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Transversal intersection of separatrices and branching of solutions as obstructions to the existence of an analytic integral in many-dimensional systems. I. Basic result: Separatrices of hyperbolic periodic points

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Abstract

It is well-known that the existence of transversally intersecting separatrices of hyperbolic periodic solutions leads, in a typical situation, to complicated and irregular dynamics. Therefore, in the case of a two-dimensional mapping or a three-dimensional flow, with this transversality property, there is no non-trivial analytic or meromorphic first integral, i.e., a function constant along each trajectory of the system under consideration. Additional robust conditions are obtained and discussed that guarantee the absence of such an integral in the many-dimensional case, regardless of the finite dimension in question (the strongest analytic non-integrability). These conditions guarantee also the absence of any non-trivial analytic one-parameter symmetry group, and, more generally, analytic or meromorphic vector fields generating a local symmetry, i.e., a local phase flow commuting with the system under consideration. Furthermore, the analytic centralizer of the system is discrete in the compact-open topology. A differential-topological structure of the invariant set of "quasi-random motions" is studied for this purpose. The approach used is essentially geometrical. Some related topics are also discussed.

Contents

It is well-known, due to G. D. Birkhoff, S. Smale, L. P. Shil′nikov, V. M. Alekseev, and Yu. I. Neĭmark, that the existence of transversally intersecting separatrices of hyperbolic periodic solutions leads, in a typical situation, to complicated and irregular dynamics. Therefore, in the case of a two-dimensional mapping or a three-dimensional flow with this transversality property there is no non-trivial analytic first integral, i.e., a function constant along each trajectory of the system under consideration. (In essence, the non-integrability in this situation was pointed out by H. Poincaré. His statement [42, Paragraph 397] concerning the complication of a network pictured by intersecting separatrices is well-known.) We will obtain and discuss conditions that guarantee the absence of such an integral in the many-dimensional case, regardless of the finite dimension in question. This is the strongest analytic non-integrability and the Hamiltonian character of the equations (which is of no importance) is not supposed here. The conditions obtained guarantee also the absence of any non-trivial analytic one-parameter symmetry group, persist under small perturbations, and can be constructively verified for concrete dynamical systems in mechanics and physics. A differential-topological structure of the set of “quasi-random motions” is studied for this purpose. An interesting construction of “nonautonomous linearization” by Y. Yomdin is used in our proof and the accompanying and related geometrical objects will be also discussed in detail.

The author avoids many clear but cumbersome formalities by replacing them by simple descriptive geometrical considerations. Rigorous mathematical arrangements of the presented proofs (if they are not completed) are easy and left to the reader. On the other hand, an emphasis on the underlying geometric constructions is pursued. Some basic concepts and geometric objects related to building symbolic dynamics in a wide class of dynamical systems will be briefly premised to the statement of the problem and discussed while proving the main result and the auxiliary propositions. These ideas will be systematically used in the sequel on two levels: to construct real symbolic dynamics as well as some invariant (with respect to the diffeomorphism under consideration) objects in the phase space, M, and in its tangent bundle, TM, and to build and analyze “nonautonomous linearizations” on stable and unstable manifolds.

Some results presented here were earlier reported by the author at a few meetings and published in Russian in 1991 (in some sources difficult to access). An English version containing the main result without a detailed proof appeared in [17][1]. A simplified result concerning the case of a single hyperbolic point was announced in [18] and will be reproduced at the end of Section 1.

The forthcoming parts of the present paper will be devoted to further developments of the basic result and to some applications in mechanical and physical problems (such as a three-component homogeneous Yang–Mills field, a spherical pendulum with the horizontally oscillating point of suspension, the planar problem of more than three point vortices in an ideal incompressible liquid, and the planar problem of more than two bodies attracting by the Newton law).

1. Preliminaries, formulation of the problem, and the statement of results

Firstly, we recall the definition of a *topological Markov chain* (TMC) and how it is determined by a directed graph. The term “topological Markov chain” was introduced by V. M. Alekseev [2, Part 1] although the corresponding mathematical object can be found in earlier papers. In the English mathematical literature it is usually referred to as *“subshift of finite type”* going back to S. Smale [47] (because there is a natural one-to-one topological equivalence between topological Markov

[1] Note that [17] contains inaccuracies in Remark 5 and Lemma 4 that are corrected in the present paper.

chains and Smale's subshifts of finite type). Consider a finite set (*"alphabet"*) $\mathcal{L}$ containing $m < \infty$ elements and equipped with the discrete topology. One can put $\mathcal{L} = \{1, \dots, m\}$. Denote by $\Omega = \prod_{n=-\infty}^{+\infty} \mathcal{L}$ the Tychonoff product of a countable collection of copies of the space $\mathcal{L}$, i.e., the space of doubly-infinite sequences $\omega = \big[\omega_n \in \mathcal{L} : -\infty < n < +\infty\big]$ equipped with the topology such that

$$\omega^{(k)} = \big[\omega_n^{(k)} : n \in \mathbb{Z}\big] \to \omega = \big[\omega_n : n \in \mathbb{Z}\big] \quad \text{as} \quad k \to +\infty \iff \\ \omega_n^{(k)} \to \omega_n \quad \text{as} \quad k \to +\infty \quad \text{for each} \quad n \in \mathbb{Z}\,,$$

and by $T\colon \Omega \to \Omega$, the shift homeomorphism by one symbol to the left. Each zero-unit matrix $\Pi = \big(\pi_{i,j}\big)_{i,j=1}^{m}$ of size $m \times m$ defines a closed T-invariant compact subset $\Omega^\Pi \subset \Omega$ by the following condition:

$$\omega = [\omega_n] \in \Omega^\Pi \iff \pi_{\omega_n, \omega_{n+1}} = 1 \quad \text{for all} \quad n.$$

DEFINITION 1. The restriction $T \mid \Omega^\Pi$ is called a TMC with m states

$$\{1, \dots, m\} = \mathcal{L}$$

and *transition matrix* Π. Each sequence $\omega \in \Omega$ is called a *"word"*, and a sequence $\omega \in \Omega^\Pi$ (containing only admissible transitions determined by the matrix), an *"admissible word"*. If $\pi_{i,j} = 1$ for all the pairs (i, j) (i.e., the transition from every state into every another one is possible) then $\Omega = \Omega^\Pi$ and the TMC $T \mid \Omega$ is called a *Bernoulli shift* on the set of m elements (states).

It is convenient to represent a TMC by the directed graph with vertex set $\mathcal{L}$ and edges $\overrightarrow{(i, j)}$ whose initial vertices (origins), i, and terminal vertices (ends), j, form pairs (i, j) such that $\pi_{i,j} = 1$. Without loss of generality one can suppose that a TMC is *"conservative"*, i.e., each vertex $i \in \mathcal{L}$ happens to be a *"transition"*. This means that there are vertices $j^+, j^- \in \mathcal{L}$ such that $\pi_{j^-,i} = \pi_{i,j^+} = 1$. Indeed, removing the remaining vertices, if any, does not lead to a change of the set Ω^Π of admissible words.

