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The Baskakov operators for functions of two variables

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Abstract

We study the Baskakov operators $A_{m,n}$ and $B_{m,n}$ in polynomial weighted spaces of continuous functions of two variables. We give theorems on the degree of approximation, the Voronovskaya type theorem and we prove some differential properties of these operators. Some properties of Baskakov operators of functions of one variable were proved in the papers [1]-[4].

1. Preliminaries

1.1. Let $\mathbb{N} := \{1, 2, ...\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := (0, +\infty)$, $\mathbb{R}_0 := \mathbb{R}_+ \cup \{0\}$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_0^2 := \mathbb{R}_0 \times \mathbb{R}_0$. Similarly as in [1], for a fixed $p \in \mathbb{N}_0$, we define the function w_p on $\mathbb{R}_0$ by

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \qquad \text{if} \qquad p \geq 1\,. \tag{1}$$

Next, for fixed $p, q \in \mathbb{N}_0$, we define the weighted function $w_{p,q}$ on $\mathbb{R}_0^2$ by

$$w_{p,q}(x, y) := w_p(x)\, w_q(y)\,, \tag{2}$$

and the weighted space $C_{p,q}$ of all real-valued continuous functions f on $\mathbb{R}_0^2$ for which $w_{p,q}f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ and the norm is given by the formula

$$\|f\|_{p,q} \equiv \|f\,(\cdot,\cdot)\,\| := \sup_{(x,y)\in\mathbb{R}_0^2} w_{p,q}(x, y)\,|f(x, y)|\,. \tag{3}$$

For $f \in C_{p,q}$, we define the modulus of continuity

$$\omega(f, C_{p,q}; t, s) := \sup_{0 \le h \le t,\, 0 \le \delta \le s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}\,, \qquad t, s \ge 0\,, \tag{4}$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ for $(x,y) \in \mathbb{R}_0^2$ and $h, \delta \in \mathbb{R}_0$. Moreover, for fixed $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$, let $C_{p,q}^m$ be the set of all functions $f \in C_{p,q}$ having the partial derivatives $\frac{\partial^k f}{\partial x^s \partial y^{k-s}} \in C_{p,q}$, $k = 1, 2, ..., m$.

1.2. The Baskakov operators are defined by

$$A_n(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)\, f\left(\frac{k}{n}\right), \qquad n \in \mathbb{N}, \tag{5}$$

for functions f on $\mathbb{R}_0$, where

$$a_{n,k}(x) := \binom{n-1+k}{k} x^k (1+x)^{-n-k}, \tag{6}$$

([1], cf. [2]).

Moreover let B_n be the Baskakov-Kantorovich operators defined by

$$B_n(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x)\, n \int_{k/n}^{(k+1)/n} f(t)dt\,, \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{7}$$

for continuous functions f on $\mathbb{R}_0$.

In the present paper, we shall consider the Baskakov operators

$$A_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)\, a_{n,k}(y)\, f\left(\frac{j}{m}, \frac{k}{n}\right), \tag{8}$$

$$B_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)\, a_{n,k}(y)\, m\, n \int_{j/m}^{(j+1)/m} dt \int_{k/n}^{(k+1)/n} f(t, z)dz, \tag{9}$$

$(x,y) \in \mathbb{R}_0^2$, $m, n \in \mathbb{N}$, defined for functions $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$.

From (5)-(9), it follows that

$$A_n(1; x) = 1 = B_n(1; x) \qquad \text{for} \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{10}$$

(11) $A_{m,n}(1;x,y) = 1 = B_{m,n}(1;x,y)$ for $(x,y) \in \mathbb{R}_0^2$, $m,n \in \mathbb{N}$.

Moreover, if $f \in C_{p,q}$ and if $f(x,y) = f_1(x)f_2(y)$ for all $(x,y) \in \mathbb{R}_0^2$, then

(12)
$$\begin{aligned} A_{m,n}(f(t,z);x,y) &= A_m(f_1(t);x)\, A_n(f_2(z);y), \\ B_{m,n}(f(t,z);x,y) &= B_m(f_1(t);x)\, B_n(f_2(z);y), \end{aligned}$$

for $(x,y) \in \mathbb{R}_0^2$ and $m,n \in \mathbb{N}$.

