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On the genus of $\mathbb{R}\mathbf{P}^3 \times \mathbb{S}^1$*

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Abstract

We continue the topological classification of closed connected orientable 4-manifolds according to the (regular) genus, as developed in a series of papers (see [3], [4], [5]). In particular, we prove that any closed prime orientable PL 4-manifold of genus six is topologically homeomorphic to a lens-fiber bundle over the 1-sphere. There are good reasons to conjecture that the genus six characterizes the topological product $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$ of the real projective 3-space by the 1-sphere among closed connected prime orientable 4-manifolds.

1. Introduction

Through the paper we shall work in the piecewise linear (PL) category [16] and represent (PL) manifolds by means of edge-colored graphs, as shown for example in [1] and [8]. We recall now the main concepts and definitions used in the paper. For more details on graph theory and on the combinatorics of colored triangulations of manifolds see for example [1], [8] and [12]. An $(n+1)$-*colored graph* is a pair (Γ, γ), where $\Gamma = (V, E)$ is a finite multigraph, regular of degree $n+1$, and $\gamma : E \to \Delta_n =$

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$\{i \in \mathbb{Z} : \quad 0 \le i \le n\}$ is a coloring of the edges of Γ, i.e. any two adjacent edges of Γ have different colors. An *n-pseudocomplex* $K(\Gamma)$ can be associated to (Γ, γ) by the following rules. We consider an n-simplex $\sigma^n(v)$ for each vertex v of Γ and label its vertices by Δ_n. If two vertices v and w are joined in Γ by a c-colored edge, then we identify the $(n-1)$-faces of the simplexes $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertex labeled by c, so that equally labeled vertices are identified. Let $\hat{c}$ denote the set $\Delta_n \backslash \{c\}$. For any subset $B \subset \Delta_n$, we set $\Gamma_B = (V, \gamma^{-1}(B))$. An $(n+1)$-colored graph (Γ, γ) is said to be a *crystallization* of a closed connected PL n-manifold M if $\Gamma_{\hat{c}}$ is connected for any color $c \in \Delta_n$ and the polyhedron $|K(\Gamma)|$, underlying $K(\Gamma)$, is (PL) homeomorphic to M, i.e. $|K(\Gamma)| \cong M$. We say that $K(\Gamma)$ is a *contracted triangulation* of M and that Γ *represents* M and every homeomorphic space. It is well-known that any closed connected PL n-manifold admits a crystallization (see for example [8]). A 2-cell embedding $f : |\Gamma| \to F$ of an $(n+1)$-colored graph (Γ, γ) into a closed connected surface F is called *regular* if there exists a cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \cdots, \epsilon_n)$ of Δ_n such that each region of f is bounded by a cycle with edges alternatively colored by ϵ_i, ϵ_{i+1} (indices mod $n+1$). The *genus* of Γ, written $g(\Gamma)$, is the minimum genus of a closed connected surface into which Γ regularly embeds. The *regular genus* $g(M)$ of a closed connected PL n-manifold M is the smallest $g(\Gamma)$ over all crystallizations Γ of M. In this paper, we construct a 5-colored graph which represents the topological product $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$. Then eliminating dipoles (see [8]) from this graph yields a crystallization of $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$, which has order 40 and genus 6. This implies that $g(\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1) \le 6$. Then we show that the regular genus of this manifold equals 6 by using some results proved in [5]. Moreover, we classify the topological structure of closed connected prime orientable 4-manifolds of genus six. Indeed, these manifolds are proved to be homeomorphic to lens-fiber bundles over the 1-sphere. If further the manifold is spin, i.e. the second Stiefel-Whitney class vanishes, then the bundle is trivial so it is homeomorphic to either $L(p,q) \times \mathbb{S}^1$, $q \ne 0$, or $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^1$. Finally we conjecture that $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$ is the unique, up to homeomorphism, closed connected prime orientable 4-manifold of genus six.

2. A crystallization of $\mathbf{L(p,q)} \times \mathbb{S}^1$

Let $L(p,q)$ be the lens space of type (p,q), where p, q are coprime integers such that $p > q \ge 1$. In this section we shall construct a simple crystallization of the topological product $L(p,q) \times \mathbb{S}^1$, whose genus is less or equal to $6p-6$. This implies that $g(\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1) \le 6$ as $\mathbb{R}\mathrm{P}^3 \cong L(2,1)$. Let $K = K(p,q)$ denote the standard contracted triangulation of the lens space $L(p,q)$, as given in [1]. We triangulate each 4-cell of $K \times I$, $I = [0, 1]$, by taking the join over a complex from an opposite vertex, in some standard way. Then we label the vertices of each 4-cell by the elements of Δ_4 in a standard way, so that all vertices of each simplex of the previous triangulation are differently labeled. It is not yet convenient to identify the top of $K \times I$ with the bottom of it because the quotient would not be a pseudocomplex. Therefore we consider a copy $K \times J$, $J = [1, 2]$, of $K \times I$, which is triangulated as the previous one and label its vertices by the elements of Δ_4 too. Then we identify the top of $K \times J$ with the bottom of $K \times I$ in order to obtain a pseudocomplex $\tilde{K} = \tilde{K}(p, q)$ triangulating the topological product $L(p, q) \times \mathbb{S}^1$. In Figure 1 we show the 5-colored graphs representing the pseudocomplexes $K \times I$ and $K \times J$ respectively, for the case $(p, q) = (2, 1)$.