For a wide class of diffeomorphisms one can establish the topological equivalence of a TMC and the restriction $S \mid A$ of the diffeomorphism S to some locally maximal invariant set A. This means the presence of a topological conjugacy $\psi\colon \Omega^\Pi \to A$, i.e., a homeomorphic mapping, ψ, such that the diagram

$$\begin{tikzcd} \Omega^\Pi \arrow[r, "T|\Omega^\Pi"] \arrow[d, "\psi"'] & \Omega^\Pi \arrow[d, "\psi"] \\ A \arrow[r, "S|A"'] & A \end{tikzcd}$$

commutes. The key idea in constructing the topological equivalence $\psi: \Omega^{\Pi} \to A$ is as follows. In the phase space, M, of the diffeomorphism $S: M \to M$, one can choose non-intersecting closed sets D_i, $i \in \mathcal{L}$, such that to each edge $\overrightarrow{(i,j)}$ of the graph Γ there corresponds a nonempty intersection, $l_{i,j} = S(D_i) \cap D_j$, and the *"Markov property"* is satisfied: the intersection

$$\bigcap_{n=-\infty}^{+\infty} S^{-n} D_{\omega_n} = \psi(\omega)$$

is nonempty and consists of a single point if $\omega \in \Omega^{\Pi}$. The so-called *"itinerary scheme"*, which will be discussed in the sequel, is constructed for this purpose. Now we consider the following situation studied by V. M. Alekseev.

Let $q_i\,(1 \le i \le l)$ be hyperbolic periodic points of a diffeomorphism S of an n-dimensional manifold M and W_i^-, W_i^+ be their outgoing and incoming invariant manifolds (separatrices)[2]. Suppose that for any i the dimensions of $W_i^{\pm}$ are equal to $n^{\pm}$ (so, $n^+ + n^- = n$), $r_j\,(1 \le j \le s)$ are transversal homo- and heteroclinic points, so that as $m \to \pm\infty$, $S^m(r_j)$ tends to the orbit of the point $q_{i^{\pm}(j)}$, and $W^-_{i^-(j)}, W^+_{i^+(j)}$ intersect transversally at the point r_j. One can assume that distinct points q_i, r_j do not belong to the same orbit under S. The orbits of the points q_i, r_j are said to form a *homoclinic structure*. Let us build the directed graph Γ which contains non-intersecting closed circuits, γ_i $(1 \le i \le l)$, and paths, π_j $(1 \le j \le s)$, consisting of N_j edges and joining origins and ends placed arbitrarily on the circuits $\gamma_{i^-(j)}$ and $\gamma_{i^+(j)}$. Moreover, the number of edges in the circuit γ_i equals to the period of q_i. By V. M. Alekseev's theorem [3], for any open set U containing the orbits of the points q_i and r_j, there exist N_j and an open set $V \subset U$ such that the restriction $S \mid A$ of S to the maximal S-invariant set A located in V is topologically conjugated to the topological Markov chain (subshift of finite type) $T \mid \Omega^{\Pi}$ determined by the graph Γ, and $q_i, r_j \in A$.

The motions on this set are said to be *quasi-random* if the topological entropy (see, for instance, [4]) of the topological Markov chain $T \mid \Omega^{\Pi}$ is positive.

Proposition 1

This condition is satisfied if and only if the graph Γ contains a connected branched subgraph Γ' (maybe identical with Γ), i.e.,

1) for any two vertices a, b of the graph Γ', there is a path on Γ' with origin a and end b, and

2) there exists a vertex being the origin of at least two distinct edges or, equivalently, there exists a vertex being the end of at least two distinct edges.

Otherwise the graph Γ consists of a finite number of circular subgraphs.

[2] It is common do denote W^- as W^u (unstable manifold) and W^+ as W^s (stable manifold) .

Proof. The sufficiency of these conditions was proven in [2, Part 1]. The necessity is easily established using the representation of the transition matrix in the *normal* block-triangle form (see, for example, [21]). Very simple details are omitted here. To a connected graph, there corresponds the so-called *indecomposable* TMC [4]. □

Let a set $A' \subset A$ correspond to the graph Γ'. In the sequel, it is the motions on the set A' that will be called to be quasi-random. Restricting Alekseev's theorem to some part of the homoclinic structure, if necessary, we can assume that the full graph, Γ, coincides with the subgraph Γ'. It is this homoclinic structure that will be considered in the sequel. In particular, one can consider the case of one homoclinic point.

It is well-known that in the two-dimensional case ($n = 2, n^{\pm} = 1$) the presence of non-coincident and intersecting (possibly, non-transversally) separatrices leads to the absence of a non-constant analytic (first) integral of the diffeomorphism S, i.e., a function F on M such that $F \circ S = F$ [8, 30][3]. In fact, this result has been originally established via the method of symbolic dynamics for the particular case where there exists a set A' of quasi-random motions [2, Part 3]. An outline of the proof [2, 8, 34] is as follows:

1) Every continuous first integral F is constant on A' because A' contains everywhere dense trajectories (the *topological transitivity*).
2) If $F \mid A' \equiv \text{const}$ and the function F is differentiable at a point $x \in A'$ then the differential dF at x is equal to zero. Indeed, any point $x \in A'$ has one-dimensional stable and unstable manifolds which intersect transversally at x and there exist two sequences of points in A' which tend to x and belong to these manifolds. Thus, for any analytic function, F, constant on A', all the partial derivatives vanish at the points of the set A', i.e., $F \equiv \text{const}$. So, the set A' is the *key set* in the sense of [8, 30] (for the class of functions, F, analytic in a neighbourhood of $A' \subset M$, whose every connected component intersects with A').

In the many-dimensional case ($n \geq 3$) point 1) remains valid but point 2) does not[4].

[3] Here and in the sequel, we restrict our consideration to a finite number of connected components, to which the homoclinic structure belongs .

[4] D. V. Anosov attached kindly the author's attention to the fact that $dF|A \equiv 0$ for any first integral $F \in C^1$. Indeed, $F|W_r^{\pm} \equiv \text{const}$ for any $r \in A$, where W_r^+ and W_r^- denote, respectively, the stable and unstable manifolds of the point r of the hyperbolic set A, and this implies the desired result. However, one cannot prove in this way that the second derivatives of F vanish at $r \in A$.

However, it is stated sometimes in the physical literature that the presence of transversally intersecting separatrices in many-dimensional systems causes non-integrability in some sense including the strongest non-integrability—the non-existence of any non-trivial integral. In the following simple example, the mapping S is shown to be able to possess integrals even in the case when transversal biasymptotic points are present. However, in this example the separatrices have also the whole manifolds of intersection points or are extendible up to the boundary of the domain of definition of the integrals, which is somewhat unnatural.