In Section 2 we shall give some auxiliary results. In Section 3 we shall prove the main theorems.

In this paper we shall denote by $M_k(a,b)$, $k = 1,2,...$, the suitable positive constants depending only on indicated parameters a, b.

2. Auxiliary results

2.1. From (5)-(7) we derive the following three lemmas, given for A_n in [1]. Let $\varphi(x) := x(1+x)$ for $x \in \mathbb{R}_0$.

Lemma 1

For all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$

$$A_n(t-x;x) = 0, \qquad B_n(t-x;x) = \frac{1}{2n},$$
$$A_n\big((t-x)^2;x\big) = \frac{\varphi(x)}{n}, \qquad B_n\big((t-x)^2;x\big) = \frac{\varphi(x)}{n} + \frac{1}{3n^2}.$$

Lemma 2

Let $L_n \in \{A_n, B_n\}$, i.e. $L_n = A_n$ for all $n \in \mathbb{N}$ or $L_n = B_n$ for all $n \in \mathbb{N}$. Then for every fixed $x_0 \in \mathbb{R}_0$ there exists a positive constant $M_1(x_0)$ such that for $n \in \mathbb{N}$, $L_n\big((t-x_0)^4;x_0\big) \le M_1(x_0)\, n^{-2}$.

Lemma 3

Let $L_n \in \{A_n, B_n\}$. Then for every $p \in \mathbb{N}_0$ there exist positive constants $M_k(p)$, $k = 2,3$, such that

$$w_p(x)\, L_n\left(\frac{1}{w_p(t)};x\right) \le M_2(p),$$
$$w_p(x)\, A_n\left(\frac{(t-x)^2}{w_p(t)};x\right) \le M_3(p)\,\frac{\varphi(x)}{n},$$
$$w_p(x)\, B_n\left(\frac{(t-x)^2}{w_p(t)};x\right) \le M_3(p)\,\frac{\varphi(x)+1}{n},$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

2.2. Applying the above results, we shall prove two lemmas.

Lemma 4

For every $p \in \mathbb{N}_0$ there exists a positive constant $M_4(p)$ such that

$$w_p(x) \sum_{k=0}^{\infty} \left| \frac{d}{dx} a_{n,k}(x) \right| \left(w_p\left(\frac{k}{n}\right)\right)^{-1} \leq M_4(p)\, n, \tag{13}$$

$$w_p(x) \sum_{k=0}^{\infty} \left| \frac{d}{dx} a_{n,k}(x) \right| n \int_{k/n}^{(k+1)/n} \frac{dt}{w_p(t)} \leq M_4(p)\, n, \tag{14}$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. We shall prove only (13) because the proof of (14) is analogous. From (1) and (6) it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} \left|a'_{n,k}(x)\right| \left(w_p\left(\frac{k}{n}\right)\right)^{-1} &\leq n(1+x)^{-1} \left\{ \sum_{k=0}^{\infty} a_{n,k}(x) \left(w_p\left(\frac{k}{n}\right)\right)^{-1} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} a_{n+1,k}(x) \left(w_p\left(\frac{k+1}{n}\right)\right)^{-1} \right\} \\ &\leq n \left\{ A_n\left(\frac{1}{w_p(t)};x\right) + 4^p A_{n+1}\left(\frac{1}{w_p(t)};x\right)\right\},\end{aligned}$$

which implies (13) by Lemma 3. □

Lemma 5

Let $L_{m,n} \in \{A_{m,n}, B_{m,n}\}$. Then for every $p, q \in \mathbb{N}_0$, there exist two positive constants $M_i(p,q)$, $i = 5, 6$, such that

$$\left\| L_{m,n}\left(\frac{1}{w_{p,q}(t,z)};\cdot,\cdot\right)\right\|_{p,q} \leq M_5(p,q), \qquad m, n \in \mathbb{N}, \tag{15}$$

and for every $f \in C_{p,q}$ and for all $m, n \in \mathbb{N}$

$$\|L_{m,n}(f;\cdot,\cdot)\|_{p,q} \leq M_5(p,q)\, \|f\|_{p,q}, \tag{16}$$

$$\left\| \frac{\partial}{\partial x} L_{m,n}(f;x,y) \right\|_{p,q} \leq M_6(p,q)\, m\, \|f\|_{p,q}, \tag{17}$$

$$\left\| \frac{\partial}{\partial y} L_{m,n}(f;x,y) \right\|_{p,q} \leq M_6(p,q)\, n\, \|f\|_{p,q}. \tag{18}$$

Hence, $L_{m,n}$ is a linear positive operator from the space $C_{p,q}$ into $C^1_{p,q}$.