We obtain now a (noncontracted) 5-colored graph $\Gamma = \Gamma(p, q)$ representing $\tilde{K}$ by using the rules discussed in Section 1. The graph Γ is isomorphic to the 1-skeleton of the dual cellular subdivision of $\tilde{K}$. Thus Γ has exactly $32p$ vertices, which are the barycenters of the n-cells of $\tilde{K}$. The coloring of Γ is obtained by assigning to each edge the color of the vertex opposite to its dual 3-cell in $\tilde{K}$. For each cyclic permutation ϵ of Δ_4, the graph Γ regularly embeds into a closed connected orientable surface of genus g, according to the following table:

ϵ	g
(01234)	$10p - 7$
(01243)	$8p - 3$
(01324)	$8p - 3$
(01342)	$5p + 1$
(01423)	$5p + 1$
(01432)	$4p + 1$
(02134)	$11p - 7$
(02143)	$8p - 3$
(02314)	$8p - 3$
(02413)	$6p + 1$
(03124)	$12p - 7$
(03214)	$11p - 7.$

We simplify now Γ by deleting some dipoles (see [8]) in order to obtain a crystallization of $L(p, q) \times \mathbb{S}^1$. Eliminating dipoles of type 1 and the induced dipoles of types 2 and 3 produces a contracted graph (crystallization) representing $L(p, q) \times \mathbb{S}^1$, as shown in Figure 2 for the case $(p, q) = (2, 1)$.

Now it is very easy to check that the genus of this crystallization equals $6p - 6$ by using the cyclic permutation $\epsilon = (01432)$. This implies that $g(L(p, q) \times \mathbb{S}^1) \le 6p - 6$.

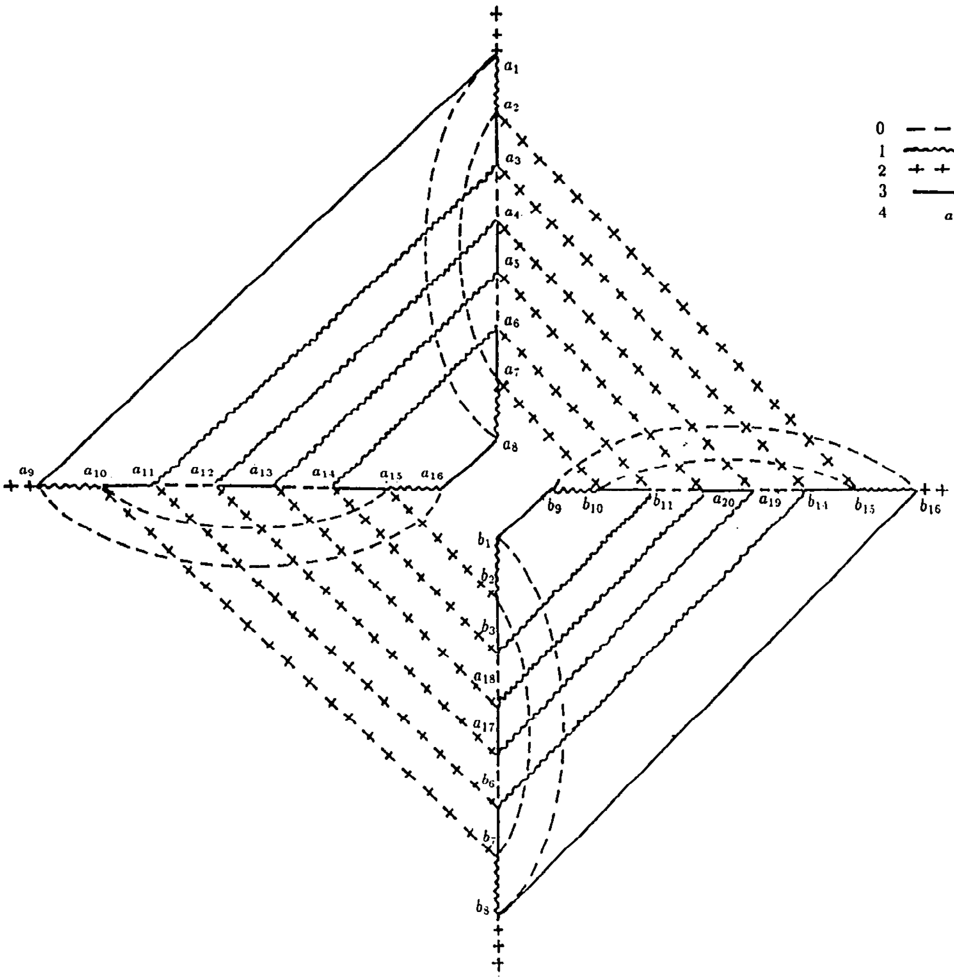

Fig. 1a: A noncontracted 5-colored graph representing $\mathbb{R}\mathrm{P}^3 \times [0,1]$

In particular, we have $g(L(2,1) \times \mathbb{S}^1) \leq 6$ as claimed. In summary, we have obtained the following result (use also [5] and [9] and Theorem 2 below).

Proposition 1

Let $L(p,q)$ be the lens space of type (p,q), $p > q \geq 1$. Then the regular genus of the topological product $L(p,q) \times \mathbb{S}^1$ satisfies the inequalities

$$6 \leq g(L(p,q) \times S^1) \leq \min\{6p-6,\, 4p+1\}.$$

Furthermore, the regular genus of $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$ is exactly 6.

