Collect. Math. **50**, 2 (1999), 211–220

Decomposable subbundles of polystable vector bundles on projective curves

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Received October 3, 1997. Revised March 3, 1998

Abstract

Let X be a smooth projective curve of genus $g \geq 4$. Here we show the existence for several numerical invariants $x > 0$, deg(E_i), rank$(E_i), 1 \leq i \leq x$, deg(F), rank(F) of semistable vector bundles $E_i, 1 \leq i \leq x, F$ on X such that $E := \oplus_{1 \leq i \leq x} E_i$ is a saturated subbundle of F and F/E is semistable. If X is either bielliptic or with general moduli we may find stable vector bundles $E_i, 1 \leq i \leq x$, and F with F/E stable.

Let X be a smooth projective curve of genus $g \geq 2$. Recently there was a lot of research concerning the existence of stable vector bundles, F, on X with a saturated subbundle E with both E and F/E stable and such that $\deg(F)$, $\mathrm{rank}(F)$, $\deg(E)$ and $\mathrm{rank}(E)$ are arbitrary with the only restrictions that $\mathrm{rank}(E) < \mathrm{rank}(F)$ and $\mu(E) := \deg(E)/\mathrm{rank}(E) < \mu(F) := \deg(F)/\mathrm{rank}(F)$ (Lange's conjecture). Now fix 6 integers r', r'', a', a'', s and b with $r' > 0, r'' > 0, s > r', s > r'', a'/r' < b/s$ and $a''/r'' < b/s$. Are there stable vector bundles E', E'' and F with $\deg(E') = a'$, $\deg(E'') = a''$, $\deg(F) = b$, $\mathrm{rank}(E') = r'$, $\mathrm{rank}(E'') = r''$, $\mathrm{rank}(F) = s$ and with E' and E'' saturated subbundles of F? If $r' + r'' < s$ and the answer to the previous question is affirmative it is natural to ask if there is such bundles E', E'' and F such that $E' \oplus E''$ is a saturated subbundle of F.

To state our results we will use the following notation; for every integer u set $\varepsilon(u) = 1$ if u is odd and $\varepsilon(u) := 2$ if u is even. In the first section we will prove the following result.

Theorem 0.1

Fix integers $g, x, a_i, 1 \leq i \leq x, r_i, 1 \leq i \leq x, b$ *and* s *with* $g \geq 4, x > 0, r_i > 0$ *for every* $i, s > r := \sum_{1 \leq i \leq x} r_i$*; set* $\varepsilon := \sum_{1 \leq i \leq r} \varepsilon(a_i)$ *and* $\beta := 2\varepsilon + 1$*. Assume* $(a_i + \varepsilon(a_i))/r_i < (b - \beta)/s$ *for every* i*. Let* X *be a bielliptic curve of genus* g*. There exist stable vector bundles* $E_i, 1 \leq i \leq x$*, and* F *on* X *with* $\mathrm{rank}(E_i) = r_i, \deg(E_i) = a_i, \mathrm{rank}(F) = s, \deg(F) = b$ *such that* $E := \oplus_{1 \leq i \leq x} E_i$ *is a saturated subbundle of* F *and* F/E *is stable.*

In Section 2 we will use the statement of 0.1 to obtain the following result for curves with general moduli.

Theorem 0.2

Fix integers $g, x, a_i, 1 \leq i \leq x, r_i, 1 \leq i \leq x, b$ *and* s *with* $g \geq 4, x > 0, r_i > 0$ *for every* $i, s > r := \sum_{1 \leq i \leq x} r_i$*; set* $\varepsilon := \sum_{1 \leq i \leq r} \varepsilon(a_i)$ *and* $\beta := 2\varepsilon + 1$*. Assume* $(a_i + \varepsilon(a_i))/r_i < (b - \beta)/s$ *for every* i*. Let* X *be a general smooth curve of genus* g*. There exist stable vector bundles* $E_i, 1 \leq i \leq x$*, and* F *on* X *with* $\mathrm{rank}(E_i) = r_i$*,* $\deg(E_i) = a_i$*,* $\mathrm{rank}(F) = s$*,* $\deg(F) = b$ *such that* $E := \oplus_{1 \leq i \leq x} E_i$ *is a saturated subbundle of* F *and* F/E *is stable.*

Then we will use specialization to obtain from the statement of 0.2 the following result concerning all smooth curves of genus g.