Proof. Let $L_{m,n} = A_{m,n}$. From (1), (2) and (12) we get

$$w_{p,q}(x,y)A_{m,n}\left(\frac{1}{w_{p,q}(t,z)};x,y\right) = \left(w_p(x)A_m\left(\frac{1}{w_p(t)};x\right)\right)\left(w_q(y)\,A_n\left(\frac{1}{w_q(z)};y\right)\right)$$

for all $(x,y) \in \mathbb{R}_0^2$ and $m,n \in \mathbb{N}$. Applying Lemma 3 and by (3), we obtain (15).

For $f \in C_{p,q}$ we get by (2), (3) and (8)

$$\|A_{m,n}(f;\cdot,\cdot)\|_{p,q} \le \|f\|_{p,q}\left\|A_{m,n}\left(\frac{1}{w_{p,q}(t,z)};\cdot,\cdot\right)\right\|_{p,q}, \quad m,n \in \mathbb{N},$$

which by (15) implies (16). Moreover, we get from (8)

$$\left|\frac{\partial}{\partial x}A_{m,n}(f;x,y)\right| \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{d}{dx}a_{m,j}(x)\right| a_{n,k}(y)\left|f\left(\frac{j}{m},\frac{k}{n}\right)\right|,$$

which implies, by (1)-(3) and Lemma 3 and Lemma 4,

$$\begin{aligned} &w_{p,q}(x,y)\left|\frac{\partial}{\partial x}A_{m,n}(f;x,y)\right| \\ &\le \|f\|_{p,q}\left\{w_p(x)\sum_{j=0}^{\infty}\left|a'_{m,j}(x)\right|\left(w_p\left(\frac{j}{m}\right)\right)^{-1}\right\} w_q(y)\,A_n\left(\frac{1}{w_q(z)};y\right) \\ &\le M_2(p)\,M_4(p)\,m\,\|f\|_{p,q}, \end{aligned}$$

which yields (17) for $L_{m,n} = A_{m,n}$.

The proof of (18) for $A_{m,n}$ and the proof of (15)-(18) for $B_{m,n}$ are analogous. □

3. The main results

3.1. First we shall prove two theorems on the degree of approximation of functions $f \in C_{p,q}$ by $A_{m,n}$ and $B_{m,n}$. We shall denote by $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ or f'_x, f''_{xx} the partial derivatives of f.

Theorem 1

Suppose that $f \in C^1_{p,q}$ with some $p,q \in \mathbb{N}_0$. Then there exists a positive constant $M_7(p,q)$ such that for all $(x,y) \in \mathbb{R}^2_+$ and $m,n \in \mathbb{N}$

$$\begin{aligned} &w_{p,q}(x,y)\,|A_{m,n}(f;x,y) - f(x,y)| \\ &\le M_7(p,q)\left\{\|f'_x\|_{p,q}\sqrt{\frac{\varphi(x)}{m}} + \|f'_y\|_{p,q}\sqrt{\frac{\varphi(y)}{n}}\right\} \end{aligned} \tag{19}$$

and

$$(20)\quad \begin{aligned} &w_{p,q}(x,y)\left|B_{m,n}(f;x,y)-f(x,y)\right| \\ &\qquad \le M_7(p,q)\left\{\|f'_x\|_{p,q}\sqrt{\frac{\varphi(x)+1}{m}}+\|f'_y\|_{p,q}\sqrt{\frac{\varphi(y)+1}{n}}\right\}. \end{aligned}$$

Proof. Let $(x,y)\in\mathbb{R}^2_+$ be a fixed point. Then we have

$$f(t,z)-f(x,y)=\int_x^t f'_u(u,z)du+\int_y^z f'_v(x,v)dv \qquad \text{for} \quad (t,z)\in\mathbb{R}^2_0.$$