EXAMPLE 1: Let $S_1\colon M_1 \to M_1$ be a mapping possessing an integral f and a hyperbolic fixed point O, and $S_2\colon M_2 \to M_2$ be a mapping possessing a homoclinic structure, H, formed by the trajectories of periodic and doubly-asymptotic points. Then the mapping $S = S_1 \times S_2$ (the direct product of S_1 and S_2) of the manifold $M = M_1 \times M_2$ into itself has the homoclinic structure $\{O\} \times H$ and the integral $f \circ \pi$, where $\pi\colon M \to M_1$ is the natural projection. If the separatrices $W^{\pm}$ of the hyperbolic fixed point of the mapping S_1 coincide[5], then the separatrices of the mapping S possess the whole manifolds $W^{\pm} \times \{r\}, r \in H$, of intersection points.

In the sequel, considering a power of the mapping S, if necessary, one can assume that all the points q_i are fixed.

Let O be a hyperbolic fixed point of the analytic mapping S, and assume that its eigenvalues, λ_j, with moduli greater (less) than unit, i.e., which correspond to the separatrix W^- (W^+), satisfy the non-resonance multiplicative conditions

$$\prod_j \lambda_j^{m_j} \neq 1$$

for all integers m_j, at least one of which is not equal to zero. (So, the spectrum $\{\lambda_j\}$ lies in the Siegel domain.) Then the mapping $S \mid W^{\pm}$ takes the linear diagonal form $x_j \mapsto \lambda_j x_j$ in some analytic (complex) coordinates on W^- or W^+. Actually, these coordinates are constructed in a neighbourhood of the point O and are then extendible over the whole separatrix. Let us call a point r on W^- (W^+) to be *in general position* if all its coordinates x_j are distinct from zero.

Theorem 1

Let a mapping S be analytic, and let the outgoing separatrix W^- and the incoming separatrix W^+ of fixed hyperbolic points, O^- and O^+, possess transversal

[5] In a typical case this condition is satisfied for integrable two-dimensional mappings, S_1, preserving a measure with a positive continuous density. In order that this invariant measure of S_1 on a compact manifold M_1 exists, it is sufficient to require the presence of such a measure for the mapping S; the proof of this result is elementary and uses the presence of a periodic point of the mapping S_2.

intersection at a point r. Moreover, suppose that the eigenvalues of the mapping S at the point O^- (respectively, O^+) with moduli greater (less) than unit, i.e.,which correspond to the separatrix W^- (W^+), satisfy the above non-resonance multiplicative conditions. Then the mapping S possesses no non-trivial analytic integral if the point r at both manifolds $W^\pm$ is in general position.

Proof. Let B be the closure of the trajectory of the point r. Every continuous first integral is constant on B. Any analytic function F that is constant on B is easily seen to be also constant on the separatrices $W^\pm$. Therefore, the differential satisfies $dF \equiv 0$ on B, which leads to the required result via the key property of the set B, due to the fact that all the partial derivatives of F vanish at the points of B. □

Theorem 1 has the following deficiency: its conditions are broken by arbitrarily small perturbations[6]. On the other hand, in the typical case among the eigenvalues can exist multiple ones if the system possesses a symmetry. Then Theorem 1 is not applicable.

Our goal is to formulate sufficient conditions which are free of the above shortcomings, are satisfied in an open set in the space of mappings, and guarantee the following property

$$F \in C^N(M), F \mid A' \equiv 0 \Rightarrow dF \mid A' \equiv 0. \tag{1}$$

This implication is completely analogous to point 2) and leads to the absence of a non-trivial first analytic integral via the key property of the set A'. For this purpose we shall consider the structure of the set A'.

Thus, in both the cases, under the conditions of Theorem 1 or under assumption (1), the absence of an integral is implied by the presence of an S-invariant topologically transitive key set C (where $C = B$ or $C = A'$). However, the key properties of these two sets are different. Let us call a neighbourhood of C, U, a *domain of the key property* (DKP) of C if C is the key set for the class of functions, F, analytic in U, i.e., $F \mid A' \equiv \text{const}$ implies $F \equiv \text{const}$. On the one hand, any neighbourhood of A' is a DKP of A' provided that all the connected components of that neighbourhood intersect A'. The same is valid if one replaces A' by the intersection of A' and a neighbourhood of any of its points. On the other hand, it is easily seen that any disconnected neighbourhood of B is not a DKP of B and any small enough neighbourhood of B is disconnected. There exists an arc, $\gamma^\pm$, that lies

[6] In practice, Theorem 1 can be used to establish the non-integrability of the mapping S depending on parameters. For example, the values of *a priori* first integrals can be considered as parameters to prove the non-existence of an additional integral (on almost all the levels of the *a priori* ones).

on $W^{\pm}$ and contains the semitrajectory of r, $\{S^{\pm t}r : t \geq 0\}$, and its limit point $O^{\pm}$. Then any neighbourhood of $\gamma = \gamma^{-} \cup \gamma^{+}$ is a DKP for B. Here, γ can be chosen to be an arc without selfintersections.

Note that topologically the set A' is organized in a very simple way. Indeed, it is a *discontinuum* (i.e. a perfect, totally disconnected compact set) and thus homeomorphic to the standard Cantor set $K \subset [0;1]$ (the *"middle-thirds set"*) [1, 7]. Moreover, the following result is valid.

Proposition 2

Let the closure of a domain $\widetilde{V} \subset \mathbb{R}^n$ be diffeomorphic to the closure of V and let $K \subset \widetilde{V} \cap \mathbb{R}^1$ be the (standard) Cantor set. Then there exists a homeomorphism $f\colon \widetilde{V} \to V$ such that it and its inverse are Hölder and $f(K) = A'$.[7] *In particular, the set A' has positive Hausdorff dimension.*

Proof. It follows easily from the analysis of the procedure of building the set A', that is, an iterative process of Suslin's A-operation type [2, Part 3] (see [1]). □

The conditions we will obtain can be constructively verified for concrete dynamical systems and persist under small perturbations.

In particular, in the space of the C^{∞}-diffeomorphisms possessing a homoclinic trajectory, there is an open everywhere dense subset of "non-integrable" ones. The same result is valid in the analytic (C^{ω}-) category if each connected component of M has a countable base, i.e., M is paracompact (the verification of the C^{ω}-density requires some effort, while the C^{ω}-openness is a direct consequence of the C^{∞}-one). The condition (1) means that the set A' is "rough enough" and does not lie on any regular C^{N}-submanifold and even on a union of countable collection of such submanifolds.

Note that the manifold M under consideration should be analytic in view of the fact that analytic first integrals are discussed. Because of the possible noncompactness of M, the C^{k}-topologies, $1 \leq k \leq \infty$, are considered in the *weak* sense, i.e., the C^{k}-convergence is understood as the C^{k}-convergence over compact sets. For the validity of the above-mentioned C^{ω}-result, the C^{ω}-topology in the space $\mathrm{Diff}^{\omega}(M)$ of real analytic diffeomorphisms on M can be chosen in different ways. Actually, it has to be not coarser than the C^{∞}-topology but not finer than some very fine topology τ such that the space $\mathrm{Diff}^{\omega}(M)$ endowed with τ is not separable [10].