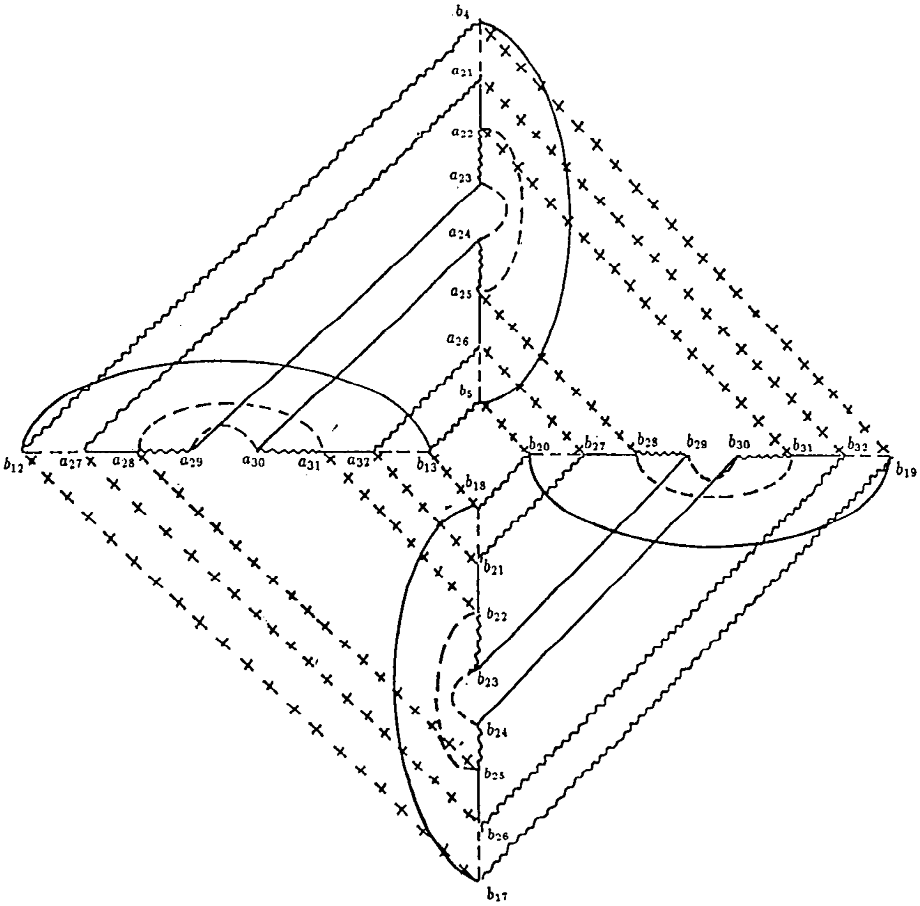

Fig. 1b: A noncontracted 5-colored graph representing $\mathbb{R}\mathrm{P}^3 \times [1, 2]$

3. Main results

In this section we shall study the topological structure of closed connected prime orientable (PL) 4-manifolds of genus 5 and 6. The closed 4-manifolds of genus $g \leq 4$ are completely classified in [3], [4], [5]. Other results concerning the classification of closed connected orientable PL 5-manifolds up to regular genus seven can be found in [2]. We state our main results.