From this and by (11) we get

$$(21)\quad \begin{aligned} A_{m,n}(f(t,z);x,y)-f(x,y)=&A_{m,n}\left(\int_x^t f'_u(t,z)du;x,y\right) \\ &+A_{m,n}\left(\int_y^z f'_v(x,v)dv;x,y\right). \end{aligned}$$

But by (1)-(3) we have

$$\begin{aligned} \left|\int_x^t f'_u(u,z)du\right| &\le \|f'_x\|_{p,q}\left|\int_x^t \frac{du}{w_{p,q}(u,z)}\right| \\ &\le \|f'_x\|_{p,q}\left(\frac{1}{w_{p,q}(t,z)}+\frac{1}{w_{p,q}(x,z)}\right)|t-x|, \end{aligned}$$

and analogously

$$\left|\int_y^z f'_v(x,v)dv\right| \le \|f'_y\|_{p,q}\left(\frac{1}{w_{p,q}(x,z)}+\frac{1}{w_{p,q}(x,y)}\right)|z-y|.$$

Using these inequalities and by (12), we get for $m,n\in\mathbb{N}$

$$\begin{aligned} &w_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du;x,y\right)\right| \\ &\quad\le w_{p,q}(x,y)A_{m,n}\left(\left|\int_x^t f'_u(u,z)du\right|;x,y\right) \\ &\quad\le \|f'_x\|_{p,q}\,w_{p,q}(x,y)\left\{A_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)};x,y\right)+A_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)};x,y\right)\right\} \\ &\quad= \|f'_x\|_{p,q}\,w_q(y)A_n\left(\frac{1}{w_q(z)};y\right)\left\{w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};x\right)+A_m\left(|t-x|;x\right)\right\}, \end{aligned}$$

and analogously

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_y^z f'_v(x,v)dv;x,y\right)\right|$$
$$\le \|f'_y\|_{p,q}\left\{w_q(y)A_n\left(\frac{|z-y|}{w_q(z)};y\right)+A_n\left(|z-y|;y\right)\right\}.$$

By the Hölder inequality, (10), Lemma 1 and Lemma 3, we get for $m\in\mathbb{N}$

$$A_m\left(|t-x|;x\right)\le\left\{A_m((t-x)^2;x)A_m(1;x)\right\}^{1/2}\le\sqrt{\frac{\varphi(x)}{m}},$$

and

$$w_p(x)\,A_m\left(\frac{|t-x|}{w_p(t)};x\right)$$
$$\le w_p(x)\left\{A_m\left(\frac{(t-x)^2}{w_p(t)};x\right)A_m\left(\frac{1}{w_p(t)};x\right)\right\}^{1/2}$$
$$\le M_8(p)\sqrt{\frac{\varphi(x)}{m}}.$$

Analogously, for $n\in\mathbb{N}$, we obtain

$$A_n\left(|z-y|;y\right)\le\sqrt{\frac{\varphi(y)}{n}},$$

$$w_q(y)A_n\left(\frac{|z-y|}{w_q(z)};y\right)\le M_9(q)\sqrt{\frac{\varphi(y)}{n}}.$$

Combining these, we derive from (21)

$$w_{p,q}\left|A_{m,n}(f(t,z);x,y)-f(x,y)\right|\le M_{10}(p,q)\left\{\|f'_x\|_{p,q}\sqrt{\frac{\varphi(x)}{m}}+\left\|f'_y\right\|_{p,q}\sqrt{\frac{\varphi(y)}{n}}\right\}$$

for all $m,n\in\mathbb{N}$. Thus the proof of (19) is completed. The proof of (20) is analogous. □

Theorem 2

Suppose that $f\in C_{p,q}$ with some $p,q\in\mathbb{N}_0$. Then there exists a positive constant $M_{11}(p,q)$ such that

$$(22)\quad w_{p,q}(x,y)\left|A_{m,n}(f;x,y)-f(x,y)\right|\le M_{11}(p,q)\,\omega\left(f,C_{p,q};\sqrt{\frac{\varphi(x)}{m}},\sqrt{\frac{\varphi(y)}{n}}\right),$$

$$\begin{aligned} &w_{p,q}(x,y)\left|B_{m,n}(f;x,y)-f(x,y)\right| \\ &\qquad \le M_{11}(p,q)\,\omega\left(f,C_{p,q};\sqrt{\frac{\varphi(x)+1}{m}},\sqrt{\frac{\varphi(y)+1}{n}}\right) \end{aligned} \tag{23}$$

for all $(x,y)\in\mathbb{R}_+^2$ *and* $m,n\in\mathbb{N}$.