The relevant geometrical objects in bundles over M will be discussed in detail. It is interesting that many of them admit the following treatment. Let $S\colon M \to M$

[7] Antoine's example ("necklace of Antoine") [1, 7, 16] shows that a homeomorphism of two discontinua can be inextensible to a homeomorphism of the ambient spaces. See also [13] and further investigation in [14].

be a diffeomorphism with a homoclinic structure, H, which generates an invariant set A by Alekseev's theorem. Then there exist some natural bundle $\pi\colon \mathfrak{M} \to M$ and an extension $\mathcal{S}\colon \mathfrak{M} \to \mathfrak{M}$ of the diffeomorphism $S\colon M \to M$ (i.e., the fiber map $\mathcal{S}$ covers the map S of the base). Furthermore, over each hyperbolic fixed point O of the mapping S with separatrices W^+ and W^-, if some conditions on the eigenvalues of S at O are valid, there lies a fixed point $\widetilde{O}$ of the mapping $\mathcal{S}$ with incoming $\widetilde{W}^+$ and outgoing $\widetilde{W}^-$ manifolds such that the subbundles $\widetilde{W}^\pm \to W^\pm$ of the restrictions of the bundle π are well-defined. If S has a high enough class of smoothness then $\mathcal{S}$ is a diffeomorphism and $\widetilde{W}^\pm$ are separatrices of the hyperbolic point $\widetilde{O}$. Under some conditions, the diffeomorphism $\mathcal{S}$ possesses a homoclinic structure, $\mathcal{H}$, which is homeomorphically projected onto H. By Alekseev's theorem, to the structure $\mathcal{H}$ there corresponds an $\mathcal{S}$-invariant set, $\mathfrak{A}$, which is also, obviously, homeomorphically projected onto A.

There is the concept of symmetry groups which is closely related to the first integrals. Recall some definitions restricting ourselves in the sequel to the case of analytic symmetries.

DEFINITION 2. The C^k-*centralizer* $Z(S) = Z^k(S)$ of a diffeomorphism S is the group of C^k-diffeomorphisms $f\colon M \to M$ commuting with S, $Z(S) = \{f : f \circ S = S \circ f\}$. These diffeomorphisms f are called C^k-*symmetries* of S. The centralizer $Z(S)$ is said to be *trivial* if $Z(S) = (S)$ where $(S) = \{S^k : k \in \mathbb{Z}\}$ is the group generated by S.

We will prove that assumption (1) or the conditions of Theorem 1 imply the non-existence of a non-trivial analytic one-parameter symmetry group (see [31] in the two-dimensional case). Moreover, the following stronger result holds: the analytic (C^ω-) centralizer $Z(S)$ is discrete in the compact-open (weak C^0-) topology. (Obviously, the discreteness of the centralizer implies the absence of a non-trivial analytic one-parameter symmetry group.)

Finally, we present the simplified version of the main result related to the case of a single hyperbolic fixed point. The Theorem A below was announced in [18].

Let q be a hyperbolic fixed point of a C^N-diffeomorphism S of an n-dimensional manifold M and let W^-, W^+ be its outgoing and incoming invariant manifolds (separatrices). Assume, as above, that the dimensions of $W^\pm$ are equal to $n^\pm$ and r_j $(1 \le j \le s)$ are some transversal homoclinic points. Let λ_j $(1 \le j \le n^+)$ and μ_j $(1 \le j \le n^-)$ be all the eigenvalues of the mapping S at the point q and $0 < |\lambda_j| < 1 < |\mu_j|$. Assume that the numbers λ_j satisfy the non-resonance multiplicative conditions

$$|\lambda_s| \neq \left| \prod_j \lambda_j^{m_j} \right| \tag{2}$$

for all indices s and all non-negative integers m_j such that $\sum_j m_j \geq 2$. (So, the spectrum $\{\lambda_j\}$ lies in the Poincaré domain.) Introduce the following equivalence relation: $\lambda_{j'} \sim \lambda_{j''}$ if and only if in all the inequalities (2) the signs persist under replacement of all factors $\lambda_{j'}$ by $\lambda_{j''}$ or all factors $\lambda_{j''}$ by $\lambda_{j'}$ in the left-hand side and right-hand side simultaneously. Let $\{\Lambda_1^+, \ldots, \Lambda_{p^+}^+\}$ be the corresponding partition of $\{\lambda_j\}$ into equivalence classes placed in the order of non-decreasing moduli. Analogously, let $\{\mu_i\}$ also satisfy the non-resonance multiplicative conditions and let $\{\Lambda_1^-, \ldots, \Lambda_{p^-}^-\}$ be the corresponding partition of $\{\mu_i\}$ into equivalence classes placed in the order of non-increasing moduli. The integer

$$N > \max \left\{ \ln \min_j |\lambda_j| / \ln \max_j |\lambda_j|, \ln \max_j |\mu_j| / \ln \min_j |\mu_j| \right\}$$

will bound from below the smoothness class of the diffeomorphism S. Consider subspaces $L_1^\pm \subset \cdots \subset L_{p^\pm}^\pm = T_q W^\pm$ such that L_i^+ (respectively, L_i^-) corresponds to the Jordan blocks for the eigenvalues $\lambda_s \in \Lambda_j^+$ (respectively, $\mu_s \in \Lambda_j^-$) where $j \leq i$. By a theorem of S. Sternberg [48], there are linearizing C^N coordinates $y^\pm \in \mathbb{R}^{n^\pm}$ on $W^\pm$, in which the mapping $S \mid W^\pm$ takes a linear form $y^\pm \mapsto J^\pm y^\pm$. Denote by $\mathfrak{N}^\pm \subset \mathbb{R}^{n^\pm}$ the collection of $y^\pm$-coordinates of the points $r_j \in W^\pm$, and by $\widetilde{\mathcal{L}}_{i,j}^\pm \subset T_{r_j} W^\pm$, the subspaces obtained from the subspace $L_i^\pm$ by parallel translations in the linearizing coordinates. Let each subspace $\widetilde{\mathcal{L}}_{i,j}^\pm$ tend to $L_i^\pm$ under the positive iterations of the tangent mapping $TS^{\mp 1}\colon TM \to TM$. For each eigenvalue ν of the mapping $J^\pm = S \mid W^\pm$ linear in coordinates $y^\pm$, the projection $\pi_\nu\colon \mathbb{C}^{n^\pm} \to \mathbb{C}^{n^\pm}/\mathrm{im}(J^\pm - \nu \cdot \mathrm{id}) = \mathrm{coker}(J^\pm - \nu \cdot \mathrm{id})$ is well-defined and let the $\mathbb{C}$-linear hull of the set $\pi_\nu(\mathfrak{N}^\pm)$ coincide with $\pi_\nu(\mathbb{C}^{n^\pm})$.

