L^2(\mathbb{R}^n)$. The Fourier transform of F is taken in the z-variable. Then, Cauchy-Schwarz's inequality implies

$$|\widehat{g}_{ab}(z)|^2 \le \left(\int_a^b |\widehat{F}(z,t)|^2 \frac{dt}{t}\right)\left(\int_a^b |\widehat{\varphi}(tz)|^2 \frac{dt}{t}\right) \le \left(\int_a^b |\widehat{F}(z,t)|^2 \frac{dt}{t}\right)$$

by (4.3). Thus

$$\|g_{ab}\|^2_{L^2(\mathbb{R}^n)} = \|\widehat{g}_{ab}\|^2_{L^2(\mathbb{R}^n)} \le \int_a^b \int_{\mathbb{R}^n} |\widehat{F}(z,t)|^2 dz \frac{dt}{t}$$

To prove the convergence of $\{g_{ab}\}$ in $L^2(\mathbb{R}^n)$ it suffices to prove the convergence of each $\{g_{a1}\}$ and $\{g_{1b}\}$. For $\{g_{a1}\}$, fixing $0 < a' < a < 1$ we have

$$\|g_{a1} - g_{a'1}\|^2_{L^2(\mathbb{R}^n)} \le \int_{a'}^a \int_{\mathbb{R}^n} |\widehat{F}(z,t)|^2 dz \frac{dt}{t} \to 0 \text{ as } a \to 0\,.$$

The same principle applies to the convergence of $\{g_{1b}\}$. This completes the proof of the Lemma 4.12. □

Remark 5. If $F(x,t) = f * \varphi_t(x)$ then $g_{ab} = \int_a^b F * \overline{\varphi}_t dt/t \to f$ in $L^2(\mathbb{R}^n)$ as $a \to 0, b \to \infty$, which is a simple consequence of (4.3) and the uniqueness of the Fourier transform.

We also point out that Lemma 4.12 can be used, instead of the square function, in the proof of the first part of Theorem 4.10.

Now we are ready to prove Lemma 4.11.

Proof of Lemma 4.11. The first step is to show that $\{g_{ab}\}$ is bounded in $CMO^{2,\lambda}$. More precisely, we will show that there is a constant $C = C(F, \varphi, \lambda) > 0$ such that for every $0 < a < b < \infty$ and each $B = B(0, R), R > 0$, there is a complex number m_{abR} satisfying

$$\left(\frac{1}{|B|}\int_B |g_{ab} - m_{abR}|^2 dx\right)^{1/2} \leq C|B|^{\lambda}.$$

For $R > 0$ fixed, we consider three cases: (i) $b \leq R$, (ii) $R \leq a$, and (iii) $a \leq R \leq b$. Actually, since $g_{ab} = g_{aR} + g_{Rb}$, case (iii) can be reduced to a combination of cases (i) and (ii).

First we consider case (i). Here we set $m_{abR} = 0$; we wish to show that

$$\left(\frac{1}{|B|}\int_B |g_{ab}|^2 dx\right)^{1/2} \leq C|B|^{\lambda}.$$

First,

$$\begin{aligned}\chi_B(x)g_{ab}(x) &= \int_a^b \left[\chi_{2B}(\cdot)F(\cdot,t) * \overline{\varphi}_t\right](x)\frac{dt}{t} \\ &= \int_a^b \int_{\mathbb{R}^n} \left[\chi_{2B}(y)F(y,t)\overline{\varphi}_t(x-y)\right]\frac{dydt}{t}\end{aligned}$$

Therefore, as in the proof of Lemma 4.12, we can write

$$\int_B |g_{ab}(x)|^2\, dx \leq \int_a^b \int_{2B} |F(y,t)|^2 \,\frac{dydt}{t}$$

which implies that

$$\int_B |g_{ab}(x)|^2\, dx \leq \left\{\sup_B \int_{T(B)} |F(y,t)|^2 \,\frac{dydt}{t}\right\} |B|^{1+2\lambda}.$$

This proves the claim for case (i).

Case (ii) is a little harder. In this case we set $m_{abR} = g_{ab}(0)$. Then

$$g_{ab}(x) - m_{abR} = \int_a^b \int_{\mathbb{R}^n} F(y,t)\left[\overline{\varphi}_t(x-y) - \overline{\varphi}_t(-y)\right]\frac{dydt}{t}\,.$$

Hence

$$\chi_B(x)\left(g_{ab}(x) - m_{abR}\right) = \int_a^b \int_{\mathbb{R}^n} \chi_{B(0,2t)}(y)F(y,t)\left[\overline{\varphi}_t(x-y) - \overline{\varphi}_t(-y)\right]\frac{dydt}{t}.$$

From the Mean Value Theorem it follows that

$$\left|\overline{\varphi}_t(x-y) - \overline{\varphi}_t(-y)\right| \leq \frac{|x|}{t^{n+1}}\left\|\nabla\varphi\right\|_{\infty}\,.$$

Therefore,

$$\begin{aligned}\int_B |g_{ab}(x) - m_{abR}|^2\, dx &\le C_\varphi \int_B |x|^2 \left(\int_a^b \int_{\mathbb{R}^n} \chi_{B(0,2t)}(y)\, |F(y,t)|\, dy \frac{dt}{t^{n+2}} \right)^2 dx \\ &= C_\varphi R^{n+2} \left(\int_a^b \frac{1}{t^\beta} \int_{\mathbb{R}^n} \chi_{B(0,2t)}(y)\, |F(y,t)|\, dy \frac{dt}{t^{n+2-\beta}} \right)^2 \\ &\le C_\varphi R^{n+2} \left[\int_a^b \left(\int_{\mathbb{R}^n} \chi_{B(0,2t)}(y)\, |F(y,t)|\, dy \right)^2 \frac{dt}{t^{2n+4-2\beta}} \right] \\ &\quad \times \left[\int_a^b \frac{1}{t^{2\beta}} \right]. \end{aligned} \tag{4.7}$$

Here $\beta \in (1/2, 1]$ is a player to be named later. The last inequality follows from Cauchy-Schwarz's inequality. Since $|x| \le R$ and

$$\int_a^b \frac{dt}{t^{2\beta}} = C_\beta \left(a^{1-2\beta} - b^{1-2\beta} \right) \le C_\beta R^{1-2\beta} \ (R \le a),$$

a second application of Cauchy-Schwarz's inequality, this time in the y-variable, shows that (4.7) can be estimated by

$$\begin{aligned} &C_{\varphi,\beta} R^2 R^{1-2\beta} |B| \int_a^b |B(0,2t)| \int_a^b \left(\int_{B(0,2t)} |F(y,t)|^2\, dy \right) \frac{dt}{t^{2n+4-2\beta}} \\ &\le C_{\varphi,\beta,n} R^{n+3-2\beta} \int_a^b \left(\int_{B(0,2t)} |F(y,t)|^2\, dy \right) \frac{dt}{t^{n+4-2\beta}}\,. \end{aligned} \tag{4.8}$$

Now set

$$G(t) = \int_a^t \int_{B(0,2s)} |F(y,s)|^2\, dy \frac{ds}{s}$$

so that

$$G'(t) = \frac{1}{t} \int_{B(0,2t)} |F(y,t)|^2\, dy\,,$$

and

$$G(t) \le C \left\{ \sup_B \int_{T(B)} |F(y,s)|^2\, dy \frac{ds}{s} \right\} t^{n(1+2\lambda)}.$$

