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Oscillation of first order neutral differential equations with positive and negative coefficients

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Abstract

We obtain some new sharp sufficient conditions for the oscillation of all solutions of the first order neutral differential equation with positive and negative coefficients of the form

$$\frac{d}{dt}\big(x(t) - R(t)x(t-r)\big) + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0$$

where $P, Q, R \in C([t_0,\infty), R^+)$, $r \in (0,\infty)$ and $\tau, \delta \in R^+$. In particular, the conditions are necessary and sufficient when the coefficients are constants. As corollaries, many known results are extended and improved in the literature.

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1. Introduction

Consider the first order neutral delay differential equations with positive and negative coefficients of the form

$$\frac{d}{dt}\big(x(t) - R(t)x(t-r)\big) + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0 \tag{1}$$

where

$$P, Q, R \in C\big([t_0,\infty), R^+\big), r \in (0,\infty) \quad \text{and} \quad \tau, \delta \in R^+, \tag{2}$$

$$\tau > \delta, \bar{P}(t) = P(t) - Q(t-\tau+\delta) \geq 0 \quad \text{and not identically zero.} \tag{3}$$

When $Q(t) \equiv 0$, Eq.(1) reduces to

$$\frac{d}{dt}\big(x(t) - R(t)x(t-r)\big) + P(t)x(t-\tau) = 0. \tag{4}$$

Let $T_0 = \max\{r, \tau, \delta\}$. By a solution of Eq.(1) we mean a function $x \in C([t_0 - T_0, \infty), R)$, for some $t_1 \geq t_0$ such that $x(t) - R(t)x(t-r)$ is continuously differentiable on $[t_1, \infty)$ and such that Eq.(1) is satisfied for $t \geq t_1$.

As is customary, a solution of Eq.(1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

The purpose of this paper is to obtain several new sharp sufficient conditions for the oscillation of all solutions of Eq.(1). This is done by using the lemmas obtained in [13, 15, 16]. In particular, these conditions are necessary and sufficient when the coefficients are constants.

2. Main results

In this section, we give some new sharp sufficient conditions for the oscillation of all solutions of Eq.(1). Before stating our main results we need the following lemmas.

Lemma 1 [15, 16]

Assume that (2) *and* (3) *hold and that*

$$R(t) + \int_{t-\tau+\delta}^{r} Q(s)ds \leq 1, \quad t \geq t_1 \geq t_0. \tag{5}$$

Let $x(t)$ be an eventually positive solution of Eq.(1), *and set*

$$y(t) = x(t) - R(t)x(t-r) - \int_{t-\tau-\delta}^{t} Q(s)x(s-\delta)ds. \tag{6}$$

Then

$$y'(t) \leq 0 \quad \text{and} \quad y(t) > 0. \tag{7}$$

Lemma 2 [6]

Assume that $\delta > 0$, $R \in C([t_0, \infty), R^+)$, $\lambda \in C([t_0 - \delta, \infty), R^+)$, *and that*

$$\lambda(t) \geq R(t) \exp\Big(\int_{t-\delta}^{t} \lambda(s) ds \Big), \quad t \geq t_0.$$

Then condition

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} R(s) ds > 0$$

implies that

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} \lambda(s) ds < \infty.$$

Theorem 1

Assume that (2), (3) *and* (5) *hold and that*

(i)

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) ds > 0. \tag{8}$$

(ii) There exists a positive continuous function $H(t)$ *such that*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} H(s) ds > 0. \tag{9}$$

(iii) either

$$\begin{aligned} \inf_{\lambda>0, t\geq T} \Big\{ & \frac{R(t-\tau)P(t)H(t-r)}{\bar{P}(t-r)H(t)} \exp\Big(\lambda \int_{t-r}^{t} H(s) ds\Big) \\ & + \frac{1}{H(t)\lambda} \bar{P}(t) \exp\Big(\lambda \int_{t-\tau}^{t} H(s) ds\Big) \\ & + \frac{\bar{P}(t)}{H(t)} \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\Big(\lambda \int_{s-\delta}^{t} H(u) du\Big) ds \Big\} > 1, \\ & \text{for} \quad \bar{P}(t) > 0, t \geq T \end{aligned} \tag{10}$$

or

$$\begin{aligned} \inf_{\lambda>0, t\geq T} \Big\{ & \frac{1}{H(t)\lambda} \exp\Big(\lambda \int_{t-\tau}^{t} H(s)\bar{P}(s) ds\Big) \\ & + \frac{H(t-r)R(t-\tau)}{H(t)} \exp\Big(\lambda \int_{t-r}^{t} H(s)\bar{P}(s) ds\Big) \\ & + \frac{1}{H(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau)H(s-\delta) \exp\Big(\lambda \int_{s-\delta}^{t} H(u)\bar{P}(u) du\Big) ds \Big\} > 1 \\ & \text{for} \quad \bar{P}(t) \geq 0, t \geq T. \end{aligned} \tag{11}$$