Theorem 0.3

Fix integers $g, x, a_i, 1 \leq i \leq x, r_i, 1 \leq i \leq x, b$ *and* s *with* $g \geq 4, x > 0, r_i > 0$ *for every* $i, s > r := \sum_{1 \leq i \leq x} r_i$*; set* $\varepsilon := \sum_{1 \leq i \leq r} \varepsilon(a_i)$ *and* $\beta := 2\varepsilon + 1$*. Assume* $(a_i + \varepsilon(a_i))/r_i < (b - \beta)/s$ *for every* i*. Let* X *be a smooth curve of genus* g*. There exist stable vector bundles* $E_i, 1 \leq i \leq x$*, and a semistable vector bundle* F *on* X *with* $\mathrm{rank}(E_i) = r_i$*,* $\deg(E_i) = a_i$*,* $\mathrm{rank}(F) = s$*,* $\deg(F) = b$ *such that* $E := \oplus_{1 \leq i \leq x} E_i$ *is a saturated subbundle of* F *and* F/E *is stable.*

In this paper we work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. We stress that 0.1, 0.2 and 0.3 are for every $g \geq 4$ even if the integers r and s are large and that $\beta \leq 4x + 1$, i.e. β is bounded by x independently from g, r and s.

This research was partially supported by MURST and GNSAGA of CNR (Italy) and by Max-Planck-Institute für Mathematik in Bonn.

1. Proof of Theorem 0.1

Let Y be a smooth projective curve and $P \in Y$. For any vector bundle A on Y, let A_P be the fiber of A over P. Let $\mathbf{K}_P$ be the skyscraper length 1 sheaf supported by P. Let U, V be vector bundles on Y. We will say that U is obtained from V making a negative elementary transformation supported by P and that V is obtained from U making a positive elementary transformation supported by P if there exists a surjection $f : V \to \mathbf{K}_P$ with $U \cong \mathrm{Ker}(f)$. Note that if such f exists, then $\mathrm{rank}(V) = \mathrm{rank}(U)$ and $\deg(V) = \deg(U) + 1$.

Remark 1.1. Let $\pi : Y \to Z$ a double covering of smooth irreducible projective curves with $p_a(Y) \geq 2p_a(Z) > 0$, i.e. assume that π is not étale. Let E be a stable vector bundle on Z. By [3], Lemma 2.2, the bundle $\pi^*(E)$ is stable.

Proposition 1.2

Fix integers $x, y, a_i, 1 \leq i \leq x, ri, 1 \leq i \leq x, b_j, 1 \leq j \leq y, s_j, 1 \leq j \leq y$, *with* $x > 0, y > 0, r_i > 0$ *for every* $i, s_j > 0$ *for every* j, a_i *and* b_j *even for every* i *and every* j, *and with* $a_i/r_i < b_j/s_j$ *for every* i *and every* j. *Set* $r := \sum_{1\leq i\leq x} r_i, s := \sum_{1\leq j\leq y} s_j$ *and assume* $r < s$. *Let* X *be a bielliptic curve. Then there are polystable vector bundles* $E_i, 1 \leq i \leq x$, *and* $F_j, 1 \leq j \leq y$, *on* X *with* $\mathrm{rank}(E_i) = r_i$, $\deg(E_i) = a_i$, $\mathrm{rank}(F_j) = s_j$, $\deg(F_j) = b_j$ *such that, setting* $E := \oplus_{1\leq i\leq x} E_i$ *and* $F := \oplus_{1\leq j\leq y} F_j, E$ *is a saturated subbundle of* F.