Therefore, (4.8) can be written as

$$CR^{n+3-2\beta}\int_a^b tG'(t)\frac{dt}{t^{n+4-2\beta}}\,.$$

Integration by parts then yields

$$\begin{aligned}\int_a^b G'(t)\frac{dt}{t^{n+3-2\beta}} &= \frac{G(b)}{b^{n+3-2\beta}} + (n+3-2\beta)\int_a^b G(t)\frac{dt}{t^{n+4-2\beta}}\\ &\le C_F b^{2n\lambda-3+2\beta} + C_{\beta,F}\int_a^b t^{2n\lambda-4+2\beta}dt\,. \qquad (4.9)\end{aligned}$$

Now, since $\lambda < 1/n$ and $1 \ge \beta > 1/2$ there is some $\varepsilon = \beta - 1/2 \in (0, 1/2)$ such that

$$2n\lambda - 3 + 2\beta = 2(n\lambda - 1 + \varepsilon) < 0\,.$$

Therefore, substituting (4.9) into (4.8) gives

$$\begin{aligned}\int_B |g_{ab}(x) - m_{abR}|^2\,dx &\le CR^{n+3-2\beta}\left[b^{2n\lambda-3+2\beta} + a^{2n\lambda-3+2\beta}\right]\\ &\le CR^{n+3-2\beta+2n\lambda-3+2\beta} = CR^{n(1+2\lambda)} = C|B|^{1+2\lambda}\,,\end{aligned}$$

where we have used the fact that $R \le a \le b$.

This proves that $\{g_{ab}\}$ is uniformly bounded in $CMO^{2,\lambda}$ by

$$C_{n,\lambda,\varphi}\left\{\sup_B \int_{T(B)} |F(y,s)|^2\,dy\frac{ds}{s}\right\}\,.$$

Consequently, $\{g_{ab}\}$ is uniformly bounded in $L^2((1+|x|)^{-n-2}dx)$ and, therefore, there exists a subnet $\{g_{a'b'}\}$ weakly convergent to some $g \in L^2((1+|x|)^{-n-2}dx)$. At the same time, $\{g_{ab} - m_{abR}\}$ converges in $L^2(B)$. Therefore, $g \in CMO^{2,\lambda}$ and $\{g_{ab}\}$ converges weakly to g in $CMO^{2,\lambda}$. This, finally, completes the proof of Lemma 4.11. □

There is an analogous result for λ-Carleson sequences. Though such sequences are just special cases of Carleson measures, their dyadic structure makes them easier to work with. This will become evident in the proof of the following proposition, but will play an even more central role in the discussion of atomic decomposition and duality in the following subsection.

Proposition 4.13

Fix $\lambda < 1/n$. If $g \in CMO^{2,\lambda}$ then the sequence of wavelet coefficients of g defines an element of $CV^{2,\lambda}$. Conversely, if $b \in CV^{2,\lambda}$ then its wavelet balayage $\sum_Q b(Q)\psi_Q$ converges to an element of $CMO^{2,\lambda}$ in the weak $(HA^{1,1/2+\lambda}, CMO^{2,\lambda})$ topology.

Proof. The argument is adapted from [28]. That the wavelet coefficient mapping injects $CMO^{2,\lambda}$ into $CV^{2,\lambda}$ is, in principle, the same argument as in Theorem 4.10. The argument for the converse differs slightly, but importantly: the sequential Carleson condition implies $|b(Q)| \leq C|Q|^{2\lambda+1}$ for each $Q \in \mathcal{Q}$.

Now, suppose that $b \in CV^{2,\lambda}$ and form $\sum_Q b(Q)\psi_Q$. We wish to show that the sum converges weakly to an element $g \in CMO^{2,\lambda}$. We break the sum up into three pieces according to a fixed ball $B = B(0,R)$.

Given $j_0 \in \mathbb{Z}$ such that $2^{-j_0} \leq R < 2^{1-j_0}$, let g_2 be the sum over all of those cubes having length at least 2^{1-j_0}, let g_{11} be the sum over those cubes having length at most 2^{-j_0} and such that the support of ψ_Q intersects B, and let g_{12} denote the sum over the remaining cubes.

To verify the $CMO^{2,\lambda}$ condition for the ball B, we note that g_{12} does not contribute to the condition over B. On the other hand, each term in g_{11} comes from a cube supported in the ball mB with radius mR, for some fixed $m > 0$.

Hence

$$\int_B |g_{11}|^2 \leq \sum_{Q\subset mB} |b(Q)|^2 \leq Cm^{n(1+2\lambda)}|B|^{1+2\lambda}$$

since $b \in CV^{2,\lambda}$. Therefore, it just remains to check the average for g_2.

Now we note that, because the wavelets have compact support, at any dyadic level $j < j_0$ there can be at most a fixed finite number M of cubes at level j whose corresponding wavelets have support intersecting $B(0,R)$. On the other hand, for each such intersecting wavelet ψ_Q at level j, the C^1 hypothesis guarantees that if $|x| < R$ then

$$\left|\psi_Q(x) - \psi_Q(0)\right| \leq C2^{nj/2}2^j|x|\,.$$

At the same time, the $CV^{2,\lambda}$ condition ensures that $|b(Q)| \leq 2^{-jn(\lambda+\ 1/2)}$.

Then $\sum_{j<j_0}\sum_{Q\in\mathcal{Q}_j} b(Q)\,[\psi_Q(x) - \psi_Q(0)]$ converges uniformly on B as long as $\lambda < 1/n$.

Thus, we have:

$$\begin{aligned}\int_B \left|\sum_{j<j_0}\sum_{Q\in\mathcal{Q}_j} b(Q)\,[\psi_Q(x)-\psi_Q(0)]\right|^2 &\leq CM^{2n}|B|2^{-2j_0}\left|\sum_{j<j_0} 2^{-jn(\lambda+\ 1/2)}2^{nj/2}2^j\right|^2 \\ &= CM^{2n}|B|^{(1+\ 2/n)}\left|\sum_{j<j_0} 2^{-j(n\lambda-1)}\right|^2 \\ &\leq CM^{2n}|B|^{(1+\ 2/n)}2^{-2j_0(n\lambda-1)} \\ &= CM^{2n}|B|^{1+\ 2\lambda}\,.\end{aligned}$$

This proves that the wavelet sum converges to g in the weak topology. This completes the proof of Proposition 4.13. □

Remark 6. We cannot conclude strong convergence, which would imply that the wavelet sum converges locally in L^2 - it does not. On the other hand, one can show by the estimates above that the integral of the wavelet sum against any central atom converges absolutely. We reiterate the fact that the exponent λ can be extended beyond $1/n$.

4.2. Atomic decomposition of $TA^{p,\alpha}$ spaces. Duality at the coefficient level

In this subsection we state the atomic decomposition of the central discrete tent spaces in the case $\alpha \geq 1/p - 1/2$. As a consequence we can characterize the preduals of the discrete central Carleson sequences. Similar results hold in the case of continuous densities, but we will only consider the discrete case.