Theorem A. *Under the conditions above, implication* (1) *is valid.*

Note that in the two-dimensional case ($n = 2, n^\pm = 1$) the conditions of Theorem A are automatically satisfied. Denote $\widehat{\mathfrak{N}^\pm} = \bigcup_{m=0}^{n^\pm - 1} (J^\pm)^m (\mathfrak{N}^\pm)$. Then the condition of Theorem A which is related to the displacement of the homoclinic points r_j on the separatrix $W^\pm$ is equivalent to that the $\mathbb{R}$-linear hull of the set $\widehat{\mathfrak{N}^\pm}$ coincides with $\mathbb{R}^{n^\pm}$ (see Lemma 3 below).

Note, as a possible application, that Theorem A allows one to build multi-dimensional non-integrable diffeomorphisms as the direct products of lower-dimensional ones. This provides a variety of multi-dimensional examples (constructed, for instance, from two-dimensional diffeomorphisms with transversal homoclinic points). We consider here the case of the direct product of the identical copies of a non-integrable diffeomorphism, postponing the general discussion until Section 8.

Theorem B. *Let a mapping* $S_1 : M_1 \to M_1$ *have a hyperbolic fixed point* q *and a set* $\mathfrak{P} = \{r_j\}$ *of homoclinic points such that the conditions of Theorem* A *are satisfied with some* N. *Denote* $\widehat{\mathfrak{P}} = \bigcup_{m=0}^{\max\{n^+,n^-\}-1} S_1^m(\mathfrak{P})$. *Then the direct product of* k *copies of* S_1, $S = \underbrace{S_1 \times \cdots \times S_1}_{k} : M \to M$, *where* $M = \underbrace{M_1 \times \cdots \times M_1}_{k}$, *has the hyperbolic fixed point* $\{q\} \times \cdots \times \{q\}$ *and the set of homoclinic points* $\widehat{\mathfrak{P}} \times \cdots \times \widehat{\mathfrak{P}}$ *satisfying the conditions of Theorem* A *with the same* N.

2. Preliminary definitions for number sets

Let complex numbers $\lambda_1, \ldots, \lambda_n$ satisfy the condition $0 < |\lambda_j| < 1$. Consider partitions of the set $\{\lambda_j\}$ into classes Λ_r.

DEFINITION 3. A partition is said to be *strongly ordered* if the classes Λ_r are sets of numbers, λ_j, whose moduli lie, respectively, in some non-intersecting intervals.

The classes Λ_r will be placed in the order of non-decreasing moduli of the numbers $\lambda \in \Lambda_r$.

Now let the numbers λ_j satisfy the non-resonance multiplicative conditions

$$|\lambda_s| \neq |\lambda^m|, \quad \text{where} \quad \lambda^m = \prod_{j=1}^{n} \lambda_j^{m_j},\ m = (m_1, \ldots, m_n) \tag{2}$$

for all indices $s\,(1 \leq s \leq n)$ and all non-negative integers $m_j\ (1 \leq j \leq n)$ such that $|m| = \sum_j m_j \geq 2$. The conditions (2) are obviously satisfied if $|m| \geq N$ where

$$N > \ln \min_j |\lambda_j| / \ln \max_j |\lambda_j| \tag{3}$$

(the collection $\{\lambda_j\}$ lies in the Poincaré domain).

DEFINITION 4. A partition is said to be *right* if the following condition is satisfied: two numbers $\lambda_{j'}, \lambda_{j''}$ belong to the same class if and only if in all inequalities (2) the signs persist under replacement of all factors $\lambda_{j'}$ by $\lambda_{j''}$ or all factors $\lambda_{j''}$ by $\lambda_{j'}$ in the left-hand side and right-hand side simultaneously.

Remark 1. This property indeed determines an equivalence relation. Thus, a right partition exists, and is unique, and is also strongly ordered. Definition 4 admits an equivalent formulation dealing with the replacement of an arbitrary part of the factors.

Analogously, one can consider complex numbers μ_j such that $|\mu_j| > 1$ and introduce strongly ordered partitions and, if the non-resonance multiplicative conditions are met, the right partitions. Obviously, this case is conjugate with the previous one by the inversion operation $\mu_j \mapsto \lambda_j = \mu_j^{-1}$. Therefore, elements of partitions for this case will be placed in the order of non-increasing moduli.

Let $\lambda_{i,j}\,(1 \le j \le n^+)$, $\mu_{i,j}\,(1 \le j \le n^-)$ be the eigenvalues of the mapping S at the point q_i and $0 < |\lambda_{i,j}| < 1 < |\mu_{i,j}|$. Suppose that there are strongly ordered partitions

$$\xi_i^+ = \{\Lambda_{i,1}^+, \dots, \Lambda_{i,p^+}^+\}, \qquad \xi_i^- = \{\Lambda_{i,1}^-, \dots, \Lambda_{i,p^-}^-\}$$

of the sets $\{\lambda_{i,j} : 1 \le j \le n^+\}$ and $\{\mu_{i,j} : 1 \le j \le n^-\}$. Moreover, assume that the quantities of classes, $p^\pm$, and the numbers of elements in each class $\mathrm{card}\Lambda_{i,s}^\pm = c_s^\pm\,(1 \le s \le p^\pm)$ do not depend on i. Here card denotes the cardinality (the quantity of elements) of a set.

Definition 5. Strongly ordered partitions $\xi_i^\pm$ which satisfy these conditions will be said to be *concordant*.

Definition 6. Concordant and right partitions $\xi_i^\pm$ will be said to be *right concordant* if in addition the following condition holds: in all inequalities (2), where $\lambda_s \in \Lambda_{i,t_0}^\pm$, $\lambda_j \in \Lambda_{i,t_j}^\pm$, the signs do not depend on i.

The last definition will be used only in a discussion of some accompanying results.

The inequalities (3) define the same number N for right concordant partitions.

3. Linearization of the mapping on separatrices. Induced map of the Grassmannian space bundle

Let $S_i^\pm = S \mid W_i^\pm$ be the restriction of a C^N-diffeomorphism S to the separatrix $W_i^\pm$ of the point q_i. This separatrix is a C^N-manifold. Suppose that $\xi_i^\pm$ is a right partition (or, more generally, that the conditions

$$\lambda_s \ne \lambda^m \tag{2'}$$

analogous to (2) are valid for the elements of sets $\{\lambda_{i,j}\}$, $\{\mu_{i,j}^{-1}\}$) and $N \ge N_i^\pm$, where

$$\begin{aligned} N_i^+ &> \ln \min_j |\lambda_{i,j}| / \ln \max_j |\lambda_{i,j}|, \\ N_i^- &> \ln \max_j |\mu_{i,j}| / \ln \min_j |\mu_{i,j}|, \end{aligned} \tag{4}$$