Proof. Let $f_{h,\delta}$ be the Steklov function of $f\in C_{p,q}$, defined by the formula

$$f_{h,\delta}(x,y):=\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv, \tag{24}$$

$(x,y)\in\mathbb{R}_0^2$ and $h,\delta\in\mathbb{R}_+$. From (24) it follows that

$$f_{h,\delta}(x,y)-f(x,y)=\frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,v}f(x,y)dv,$$

$$\begin{aligned} \frac{\partial}{\partial x}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^\delta \Delta_{h,0}f(x,y+v)dv \\ &= \frac{1}{h\delta}\int_0^\delta \left(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y)\right)dv, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^h \Delta_{0,\delta}f(x+u,y)du \\ &= \frac{1}{h\delta}\int_0^h \left(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y)\right)du. \end{aligned}$$

Thus, by (3), (4) we get

$$\left\|f_{h,\delta}-f\right\|_{p,q}\le\omega\left(f,C_{p,q};h,\delta\right), \tag{25}$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{p,q}\le 2h^{-1}\omega\left(f,C_{p,q};h,\delta\right), \tag{26}$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{p,q}\le 2\delta^{-1}\omega\left(f,C_{p,q};h,\delta\right), \tag{27}$$

for all $h, \delta > 0$. Hence, we can write

$$\begin{aligned} w_{p,q}(x,y)\,&|A_{m,n}(f(t,z);x,y) - f(x,y)| \\ &\le w_{p,q}(x,y)\,\{|A_{m,n}\,(f(t,z) - f_{h,\delta}(t,z);x,y)| \\ &\quad + |A_{m,n}\,(f_{h,\delta}(t,z);x,y) - f_{h,\delta}(x,y)| \\ &\quad + |f_{h,\delta}(x,y) - f(x,y)|\} := S_1 + S_2 + S_3. \end{aligned} \tag{28}$$

By (3), (16) and (25) it follows that

$$\begin{aligned} S_1 \le \|A_{m,n}\,(f - f_{h,\delta};\cdot,\cdot)\|_{p,q} &\le M_5(p,q)\,\|f - f_{h,\delta}\|_{p,q} \\ &\le M_5(p,q)\,\omega\,(f, C_{p,q}; h, \delta)\,, \end{aligned}$$

$$S_3 \le \omega\,(f, C_{p,q}; h, \delta)\,.$$

Applying Theorem 1 and (26) and (27), we get

$$\begin{aligned} S_2 &\le M_7(p,q)\left\{\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{p,q}\sqrt{\frac{\varphi(x)}{m}} + \left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{p,q}\sqrt{\frac{\varphi(y)}{n}}\right\} \\ &\le 2M_7(p,q)\omega\,(f, C_{p,q}; h, \delta)\left\{h^{-1}\sqrt{\frac{\varphi(x)}{m}} + \delta^{-1}\sqrt{\frac{\varphi(y)}{n}}\right\}. \end{aligned}$$

Consequently, we derive from (28)

$$\begin{aligned} w_{p,q}(x,y)\,&|A_{m,n}(f;x,y) - f(x,y)| \\ &\le M_{12}(p,q)\omega\,(f, C_{p,q}; h, \delta)\left\{1 + h^{-1}\sqrt{\frac{\varphi(x)}{m}} + \delta^{-1}\sqrt{\frac{\varphi(y)}{n}}\right\} \end{aligned} \tag{29}$$

for all $(x,y) \in \mathbb{R}^2$, $m, n \in \mathbb{N}$ and $h, \delta > 0$. Now, setting $h = \sqrt{\frac{\varphi(x)}{m}}$ and $\delta = \sqrt{\frac{\varphi(y)}{n}}$ to (29), we immediately obtain (22).

The proof of (23) is identical. □

Theorem 2 implies the following

Corollary 1

Let $L_{m,n} \in \{A_{m,n}, B_{m,n}\}$ and let $f \in C_{p,q}$ with some $p, q \in \mathbb{N}_0$. Then for every $(x,y) \in \mathbb{R}^2_+$,

$$\lim_{m,n\to\infty} L_{m,n}(f;x,y) = f(x,y). \tag{30}$$

Moreover, the assertion (30) holds uniformly on every rectangle $0 \le x \le a$, $0 \le y \le b$.