DEFINITION 4.14. A sequence $s = \{s(Q)\}_{Q\in\mathcal{Q}}$ indexed by the family $\mathcal{Q}$ of dyadic cubes belongs to the discrete tent space $T_dA^{p,\alpha}$ provided the dyadic square function

$$S(s)(x) = \left(\sum_{x\in Q\in\mathcal{Q}} |s(Q)|^2/|Q|\right)^{1/2} \in A^{p,\alpha}\,.$$

Remark 7. By [21], if $f \in HA^{p,\alpha}$ then its wavelet coefficient sequence belongs to $T_dA^{p,\alpha}$.

DEFINITION 4.15. A sequence $a = \{a(Q)\}_{Q\in\mathcal{Q}}$ is called a discrete central type $(2,\alpha)$-atom provided there is a ball B centered at the origin such that $a(Q) = 0$ if $Q \not\subseteq B$ and $\sum_{Q\subset B} |a(Q)|^2 \leq |B|^{-2\alpha}$.

Remark 8. If $\{a(Q)\}_{Q\in\mathcal{Q}}$ is a discrete central type $(2,\alpha)$-atom, and $\{\psi_Q\}$ is a $(1+\gamma)$-regular compactly supported wavelet basis with $\gamma \geq \alpha$, then $\sum_Q a(Q)\psi_Q$ is a central type $(2,\alpha)$-atom supported in mB where B supports $\{a(Q)\}$.

On the other hand, if $a(x)$ is a central type $(2,\alpha)$-atom, it is not necessarily the case that $\{\langle a, \psi_Q\rangle\}_Q$ is supported in a fixed ball.

We contrast the present definition of discrete atom with the definition of an atom sequence in [21].

We wish to show that any element of $T_dA^{p,\alpha}$ has an atomic decomposition in terms of the atoms introduced in Definition 4.15:

Theorem 4.16

If the sequence s belongs to $T_dA^{p,\alpha}$ then s has a decomposition $s = \sum_{j=1}^{\infty} \lambda_j a_j(Q)$ into discrete central type $(2,\alpha)$-atoms $a_j(Q)$ such that $\sum_{j=1}^{\infty} |\lambda_j|^p \leq C\,\|S(s)\|^p_{A^{p,\alpha}}$.

The next result relates the tent space to the space downstairs:

Proposition 4.17

If $a = \{a(Q)\}_{Q\in\mathcal{Q}}$ is a central type $(2,\alpha)$-atom with $\alpha > 0$, then $S(a) \in A^{p,\alpha}$ and $\|S(a)\|^p_{A^{p,\alpha}} \le C$, where C does not depend on a.

Proof. To prove this proposition, let the "supporting ball" B of a have radius R comparable with 2^{k_0} for some k_0.

Then

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} \left(2^{nk\alpha} \|S(a)(x)\chi_{\Delta_k}\|_{L^2}\right)^p \\
&\le \sum_{k\in\mathbb{Z}} 2^{nk\alpha p} \left[\int \left(\sum_{x\in Q\subset B} |a(Q)|^2/|Q|\right) \chi_{\Delta_k} dx\right]^{p/2} \\
&= \sum_{k\le k_0} 2^{nk\alpha p} \left[\sum_{Q\subset B} |a(Q)|^2\right]^{p/2} \le |B|^{-\alpha p} \sum_{k\le k_0} 2^{nk\alpha p} \le C\,.
\end{aligned}
$$

The condition $\alpha > 0$ implies that the sum converges.

Remark 9. What this proposition really shows is that our atoms are a proper subset of the atom sequences in the sense of [21].

Corollary 4.18

Under the hypothesis of Proposition 4.17, we have

$$
\inf \sum_{j=1}^{\infty} |\lambda_j| \sim C\,\|S(s)\|_{A^{1,\alpha}}
$$

where the infimum is taken over all atomic decompositions into central type $(2,\alpha)$-atoms.

Proof. Let $s = \sum_{j=1}^{\infty} \lambda_j a_j(Q)$. Then

$$
\|S(s)\|_{A^{1,\alpha}} \le \sum_j |\lambda_j|\,\|S(a_j)\|_{A^{1,\alpha}} \le C \sum_j |\lambda_j|\,.
$$

The reverse inequality follows from Theorem 4.16. Thus, the proof of the corollary is completed. □

The corollary still holds for $p < 1$, with atoms defined appropriately [21]. However, the argument above does not apply because we cannot use the triangle inequality. Analogously, the space $HA^{p,\alpha}$ can be defined as the space of functions whose wavelet coefficients belong to $T_dA^{p,\alpha}$ (cf. [21]).

We shall include the proof of Theorem 4.16 in an appendix , since the technique, adapted from [28], p. 150, is similar to the one used in the proof of Theorem 2.1 in [24], page 107.

Theorem 4.19

Set $\alpha = \lambda - 1/2$ and assume $\alpha \geq 0$. Then the dual of the atomic space $T_dA^{1,\alpha}$ is the space $CV^{2,\lambda}$ of λ-central Carleson sequences.

Proof. First, we show that $CV^{2,\lambda} \subset \left(T_dA^{1,\alpha}\right)^*$.

Because of the atomic norm of $T_dA^{1,\alpha}$, it suffices to show that summation of $b \in CV^{2,\lambda}$ against any central type $(2,\alpha)$-atom is bounded by a fixed constant.

But if $a(Q)$ is supported inside $B = B(0,R)$ and $\sum_{Q\subset B} |a(Q)|^2 \leq |B|^{-2\alpha}$, it follows that

$$\sum_Q a(Q)b(Q) \leq \left(\sum_{Q\subset B} |a(Q)|^2\right)^{1/2} \left(\sum_{Q\subset B} |b(Q)|^2\right)^{1/2} \leq C|B|^{-\alpha}|B|^{1/2\ +\lambda} \leq C$$

since $\alpha = 1/2\ +\lambda$.

To show that $CV^{2,\lambda} \supset \left(T_dA^{1,\alpha}\right)^*$, we note that since $T_dA^{1,\alpha}$ contains any central type $(2,\alpha)$-atom, by the Riesz representation theorem any linear functional on $T_dA^{1,\alpha}$ is locally square summable. Here locally means restricting to those cubes inside a ball centered at the origin. This representation is consistent as one passes from smaller balls to larger. Finally, by continuity, the sequence representing the linear functional must satisfy the Carleson sequence condition. This completes the proof of Theorem 4.19, proving duality in the special case where $p = 1$. □

Remark 10. As with $A^{p,\alpha}$ and $HA^{p,\alpha}$, convexity shows that for a fixed value of α, the spaces $T_dA^{p,\alpha}$ are increasing with p. In particular, if $p < 1$ and $\alpha \geq 1/p - 1/2$ are fixed, then $CV^{2,\lambda}$ is the dual of $T_dA^{p,\alpha}$ by the same argument as above, along with convexity.

Corollary 4.20

With $\alpha \geq 1/p - 1/2$, $T_dA^{1,\alpha}$ is the containing Banach space of $T_dA^{p,\alpha}$.

5. Continuity of singular integral operators on Herz-type spaces

Several authors have obtained continuity results on various homogeneous and inhomogeneous versions of the Herz-type spaces. For instance, we mention the work of García-Cuerva [14], Lu and Yang [24], [26], [27], and Li and Yang [23]. Collectively, these results show that there are significant differences, depending on the values of the parameters. They also show that the behavior of the homogeneous version of the spaces greatly differ from the inhomogeneous version.