Proof. Let $\pi : X \to C$ be a double covering with C smooth elliptic curve. Fix polystable vector bundles $A_i, 1 \leq i \leq x$, and $B_j, 1 \leq j \leq y$, on C with $\deg(A_i) = a_i/2$, $\mathrm{rank}(A_i) = r_i$, $\deg(B_j) = b_j/2$ and $\mathrm{rank}(B_j) = s_j$. Set $A := \oplus_{1\leq i\leq x} A_i$ and $B := \oplus_{1\leq j\leq y} B_j$. Assume that no two among the indecomposable factors of A and B are isomorphic. There exist such bundles by Atiyah's classification of vector bundles on C ([1], Part II). By [2], Theorem 0.2, par (b), A is isomorphic to a saturated subbundle of B. Set $E_i := \pi^*(A_i)$ and $F_j := \pi^*(B_j)$. By Remark 1.1 E_i and F_j are polystable. Hence we conclude. □

Remark 1.3. Let $\pi : X \to C$ be a double covering of smooth projective curves and $\sigma \in \mathrm{Aut}(X)$ the involution determined by π. Let B be a vector bundle on X. By descent theory B is of the form $\pi^*(A)$ with A vector bundle on C if and only if the following two conditions are satisfied:

(i) B is σ-invariant:

(ii) for every ramification point Q of $\pi\sigma$ acts as the identity on the fiber B_Q of B at Q.

Note that if $B = \pi^*(A)$, then condition (ii) is satisfied by every subbundle of B. Hence every σ-invariant subbundle of $\pi^*(A)$ is of the form $\pi^*(J)$. Hence the saturation K of a σ-invariant subsheaf D of $\pi^*(B)$ has $\deg(K)$ - $\deg(D)$ even. In particular if D is not saturated, then $\deg(K) \geq \deg(D) + 2$.

Lemma 1.4

Let $\pi : X \to C$ be a double covering with C elliptic curve and X smooth curve of genus g. Fix a polystable vector bundle A on C such that no two among the indecomposable factors of A are isomorphic. Set $B := \pi^*(A)$. Fix general $P, P' \in X$. Let U be the general bundle obtained from B making a general positive elementary transformation supported by P and let V be the general bundle obtained from B making two general positive elementary transformations, one of them supported by P and the other one supported by P'. If $g \geq 3$, then U is stable. If $g \geq 4$, then V is stable.

Proof. This result was essentially proved in [5], Steps 2 and 5 of the proof of Theorem A1. The semistability of U for $g \geq 2$ was checked in [4] proof of 0.1. We want to reproduce (just with trivial modifications) the proof given in [5] for the stability of V. The proof of the stability of U is the same, just with simpler notations. We will use that P and P' are general only at the very end of the proof and for a very minor reason. In the rest of the proof we will need only that $\pi(P) \neq \pi(P')$ and that π is étale at P and at P'. Let $\sigma \in \mathrm{Aut}(X)$ be the involution associated to π. By [4], Proposition 1.6, there are polystable bundles A' and A'' on C with $\mathrm{rank}(A') = \mathrm{rank}(A'') = \mathrm{rank}(A)$, $\det(A') \cong \det(A)(\pi(P))$, $\det(A'') \cong \det(A')(\pi(P'))$, such that no two indecomposable factors of A' and A'' are isomorphic, and such that there is an inclusion of A into A', i.e. such that A' is obtained from A making a positive elementary transformation supported by $\pi(P)$, and an inclusion of A' into A'', i.e. such that A'' is obtained from A' making a positive elementary transformation supported by $\pi(P')$. Set $B' := \pi^*(A')$ and $B'' := \pi^*(A'')$. We want to prove that if $g \geq 4$ every bundle, V, on X with $B \subset V \subset B''$ and such that $\deg(V) = \deg(B) + 2$ and $\deg(V \cap B') = \deg(B) + 1$ is stable. By the openness of stability this implies 1.4. We assume that every vector bundle U on X with $B \subset U \subset B', U \neq B$ and $U \neq B'$ is stable; indeed the proof below will give verbatim this result for $g \geq 3$. By 1.2 the bundles B, B' and B'' are polystable. Set $m := \mathrm{rank}(A)$ and $a := \deg(A)$. Hence $\deg(V) = 2a + 2$. In order to obtain a contradiction we assume the existence of a proper subbundle T of V with $\mu(T) \geq (2a+2)/m$. Taking $\mathrm{rank}(T)$ minimal we may assume that T is stable. By assumption the subsheaf $B' \cap V$ of B'' is a stable vector bundle of degree $2a + 1$. Since $\sigma \in \mathrm{Aut}(X)$, the subsheaf $\sigma(B' \cap V)$ of B'' is stable.