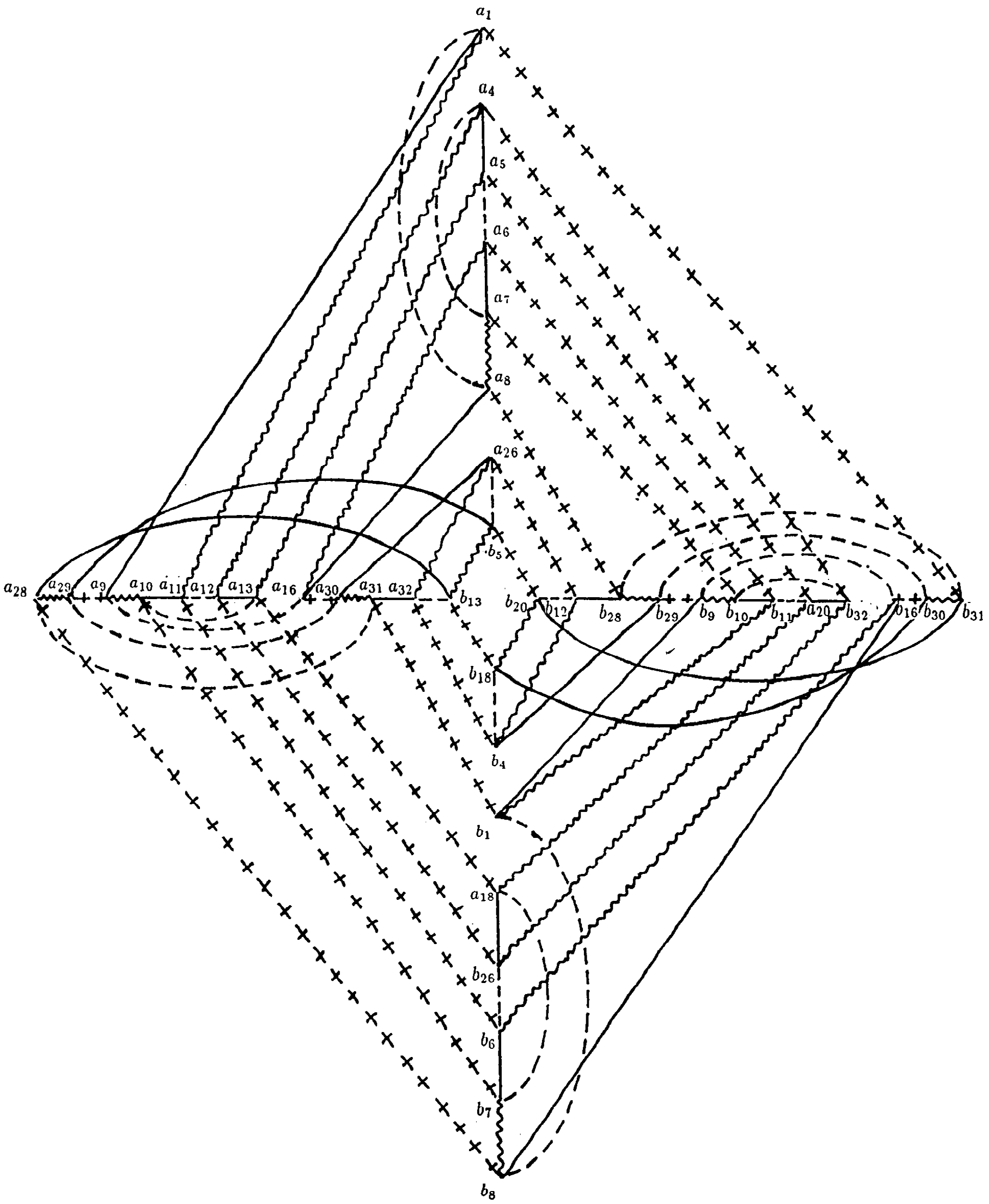

Fig. 2: A genus six crystallization of $\mathbb{R}\mathbf{P}^3 \times \mathbb{S}^1$

Theorem 2

Let M be a smooth closed connected orientable prime 4-manifold of genus 5. Then M is homeomorphic to one of the following manifolds: $\#_5(\mathbb{S}^1 \times \mathbb{S}^3)$, $\#_3(\mathbb{S}^1 \times \mathbb{S}^3)\#\mathbb{C}P^2$, $\mathbb{C}P^2\#\mathbb{C}P^2\#(\mathbb{S}^1 \times \mathbb{S}^3)$, $(\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^3)$, $(\mathbb{S}^2 \times \mathbb{S}^2)\#(\mathbb{S}^1 \times \mathbb{S}^3)$.

Theorem 3

Let M be a smooth closed connected orientable prime 4-manifold. If $g(M) = 6$, then M is homeomorphic to a lens-fiber bundle over the 1-sphere. If further M is spin, then M is homeomorphic to the topological product $L(p, q) \times \mathbb{S}^1$, $q \neq 0$, possibly including the case $L(0, 1) = \mathbb{S}^1 \times \mathbb{S}^2$.

In order to prove these theorems, we recall some constructions and results given in [3], [4], [5]. Let M be a closed connected orientable smooth (or PL) 4-manifold. Let (Γ, γ) be a crystallization of M and $\{v_i \,|\, i \in \Delta_4\}$ the vertex-set of $K = K(\Gamma)$. If $\{i, j\} = \Delta_4 \backslash \{r, s, t\}$, then $K(i, j)$ (resp. $K(r, s, t)$) represents the subcomplex of K generated by the vertices v_i, v_j (resp. v_r, v_s, v_t). By g_{rst} (resp. g_{ij}) we denote the number of edges (resp. triangles) of $K(i, j)$ (resp. $K(r, s, t)$). Note that g_{rst} and g_{ij} also represent the number of components of the subgraphs $\Gamma_{\{r,s,t\}}$ and $\Gamma_{\{i,j\}}$ respectively. If $N = N(i, j)$ and $N' = N(r, s, t)$ are regular neighborhoods of $K(i, j)$ and $K(r, s, t)$ respectively, then N and N' are complementary bordered 4-manifolds, i.e. $M = N \cup N'$ and $N \cap N' = \partial N = \partial N'$. Following [3] and [4], we can always assume that (Γ, γ) regular embeds into the closed orientable surface of genus $g = g(M)$ and of Euler characteristic $\chi(M) = g_{01} + g_{12} + g_{23} + g_{34} + g_{40} - 3p$, where p is the order of Γ divided by 2. As proved in [3], we have the following relations:

1) $g_{013} = 1 + g - g_{\hat{2}} - g_{\hat{4}}$	6) $g_{14} = g_{014} + g - g_{\hat{0}}$
2) $g_{023} = 1 + g - g_{\hat{1}} - g_{\hat{4}}$	7) $g_{02} = g_{012} + g - g_{\hat{1}}$
3) $g_{024} = 1 + g - g_{\hat{1}} - g_{\hat{3}}$	8) $g_{13} = g_{123} + g - g_{\hat{2}}$
4) $g_{124} = 1 + g - g_{\hat{0}} - g_{\hat{3}}$	9) $g_{24} = g_{234} + g - g_{\hat{3}}$
5) $g_{134} = 1 + g - g_{\hat{0}} - g_{\hat{2}}$	10) $g_{03} = g_{034} + g - g_{\hat{4}}$.

Furthermore, it was also proved that $\chi(M) = 2 - 2g + \sum_i g_{\hat{i}}$, where $g_{\hat{i}}$ ($0 \leq g_{\hat{i}} \leq g$) is the genus of an orientable closed surface into which the subgraph $\Gamma_{\hat{i}}$ ($i \in \Delta_4$) regularly embeds.