Our purpose is to make these observations more precise by working with some relevant particular cases. Indeed, we will consider the inhomogeneous spaces $B^{q,\lambda}$, $CMO^{q,\lambda}$, and $HA_q^{p,\alpha}$. We will study the action on these spaces of non-convolution singular integral operators satisfying fairly general conditions. In particular, these conditions allow for operators that are more singular than Calderón-Zygmund operators, such as pseudo-differential operators in the Hörmander class $L^m_{\rho,\delta}$ [19].

Definition 5.1. Let $T : C_0^\infty \longrightarrow \mathcal{D}'$ be a linear and continuous operator. Given $1 < q < \infty$, $\lambda \in \mathbb{R}$, we say that T is a (q, λ)-central singular integral operator if T satisfies the following conditions:

a) T extends to a continuous operator on L^q.

b) The distribution kernel of T coincides in the complement of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ with a locally integrable function $k(x, y)$. Moreover, if $C_j\,(0, R) = B(0, 2^{j+1}R) \setminus B(0, 2^j R)$, $j = 1, 2, ...$, the function $k(x, y)$ satisfies the estimate

$$\sup_{R\geq 1}\sup_{|x|<R}\left[|C_j\,(0,R)|^{q'-1}\int_{C_j(0,R)}|k(x,y)-k(0,y)|^{q'}\,dy\right]^{1/q'}\leq d_j \quad (5.1)$$

with $\sum 2^{jn\lambda} d_j < \infty$.

c) If f, $g \in C_0^\infty$ and $\mathrm{supp}(f) \cap\ \mathrm{supp}(g) = \emptyset$, the operator T has the integral representation

$$(Tf, g) = \iint k(x, y) f(y) g(x) dy dx\,.$$

Remark 11. When $\lambda = 0$, estimate (5.1) is the inhomogeneous central version of the integral condition introduced by Rubio de Francia, Ruiz, and Torrea in [30]. In particular, it is implied by the pointwise condition

$$\left|k(x,y)-k(0,y)\right| \leq C\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon/b}} \quad (5.2)$$

if $2\,|x| < |y|$, for some $0 < \varepsilon, b \leq 1$.

In fact, using (5.2) we can obtain (5.1) with any $1 < q < \infty$, $d_j = 2^{-j\varepsilon/b}$. When $\varepsilon = b = 1$, condition (5.2) gives standard kernels as defined in [8], p. 78. Thus, Calderón-Zygmund operators in the sense of Coifman and Meyer [8] p. 78, are (q, λ)-central singular integral operators for $1 < q < \infty$, $\lambda < 1/n$. We can have larger values of λ if we modify (5.1) to include higher order increments. Accordingly, this means to consider standard kernels with continuous derivatives satisfying an appropriate version of (5.2). We encountered in Section 4 the dual situation, when we needed to consider atoms with higher order vanishing moments in order to lift the restriction $\lambda < 1/n$.

Weakly-strongly singular operators [3] satisfy Definition 5.1 with any $1 < q < \infty$ and $\lambda < \varepsilon/nb$. These are singular integral operators associated to kernels that satisfy (5.2). Thus, the kernels are more singular at the diagonal than standard kernels, but they have faster decay at ∞. Moreover, the operators are still continuous on L^q for $1 < q < \infty$. This is the reason for the name weakly-strongly singular.

Pseudo-differential operators in the Hörmander class $L^m_{\rho,\delta}$ ([19]), provide important examples of the classes mentioned above. In general, L has the form

$$L\left(f\right)(x) = \int e^{2\pi i x\cdot\xi} p(x,\xi)\widehat{f}(\xi)d\xi,\ f \in \mathcal{S},$$

where the symbol $p(x, \xi)$ belongs to the class $S^m_{\rho,\delta}$. That is to say,

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| \leq C_{\alpha\beta}\left(1 + |\xi|\right)^{m-\rho|\beta|+\delta|\alpha|}$$

for some $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$, and for every n-tuples α, β.

The estimates for the kernel and the (L^p, L^q)-continuity properties proved in [3] show that one can identify, within $L^m_{\rho,\delta}$, Calderón-Zygmund operators and weakly-strongly singular operators, depending on the values of the parameters m, ρ, and δ. Roughly speaking, the order $m = (n/2)(1-\rho)$ is the threshold. When $m > -n/2(1-\rho)$, the classical multiplier of Hardy, Hirschman, and Wainger shows that pseudo-differential operators in the class $L^m_{\rho,\delta}$ fail to be continuous on L^q for some or all values of $q \neq 2$. These operators were named strongly singular by C. Fefferman [12], [11].

It is certainly of interest to classify in this way pseudo-differential operators. However, it is the inhomogeneous nature of (5.1) that allows for operators associated to kernels that, although are more singular at the diagonal, have better decay at infinity. This will be made clear in Corollary 5.3.

The pointwise condition (5.2) was considered in [24] for $b = 1$. That is, for the Calderón-Zygmund case.

Proposition 5.2

Given $1 < q < \infty$, $\lambda \in \mathbb{R}$, let T be a (q, λ)-central singular integral operator. Then, T is continuous from $B^{q,\lambda}$ into $CMO^{q,\lambda}$.

Proof. The proof of this result is straightforward and it resembles the classical proof of the (L^∞, BMO)-continuity of a Calderón-Zygmund operator.

Indeed, give $f \in B^{q,\lambda}$ and given $R \geq 0$, we write

$$f = f\chi_{B(0,2R)} + f\left(1 - \chi_{B(0,2R)}\right) = f_1 + f_2.$$

Thus,

$$\begin{aligned}
&\left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |Tf(x) - Tf_2(0)|^q\, dx\right]^{1/q} \\
&\leq \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |Tf_1(x)|^q\, dx\right]^{1/q} \\
&\quad + \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |Tf_2(x) - Tf_2(0)|^q\, dx\right]^{1/q} \\
&= I_1 + I_2.
\end{aligned}$$

To estimate I_1, we use the continuity of T on L^q. Thus,

$$I_1 \leq C\,\|f\|_{B^{q,\lambda}}\,|B(0,R)|^\lambda\,.$$

To estimate I_2 we first obtain a pointwise estimate of the difference $|Tf_2(x) - Tf_2(0)|$ for $|x| < R$.

$$\begin{aligned}
|Tf_2(x) - Tf_2(0)| &\leq \int_{\mathbb{R}^n\setminus B(0,R)} |k(x,y) - k(0,y)|\,|f(y)|\,dy \\
&= \sum_{j=1}^{\infty}\int_{C_j(0,R)} |k(x,y) - k(0,y)|\,|f(y)|\,dy \\
&\leq \sum_{j=1}^{\infty}\left[\int_{C_j(0,R)} |k(x,y) - k(0,y)|^{q'}\,dy\right]^{1/q'} \\
&\quad\times \left[\int_{B(0,2^{j+1}R)} |f(y)|^q\,dy\right]^{1/q} \\
&\leq C\,\|f\|_{B^{q,\lambda}}\sum_{j=1}^{\infty} d_j\left|B(0,2^{j+1}R)\right|^{-1+\ 1/q'\ +\ 1/q\ +\lambda} \\
&= C\,\|f\|_{B^{q,\lambda}}\left(\sum_{j=1}^{\infty} 2^{jn\lambda} d_j\right)|B(0,R)|^\lambda\,.
\end{aligned}$$

Thus,

$$I_2 \leq C \|f\|_{B^{q,\lambda}} |B(0,R)|^{\lambda} .$$

This completes the proof of the proposition. □

Remark 12. Proposition 5.2 was proved by García-Cuerva [14] when $\lambda = 0$ and T is a Calderón-Zygmund operator.