Then every solution of Eq.(1) *is oscillatory.*

Proof. Without loss of generality, assume that Eq.(1) has an eventually positive solution $x(t)$. Let $y(t)$ be defined by (6). Then by Lemma 1 we have

$$y'(t) \le 0, \quad y(t) > 0, \quad t \ge t_1 \ge t_0.$$

¿From (1) we have

$$\begin{aligned} y'(t) &= -\bar{P}(t)x(t-\tau) \\ &= -\bar{P}(t)\Big[y(t-\tau) + R(t-\tau)x(t-\tau-r) + \int_{t-\tau+\delta}^{t} Q(s-\tau)x(s-\tau-\delta)ds\Big] \\ &= -\bar{P}(t)y(t-\tau) + \frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}y'(t-r) \\ &\quad + \bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}y'(s-\delta)ds. \end{aligned} \tag{12}$$

Assuming condition (10) holds, set

$$\lambda(t)H(t) = -\frac{y'(t)}{y(t)}.$$

Then (12) reduces to

$$\begin{aligned} \lambda(t)H(t) &= \bar{P}(t)\exp\Big(\int_{t-\tau}^{t}\lambda(s)H(s)ds\Big) \\ &\quad + \lambda(t-r)H(t-r)\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}\exp\Big(\int_{t-r}^{t}\lambda(s)H(s)ds\Big) \\ &\quad + \bar{P}(t)\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)}{\bar{P}(s-\delta)}\lambda(s-\delta)H(s-\delta)\exp\Big(\int_{s-\delta}^{t}\lambda(u)H(u)du\Big)ds. \end{aligned} \tag{13}$$

It is obvious that $\lambda(t)H(t) > 0$ for $t \ge t_0$. From (13) we have

$$\lambda(t)H(t) \ge \bar{P}(t)\exp\Big(\int_{t-\tau}^{t}\lambda(s)H(s)ds\Big).$$

In view of (8) and Lemma 2, we get

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}\lambda(s)H(s)ds < \infty,$$

which implies, by using (9), that $\lim_{t\to\infty}\inf \lambda(t) < \infty$. Now we show that

$$\liminf_{t\to\infty} \lambda(t) > 0.$$

In fact, if $\liminf_{t\to\infty} \lambda(t) = 0$, then there exists a sequence $\{t_n\}$ such that $t_n \geq t_1$, $t_n \to \infty$ as $n \to \infty$, and $\lambda(t_n) \leq \lambda(t)$ for $t \in [t_1, t_n]$. From (13) we have

$$\begin{aligned}\lambda(t_n)H(t_n) \geq \bar{P}(t_n)\exp\Big(\lambda(t_n)\int_{t_n-\tau}^{t_n} H(s)ds\Big)\\ + \lambda(t_n)H(t_n-r)\frac{R(t_n-\tau)\bar{P}(t_n)}{\bar{P}(t_n-r)}\exp\Big(\lambda(t_n)\int_{t_n-r}^{t_n} H(s)ds\Big)\\ + \bar{P}(t_n)\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)}{\bar{P}(s-\delta)}\lambda(t_n)H(s-\delta)\exp\Big(\lambda(t_n)\int_{s-\delta}^{t_n} H(u)du\Big)ds.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{\lambda(t_n)H(t_n)}\bar{P}(t_n)\exp\Big(\lambda(t_n)\int_{t_n-\tau}^{t_n} H(s)ds\Big)\\ &+\frac{H(t_n-r)R(t_n-r)\bar{P}(t_n)}{H(t_n)\bar{P}(t_n-r)}\exp\Big(\lambda(t_n)\int_{t_n-r}^{t_n} H(s)ds\Big)\\ &+\frac{\bar{P}(t_n)}{H(t_n)}\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(\lambda(t_n)\int_{s-\delta}^{t_n} H(u)du\Big)ds \leq 1,\end{aligned}$$