Since $(2a+2)/m > (2a+1)/m$, we have $T \cap (B' \cap V) \neq T$ and $T \cap (\sigma(B' \cap V)) \neq T$. This implies $\deg(T \cap B) = \deg(T) - 2$ and that T is saturated in B'' at the points P and P'. Set $N := \sigma^*(V)$ (hence $B \subset N \subset B''$ and $N \cap V = B$ as subsheaves of B''). Since B'' and B' come from Y, we have $\deg(\sigma(T) \cap B) = \deg(T) - 2$ and $\sigma(T)$ is saturated in B'' at the points $\sigma(P)$ and $\sigma(P')$. Since $\mu(B'') = (2a+4)/m$ and B'' is semistable, if $\mathrm{rank}(T) < m/2$, then T is saturated in B''. First we will assume that T is saturated in B''. Let K be the saturation in B'' of $T + \sigma(T)$. By Remark 1.3 the saturation in B'' of every σ-invariant non saturated subsheaf, L, has degree $\geq \deg(L) + 2$. Hence if $T + \sigma(T)$ is not saturated, then $\deg(K) \geq \deg(T + \sigma(T)) + 2$ and by the stability of T we have $\mu(K) \geq \mu(T) + 2/(\mathrm{rank}(T + \sigma(T))$. This inequality contradicts the semistability of B'' except in the case $\mathrm{rank}(T + \sigma(T)) = m$ and $\deg(T+\sigma(T)) = 2a+2$. Since $T+\sigma(T)$ is a quotient of the polystable bundle $T \oplus \sigma(T)$, this is possible only if $\mathrm{rank}(T) = m/2$, $\deg(T) = a + 1$ and $T + \sigma(T) \cong T \oplus \sigma(T)$. Assume $T + \sigma(T)$ saturated in B''. By Remark 1.3 there is a vector bundle M on C with $\pi^*(M) \cong T + \sigma(T)$. Assume $T + \sigma(T) \cong T \oplus \sigma(T)$. Thus $\pi^*(M)$ splits. We may assume M indecomposable because $\mathrm{rank}(T)$ is minimal. Note that by Riemann - Hurwitz the branch locus $B(\pi)$ of π has degree $2g - 2 > 4$ because $g \geq 4$. By the projection formula we have $\pi^*(\mathrm{End}(\pi^*(M)) \cong \mathrm{End}(M) \oplus \mathrm{End}(M)(-B(\pi))$. Let J be a vector bundle on C and set $t(J) := 0$ if J is semistable, while if J is not semistable let $t(J) > 0$ be the maximal difference between the slopes of two direct summands of the graded bundle associated to J. Since C is an elliptic curve we have $h^0(C, \mathrm{End}(J)(-D)) = 0$ for every divisor D on Y with $\deg(D) > t(J)$. Since T is stable, $\sigma^*(T)$ is a stable bundle with $\mu(\sigma(T)) = \mu(T)$. Hence M is polystable. Since $\deg(B(\pi)) > 0 = t(M)$, we obtain that every endomorphism of $\pi^*(M)$ comes from C. Since $T + \sigma(T)$ splits and the splitting does not come from C, we obtain a contradiction. Now assume $\mathrm{rank}(T + \sigma(T)) < 2(\mathrm{rank}(T))$, i.e. $T \cap \sigma(T) \neq 0$; in particular in this case we have seen that $T + \sigma(T)$ is always saturated in B''. Since $T \cap \sigma(T)$ is the kernel of the map $T \oplus \sigma(T) \to T + \sigma(T), T \cap \sigma(T)$ is saturated in T. By Remark 1.3 we have $T \cap \sigma(T) = \pi^*(L)$ with L saturated subbundle of M. We claim that $t(M/L) < 6$. Since $\pi^*(M)$ is a quotient of the polystable bundle $T \oplus \sigma(T)$, every direct factor of M/L has slope $\geq \mu(T)/2$. In order to obtain a contradiction we assume the existence of a direct factor, D, of M/L with $\mu(D) \geq \mu(T)/2 + 6$. Write $M/L \cong D \oplus D'$. We have $\deg(M/L) \geq \mu(T) \cdot \mathrm{rank}(D)/2 + 2(\mathrm{rank}(D)) + \mu(T) \cdot \mathrm{rank}(D') \geq \mu(T) \cdot \mathrm{rank}(M/L)/2 + 6$. Thus $\deg(T + \sigma(T)) \geq \mu(T) \cdot \mathrm{rank}(T + \sigma(T)) + 6$. Since $\mathrm{rank}(T + \sigma(T)) \leq m$ and $\mu(T) \geq \mu(B'') - 2/m$, we obtain $\mu(T + \sigma(T)) > \mu(B'')$, contradicting the semistability of B''. Hence the claim is true. Since $T \neq \sigma(T)$ (as subsheaves of B'') the projection of $\pi^*(M/L)$ into its factor $T/(T \cap \sigma(T))$ does not come from an element of $H^0(C, \mathrm{End}(M/L))$. Since $g \geq 4$, we have $B(\pi) = 2g - 2 > 6 \geq t(M/L)$ and hence $h^0(X, \mathrm{End}(\pi^*(M/L))) = h^0(C, \mathrm{End}(M/L)) + h^0(C, \mathrm{End}(M/L)(-B(\pi))) = h^0(C, \mathrm{End}(M/L))$ by the projection formula. Thus we obtained a contradiction. Now we assume that $T + \sigma(T)$ is not saturated in B''. In particular we assume $\mathrm{rank}(T) = m/2, T + \sigma(T) \cong T \oplus \sigma(T)$ and that B'' is obtained from $T + \sigma(T)$ making the composition of two positive elementary transformations. Since $T + \sigma(T)$ is σ-invariant, this implies $T + \sigma(T) = \pi^*(J)$ for some bundle J on C (Remark 1.3). Furthermore, J is obtained from the semistable bundle A making 3 positive elementary transformations. Hence $t(J) \leq 3 < \mathrm{card}(B(\pi))$ and again we obtain that $H^0(X, \mathrm{End}((T + \sigma(T))/(T \cap \sigma(T))))$ cannot contain the projection onto the factor $T/(T \cap \sigma(T))$. Now we assume that T is not saturated in B''. Call T' its saturation. We have $\deg(T') > \deg(T)$ and hence $\mu(T') \geq \mu(T) + 1/\mathrm{rank}(T) \geq \mu(V) + 1/\mathrm{rank}(T) \geq (2a+2)/m + 1/\mathrm{rank}(T)$. Since B'' is semistable and $\mu(B'') = (2a+4)/m$ we obtain $\mathrm{rank}(T) \geq m/2$ and $\deg(T') = \deg(T) + 1$. Let Δ be the saturation of $T' + \sigma(T')$. By Remark 1.3 we obtain $\mu(\Delta) \geq \mu(T) + 2/\mathrm{rank}(T + \sigma(T))$ and hence we obtain a contradiction, unless $\mathrm{rank}(T + \sigma(T)) = m$ and $\deg(T') = \deg(T) + 1$. We have $\mathrm{Supp}(T'/T) \subseteq \{\sigma(P), \sigma(P')\}$ and $\deg(T') = \deg(T) +$ $\mathrm{card}(\mathrm{Supp}(T'/T))$. Hence either $\mathrm{Supp}(T'/T) = \{\sigma(P)\}$ or $\mathrm{Supp}(T'/T) = \{\sigma(P')\}$. Moving $\{P, P'\}$ in a Zariski open (hence integral) subset of the symmetric product $S^2(X)$, we would obtain $\{\sigma(P), \sigma(P')\} \subseteq \mathrm{Supp}(T'/T)$ and hence obtain a contradiction. □