Observe that the better behavior of the kernel of T as λ increases, implies that we do not need to subtract any moments in the definition of $CMO^{q,\lambda}$.

We have the following result for pseudo-differential operators.

Corollary 5.3

Let $T \in L^m_{\rho,\delta}$ be a pseudo-differential operator with $\rho > 0$. Assume that T is bounded on L^q for some $1 < q < \infty$. Then, given $\lambda \in \mathbb{R}$, T maps continuously $B^{q,\lambda}$ into $CMO^{q,\lambda}$.

Proof. According to Proposition 5.2, it suffices to show that the kernel $k(x,y)$ of T satisfies condition (5.1).

Let

$$Tf(x) = \int e^{2\pi i x\cdot\xi} p(x,\xi)\widehat{f}(\xi)d\xi,$$

where $p \in S^m_{\rho,\delta}$, $f \in \mathcal{S}$.

If $\eta \in C_0^\infty(\mathbb{R}^n)$ is a usual cut-off function, we can write

$$Tf(x) = \lim_{\varepsilon\to 0} \int\int e^{2\pi i(x-y)\cdot\xi} p(x,\xi)\eta(\varepsilon\xi)f(y)dyd\xi.$$

Thus, the kernel $k(x,y)$ of T is given by

$$k(x,y) = \lim_{\varepsilon\to 0^+} \int e^{-2\pi i(x-y)\cdot\xi} p(x,\xi)\eta(\varepsilon\xi)d\xi.$$

It is a classical result (see [19], p. 140) that $k(x,y)$ is a smooth function outside the diagonal, if $\rho > 0$. Moreover, each derivative $\partial_x^\alpha \partial_y^\beta k(x,y)$ is rapidly decreasing at infinity, if $\rho > 0$. This can be seen using the substitution

$$e^{-2\pi i(x-y)\cdot\xi} = \left(-4\pi^2 |x-y|^2\right)^l \triangle_\xi^l \, e^{-2\pi i(x-y)\cdot\xi}$$

with l sufficiently large, integrating by parts, and taking the limit as $\varepsilon \to 0^+$.

Thus,

$$\left|\partial_x^\alpha \partial_y^\beta k(x,y)\right| \leq \frac{C_{\alpha\beta l}}{|x-y|^{2l}}\,. \tag{5.3}$$

We observe that in condition (5.1) we can assume that $|x-y|>2$. Moreover,

$$k(x,y)-k(0,y)=\int_0^1 \left(\nabla_x k\right)(tx,y)\cdot x dt$$

and hence

$$|k(x,y)-k(0,y)| \leq C_l \int_0^1 \frac{|x|}{|tx-y|^{2l}} dt\,.$$

Since $|x|<R$ and $|y|>2^j R>2\,|x|$ we can write

$$|k(x,y)-k(0,y)| \leq C_l \frac{R}{|y|^{2l}}\,.$$

We substitute this estimate in the left-hand side of (5.1) and we obtain

$$\begin{aligned} & C_l \left[\sup_{R\geq 1} \left(2^j R\right)^{n\left(q'-1\right)} R^{q'} \int_{2^j R}^{\infty} t^{-2lq'} t^{n-1} dt\right]^{1/q'} \\ &= C_l \sup_{R\geq 1} \left[R^{1+n-2l} \ \left(2^j\right)^{n/q \ + \ n/q' \ -2l}\right] \end{aligned} \tag{5.4}$$

if $l>n/\left(2q'\right)$.

Since $R\geq 1$, we can estimate (5.4) with

$$C_l \left(2^j\right)^{n/q \ +n-2l}=d_j\,.$$

if $l>(n+1)/2$.

Finally, given $\lambda\in\mathbb{R}$, we can fix l sufficiently large such that $\sum_j 2^{jn\lambda}\left(2^j\right)^{n/q+n-2l}<\infty$. Thus, T is a (q,λ)-central singular integral operator.

This completes the proof of the Corollary 5.3. □

Lu and Yang have obtained in [27] several continuity results for the linear commutator of a BMO function with various singular integral operators.

We now study the linear commutator of a $CMO^{q,\lambda}$ function with an operator satisfying Definition 5.1.

Proposition 5.4

Given $1 < p < q < \infty$, $\lambda \in \mathbb{R}$, *let* T *be a* (p,λ)*-central singular integral operator. Assume also that* T *is continuous on* L^q. *Let* $1/s = 1/p - 1/q$. *Then, given* $b \in CMO^{s,\mu}$, $\mu > 0$, *the linear commutator*

$$[b,T](f) = bT(f) - T(bf)$$

maps continuously $B^{q,\lambda-\mu}$ *into* $CMO^{p,\lambda}$. *Moreover,*

$$\|[b,T](f)\|_{CMO^{p,\lambda}} \le C\,\|b\|_{CMO^{s,\mu}}\,\|f\|_{B^{q,\lambda-\mu}}\ .$$

Proof. Given $f \in B^{q,\lambda-\mu}$ and given $R \ge 1$, we write

$$f = f\chi_{B(0,2R)} + f\left(1 - \chi_{B(0,2R)}\right) = f_1 + f_2\,.$$

Now, given $c \in \mathbb{R}$ fixed, we have

$$[b,T](f) = (b-c)Tf - T\big((b-c)f_1\big) - T\big((b-c)f_2\big)\,.$$

Thus

$$\left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |[b,T](f) - T((b-c)f_2)(0)|^p\,dx\right]^{1/p} \le I_1 + I_2 + I_3$$

where

$$I_1 = \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} \left|(b-c)Tf\right|^p dx\right]^{1/p},$$

$$I_2 = \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} \left|T((b-c)f_1)\right|^p dx\right]^{1/p},$$

and

$$I_3 = \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} \left|T((b-c)f_2) - T((b-c)f_2)(0)\right|^p dx\right]^{1/p}.$$

We estimate I_1 as follows:

$$I_1 \le \left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |b-c|^s\,dx\right]^{1/s}\cdot\left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |Tf|^q\,dx\right]^{1/q}$$

$$\le C\,\|f\|_{B^{q,\lambda-\mu}}\left[\frac{1}{|B(0,R)|}\int_{B(0,R)} |b-c|^s\,dx\right]^{1/s} |B(0,R)|^{\lambda-\mu}\,.$$

We know consider I_2,

$$I_2 \le C \left[\frac{1}{|B(0,R)|} \int_{B(0,2R)} |(b-c)f|^p \, dx \right]^{1/p}$$
$$\le C \left[\frac{1}{|B(0,2R)|} \int_{B(0,2R)} |b-c|^s \, dx \right]^{1/s} \cdot \left[\frac{1}{|B(0,2R)|} \int_{B(0,2R)} |f|^q \, dx \right]^{1/q}$$
$$\le C \|f\|_{B^{q,\lambda-\mu}} \left[\frac{1}{|B(0,2R)|} \int_{B(0,2R)} |b-c|^s \, dx \right]^{1/s} |B(0,R)|^{\lambda-\mu} .$$