which contradicts (10) and therefore

$$0 < \liminf_{t\to\infty} \lambda(t) = h < \infty. \tag{14}$$

¿From (10) there exists an $\alpha \in (0,1)$ such that

$$\begin{aligned}\alpha \inf_{\lambda>0,t\geq T}\Big\{&\frac{R(t-\tau)\bar{P}(t)H(t-r)}{\bar{P}(t-r)H(t)}\exp\Big(\lambda\int_{t-r}^{t} H(s)ds\Big)\\ &+\frac{1}{H(t)\lambda}\bar{P}(t)\exp\Big(\lambda\int_{t-\tau}^{t} H(s)ds\Big)\\ &+\frac{\bar{P}(t)}{H(t)}\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(\lambda\int_{s-\delta}^{t} H(u)du\Big)ds\Big\} > 1\end{aligned} \tag{15}$$

On the other hand, in view of the definition of $\liminf_{t\to\infty} \lambda(t) = h$, there exists a $t_2 > t_1$ such that

$$\lambda(t) > \alpha h, \quad t \geq t_2. \tag{16}$$

Substituting (16) into (13), we obtain

$$\begin{aligned}\lambda(t)H(t) \geq\ & \bar{P}(t)\exp\Big(h\alpha\int_{t-\tau}^{t}H(s)ds\Big)\\ & + h\alpha\frac{H(t-r)R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}\exp\Big(h\alpha\int_{t-r}^{t}H(s)ds\Big)\\ & + \bar{P}(t)\alpha h\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(h\alpha\int_{s-\delta}^{t}H(u)du\Big)ds\end{aligned}$$

for $t \geq t_2 + T_0$. Hence

$$\begin{aligned}h \geq \liminf_{t\to\infty} & \bigg\{\frac{\bar{P}(t)}{H(t)}\exp\Big(\alpha h\int_{t-\tau}^{t}H(s)ds\Big)\\ & + \alpha h\frac{H(t-r)R(t-r)\bar{P}(t)}{H(t)\bar{P}(t-r)}\exp\Big(\alpha h\int_{t-\tau}^{t}H(s)ds\Big)\\ & +\frac{\bar{P}(t)}{H(t)}\alpha h\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(\alpha h\int_{s-\delta}^{t}H(u)du\Big)ds\bigg\},\end{aligned}$$

which implies that there exists a sequence $\{\bar{t}_n\}$ such that $\bar{t}_n \geq \max\{T, t_2 + T_0\}$, $\bar{t}_n \to \infty$ as $n \to \infty$, and

$$\begin{aligned}\lim_{n\to\infty} & \bigg\{\frac{\bar{P}(\bar{t}_n)}{H(t_n)}\exp\Big(\alpha h\int_{t-\tau}^{t}H(s)ds\Big)\\ & + \alpha h\frac{H(\bar{t}_n-r)R(\bar{t}_n-\tau)\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\bar{P}(\bar{t}_n-r)}\exp\Big(\alpha h\int_{\bar{t}_n-r}^{\bar{t}_n}H(s)ds\Big)\\ & +\frac{\bar{P}(t_n)}{H(\bar{t}_n)}\alpha h\int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(\alpha h\int_{s-\delta}^{\bar{t}_n}H(u)du\Big)ds\bigg\} = \bar{h} \leq h.\end{aligned}$$

Set $\lambda = h\alpha$, then

$$\begin{aligned}\alpha\lim_{n\to\infty} & \bigg\{\frac{\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\lambda}\exp\Big(\lambda\int_{t-\tau}^{r}H(s)ds\Big)\\ & + \frac{H(\bar{t}_n-r)R(\bar{t}_n-\tau)\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\bar{P}(\bar{t}_n-r)}\exp\Big(\lambda\int_{\bar{t}_n-r}^{\bar{t}_n}H(s)ds\Big)\\ & +\frac{\bar{P}(t_n)}{H(\bar{t}_n)}\int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\Big(\lambda\int_{s-\delta}^{\bar{t}_n}H(u)du\Big)ds\bigg\} \leq 1,\end{aligned}$$

which contradicts (15) and completes the proof of this theorem under condition (10).