i.e., the numbers N_i^+ and N_i^- are defined by (3) for the sets $\{\lambda_{i,j}\}$ and $\{\mu_{i,j}^{-1}\}$. Then, by S. Sternberg's theorem [26, 48], there exists a C^N-linearization $l_i^\pm: W_i^\pm \to \mathbb{R}^{n^\pm}$ of the map $S_i^\pm$, i.e., in $\mathbb{R}^{n^\pm}$-coordinates determined on $W_i^\pm$ by the diffeomorphism $l_i^\pm$, the mapping $S_i^\pm$ has the linear form $\left(l_i^\pm\right) \circ S_i^\pm \circ \left(l_i^\pm\right)^{-1} = J_i^\pm: \mathbb{R}^{n^\pm} \to \mathbb{R}^{n^\pm}$, where $J_i^\pm$ is the matrix of the linear part of $S_i^\pm$ at the point q_i. Note that although the linearizing diffeomorphism $l_i^\pm$ is constructed in a neighbourhood of the point $q_i \in W_i^\pm$ it is extendible over the whole separatrix [48]. Denote by $L_{i,s}^\pm$ $(1 \le s \le p^\pm)$ the linear $a_s^\pm$-dimensional invariant subspace in $\mathbb{R}^{n^\pm}$ corresponding to all the Jordan blocks for the eigenvalues $\lambda_{i,j}$, $\mu_{i,j}$ which are included in the classes $\Lambda_{i,t}^\pm$ for $t \le s$. Obviously, $a_s^\pm = c_1^\pm + \cdots + c_s^\pm$. The diffeomorphism $l_i^\pm$ allows one to define in a natural way, for any point $r \in W_i^\pm$, an $a_s^\pm$-dimensional direction (subspace) $\widetilde{\mathcal{L}}_{i,s}^\pm(r)$ in the tangent space $T_r W_i^\pm \subset T_r M$ to the separatrix $W_i^\pm$ at the point r: $\widetilde{\mathcal{L}}_{i,s}^\pm(r) = \left(\dot{l}_i^\pm\right)^{-1} \big|_y \left(L_{i,s}^\pm\right)$, where $\dot{l}\,|_y$ is the differential of a map l at the point y, $y = l_i^\pm(r)$. (For each r, the subspace $\widetilde{\mathcal{L}}_{i,s}^\pm(r)$ is obtained from the subspace $L_{i,s}^\pm$ by parallel translation in the linearizing coordinates on $W_i^\pm$.) Obviously, $L_{i,p^\pm}^\pm = \mathbb{R}^{n^\pm}$, $\widetilde{\mathcal{L}}_{i,p^\pm}^\pm(r) = T_r W_i^\pm$, $a_{p^\pm}^\pm = n^\pm$. Denote by $G_m(r)$ the Grassmannian manifold of the m-dimensional tangent linear subspaces at the point $r \in M$. So, $\widetilde{\mathcal{L}}_{i,s}^\pm(r) \in G_{a_s^\pm}(r)$. For any point $r \in M$ the differential $\dot{S}\,\big|_r$ of the diffeomorphism S at r induces a diffeomorphism $G_m(r) \to G_m\big(S(r)\big)$ of the Grassmannian manifolds which will be denoted again by $\dot{S}\,\big|_r$. The set $\mathfrak{M}_m = \bigcup\limits_{r \in M} G_m(r)$ has the natural structure of a manifold whose class of smoothness is one unit less than that of M and it is the total space of the bundle with base M and fiber G_m, where G_m is the Grassmannian manifold of the m-dimensional linear subspaces in $\mathbb{R}^n$. Points of $\mathfrak{M}_m$ will be denoted as $z = (r, \sigma_r)$, where $\sigma_r \in G_m(r)$. Consider the skew product $\mathcal{S}$ of the maps S and $\dot{S}\,\big|_r$ which acts on $\mathfrak{M}_m$: $\mathcal{S}(r, \sigma_r) = \big(S(r), \dot{S}\,\big|_r(\sigma_r)\big)$, i.e., the mapping of the bundle generated by the mapping of the base. Then $\mathcal{S} \in C^{N-1}$ if $S \in C^N$.

Lemma 1

Let O be a fixed hyperbolic point of a C^N-diffeomorphism S of an n-dimensional manifold M, γ_j $(1 \le j \le n)$ be all the eigenvalues placed in the order of non-increasing moduli, n^+ and n^- be, respectively, the quantities of numbers γ_j whose moduli are less and greater than unit. Suppose in addition that m is a number such that $1 \le m < n$ and $|\gamma_j| > \varepsilon$ for $j \le m$, $|\gamma_j| < \varepsilon$ for $j > m$ (in particular, one can put $m = n^-$, $\varepsilon = 1$, if both numbers $n^\pm > 0$); $\mathfrak{M}_s^\pm = \bigcup\limits_{r \in W^\pm} G_s(r) \subset \mathfrak{M}_s$ is a C^{N-1}-manifold defined, respectively, if $n^\pm > 0$ and which is the trivial bundle whose base is the separatrix $W^\pm$ and whose fiber is the Grassmannian manifold G_s.

To the eigenvalues γ_j $(j \le m)$, there corresponds a linear m-dimensional subspace in $T_O M$, i.e., an element $\alpha_m^+ \in G_m(O)$. Analogously, to the eigenvalues γ_j $(j > m)$, there corresponds an element $\alpha_m^- \in G_{n-m}(O)$. Let $x^\pm = (O, \alpha_m^\pm)$. Then

1) for almost all $z^+ = (r, \sigma_r) \in \mathfrak{M}_m^+$, $z^- = (r, \sigma_r) \in \mathfrak{M}_{n-m}^-$ forming open everywhere dense sets, the following holds true:

$$\lim_{p \to \pm\infty} \mathcal{S}^p(z^\pm) = (O, \alpha_m^\pm) = x^\pm. \tag{5}$$

(By some misuse of language, one can rewrite this condition in the form $\lim\limits_{p\to\pm\infty} \big(\dot{S}^p\big)\big|_r (\sigma_r) = \alpha_m^\pm$, where $\big(\dot{S}^p\big)\big|_r = \dot{S}\,\big|_{S^{p-1}(r)} \circ \cdots \circ \dot{S}\big|_r$ for $p \ge 0$ and $\big(\dot{S}^p\big)\big|_r = \big(\dot{S}^{-p}\,\big|_{S^p(r)}\big)^{-1}$ for $p \le 0$ because the manifolds $G_s(r)$, $G(O)$ can be identified and $\lim\limits_{p\to\pm\infty} S^p(r) = O$.)

In particular, the point $x^\pm$ is asymptotically stable with respect to the restriction of the mapping $\mathcal{S}$ to $\mathfrak{M}_m^+$ or $\mathcal{S}^{-1}$ to $\mathfrak{M}_{n-m}^-$.