Proof of Theorem 2. If $g=5$, then the sum $R=g_{013}+g_{023}+g_{024}+g_{124}+g_{134}=5+5g-2\sum_i g_{\hat{i}}$ belongs to the set $\{2h : 3\leq h\leq 15,\ h\in\mathbb{N}\}$. For $12\leq R\leq 30$, the manifolds are topologically classified in [5]. In particular, if $R=30$, then $M\cong \#_5\mathbb{S}^1\times\mathbb{S}^3$; if $R=20$, then $M\cong \#_3\mathbb{S}^1\times S^3\#\mathbb{C}P^2$. The other cases in that range give a contradiction. We are going to consider the cases $R\in\{6,\,8,\,10\}$. If $R=6$, then $\sum_i g_{\hat{i}}=12$ and $\chi(M)=4$. Because at least one of the g_{ijk}'s in R equals 1, the 4-manifold M is simply-connected, hence $\chi(M)=4$ implies that $\beta_2(M)=2$. Here $\beta_i(M)$ denotes the i-th Betti number of M. Now we consider the intersection form λ_M as a pairing $H^2(M)\otimes H^2(M)\to\mathbb{Z}$ so defined: $\lambda_M(x,y)=< x\bigcup y,[M]>$, where $\cup$ and $[M]$ denote the cup product and the fundamental class of M respectively. By Donaldson's theorems and Freedman's classification of simply-connected 4-manifolds (see for example [10], [11], [15]), we may have only the following cases:

1) If λ_M is positive (resp. negative) definite, then λ_M is isomorphic over the integers to $(1)\oplus(1)$ (resp. $(-1)\oplus(-1)$). Thus M is (TOP) homeomorphic to either $\mathbb{C}P^2\#\mathbb{C}P^2$ or $(-\mathbb{C}P^2)\#(-\mathbb{C}P^2)$ respectively.
2) If λ_M is an odd indefinite form, then λ_M is isomorphic to $(1)\oplus(-1)$, hence $M\cong\mathbb{C}P^2\#(-\mathbb{C}P^2)\cong\mathbb{S}^2\underset{\sim}{\times}\mathbb{S}^2$.
3) If λ_M is an even indefinite form, then λ_M is isomorphic to the form

$$\omega=2aE_8+b\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

where rank (ω)=$16|a|+2|b|$. Since rank (λ_M)= rank(ω)= rank$H_2(M)$=2, we obtain $a=0$ and $b=1$, i.e. $\lambda_M\cong\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Now the Freedman theorem (see [10], [11]) implies that M is (TOP) homeomorphic to $\mathbb{S}^2\times\mathbb{S}^2$. All the cases give a contradiction because the above 4-manifolds have genus 4, as shown in [5].

If $R=8$, then $\sum_i g_{\hat{i}}=11$ and $\chi(M)=3$. Because at least one of g_{ijk}'s in R equals 1, the 4-manifold M is simply-connected. The relation $\chi(M)=3$ implies that $\beta_2(M)=1$, i.e. $H_2(M)\simeq H^2(M)\simeq FH_2(M)\simeq\mathbb{Z}$. Thus $\lambda_M\cong(\pm1)$, hence $M\cong\pm\mathbb{C}P^2$. This gives a contradiction because $g(\mathbb{C}P^2)=2$, as proved in [4].

If $R=10$, then $\sum_i g_{\hat{i}}=10$ and $\chi(M)=2$. If at least one of g_{ijk}'s in R equals 1, then $\Pi_1(M)=0$, hence $\chi(M)=2$ and $\beta_2(M)=0$. Thus $H_2(M)=0$, $\lambda_M\cong 0$ and M is homeomorphic to the 4-sphere $\mathbb{S}^4$. This gives a contradiction because $g(\mathbb{S}^4)=0$.

If $g_{ijk}\geq 2$, then we obtain $g_{013}=g_{023}=g_{024}=g_{124}=g_{134}=2$. Thus $1\leq \mathrm{rank}\Pi_1(M)\leq 1=g_{013}-1$, i.e. we have either $\Pi_1(M)\cong\mathbb{Z}$ or $\Pi_1(M)\cong\mathbb{Z}_n$. If $\Pi_1(M)\cong\mathbb{Z}\cong H_1(M)$, then $\chi(M)=2$ implies that $\beta_2(M)=2$, hence $H_2(M)=\mathbb{Z}\oplus\mathbb{Z}$. By (1), $\cdots$, (5) we obtain $g_{\hat{0}}=g_{\hat{1}}=g_{\hat{2}}=g_{\hat{3}}=g_{\hat{4}}=2$. By (6), $\cdots$, (10) it follows that $g_{14} = g_{014} + 3$, $g_{02} = g_{012} + 3$, $g_{13} = g_{123} + 3$, $g_{24} = g_{234} + 3$ and $g_{03} = g_{034} + 3$. Since $g_{024} = 2$, then $K(1,3)$ is formed by two vertices joined by exactly two edges, hence $N(1,3)$ is homeomorphic to $\mathbb{S}^1 \times B^3$. Furthermore, $K(0,2)$ and $K(2,4)$ are formed by two edges each one as $g_{134} = g_{013} = 2$. Because $g_{13} = g_{123} + 3$, the pseudocomplex $K(0,2,4)$ has many triangles, but three, as there are edges in $K(0,4)$. The Mayer-Vietoris sequence of the triple (M, N, N') becomes $0 = H_3(N) \oplus H_3(N') \to H_3(M) \cong \mathbb{Z} \to H_2(\partial N) \cong \mathbb{Z} \to H_2(N) \oplus H_2(N') \to H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(\partial N) \cong \mathbb{Z} \to H_1(N) \oplus H_1(N') \to H_1(M) \cong \mathbb{Z} \to 0$. Since $H_2(N) \cong 0$ and $H_1(N) \cong \mathbb{Z}$, it follows that $H_2(N') \cong \mathbb{Z}$. Now the arguments discussed in [5] implies that M is homeomorphic to the connected sum $\mathbb{S}^1 \times \mathbb{S}^3 \# \mathbb{C}P^2 \# \mathbb{C}P^2$.