3.2. In this part we shall prove the Voronovskaya type theorem.

Theorem 3

Suppose that $f \in C^2_{p,q}$ *with some* $p, q \in \mathbb{N}_0$. *Then for every* $(x, y) \in \mathbb{R}^2_+$

$$\lim_{n\to\infty} n \left\{A_{n,n}(f; x, y) - f(x, y)\right\} = \frac{\varphi(x)}{2} f''_{xx}(x, y) + \frac{\varphi(y)}{2} f''_{yy}(x, y), \tag{31}$$

and

$$\begin{aligned}&\lim_{n\to\infty} n \left\{B_{n,n}(f; x, y) - f(x, y)\right\} \\ &\qquad = \frac{1}{2}\left\{f'_x(x, y) + f'_y(x, y) + \varphi(x)f''_{xx}(x, y) + \varphi(y)f''_{yy}(x, y)\right\}.\end{aligned} \tag{32}$$

Proof. We shall prove only (31) because the proof of (32) is similar. Let (x, y) be a fixed point in $\mathbb{R}^2_+$. By the Taylor formula for $f \in C^2_{p,q}$, we have

$$\begin{aligned}f(t, z) =& f(x, y) + f'_x(x, y)(t - x) + f'_y(x, y)(z - y) \\ &+ \frac{1}{2}\left\{f''_{xx}(x, y)(t - x)^2 + 2f''_{xy}(x, y)(t - x)(z - y) + f''_{yy}(x, y)(z - y)^2\right\} \\ &+ \psi(t, z; x, y)\sqrt{(t - x)^4 + (z - y)^4},\end{aligned}$$

for $(t, z) \in \mathbb{R}^2_0$, where $\psi(\cdot, \cdot; x, y) \equiv \psi(\cdot, \cdot) \in C_{p,q}$ and $\psi(x, y) = 0$. Thus, we get

$$\begin{aligned}A_{n,n}\left(f(t, z); x, y\right) &= f(x, y) + f'_x(x, y)A_n(t - x; x) + f'_y(x, y)A_n(z - y; y) \\ &+ \frac{1}{2}\left\{f''_{xx}(x, y)A_n((t - x)^2; x) + 2f''_{xy}(x, y)A_n(t - x; x)A_n(z - y; y)\right. \\ &\left.+ f''_{yy}(x, y)A_n((z - y)^2; y)\right\} \\ &+ A_{n,n}\left(\psi(t, z)\sqrt{(t - x)^4 + (z - y)^4}; x, y\right).\end{aligned} \tag{33}$$

Applying the Hölder inequality, we have

$$\begin{aligned}&\left|A_{n,n}\left(\psi(t, z)\sqrt{(t - x)^4 + (z - y)^4}; x, y\right)\right| \\ &\le \left\{A_{n,n}\left(\psi^2(t, z); x, y\right)\right\}^{1/2}\left\{A_{n,n}\left((t - x)^4 + (z - y)^4; x, y\right)\right\}^{1/2} \\ &= \left\{A_{n,n}\left(\psi^2(t, z); x, y\right)\right\}^{1/2}\left\{A_n\left((t - x)^4; x\right) + A_n\left((z - y)^4; y\right)\right\}^{1/2}.\end{aligned} \tag{34}$$

Corollary 1 implies

$$\lim_{n\to\infty} A_{n,n}\left(\psi^2(t, z); x, y\right) = \psi^2(x, y) = 0. \tag{35}$$

Using (35) and Lemma 2, we obtain from (34)

$$\lim_{n\to\infty} n\, A_{n,n} \left(\psi(t,z)\sqrt{(t-x)^4+(z-y)^4}; x,y\right) = 0. \tag{36}$$

Using (36) and Lemma 1, we derive (31) from (33). Thus the proof is completed. □

3.3. Now we shall prove some analogue of (30) for derivatives of $L_{n,n}$.

Theorem 4

Let $L_{n,n} \in \{A_{n,n}, B_{n,n}\}$ and let $f \in C^1_{p,q}$ with some $p,q \in \mathbb{N}_0$. Then for every $(x,y) \in \mathbb{R}^2_+$

$$\lim_{n\to\infty} \frac{\partial}{\partial x} L_{n,n}(f;x,y) = \frac{\partial f}{\partial x}(x,y), \tag{37}$$

$$\lim_{n\to\infty} \frac{\partial}{\partial y} L_{n,n}(f;x,y) = \frac{\partial f}{\partial y}(x,y). \tag{38}$$