Finally, to estimate I_3 we first obtain a pointwise estimate for $|x| < R$ of

$$|T((b-c)f_2)(x) - T((b-c)f_2)(0)|$$

by

$$\sum_{j=1}^{\infty} \int_{C_j(0,R)} |k(x,y) - k(0,y)| \, |b(y)-c| \, |f_2(y)| \, dy$$
$$\le \sum_{j=1}^{\infty} \left[\int_{C_j(0,R)} |k(x,y)-k(0,y)|^{p'} \, dy \right]^{1/p'}$$
$$\times \left[\int_{B(0,2^{j+1}R)} |b(y)-c|^p \, |f_2(y)|^p \, dy \right]^{1/p}$$
$$\le \sum_{j=1}^{\infty} d_j \left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |b(y)-c|^s \, dy \right]^{1/s}$$
$$\times \left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |f(y)|^q \, dy \right]^{1/q}$$
$$\le C \|f\|_{B^{q,\lambda-\mu}} \sum_{j=1}^{\infty} d_j \left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |b(y)-c|^s \, dy \right]^{1/s} \left| B(0,2^{j+1}R) \right|^{\lambda-\mu} .$$

We now choose $c = b_{2R}$, the average of the function b over the ball $B(0,2R)$. Thus, I_1 can be estimated by

$$I_1 \le C \|f\|_{B^{q,\lambda-\mu}} \left[\frac{2^n}{|B(0,2R)|^{1+s\mu}} \int_{B(0,2R)} |b - b_{2R}|^s \, dx \right]^{1/s} |B(0,R)|^{\lambda}$$
$$\le C \|f\|_{B^{q,\lambda-\mu}} \|b\|_{B^{s,\mu}} |B(0,R)|^{\lambda} .$$

Likewise,

$$I_2 \le C \|f\|_{B^{q,\lambda-\mu}} \|b\|_{CMO^{s,\mu}} |B(0,R)|^{\lambda} \ .$$

Finally, we have to consider I_3. We observe that

$$\begin{aligned}
&\left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |b(y) - b_{2R}|^s \, dy\right]^{1/s} \\
&\le \left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |b(y) - b_{2^{j+1}R}|^s \, dy\right]^{1/s} \\
&\qquad + \left[\frac{1}{B(0,2^{j+1}R)} \int_{B(0,2^{j+1}R)} |b_{2^{j+1}R} - b_{2R}|^s \, dy\right]^{1/s} \\
&\le C \|b\|_{B^{s,\mu}} \left|B(0,2^{j+1}R)\right|^{\mu} + |b_{2^{j+1}R} - b_{2R}| \ .
\end{aligned}$$

Moreover,

$$\begin{aligned}
|b_{2^{j+1}R} - b_{2R}| &\le \sum_{k=1}^{j} |b_{2^{k+1}R} - b_{2^k R}| \\
&\le \sum_{k=1}^{j} \frac{1}{B(0,2^k R)} \int_{B(0,2^k R)} |b(y) - b_{2^{k+1}R}| \, dy \\
&\le 2^n \sum_{k=1}^{j} \frac{1}{B(0,2^{k+1}R)} \int_{B(0,2^{k+1}R)} |b(y) - b_{2^{k+1}R}| \, dy \\
&\le 2^n \left[\sum_{k=1}^{j} \frac{1}{B(0,2^{k+1}R)} \int_{B(0,2^{k+1}R)} |b(y) - b_{2^{k+1}R}|^s \, dy\right]^{1/s} \\
&\le C R^{n\mu} \|b\|_{CMO^{s,\mu}} \sum_{k=1}^{j} 2^{n\mu k} \, .
\end{aligned}$$

Thus,

$$I_3 \le C R^{n\mu} \|b\|_{CMO^{s,\mu}} \|f\|_{B^{q,\lambda-\mu}} R^{n(\lambda-\mu)} \sum_{j=1}^{\infty} d_j 2^{jn(\lambda-\mu)} \left(\sum_{k=1}^{j} 2^{n\mu k}\right) .$$

Since $\mu > 0$, we can estimate

$$\sum_{k=1}^{j} 2^{n\mu k} \le C 2^{n\mu j}.$$

So, we can write

$$I_3 \leq C \|b\|_{CMO^{s,\mu}} \|f\|_{B^{q,\lambda-\mu}} \left(\sum_{j=1}^{\infty} d_j 2^{jn\lambda} \right) |B(0,R)|^{\lambda} .$$

This completes the proof of Proposition 5.4. □

Remark 13. When $\mu < 0$, the conclusion of Proposition 5.4 still holds, if we assume that the operator T is a $(p, \lambda - \mu)$-central singular integral operator. In this case, the proof above only needs a very minor modification. Finally, when $\mu = 0$, we need to assume that the sequence $\{d_j\}$ in Definition 5.1 satisfies the stronger condition $\sum_{j=1}^{\infty} j d_j 2^{jn\lambda} < \infty$. The proof above then applies again with a very minor modification.

When T is a pseudo differential operator in the class $L^m_{\rho,\delta}$ with $\rho > 0$, we can obtain an appropriate version of Proposition 5.4:

Corollary 5.5

Let $T \in L^m_{\rho,\delta}$ be a pseudo-differential operator with $\rho > 0$. Assume that T is continuous on L^r for $1 \leq p \leq r \leq q < \infty$. Then, given $b \in CMO^{s,\mu}$, $1/s = 1/p - 1/q$, $\mu \in \mathbb{R}$, the linear commutator $[b, T]$ maps continuously $B^{q,\lambda-\mu}$ into $CMO^{p,\lambda}$, for any $\lambda \in \mathbb{R}$.

The proof of this result follows from Proposition 5.4 using the proof of Corollary 5.3.

Lu and Yang [24] have investigated conditions under which a non-convolution singular integral operator T maps the Hardy space $HA^{p,\alpha}_q$ into itself. Lu and Yang assumed that the distribution kernel $k(x,y)$ of T satisfies the pointwise condition

$$|k(x,y) - k(x,0)| \leq C \frac{|y|^{\varepsilon}}{|x|^{n+\varepsilon}} \tag{5.5}$$

if $2|y| < |x|$, for some $0 < \varepsilon \leq 1$. Additionally, they assumed that T is continuous on L^q and satisfies a cancellation condition.

We observe that if $k(x,y)$ satisfies (5.5), then its transpose $k(y,x)$ satisfies (5.2) with $b = 1$. Based on this observation, we consider a class of operators that act continuously on $HA^q = HK^{1/q',1}_q$. For simplicity, we are restricting ourselves to this particular case. However, a minor modification in the integral condition stated below, would allow us to consider the space $HA^{p,\alpha}_q$ in general.