The proof of Lemma 1.4 gives the following more precise result.

Lemma 1.5

Let $\pi : X \to C$ be a double covering with C elliptic curve and X smooth curve of genus g. Fix general $P, P' \in X$. Fix polystable vector bundles A, A' and A'' on C with $\mathrm{rank}(A) = \mathrm{rank}(A') = \mathrm{rank}(A'')$, $\det(A'') \cong \det(A')(\pi(P')) \cong \det(A)(\pi(P) + \pi(P''))$ and such that no two among the indecomposable factor of A, A' and A'' are isomorphic. There are inclusions of A into A' and inclusions of A' into A''. We fix an inclusion of A into A' and an inclusion of A' into A''. Set $B := \pi^*(A), B' := \pi^*(A')$ and $B'' := \pi^*(A'')$. Hence $B \subset B' \subset B''$. Let U be a vector bundle on X with $B \subset U \subset B', U \neq B$ and $U \neq B'$. If $g \geq 3$, then U is stable. Let V be a vector bundle on X with $B \subset V \subset B''$ and such that $\deg(V) = \deg(B) + 2, \deg(V \cap B') = \deg(B) + 1$. If $g \geq 4$, then V is stable.

Lemma 1.6

Fix integers z, w with $w > z > 0$. Let $\pi : X \to C$ be a double covering of smooth projective curves with C elliptic and X of genus $g \geq 4$. Fix z general points $P(j), 1 \leq j \leq z$ of X and $z+1$ polystable vector bundles $A(k), 0 \leq k \leq z$, on C with $\mathrm{rank}(A(k)) = \mathrm{rank}(A(0))$ and $\det(A(j)) \cong \det(A(j-1))(\pi(P(j)))$ for every $j, 1 \leq j \leq z$, and such that no two among the indecomposable factors of any $A(k), 0 \leq k \leq z$, are isomorphic. Set $B(k) := \pi^*(A(k))$. For every integer j with $1 \leq j \leq z$ there is an inclusion of $A(j-1)$ into $A(j)$; we fix any such inclusion and hence we fix an inclusion of $B(j-1)$ into $B(j)$. Let U be a vector bundle on X with $\mathrm{rank}(U) = \mathrm{rank}(A(0)), \deg(U) = 2\deg(A(0)) + z, U$ containing $B(0)$ and contained in $B(z)$ and such that for every integer k with $1 \leq k \leq z$ we have $\deg(B(k) \cap U)) = \deg(B(k-1) \cap U) + 1$ and $\mathrm{Supp}(B(k) \cap U / B(k-1) \cap U) = \{P(k)\}$. Let M be the general vector bundle obtained from U making w general positive elementary transformations. Then M is stable.

Proof. The existence of the inclusions $A(k-1) \subset A(k)$ was proved in [4], Proposition 1.6. Note that U is obtained from $B(0)$ making z positive elementary transformations. Hence, since stability is an open condition, to prove 1.6 we may increase z and decrease w keeping $z + w$ fixed. In this way it is sufficient to prove 1.6 in the case $w = z+1$ and $w = z+2$. Note that $B(z)$ is obtained from U making z positive elementary transformations. Hence to prove 1.6 it is sufficient to prove that the general bundle obtained from $B(z)$ making one or two general positive elementary transformations is stable. Since $g \geq 4$, this is true by 1.4. □

Proof of Theorem 0.1. Fix β general points $P(j), 1 \leq j \leq \beta$, of X. Note that $a_i - \varepsilon(a_i)$ is even. Let A_i, A_i' and $A_i'', 1 \leq i \leq x$, be polystable bundles on C with $\mathrm{rank}(A_i) = \mathrm{rank}(A_i') = \mathrm{rank}(A_i'') = r_i, \deg(A_i'') = \deg(A_i') + 1 = \deg(A_i) + 2 = (a_i - \varepsilon(a_i))/2 + 2$ and such that no two among their indecomposable factors are isomorphic. Set $B_i := \pi^*(A_i), B_i' := \pi^*(A_i')$ and $B_i'' := \pi^*(A_i'')$. By [4], Proposition 1.6, there are inclusion $A_i \subset A_i' \subset A_i''$. We fix any such inclusions and we obtain inclusions $B_i \subset B_i' \subset B_i''$. If a_i is odd, let E_i be a subsheaf of B_i' containing B_i and with $\deg(E_i) = \deg(B_i) + 1$ and with one of the points $P(j), 1 \leq j \leq \beta$, as support of E_i/B_i. If a_i is even, let E_i be a subsheaf of B_i'' containing B_i and such that $\deg(E_i) = \deg(E_i \cap B_i') + 1 = \deg(B_i) + 2$ and such that $\mathrm{Supp}(E_i/B_i)$ is given by two of the points. By Lemmas 1.3 or 1.4 we may assume that each E_i is stable. Furthermore, we may assume that $\cup \mathrm{Supp}(E_i/B_i)$ is given by the points $P(j)$ with $1 \leq j \leq \varepsilon$. Let A be a polystable vector bundle on C with $\mathrm{rank}(A) = s, \deg(M) = (b - \beta)/2$ and such that no two among the indecomposable factors of A are isomorphic. Set $D := \oplus_{1 \leq i \leq x} B_i, F :=$