If $\Pi_1(M) \cong \mathbb{Z}_n \cong H_1(M)$, then we have $H_3(M) \cong H^1(M) \cong FH_1(M) \oplus TH_0(M) \cong 0$. Since $\chi(M) = 2$, it follows that $\beta_2(M) = 0$, hence $H_2(M) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M) \cong \mathbb{Z}_n$. The Mayer-Vietoris sequence yields $H_3(M) \cong 0 \to H_2(\partial N) \cong \mathbb{Z} \to H_2(N) \oplus H_2(N') \cong \mathbb{Z} \oplus \mathbb{Z} \to H_2(M) \cong \mathbb{Z}_n \xrightarrow{\alpha} H_1(\partial N) \cong \mathbb{Z} \to H_1(N) \oplus H_1(N') \cong \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_n \to 0$. This easily implies that $\alpha = 0$. Now the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_n \to 0$ yields $n = 1$, i.e. $\Pi_1(M) \cong \mathbb{Z}_1 \cong 0$. It follows that M is homeomorphic to $\mathbb{S}^4$, which is a contradiction. This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. If $g = 6$, then the sum $R = 5 + 5g - 2\sum_i g_{\hat{i}}$ belongs to the set $\{2h + 1 : 2 \leq h \leq 17, h \in \mathbb{N}\}$. In [5], it was shown that: if $R = 35$, then $M \cong \#_6 \mathbb{S}^1 \times \mathbb{S}^3$; if $R = 25$, then $M \cong \#_4(\mathbb{S}^1 \times \mathbb{S}^3) \# \mathbb{C}P^2$; if $15 < R < 35$ and $R \neq 25$, there is a contradiction. So we have only to examine the cases $R \in \{5, 7, 9, 11, 13, 15\}$.

If $R = 5$, then $\sum_i g_{\hat{i}} = 15$ and $\chi(M) = 5$. Because at least one of the g_{ijk}'s in R equals 1, the manifold M is simply-connected, hence $\chi(M) = 5$ implies that $\beta_2(M) = 3$. Thus we have $H_2(M) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M) = FH_2(M)$, i.e. $H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The only possible values for the addendum of R are $g_{013} = g_{023} = g_{024} = g_{124} = g_{134} = 1$. Then the relations $(1), \cdots, (5)$ imply that $g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 3$. By $(6), \cdots, (10)$ we obtain $g_{14} = g_{014} + 3$, $g_{02} = g_{012} + 3$, $g_{13} = g_{123} + 3$, $g_{24} = g_{234} + 3$ and $g_{03} = g_{034} + 3$. Since $g_{023} = 1$, the complex $K(1,4)$ consists of one edge, hence $N(1,4)$ is a 4-cell. Furthermore, $K(0,2)$ and $K(0,3)$ are formed by one edge each one as $g_{134} = g_{124} = 1$. Because $g_{14} = g_{014} + 3$, the complex $K(0,2,3)$ contains many triangles, but three, as there are edges in $K(2,3)$. The Mayer-Vietoris sequence of the triple (M, N, N') yields $0 \to H_2(N) \oplus H_2(N') \to H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_1(\partial N) \cong 0 \to H_1(N) \oplus H_1(N') \to H_1(M) \cong 0$, hence $H_1(N') \cong 0$ and $H_2(N') \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Then the arguments developed in [3], [4] and [5] imply that M is homeomorphic to the connected sum $\#_3(\pm \mathbb{C}P^2)$.

If $R = 7$, then $\sum_i g_{\hat{i}} = 14$ and $\chi(M) = 4$. Because at least one of g_{ijk}'s in R equals 1, we have $\Pi_1(M) \cong 0$, hence $\chi(M) = 4$ implies that $\beta_2(M) = 2$. Thus it follows that $H_2(M) \cong H^2(M) \cong \mathbb{Z}\oplus\mathbb{Z}$, and hence M is homeomorphic to $\#_2(\pm\mathbb{C}P^2)$ by [10], [11] and [15].

If $R = 9$, then $\sum_i g_{\hat{i}} = 13$ and $\chi(M) = 3$. Because at least one of the g_{ijk}'s in R equals 1, we have $\Pi_1(M) \cong 0$, hence $\chi(M) = 3$ implies that $\beta_2(M) = 1$. Thus we obtain $H_2(M) \cong FH_2(M) \cong \mathbb{Z}$. The addendum of R may assume the following values (up to circular permutations):

case	g_{013}	g_{023}	g_{024}	g_{124}	g_{134}
9.1	1	1	1	1	5
9.2	3	3	1	1	1
9.3	3	1	3	1	1
9.4	4	2	1	1	1
9.5	4	1	2	1	1
9.6	3	2	2	1	1
9.7	3	2	1	2	1
9.8	3	1	2	2	1
9.9	3	2	1	1	2
9.10	2	2	2	2	1

Case 9.1). We have the relations $g_{\hat{0}} = g_{\hat{1}} = g_{\hat{2}} = 1$, $g_{\hat{3}} = g_{\hat{4}} = 5$, $g_{14} = g_{014} + 5$, $g_{02} = g_{012} + 5$, $g_{13} = g_{123} + 5$, $g_{24} = g_{234} + 1$ and $g_{03} = g_{034} + 1$. Since $g_{013} = 1$, the pseudocomplex $K(2,4)$ consists of only one edge, hence $N(2,4)$ is a 4-cell. Furthermore, $K(0,3)$ and $K(1,3)$ are also formed by one edge each one as $g_{124} = g_{024} = 1$. Thus all triangles of $K(0,1,3)$ have two edges in common. Because $g_{24} = g_{234} + 1$, the complex $K(0,1,3)$ has many triangles, but one, as there are edges in $K(0,1)$. Therefore $K(0,1,3)$ collapses to a combinatorial 2-sphere formed by exactly two triangles T_1, T_2 with common boundary. Thus M is homeomorphic to $\pm\mathbb{C}P^2$ as proved in [4]. This gives a contradiction as $g(\mathbb{C}P^2) = 2$. Now one can easily verify that the other cases yield the same result.