For the condition (11), let

$$\lambda(t)H(t)\bar{P}(t) = -\frac{y'(t)}{y(t)}.$$

Then (12) becomes

$$\begin{aligned}\lambda(t)H(t) &= \exp\Big(\int_{t-\tau}^{t} \lambda(s)H(s)\bar{P}(s)ds\Big)\\ &\quad + \lambda(t-r)H(t-r)R(t-\tau)\exp\Big(\int_{t-r}^{t} \lambda(s)H(s)\bar{P}(s)ds\Big)\\ &\quad + \int_{t-\tau+\delta}^{t} Q(s-\tau)\lambda(s-\delta)H(s-\delta)\exp\Big(\int_{s-\delta}^{t} \lambda(u)H(u)\bar{P}(u)du\Big)ds.\end{aligned} \tag{17}$$

As before, by Lemma 2 we can prove that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \lambda(s)H(s)\bar{P}(s)ds < \infty. \tag{18}$$

¿From (8), (9) and (18), we may conclude that $\lim_{t\to\infty} \inf \lambda(t) < \infty$. In view of (17), $\lambda(t) \geq 1$, and so

$$0 < \liminf_{t\to\infty} \lambda(t) = h < \infty.$$

¿From (11), there exists $\alpha \in (0,1)$ such that

$$\begin{aligned}\alpha \inf_{\lambda>0, t\geq T} \Big\{ &\frac{1}{\lambda H(t)} \exp\Big(\int_{t-\tau}^{t} H(s)\bar{P}(s)ds\Big)\\ &+ \frac{H(t-r)R(t-\tau)}{H(t)} \exp\Big(\lambda \int_{t-r}^{t} H(s)\bar{P}(s)ds\Big)\\ &+ \frac{1}{H(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau)H(s-\delta)\exp\Big(\lambda \int_{s-\delta}^{t} H(u)\bar{P}(u)du\Big)ds \Big\} > 1.\end{aligned}$$

By using a similar method as in the first part of the proof, we can derive a contradiction. The proof is complete.

Remark 1. Conditions (10) and (11) are equivalent when $\bar{P}(t) > 0$ for $t \geq T$. In fact, if $\bar{P}(t) > 0$ for $t \geq T$, set $K(t) = H(t)\bar{P}(t)$, then condition (11) becomes (10). Conversely, if we let $K(t) = H(t)/\bar{P}(t)$, then (10) reduces to (11).

Since $e^x \geq ex$, $e^x \geq 1$ for $x \geq 0$, (10) and (11) lead to the following corollary.

Corollary 1

Assume taht (2), (3), (5), (8) *and* (9) *hold. Further assume that either*

$$\liminf_{t\to\infty}\left\{\frac{R(t-\tau)\bar{P}(t)H(t-r)}{\bar{P}(t-r)H(t)}+\frac{e\bar{P}(t)}{H(t)}\int_{t-\tau}^{t}H(s)ds\right.$$
$$\left.+\frac{\bar{P}(t)}{H(t)}\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}ds\right\}>1,\quad \text{for}\quad \bar{P}(t)>0, t\geq T \tag{19}$$

or

$$\liminf_{t\to\infty}\left\{\frac{e}{H(t)}\int_{t-\tau}^{t}H(s)\bar{P}(s)ds+\frac{H(t-r)R(t-\tau)}{H(t)}\right.$$
$$\left.+\frac{1}{H(t)}\int_{t-\tau+\delta}^{t}Q(s-\tau)H(s-\delta)ds\right\}>1. \tag{20}$$

Then every solution of (1) *is oscillatory.*

Remark 2. Corollary 1 implies the main result of Yu [15] if we select $H(t)=1$.

¿From Corollary 1 we can obtain different sufficient conditions for oscillation of (1) by different choices of $H(t)$. For instance, if we choose $H(t)=\bar{P}(t)>0$ for $t\geq T$ or $H(t)=1$, then (19) becomes

$$\liminf_{t\to\infty}\left\{R(t-\tau)+e\int_{t-\tau}^{t}\bar{P}(s)ds+\int_{t-\tau+\delta}^{t}Q(s-\tau)ds\right\}>1 \tag{21}$$

or

$$\liminf_{t\to\infty}\left\{\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}+e\bar{P}(t)\tau\right.$$
$$\left.+\bar{P}(t)\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)}{\bar{P}(s-\delta)}\right\}>1,\quad \text{for}\quad \bar{P}(t)>0, t\geq T. \tag{22}$$