2) There exist sections of the bundles $\mathfrak{M}_{n-m}^+$ and $\mathfrak{M}_m^-$, i.e., maps

$g_m^+ = (id, f_m^+)\colon W^+ \to \mathfrak{M}_{n-m}^+$, *where* $f_m^+(r) \in G_{n-m}(r)$ *and*

$g_m^- = (id, f_m^-)\colon W^- \to \mathfrak{M}_m^-$, *where* $f_m^-(r) \in G_m(r)$,

which possess the following properties:

a) $\lim_{p\to\pm\infty} \mathcal{S}^p(z^\pm) = x^\mp$ *for $z^+ \in \mathfrak{M}_{n-m}^+$ or $z^- \in \mathfrak{M}_m^-$ if and only if $z^\pm \in g_m^\pm(W^\pm)$,*

b) $g_m^\pm \in C^{N-1}$. *Moreover, the following statement holds: the local functional map from the space of hyperbolic C^N-diffeomorphisms S into the space of regular C^{N-1}-submanifolds in $\mathfrak{M}_{n-m}$ or $\mathfrak{M}_m$, which determines $g_m^\pm(W^\pm)$, is continuous* [8]*,*

c) *$f_{n^-}^\pm(r) = T_r W^\pm$ for $r \in W^\pm$; if $m_1 > m_2$ then $f_{m_1}^+(r) \subset f_{m_2}^+(r)$ for $r \in W^+$ and $f_{m_1}^-(r) \supset f_{m_2}^-(r)$ for $r \in W^-$,*

d) $f_m^\pm(O) = \alpha_m^\mp$,

e) *for $z^+ = (r, \sigma_r) \in \mathfrak{M}_m^+$ or $z^- = (r, \sigma_r) \in \mathfrak{M}_{n-m}^-$ the condition* (5) *is not valid if and only if the linear m- and $(n-m)$-dimensional subspaces σ_r and $f_m^\pm(r)$ in $T_r M$ have a non-trivial intersection,*

f) *if there exists a C^1-linearization $l^+\colon W^+ \to \mathbb{R}^{n^+} = T_O W^+$ of the mapping $S \mid W^+$ and $m \ge n^-$, or there exists a C^1-linearization $l^-\colon W^- \to \mathbb{R}^{n^-} = T_O W^-$ of the mapping $S \mid W^-$ and $m \le n^-$, then, in accordance with these two cases, $f_m^\pm(r) = \mathcal{L}_m^\pm(r)$, where the $(n-m)$-dimensional subspace $\mathcal{L}_m^+(r)$ in the tangent space $T_r W^+$ and the m-dimensional subspace $\mathcal{L}_m^-(r)$ in the tangent space $T_r W^-$ are defined as $\mathcal{L}_m^\pm(r) = (\dot{l}^\pm)^{-1}\,\big|_y \circ (\dot{l}^\pm)\,\big|_O\,(\alpha_m^\mp)$ for $y = l^\pm(r)$.*

[8] Because of the non-compactness of the immersed submanifolds $W^\pm$, the C^{N-1}-topology under discussion is understood in the weak sense, even if M is compact.

Lemma 1 and its proof have a clear geometrical sense. In fact, this Lemma generalizes the Hadamard–Perron theorem on the existence of separatrices of a hyperbolic fixed point. So, we can use the below-described geometrical construction of the manifolds $g^{\pm}_m(W^{\pm})$ in the spirit of J. Hadamard's approach [24] to the proof of this theorem (see also an exposition in [36]; J. Hadamard considered only the two-dimensional case, but this is of no importance; the problem of the smoothness of the separatrices obtained is discussed in [27, 37]). Let $(\mathrm{id}, h)\colon W^{\pm} \to \mathfrak{M}^-_m$ be a C^{N-1}-section such that $h(O) \cap \alpha^-_m = \{0\}$. Then, under the iterations, $\mathcal{S}^p$, of the mapping $\mathcal{S}$, the manifold $(\mathrm{id}, h)(W^-)$ is transformed into a manifold which tends to $g^-_m(W^{\pm})$ in the C^{N-1}-norm as $p \to +\infty$ (this is a kind of λ-lemma, see Remark 16 below). Constructing $g^+_m(W^{\pm})$ is quite analogous. To prove this it is convenient to use a Riemannian metric such that the inequalities

$$\|T^{-1} \mid \alpha^+_m\|^{-1} > \|T \mid \alpha^-_m\|, \qquad \|T^{-1} \mid T_O W^-\|^{-1} > 1 > \|T \mid T_O W^+\|$$

are satisfied for the norms of the restriction of the operator $T = \dot{S} \mid_O$ to the corresponding invariant orthogonal subspaces in $T_O M$. Then, near the point O, the mapping $S \mid W^-$ is an expansion and, moreover, an expansion (a contraction) along the subspace α^+_m more rapid (slow) than that along α^-_m occurs under the action of the mapping S. Fix $\rho > 0$ and identify the operators $\alpha^-_m \to \alpha^+_m$ whose norm is $\leq \rho$ with their graphs. So, we equip a closed subset $K^- \subset G^-_m(O)$ with the corresponding structure of the ρ-ball in the linear normed space. A neighbourhood of O in M can be identified with a vicinity of zero in the tangent space $T_O M$ using a chart. Let $V \subset W^-$ be a small enough ball centered at the point O. If B is the set of the C^{N-1}-bundles over V whose fibers are elements of $K^- \subset G_m(x)$, $x \in V$ (i.e., the set of C^{N-1}-sections of the bundle $\bigcup_{x\in V} K^- \to V$) then $\mathcal{S}$ transfers each element of B into some section over $S(V) \supset V$ whose restriction to V belongs also to B. So, we define a mapping of B into itself that occurs to be contractive in the C^{N-1}-norm. The corresponding attracting point is $g^-_m \mid V$; g^-_m is extendible onto the whole separatrix W^- by iterations of $\mathcal{S}$. The original Hadamard's approach needs the *hyperbolicity*, i.e., the presence of expanding and contractive invariant subspaces. In the case under consideration, the roles of expanding and contractive subspaces are played by α^+_m and α^-_m, respectively. However, the extension along $T_O W^-$ is needed for the induced map of B into itself to be well-defined and contractive (see also a relevant discussion in [27]). Note that the above construction can be performed more directly, without the preliminary knowledge of the manifold W^-. Indeed, let a Riemannian norm be as above, and a neighbourhood of O in M be identified with a vicinity of zero in $T_O M$. Given $\rho > 0$, consider the "rectangle"

$D^+ \times D^-$ whose sides $D^\pm$ are small closed balls in $T_O W^\pm$. Consider the space Σ^- of continuous mappings $f^-: D^- \to D^+$ whose Lipschitz constants are $\leq \rho$, and identify functions f^- and their graphs. Then the correspondence $\alpha \mapsto S(\alpha) \cap D$ for $\alpha \in \Sigma^-$ will determine a well-defined mapping, $\mathfrak{S}^-$, which is contractive in the C^0-norm. Its unique fixed point is just the piece of the separatrix W^- inside D. If one considers only C^{N-1}-functions f^- then this map is contractive in an appropriate C^{N-1}-norm. The latter norm is built of the C^0-norms of the j-th derivatives with corresponding weight coefficients κ_j that decrease quickly as j rises, $0 \leq j < N$ (cf. the entirely analogous construction in the proof of Proposition 5 in Subsection 6.2). Equip each manifold $\alpha \in \Sigma^-$ with the space B^-_α of the C^{N-1}-sections of the bundle $\bigcup\limits_{x\in\alpha} K^- \to \bigcup\limits_{x\in\alpha} \{x\} = \alpha$. Then the mapping $\mathcal{S}$ induces a mapping of $\bigcup\limits_{\alpha\in\Sigma^-} B^-_\alpha$ into itself which lies over the mapping $\mathfrak{S}^-$ and has a unique attracting point—the required section $g^-_m \mid D \cap W^-$. In particular, one can prove in this way that the separatrix W^- is a C^N-manifold [27, 37] because $f^-_{n^-}(r) = T_r W^-$ for $r \in W^-$.