If $R = 11$, then $\sum_i g_{\hat{i}} = 12$ and $\chi(M) = 2$. If at least one of the g_{ijk}'s in R equals 1, then we have $\Pi_1(M) \cong 0$, hence $\chi(M) = 2$ implies that $\beta_2(M) = 0$, i.e. $H_2(M) \cong FH_2(M) \cong 0$. Thus M is homeomorphic to $\mathbb{S}^4$ which is a contradiction as $g(\mathbb{S}^4) = 0$. If $g_{ijk} \geq 2$, then we have the unique case $g_{013} = g_{024} = g_{124} = g_{134} = 2$ and $g_{023} = 3$ (up to circular permutations). Thus it follows that $1 \leq \operatorname{rank}\Pi_1(M) \leq 1 = g_{ijk} - 1$, hence we have either $\Pi_1(M) \cong \mathbb{Z}$ or $\Pi_1(M) \cong \mathbb{Z}_n$. If $\Pi_1(M) \cong H_1(M) \cong \mathbb{Z}$, then $\chi(M) = 2$ implies that $\beta_2(M) = 2$, hence $H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let M' be the closed 4-manifold obtained by killing the generator of $\Pi_1(M)$. It is well-known that the intersection forms $\lambda_{M'}$ and λ_M are isomorphic (see for example[6]). Since $H_2(M') \cong H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\Pi_1(M') \cong 0$, the Freedman-Donaldson theorems imply that either $M' \cong (\pm\mathbb{C}P^2)\#(\pm\mathbb{C}P^2)$ or $M' \cong \mathbb{S}^2 \times \mathbb{S}^2$. Now it was proved in [7] that M is homeomorphic to the connected sum $M'\#(\mathbb{S}^1 \times \mathbb{S}^3)$. If $\Pi_1(M) \cong H_1(M) \cong \mathbb{Z}_n$, then we have $H_3(M) \cong H^1(M) \cong FH_1(M) \oplus TH_0(M) = 0$. Since $\chi(M) = 2$, we obtain $\beta_2(M) = 0$, that is $H_2(M) \cong H^2(M) \cong FH_2(M) \oplus TH_1(M) \cong \mathbb{Z}_n$. These facts produce a contradiction as shown in the proof of Theorem 2.

If $R = 13$, then $\sum_i g_{\hat{i}} = 11$ and $\chi(M) = 1$. If at least one of the g_{ijk}'s in R equals 1, then we have $\Pi_1(M) \cong 0$ and $H_3(M) \cong 0$. Thus it follows that $\chi(M) = 2+\beta_2(M) \geq 2 \neq 1$, which is a contradiction. Therefore $g_{ijk} \geq 2$, rank$\Pi_1(M) \leq 1$ and either $\Pi_1(M) \cong \mathbb{Z}$ or $\Pi_1(M) \cong \mathbb{Z}_n$. If $\Pi_1(M) \cong \mathbb{Z}_n$, then we obtain a contradiction as before. If $\Pi_1(M) \cong H_1(M) \cong \mathbb{Z}$, then $\chi(M) = 1$ implies that $\beta_2(M) = 1$, hence $H_2(M) \cong FH_2(M) \cong \mathbb{Z}$. The addendum of R may only assume the following values:

case	g_{013}	g_{023}	g_{024}	g_{124}	g_{134}
13.1	5	2	2	2	2
13.2	2	2	3	3	3
13.3	2	3	2	3	3
13.4	4	3	2	2	2
13.5	4	2	3	2	2

Case 13.1). We have $g_{\hat{0}} = g_{\hat{1}} = 4$, $g_{\hat{2}} = g_{\hat{3}} = g_{\hat{4}} = 1$, $g_{14} = g_{014} + 2$, $g_{02} = g_{012} + 2$, $g_{13} = g_{123} + 5$, $g_{24} = g_{234} + 5$ and $g_{03} = g_{034} + 5$. Since $g_{023} = 2$, the pseudocomplex $K(1, 4)$ consists of two edges, hence $N(1, 4)$ is homeomorphic to $\mathbb{S}^1 \times B^3$. Furthermore, $K(0, 2)$ and $K(0, 3)$ are also formed by two edges each one as $g_{134} = g_{124} = 2$. Because $g_{14} = g_{014} + 2$, the pseudocomplex $K(0, 2, 3)$ contains many triangles, but two, as there are edges in $K(2, 3)$. The Mayer-Vietoris sequence of the triple (M, N, N') yields $H_2(N') \cong \mathbb{Z}$. Thus $K(0, 2, 3)$ collapses to a combinatorial 2-sphere $\mathbb{S}^2$, formed by exactly two triangles T_1, T_2 of $K(0, 2, 3)$ with common boundary plus an edge e such that Int $e \cap \mathbb{S}^2 = \emptyset$ and $e \cap \mathbb{S}^2 = \partial e$. Following [4] we obtain that M is homeomorphic to the connected sum $(\pm\mathbb{C}P^2)\#(\mathbb{S}^1 \times \mathbb{S}^3)$ which is a contradiction as this manifold has genus 3. Now one can verify that the other cases give the same contradiction.