DEFINITION 5.6. Let $T : C_0^\infty \longrightarrow \mathcal{D}'$ be a linear and continuous operator. Given $\theta \in \mathbb{R}$, $1 < q < \infty$, we say that T is a $(q,\theta)^t$-central singular integral operator if T satisfies the following conditions.

a) T extends to a continuous operator on L^q.

b) The distribution kernel of T coincides in the complement of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ with a locally integrable function $k(x,y)$. Moreover, if $C_j(0,R) = B(0,2^{j+1}R)\backslash B(0,2^jR)$, $j = 1,2,...$, this function $k(x,y)$ satisfies the integral condition

$$\sup_{R\geq 1}\sup_{|y|<R}\left[\frac{1}{|C_j(0,R)|^{\theta/n\ +1-q}}\int_{C_j(0,R)}|k(x,y)-k(x,0)|^q\,|x|^\theta\,dx\right]^{1/q} \leq e_j \tag{5.6}$$

with $\sum_j e_j < \infty$.

c) If $f,\ g \in C_0^\infty$ and supp$(f)\cap$ supp$(g) = \emptyset$, the operator T has the integral representation

$$(Tf,g) = \int\!\!\int k(x,y)f(y)g(x)dydx.$$

Remark 14. When $\theta = 0$, (5.6) is a dual version of (5.1) with $\lambda = 0$.

The pointwise condition (5.5), or more generally,

$$|k(x,y)-k(x,0)| \leq C\frac{|y|^\varepsilon}{|x|^{n+\ \varepsilon/d}} \tag{5.7}$$

if $2\,|y| < |x|$, for some $0 < \varepsilon, d \leq 1$, implies (5.6) with $e_j = 2^{-j\varepsilon/d}$ and any $1 < q < \infty$, provided that $\theta < q\,(n+\ \varepsilon/d) - n$. Thus, appropriate pseudo-differential operators satisfy Definition 5.6 as discussed above.

On the other hand, if $\theta > n(q-1)$, (5.6) implies the central Hörmander condition

$$\sup_{R\geq 1}\sup_{|y|<R}\int_{\mathbb{R}^n\backslash B(0,2R)}|k(x,y)-k(x,0)|\,dx < \infty\,. \tag{5.8}$$

This condition suffices to define the action of T^t on $C^\infty \cap L^\infty$. Indeed, given $f \in C^\infty \cap L^\infty$, $T^t\,(f)$ can be defined as a linear and continuous functional on the subspace of C_0^∞, (see [10]),

$$\mathcal{D}_0 = \left\{ g \in C_0^\infty : \int g = 0 \right\}.$$

Moreover, the proof of Proposition 5.2 shows that if the kernel $k(x, y)$ satisfies (5.8) and the operator T is continuous in L^q for some $1 < q < \infty$, then the operator T^t is defined and continuous from L^∞ into $CMO^{q'}$.

Proposition 5.7

Let T be a $(q, \theta)^t$-central singular integral operator for some $1 < q < \infty$, $\theta > n(q-1)$. Assume that $T^t(1) = 0$ in the sense of $CMO^{q'}$. Then, the operator T maps continuously HA^q into itself.

To prove Proposition 5.7 it suffices to show that there exists a constant $C > 0$ such that given a central $(1, q)$-atom a in HA^q, (see Definition 4.1), $T(a) \in HA^q$ and $\|T(a)\|_{HA^q} \leq C$.

The standard way to go about proving this assertion, is to introduce in HA^q an appropriate notion of molecule and to show that T maps atoms into molecules.

Definition 5.8. Let $1 < q < \infty$, $\theta > n(q-1)$. A function $M \in L^q$ is called a central (q, θ)-molecule if there exists a ball $B = B(0, R)$, $R \geq 1$, and a constant $C > 0$ not depending on M or B, such that

a) $\|M\|_q \leq C\,|B|^{-1+\,1/q}$.

b) $\left\|M\,|x|^\theta\right\|_q \leq C\,|B|^{\theta/qn\;-1+\,1/q}$.

c) $\int M(x)dx = 0$.

Remark 15. Conditions a) and b) in Definition 5.8 imply that $M \in L^1$. Thus, condition c) is well defined.

Lemma 5.9

If M is a central (q, θ)-molecule, then $M \in HA^q$. Moreover $\|M\|_{HA^q} \leq C$.

The proof of Lemma 5.9 is straightforward and we will omit it.

Once we have the appropriate notion of molecule, the proof of Proposition 5.7 follows the usual pattern. We will omit the details.

By duality ([9], [14]) we immediately obtain the following consequence of Proposition 5.7.

Corollary 5.10

Let T be a (q, θ)-central singular integral operator for some $1 < q < \infty$, $\theta > n(q-1)$. Assume that $T^t(1) = 0$ in the sense of $CMO^{q'}$. Then, the operator T^t maps continuously $CMO^{q'}$ into itself.

With obvious modifications, these results apply also to the homogeneous versions of HA^q and $CMO^{q'}$.

6. Appendix: Proof of the atomic decomposition

We include the proof of Theorem 4.16 in this appendix for the sake of completeness. It bears many similarities to the proof given in [24], page 107.

In what follows, we shall assume that the wavelet ψ is C^1 and compactly supported. We let R_Q denote the Q-translate of a subset $R \subset [0,1)^n$ such that $|R| > 0$ and such that $|\psi| > \gamma > 0$ on R. By Theorem 4.4

Proposition 6.1 ([21])

Given $s \in T_d A^{p,\alpha}$, the quantities $\|\sigma(s)\|_{A^{p,\alpha}}$ and $\|\{s(Q)\}\|_{T_d A^{p,\alpha}}$ provide equivalent norms on $T_d A^{p,\alpha}$. We recall that

$$\sigma(s)(x) = \left(\sum_{x \in Q \in \mathcal{Q}} |s(Q)|^2 \frac{\chi_{R_Q}}{|Q|} \right)^{1/2} .$$

At this stage we shall set to work on the atomic decomposition of the coefficient space.

Consider

$$E_{jk} = \left\{ x \in C_k : \sigma(s)(x) > 2^j \right\} .$$

Then for each k fixed, $E_{j+1,k} \subset E_{jk}$ and a simple distribution function argument shows that

$$\sum_{j=-\infty}^{\infty} 2^{2j} |E_{jk}| \leq 2 \left\| \sigma(s)(x) \chi_{_k} \right\|_2^2 .$$

Next, fix $0 < \gamma < \eta$ where $|R_Q| \geq \eta |Q|$. Define subcollections $\mathcal{C}_{jk}$ of $\mathcal{Q}$ to be those cubes in the definition of global γ-density of E_{jk}. That is,

$$\cup_{Q \in \mathcal{C}_{jk}} Q = E^*_{jk}$$

where

$$E^*_{jk} = \left\{ x : M_{\mathcal{Q}}(\chi_{E_{jk}})(x) > \gamma \right\} .$$

$M_{\mathcal{Q}}$ denotes the dyadic maximal function.

First of all, we can see that

$$\operatorname{supp}(s) = \mathcal{C} = \cup_{jk}\mathcal{C}_{jk}.$$

Indeed, if $s(Q) \neq 0$ then, for some integer j, $|s(Q)|^2/|Q| > 2^j$. This implies that $\sigma(s)(x) > 2^j$ for each $x \in R_Q$.

Hence, assuming that $Q \subset C_k$, we have

$$|E_{jk} \cap Q| \geq \eta|Q| > \gamma|Q|.$$

On the other hand, if $Q \not\subseteq C_k$, but

$$|E_{jk} \cap Q| > \gamma|Q|$$

then

$$|C_k \cap Q| > \gamma|Q| \text{ so } |C_k| > \gamma|Q|.$$

In fact, Q must be a dyadic cube having vertex at the origin.