If we select $H(t)=\bar{P}(t)>0$ for $t\geq T$, then (20) becomes

$$\liminf_{t\to\infty}\left\{\frac{e}{\bar{P}(t)}\int_{t-\tau}^{t}[P(s)]^2ds+\frac{\bar{P}(t-r)R(t-\tau)}{\bar{P}(t)}\right.$$
$$\left.+\frac{1}{\bar{P}(t)}\int_{t-\tau+\delta}^{t}Q(s-\tau)\bar{P}(s-\delta)ds\right\}>1,\quad \text{for}\quad \bar{P}(t)>0, t\geq T. \tag{23}$$

Corollary 2

Assume that (2), (3), (5) *and* (21) *hold. Then every solution of* (1) *is oscillatory.*

Corollary 3

Assume that (2), (3), (5) *and* (22) *hold. Then every solution of* (1) *is oscillatory.*

Corollary 4

Assume that (2), (3), (5) and (23) hold. Then every solution of (1) is oscillatory.

Remark 3. Corollaries 2 and 3 extend and improve the main results of [4, 5, 9, 11, 14, 17] when $Q(t) = 0$.

Remark 4. Condition (21) is that of Theorem 1 obtained by different technique, by Yu [15]. But we should point out that the proof of Theorem 1 of Yu [15] is wrong (see, Zhang [18]). On the other hand, if $\bar{P}(t) > 0$ is nonincreasing (nondecreasing), then it is easy to see that

$$\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)} \leq (\geq) R(t-\tau), \quad \bar{P}(t)\tau \leq (\geq) \int_{t-\tau}^{t} \bar{P}(s)ds,$$

and

$$\bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)} \leq (\geq) \frac{\bar{P}(t)}{\bar{P}(t-\delta)} \int_{t-\tau+\delta}^{t} Q(s-\tau)ds \leq (\geq) \int_{t-\tau+\delta}^{t} Q(s-\tau)ds.$$

It follows that

$$\begin{aligned} &\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)} + e\bar{P}(t)\tau + \bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}ds \\ &\qquad \leq \ (\geq) R(t-\tau) + e\int_{t-\tau}^{t} \bar{P}(s)ds + \int_{t-\tau+\delta}^{t} Q(s-\tau)ds. \end{aligned}$$

Therefore, our result is new.

Example 1: Consider the neutral differential equations

$$\frac{d}{dt}\left(x(t) - \frac{t-1}{2t}x(t-1)\right) + \left(\frac{1}{8e} - \frac{1}{t}\right)x(t-2) - \frac{1}{t-1}x(t-1) = 0, \quad t \geq 4. \quad (24)$$

Let $R(t) = \frac{t-1}{2t}$, $P(t) = \frac{1}{8e} - \frac{1}{t}$, $Q(t) = \frac{1}{t+1}$, $\tau = 2$, $\delta = 1$, then $\bar{P}(t) = \frac{1}{8e}$. It is easy to verify that the assumption of Corollary 3 are satisfied. Therefore, every solution of (24) is oscillatory. But the related results of [9, 14] can not applicable to equation (24).

Corollary 5 [15]

Assume that $R(t) = r_0 \geq 0$, $P(t) = p > 0$, $Q(t) = q \geq 0$, $p > q$, $\tau > \delta$ *and* $r_0 + q(\tau - \delta) \leq 1$. *Then every solution of Eq.*(1) *oscillates if and only if*

$$f_1(\lambda) = -\lambda(p-q) + qe^{\lambda p\tau} + \lambda(p-q)r_0 e^{\lambda(p-q)r} - qe^{\lambda(p-q)\delta} > 0, \quad \lambda > 0.$$

Corollary 6

Assume that the assumptions of Corollary 3 hold. Then every solution of Eq.(1) oscillates if and only if

$$f_2(\lambda) = -\lambda + \lambda r_0 r^{\lambda\tau} + pe^{\lambda\tau} - qe^{\lambda\delta} > 0, \quad \lambda > 0. \tag{25}$$

Proof. Sufficiency is obvious. We will prove the necessity. Assume that the condition (25) is false, then there exists $\lambda_0 > 0$ such that $f_2(\lambda_0) \leq 0$, and $f_2(0) = p - q > 0$. Thus there exists $\lambda_1 \in (0, \lambda_0]$ such that $f_2(\lambda_1) = 0$. In fact, $x(t) = \exp(-\lambda_1 t)$ is a nonoscillatory solution of Eq.(1). This is a contradiction and the proof is complete.

Remark 5. Corollaries 5 and 6 imply that the conditions of Theorem 1 are sharp.

Remark 6. Corollary 6 includes the main results of Wang [13].

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