Proof. We shall prove only (37) because the proof of (38) is identical.

a) Let $(x,y) \in \mathbb{R}^2_+$ be a fixed point. From (8) and (6) it follows that

$$\begin{aligned}
\frac{\partial}{\partial x} A_{n,n}(f;x,y) &= -n(1+x)^{-1} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y)\, f\left(\frac{j}{n},\frac{k}{n}\right) \\
&+ \frac{1}{x(1+x)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y)\, j\ f\left(\frac{j}{n},\frac{k}{n}\right) \\
&= -\frac{n}{1+x} A_{n,n}\left(f(t,z);x,y\right) + \frac{n}{x(1+x)} A_{n,n}\left(tf(t,z);x,y\right), \quad n \in \mathbb{N}.
\end{aligned}$$

By the Taylor formula for $f \in C^1_{p,q}$, we have

$$\begin{aligned}
f(t,z) =& f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(z-y) \\
&+ \chi(t,z;x,y)\sqrt{(t-x)^2+(z-y)^2} \qquad \text{for} \quad (t,z) \in \mathbb{R}^2_0,
\end{aligned} \tag{39}$$

where $\chi(\cdot,\cdot;x,y) \equiv \chi(\cdot,\cdot) \in C_{p,q}$ and $\chi(x,y) = 0$. From this and by (10)-(12) we get

$$\begin{aligned}
\frac{\partial}{\partial x} A_{n,n}(f(t,z);x,y) &= -\frac{n}{1+x}\left\{f(x,y) + f'_x(x,y)\, A_n(t-x;x)\right. \\
&\left. + f'_y(x,y)\, A_n(z-y;y) + A_{n,n}\left(\chi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\right)\right\} \\
&+ \frac{n}{x(1+x)}\left\{f(x,y)A_n(t;x) + f'_x(x,y)\, A_n(t(t-x;x))\right. \\
&+ f'_y(x,y)\, A_n(t;x)A_n(z-y;y) \\
&\left. + A_{n,n}\left(t\chi(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\right)\right\}, \quad n \in \mathbb{N}.
\end{aligned}$$

But by Lemma 1 we have for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

$$\begin{aligned} &A_n(t;x) = x, \\ &A_n\big(t(t-x);x\big) = A_n\big((t-x)^2;x\big) + xA_n(t-x;x) = \frac{x(1+x)}{n}. \end{aligned} \tag{40}$$

Consequently, we have

$$\begin{aligned} &\frac{\partial}{\partial x} A_{n,n}(f(t,z);x,y) \\ &= f'_x(x,y) + \frac{n}{x(1+x)}\, A_{n,n}\left(\chi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right). \end{aligned} \tag{41}$$

By the Hölder inequality, (10), (12), Lemma 1 and Lemma 2 it follows that

$$\begin{aligned} &\left|A_{n,n}\left(\chi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right)\right| \\ &\leq \left\{A_{n,n}\left(\chi^2(t,z);x,y\right)\right\}^{1/2}\left\{A_n\left((t-x)^4;x\right) + A_n\left((t-x)^2;x\right) A_n\left((z-y)^2;y\right)\right\}^{1/2} \\ &\leq M_{13}(x,y)\, n^{-1}\left\{A_{n,n}\left(\chi^2(t;z);x,y\right)\right\}^{1/2}. \end{aligned}$$

Corollary 1 yields

$$\lim_{n\to\infty} A_{n,n}\left(\chi^2(t,z);x,y\right) = \psi^2(x,y) = 0,$$

and so

$$\lim_{n\to\infty} nA_{n,n}\left(\chi(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right) = 0. \tag{42}$$

From (41) and (42) we obtain (37) for $A_{n,n}$.