$\oplus_{1\le i\le x}E_i$ and $N := \pi^*(A)$. By [4], Proposition 1.6, $\oplus_{1\le i\le x}A_i$ is a subbundle of A and hence D is a subbundle of N. F is obtained from D making ε suitable positive elementary transformations. These positive elementary transformations are induced by ε positive elementary transformations; these positive elementary transformations are not uniquely determined and are far from being general, but they exists. Call N' the corresponding bundle of degree $b-\beta+\varepsilon$ and rank s. By construction E is a saturated subbundle of N'. Let F be obtained from N' making $\beta-\varepsilon$ general positive elementary transformations. Since $\beta-\varepsilon>\varepsilon$, the bundle F is stable by Lemma 1.6. $\square$

2. Proof of Theorems 0.2 and 0.3.

In this section we will prove Theorems 0.2 and 0.3 and give a related result (see Proposition 2.1).

Proof of Theorem 0.2. We will follow the proof of [4], Theorem 0.2. Let Y be any smooth curve of genus g and $E_i, 1\le i\le r, M$ semistable bundles on Y with $\deg(E_i)=a_i, \mathrm{rank}(E_i)=r_i, \deg(M)=b-\sum_{1\le i\le r}a_i, \mathrm{rank}(M)=s-r$. Set $E := \oplus_{1\le i\le r}E_i$. Since $\mu(E_i)<\mu(M)$ for every i we have $h^0(Y,\mathrm{Hom}(M,E))=0$. Hence by Riemann - Roch $\dim(\mathrm{Ext}^1(M,E)) = rs(\mu(M)-\mu(E)+g-1)$, i.e. $\dim(\mathrm{Ext}^1(M,E))$ does not depend on the choice of X and of the semistable bundles E_i, and M. Let $\mathbf{U}\to M_g$ be the moduli scheme parameterizing direct sums of $r+1$ stable vector bundles, one of rank $s-r$ and degree $b-\sum_{1\le i\le r}a_i$, the remaining ones of rank r_i and degree a_i. By the theory of the global Ext-functor ([6] or [8]) there is a ramified covering, T, of $\mathbf{U}$ such that on T there is a "universal" family of extensions of a stable vector bundle of rank $s-r$ and degree $b-\sum_{1\le i\le x}a_i$ with a direct sum of r stable vector bundles of rank $r_i$and degree $a_i, 1\le i\le r$. If $Y\in M_g$ is bielliptic, then the general middle term of such extensions over Y is stable by Theorem 0.1. Hence we conclude by the openness of stability. By construction we have an embedding $\mathbf{i}: E\to F$ with $\mathrm{Coker}(\mathbf{i})$ locally free. We have to check that for a general triple $(E,F,\mathbf{i})$ the bundle $\mathrm{Coker}(\mathbf{i})$ is stable. Since F is stable, at least we know that $\mu_(F/E)>\mu(F)>\mu_+(E)$. Hence $h^0(X,\mathrm{Hom}(F/E,E))=0$, i.e. $\dim(\mathrm{Ext}^1(X,F/E,E))$ has the minimal possible value, $r(s-r)(\mu(F/E)-\mu(E)+g-1)$, given Riemann - Roch. Consider the set of extensions of bundles, Z, near F/E by the bundle $E := \oplus_{1\le i\le x}E_i$. Since every vector bundle on a smooth curve of genus ≥ 2 is the flat limit of a family of stable vector bundles ([9], Proposition 2.6, or [7], Corollary 2.2) we obtain an integral family of extensions of deformations of F/E by E which is parameterized by an integral variety and containing an extension

$$0 \to E \to F \to F/E \to 0$$

in which the middle term, F, is stable. Hence we conclude by the openness of stability. $\square$