If $R = 15$, then $\sum_i g_{\hat{i}} = 10$ and $\chi(M) = 0$. If all the g_{ijk}'s in R are greater than 3, then $R \geq 20$, which is a contradiction. Thus at least one of the g_{ijk}'s in R is less or equal 3, hence $\beta_1(M) \leq 2$ and rank $\Pi_1(M) \leq 2$. If $\beta_1(M) = 0$, then $\chi(M) = 2 + \beta_2(M) \geq 2 \neq 0$ which contradicts the relation $\chi(M) = 0$. If $\beta_1(M) = 1$, then $FH_1(M) \cong \mathbb{Z}$ so $\chi(M) = 0$ implies $\beta_2(M) = 0$. Since there is an epimorphism $\Pi_1(M) \to \mathbb{Z}$, the fundamental group $\Pi_1(M)$ is an extension of $\mathbb{Z}$ by a normal cyclic (finite or not) subgroup $\mathbb{Z}_n$ as rank $\Pi_1(M) \leq 2$ (note that $\mathbb{Z}_0 \cong \mathbb{Z}$). But such an extension splits: a choice of element $t \in \Pi_1(M)$ which projects to a generator of

$\mathbb{Z}$ determines a right inverse to the epimorphism $\Pi_1(M) \to \mathbb{Z}$. Let $\theta \in Aut(\mathbb{Z}_n)$ determined by conjugation by t in $\Pi_1(M)$. Then $\Pi_1(M)$ is isomorphic to either the semidirect product $\mathbb{Z}_n \times_\theta \mathbb{Z}$, $n \neq 0$, or $\mathbb{Z} \times \mathbb{Z}$ as M is orientable. Thus $\Pi_1(M)$ has exactly two (resp. one) ends if $\Pi_1(M) \cong \mathbb{Z}_n \times_\theta \mathbb{Z}$, $n \neq 0$ (resp. $\Pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}$). If $\Pi_1(M)$ has two ends, then the universal covering space $\tilde{M}$ of M is homotopy equivalent to $\mathbb{S}^3$ as $\chi(M) = 0$ (see [13], Theorem 10). Let $\hat{M}$ be the n-fold covering space of M. Since $\hat{M}$ is a closed connected orientable 4-manifold with $\chi(\hat{M}) = 0$, $\Pi_1(\hat{M}) \cong \mathbb{Z}$ and $\Pi_2(\hat{M}) \cong \Pi_2(\tilde{M}) \cong 0$, it was proved in [6] that $\hat{M}$ is homotopy equivalent to $\mathbb{S}^1 \times \mathbb{S}^3$. Now the results of [7] imply that $\hat{M}$ is s-cobordant to $\mathbb{S}^1 \times \mathbb{S}^3$, and hence these manifolds are also topologically homeomorphic by [10]. Now the only possibilities for M are the finite quotients of $\mathbb{S}^1 \times \mathbb{S}^3$, i.e. M is topologically homeomorphic to a lens-fiber bundle over the 1-sphere as claimed. In particular, if $\Pi_1(M) \cong \mathbb{Z}$, then M is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ by [10]. This fact gives a contradiction as $g(\mathbb{S}^3 \times \mathbb{S}^1) = 1$ (see [3]). If $\beta_1(M) = 2$, then $\Pi_1(M) \cong H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\chi(M) = 0$ implies $\beta_2(M) = 2$. Since $\chi(M) = 0$ and $\Pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ has one end, it follows from [13] that $H^1(M; \Lambda) \cong H_3(M; \Lambda) \cong H_3(\tilde{M}) \cong 0$ and $H^2(M; \Lambda) \cong H_2(M; \Lambda) \cong H_2(\tilde{M}) \cong \Pi_2(\tilde{M}) \cong \mathbb{Z}$, where $\Lambda = \mathbb{Z}[\Pi_1]$ is the integral group ring of $\Pi_1(M)$. Thus the universal covering space $\tilde{M}$ is homotopy equivalent to the standard 2-sphere $\mathbb{S}^2$ (see [13]). Furthermore, the manifold M is homotopy equivalent to an $(\mathbb{S}^1 \times \mathbb{S}^2)$-bundle over $\mathbb{S}^1$, as shown in [13], Corollary C, p.35. Now the results of [6] and [7] imply that M is also s-cobordant to an $(\mathbb{S}^1 \times \mathbb{S}^2)$-bundle over $\mathbb{S}^1$. Since $\Pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}$ is a polycyclic group, the orientable manifold M is just homeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^1$ since any s-cobordism is topologically a product for this class of fundamental groups. Thus, if M is a prime spin closed orientable 4-manifold of genus 6, then M is topologically homeomorphic to the product $L(p,q) \times \mathbb{S}^1$, $q \neq 0$, possibly including the case $L(0,1) = \mathbb{S}^1 \times \mathbb{S}^2$. This completes the proof of Theorem 3. $\square$

Finally, we conjecture that the unique closed connected orientable prime 4-manifold of genus six is really the topological product $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$. In fact, nowadays we are not able to construct a genus six crystallization for any lens-fiber bundles over $\mathbb{S}^1$ which is different from $\mathbb{R}\mathrm{P}^3 \times \mathbb{S}^1$.

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