Finally, we note that

$$\sum_j 2^{2j}|E^*_{jk}| \leq \frac{2}{\gamma}\sum_j 2^{2j}|E_{jk}| \leq \frac{2}{\gamma}\,\|\sigma(s)\chi_{C_k}\|_2^2\,.$$

Next, denote by $\mathcal{Q}_{jk}$ the collection of dyadic maximal cubes $Q(j,k,l) \in \mathcal{C}_{jk}$. These maximal cubes form a partition of E^*_{jk}. For each j,k, the parameter l has finite index. Set $\mathcal{D}_{jk} = \mathcal{C}_{jk} \setminus \mathcal{C}_{j+1,k}$ and denote by $\mathcal{D}_{jkl}$ those $\mathcal{D}_{jk}$ cubes that are also $Q(j,k,l)$ cubes. Therefore $\mathcal{D}_{jkl}$ partitions $\mathcal{D}_{jk}$. Hence, $\operatorname{supp}(s) = \mathcal{C} = \cup_{jk}\mathcal{C}_{jk}$ is the disjoint union of these $\mathcal{D}_{jkl}$. Finally, we shall focus on those nonempty $\mathcal{D}_{jkl}$ that contain a supporting cube.

Lemma 6.2

With the notations above, if one restricts s to each of the disjoint $\mathcal{D}_{jkl}$ comprising $\operatorname{supp}(s)$ *then one obtains*

$$s = \sum\sum \mu(j,k,l)a_{jkl}(Q)$$

where the a_{jkl} are central type $(2,\alpha)$-atoms with base contained in $Q(j,k,l)$. In particular, if $\sigma(x) \in A^{p\,,\,\alpha}$ then $s \in T_d A^{p\,,\,\alpha}$.

Proof. We need to prove that

$$\sum_{Q\in\mathcal{D}_{jkl}} |s(Q)|^2 \leq \frac{1}{\eta-\gamma}\int_{Q(j,k,l)\setminus E_{j+1,k}} \sigma(x)^2dx \leq \frac{1}{\eta-\gamma}2^{(2j+2)}|Q(j,k,l)|. \tag{6.1}$$

First, the inequality on the right hand side follows immediately from the definition of $Q(j,k,l)\setminus E_{j+1,k}$.

To prove the inequality on the left hand side, we observe that if $Q\in\mathcal{D}_{jkl}$, then $Q\notin\mathcal{C}_{j+1,k}$. Thus,

$$\begin{aligned}
|Q\cap E_{j+1,k}| &< \gamma|Q| \\
|R_Q\cap E_{j+1,k}| &< \gamma|Q| \leq \frac{\gamma}{\eta}|R_Q| \\
|R_Q\setminus E_{j+1,k}| &\geq \Big(1-\frac{\gamma}{\eta}\Big)|R_Q| \geq (\eta-\gamma)|Q|.
\end{aligned}$$

But $x\in Q$ implies

$$\begin{aligned}
\sigma^2(x) &\geq \sum_{Q\in\mathcal{D}_{jkl}} |s(Q)|^2\frac{\chi_{R_Q}}{|Q|} \\
\int_{Q(j,k,l)\setminus E_{j+1,k}} \sigma(x)^2dx &\geq \sum_{Q\in\mathcal{D}_{jkl}} |s(Q)|^2\frac{|R_Q\setminus E_{j+1,k}|}{|Q|} \\
&\geq (\eta-\gamma)\sum_{Q\in\mathcal{D}_{jkl}} |s(Q)|^2.
\end{aligned}$$

This establishes (6.1) and it completes the proof of the lemma.

With the notation above, we define

$$a_{jkl}(Q) = \frac{s(Q)}{\mu(j,k,l)}\chi_{\mathcal{D}_{jkl}}(Q)$$

where the coefficients $\mu(j,k,l)$ will be chosen such that

$$\sum \mu(j,k,l)^p \leq C\,\|\sigma(s)\|_{A^{p,\alpha}}\,.$$

Assuming that this has been done, we can write s as a sum of atoms via

$$s(Q) = \sum_{j,k,l}\mu(j,k,l)\left[\frac{s(Q)}{\mu(j,k,l)}\chi_{\mathcal{D}_{jkl}}(Q)\right] = \sum_{j,k,l}\mu(j,k,l)a_{jkl}(Q).$$

In order to guarantee that $a_{jkl}(Q)$ is an atom, therefore, we set

$$\mu(j,k,l) = |Q(j,k,l)|^{\alpha}\left(\sum_{Q\in\mathcal{D}_{jkl}}|s(Q)|^2\right)^{1/2}.$$

Next we need to show that we get the desired norm estimate in terms of the coefficients.

In fact

$$
\begin{aligned}
\sum \mu(j,k,l)^p &= \sum_k \left[\sum_{j,l} \mu(j,k,l)^p \right] \\
&= \sum_k \sum_{j,l} \left[\left(\sum_{Q \in \mathcal{D}_{jkl}} |s(Q)|^2 \right)^{1/2} |Q(j,k,l)|^\alpha \right]^p \\
&= \sum_k \left[\sum_{j,l} \left(\sum_{Q \in \mathcal{D}_{jkl}} |s(Q)|^2 \right)^{p/2} |Q(j,k,l)|^{\alpha p} \right] \\
&\leq \sum_k \left[\left(\sum_{j,l} \sum_{Q \in \mathcal{D}_{jkl}} |s(Q)|^2 \right)^{p/2} \left(\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} \right)^{1/(2/p)'} \right] \\
&\leq C \sum_k \left[\left(\sum_{j,l} 2^{2j} |Q(j,k,l)| \right)^{p/2} \left(\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} \right)^{(2-p)/2} \right] \\
&\leq C \sum_k \|\sigma \chi_{\Delta_k}\|_2^p \left(\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} \right)^{(2-p)/2}.
\end{aligned}
$$

Now, in the case where $\alpha = 1/p - 1/2$ we have

$$
\left(\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} \right)^{(2-p)/2} = \left(\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} \right)^{p\alpha} \leq C |\Delta_k|^{p\alpha} = 2^{nkp\alpha}.
$$

This gives the desired estimate in that case.

In the general case where $\alpha > 1/p - 1/2$

$$
\sum_{j,l} |Q(j,k,l)|^{\alpha p (2/p)'} = \sum_{j,l} |Q(j,k,l)|^r
$$

for some $r > 1$.

Now we use the fact that $l^r \subset l^1$ just above. We have,

$$
\left(\sum_{j,l} |Q(j,k,l)|^r \right)^{1/r} \leq \sum_{j,l} |Q(j,k,l)|.
$$

Keeping track of the exponents, we obtain the same conclusion. That is to say,

$$\left(\sum_{j,l} |Q(j,k,l)|^{\alpha p(2/p)'}\right)^{(2-p)/2} \leq C|\Delta_k|^{p\alpha} = 2^{nkp\alpha}.$$

This proves the atomic decomposition. Note that in the estimate $\sum_{j,l} |Q(j,k,l)| \leq C|\Delta_k|$ we also used the fact that at most a fixed number of the "long" cubes can occur as $Q(j,k,l)$ cubes.

This completes the proof of Theorem 4.16. □

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