b) Let $(x,y) \in \mathbb{R}^2_+$ be a fixed point. Arguing as in the case of the operator $A_{n,n}$, we get

$$\begin{aligned} &\frac{\partial}{\partial x} B_{n,n}\left(f(t,z);x,y\right) = -\frac{n}{1+x} B_{n,n}(f(t,z);x,y) \\ &\quad + \frac{n}{x(1+x)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y) jn \int_{j/n}^{(j+1)/n}\int_{k/n}^{(k+1)/n} f(t,z)dtdz \\ &= f(x,y)\left\{-\frac{n}{1+x} + \frac{n}{x(1+x)} A_n(t;x)\right\} \\ &\quad + f'_x(x,y)\left\{-\frac{n}{1+x} B_n(t-x;x) + \frac{n}{x(1+x)} \sum_{j=0}^{\infty} a_{n,j}(x) j \int_{j/n}^{(j+1)/n} (t-x)dt\right\} \\ &\quad + f'_y(x,y)\left\{-\frac{n}{1+x} B_n(z-y;y) + \frac{n}{x(1+x)} A_n(t;x) B_n(z-y;y)\right\} \\ &\quad + \frac{n}{x(1+x)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y)\left(\frac{j}{n} - x\right) n^2\, I_{j,k,n}(x,y) \end{aligned}$$

for every $n \in \mathbb{N}$, where

$$I_{j,k,n}(x,y) = \int_{j/n}^{(j+1)/n} dt \int_{k/n}^{(k+1)/n} \chi(t,z)\sqrt{(t-x)^2+(z-y)^2}\, dz.$$

Now, using Lemma 1, (10)-(12) and (40) and by

$$\begin{aligned}\sum_{j=0}^{\infty} a_{n,j}(x)\, j \int_{j/n}^{(j+1)/n} (t-x)dt &= A_n(t(t-x);x) + \frac{1}{2n}A_n(t;x)\\ &= \frac{x(1+x)}{n} + \frac{x}{2n},\end{aligned}$$

we obtain

$$\frac{\partial}{\partial x} B_{n,n}\left(f(t,z);x,y\right) = f'_x(x,y) + \frac{n}{x(1+x)} Z_n(x,y), \tag{43}$$

where

$$Z_n(x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y)\left(\frac{j}{n}-x\right) n^2 I_{j,k,n}(x,y).$$

Moreover, by the Hölder inequality we have

$$|I_{j,k,n}(x,y)| \le n^{-1}\left\{\int_{j/n}^{(j+1)/n} dt \int_{k/n}^{(k+1)/n} \chi^2(t,z)\left[(t-x)^2+(z-y)^2\right] dz\right\}^{1/2}.$$

Hence, using the Hölder inequality and (10)-(12), we get

$$\begin{aligned}&|Z_n(x,y)| \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x)\, a_{n,k}(y)\left|\frac{j}{n}-x\right| n^2\, |I_{j,k,n}(x,y)|\\ &\le \left\{A_{n,n}\left((t-x)^2;x,y\right)\right\}^{1/2}\left\{B_{n,n}\left(\chi^2(t,z)\left[(t-x)^2+(z-y)^2\right];x,y\right)\right\}^{1/2}\\ &\le \left\{A_n\left((t-x)^2;x\right)\right\}^{1/2}\left\{B_{n,n}\left(\chi^4(t,z);x,y\right)\left(B_n((t-x)^4;x)\right.\right.\\ &\quad \left.\left. + 2B_n\left((t-x)^2;x\right) B_n\left((z-y)^2;y\right) + B_n\left((z-y)^4;y\right)\right)\right\}^{1/4},\end{aligned}$$

which, by Lemma 1 and Lemma 2, yields

$$|Z_n(x,y)| \le M_{14}(x,y)n^{-1}\left(B_{n,n}\left(\chi^4(t,z);x,y\right)\right)^{1/4}. \tag{44}$$

Corollary 1 implies

$$\lim_{n\to\infty} B_{n,n}\left(\chi^4(t,z);x,y\right) = \psi^4(x,y) = 0, \qquad \text{for} \quad (x,y)\in\mathbb{R}^2_+,$$

which used to (44) gives

$$\lim_{n\to\infty} nZ_n(x,y) = 0.$$

Consequently, we obtain from (43)

$$\lim_{n\to\infty} \frac{\partial}{\partial x} B_{n,n}(f(t,z);x,y) = f'_x(x,y).$$

This completes the proof of (37). □

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