Proof of Theorem 0.3. We will follow the proof of [10], Theorem 0.2. Let Y be any smooth curve of genus g and $E_i, 1 \le i \le x, M$ semistable bundles on Y with $\deg(E_i) = a_i, \mathrm{rank}(E_i) = r_i, \deg(M) = b - \sum_{1\le i\le x} a_i, \mathrm{rank}(M) = s - r$. Set $E := \oplus_{1\le i\le x} E_i$. Since $\mu(E_i) < \mu(M)$ for every i we have $h^0(Y, \mathrm{Hom}(M, E)) = 0$. Hence by Riemann - Roch $\dim(\mathrm{Ext}^1(M, E)) = r(s-r)(\mu(M) - \mu(E) + g - 1)$, i.e. $\dim(\mathrm{Ext}^1(M, E))$ does not depend on the choice of X and of the semistable bundles E_i, and M. Let $\mathbf{U} \to M_g$ be the moduli scheme parameterizing direct sums of $r + 1$ equivalence classes of semistable vector bundles, one of rank $s - r$ and degree $b - \sum_{1\le i\le x} a_i$, the remaining ones of rank r_i and degree a_i. We fix one such equivalence class over X; we may find a variety $\mathbf{V}$ with a map $\mathbf{V} \to U$ containing a neighborhood of this element and such that on $\mathbf{V}$ we have relative families of $(x + 1)$-ples of semistable vector bundles. By the theory of the global Ext-functor ([6] or [8]) there is a ramified covering, T, of $\mathbf{V}$ such that on T there is a "universal" family of extensions of a stable vector bundle of rank $s-r$ and degree $b-\sum_{1\le i\le x} a_i$ with a direct sum of r stable vector bundles of rank r_i and degree $a_i, 1 \le i \le x$. If Y has general moduli then the general middle term of such extensions over Y is stable by Theorem 0.2. The main observation is that by the properness of the relative moduli scheme of semistable bundles there are surjective quotient map which $F^* \to E^* := \oplus_{1\le i\le r} E_i^*$ for every $(F, E_1, ..., E_x) \in \mathbf{V}$ and in particular by the openness of stability and semistability for general stable bundle $E_i, 1 \le i \le x$, on X there is a semistable bundle F on X such that the general map $\mathbf{i} : E := \oplus_{1\le i\le x} E_i \to F$ is an inclusion with $\mathrm{Coker}(\mathbf{i})$ locally free. By the openness of stability and the fact that every vector bundle on a smooth curve of genus ≥ 2 is a flat limit of a flat family of stable vector bundles ([9], Proposition 2.6, or [7], Corollary 2.2) we may impose that $\mathrm{Coker}(\mathbf{i})$ is stable and locally free, as wanted. $\square$

For every $u \in \mathbf{Z}$, set $\Psi(u) := 2 - \varepsilon(u)$, i.e. set $\Psi(u) := 0$ if u is even and $\Psi(u) = 1$ if u is odd. We have the following result.

Proposition 2.1

Fix integers $g, x, a_i, 1 \leq i \leq x, r_i, 1 \leq i \leq x, b$ and s with $g \geq 4, x > 0, r_i > 0$ for every $i, s > r := \sum_{1 \leq i \leq x} r_i$; set $\Psi := \sum_{1 \leq i \leq r} \Psi(a_i)$. Assume $(a_i + \Psi(a_i))/r_i < (b - 2\Psi)/s$ for every i. Let X be smooth curve of genus g. There exist semistable vector bundles $E_i, 1 \leq i \leq x$, and F on X with rank$(E_i) = r_i, \deg(E_i) = a_i, \mathrm{rank}(F) = s, \deg(F) = b$ such that $E := \oplus_{1 \leq i \leq x} E_i$ is a saturated subbundle of F and F/E is semistable.

Proof. The proof of 0.1 and Lemmas 1.4, 1.5 and 1.6 give the case "X bielliptic" of 2.1. Then the proof of 0.2 gives the case"X with general moduli" of 2.1. Then the proof of 0.3 gives the case in which X is an arbitrary smooth curve of genus g. □

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