Now we explain briefly how to prove the second part of item 2b). The basic fact that will be used many times in the sequel is that for any integer $k \geq 0$ the composition $(f, g) \mapsto f \circ g$ defines a continuous map $(\cdot,\cdot): C^k \times C^k \to C^k$ (under an appropriate choice of the domains and ranges of the mappings f and g). Using the implicit function theorem, one can obtain the following result. The contractive mapping $\mathfrak{S}^-$ in the space of C^{N-1}-functions, f, is well-defined and depends continuously on $S \in C^N$ (in fact, it is Lipschitz) under small perturbations of the latter. Moreover, its Lipschitz constant can be chosen to be locally uniform, being < 1. Therefore, the fixed point $f \in C^{N-1}$ identified with $W^-_{loc} \equiv W^- \cap D$ depends also continuously on $S \in C^N$. For simplicity, one can identify W^-_{loc} with D^- using the projection along D^+. Then the map $S^{-1} \mid W^-_{loc}$ is represented as a C^{N-1}-map $r: D^- \to D^-$. The contractive mapping $G: B \to B$ defined above, $B = B^-_{W^-_{loc}}$, acts by the formula $(Gl)(z) = \dot{S}\mid_{x(z)} l\big(r(z)\big)$, where $x(z) = \Big(r(z), f\big(r(z)\big)\Big)$ and $l(z) \in K^-$, $z \in D^-$. It follows immediately from this formula and the definition of r that for each $l \in B$, $Gl \in B$ depends continuously on $S \in C^N$. Finally, a fixed point of G, $g_m \mid W^-_{loc} \in C^{N-1}$, depends continuously on $S \in C^N$ because the Lipschitz constant < 1 of the contraction map G can be chosen to be uniform under small perturbations of S. Indeed, let S' be a sufficiently small perturbation of S, let G and G' be the corresponding mappings $B \to B$, and let l and l' be fixed points of G and G', respectively. Then $G'l$ is close to $Gl = l$ and, therefore, the fixed point of G', l', is close to l.

Some comments concerning an elementary proof of items 1) and 2e) will be given in Appendix B in a slightly more general context. Other items are obvious. $\square$

This geometrical Hadamard's approach was denoted as *"graph transform"* in [27].

Remark 2. Let $\widetilde{W}^-_{x^+} = g^-_m(W^-)$, $\widetilde{W}^+_{x^-} = g^+_m(W^+)$,

$$\begin{aligned} \widetilde{W}^+_{x^+} &= \big\{(r,\sigma_r) \in \mathfrak{M}^+_m : \sigma_r \cap f^+_m(r) = \{0\}\big\}, \\ \widetilde{W}^-_{x^-} &= \big\{(r,\sigma_r) \in \mathfrak{M}^-_{n-m} : \sigma_r \cap f^-_m(r) = \{0\}\big\}. \end{aligned}$$

Then $\widetilde{W}^+_{x^+}$ and $\widetilde{W}^-_{x^+}$ ($\widetilde{W}^+_{x^-}$ and $\widetilde{W}^-_{x^-}$, respectively) are the incoming and outgoing manifolds of the fixed point x^+ (x^-) of the homeomorphic C^{N-1}-mapping $\mathcal{S}\colon \mathfrak{M}_m \to \mathfrak{M}_m$ ($\mathcal{S}\colon \mathfrak{M}_{n-m} \to \mathfrak{M}_{n-m}$). If $N \geq 2$ then x^+ (x^-) is a hyperbolic point of the diffeomorphism $\mathcal{S}$ and, thus, the manifolds under discussion are separatrices. This fact is easily proven using the following simple statement: the spectrum at a fixed point (α_0, β_0) of a fiber mapping (i.e., one covering some mapping of the base) $F(\alpha,\beta) = \big(f(\alpha), g_\alpha(\beta)\big)$ of a bundle is the union of the corresponding spectra at the fixed point β_0 of the mapping g_{α_0} of the invariant fiber and at fixed point α_0 of the mapping f of the base.

DEFINITION 7. An element $\sigma_r \in G_m(r)$, where $r \in W^+$ (respectively, an element $\sigma_r \in G_{n-m}(r)$, where $r \in W^-$), is said to be *in general position with respect to* W^+ (W^-) if (5) is satisfied.

In particular, in our case the following substitutions can be done. The point r can be any point on W^+_i (respectively, on W^-_i), the set $\{\gamma_1, \dots, \gamma_n\}$ can be the set $\{\mu_{i,1}, \dots \mu_{i,n^-}, \lambda_{i,n^+}, \dots, \lambda_{i,1}\}$, the number m (respectively, the number $n-m$) can be the number a^-_s, where $s \leq p^-$ (respectively, the number a^+_s, where $s \leq p^+$), and the set $\{\gamma_j, j \leq m\}$ (respectively, the set $\{\gamma_j, j > m\}$) can be the set $\bigcup_{t\leq s} \Lambda^-_{i,t}$ (respectively, the set $\bigcup_{t\leq s} \Lambda^+_{i,t}$). Denote by $\widetilde{\mathcal{L}}^+_{i,s}(r)$, where $r \in W^+_i$ and $s \leq p^+$ (respectively, by $\widetilde{\mathcal{L}}^-_{i,s}(r)$, where $r \in W^-_i$ and $s \leq p^-$), the element $f^+_{n-a^+_s}(r)$ ($f^-_{a^-_s}(r)$) for the hyperbolic point $O = q_i$. If the partition $\xi^\pm_i$ is right then there is a linearization $l^\pm_i$ on the separatrix $W^\pm_i$ which will determine the element $\widetilde{\mathcal{L}}^\pm_{i,s}(r)$ by the way discussed above.

4. A lemma on a linear contraction map

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear nondegenerate operator with eigenvalues whose moduli are less than the unit, and $\mathfrak{N} \subset \mathbb{R}^n$ be some set. We shall formulate conditions under which the functions F, defined in a neighborhood of the origin $0 \in \mathbb{R}^n$, possess the property

$$F \in C^N(0), \quad F \mid \mathfrak{N}_\infty \equiv 0 \Rightarrow dF(0) = 0, \tag{6}$$

where $\mathfrak{N}_l = \bigcup_{k=0}^{l} T^k(\mathfrak{N})$.

Let λ_r be all distinct eigenvalues of the linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$, $0 < |\lambda_r| < 1$. In the Jordan (maybe complex) form of T, to each number λ_r there correspond the following blocks (cells) of sizes $s(u) \times s(u)$:

$$T_{r,u} = \lambda_r \begin{pmatrix} 1 & 1 & & & \\ & \cdot & \cdot & 0 & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & 0 & & \cdot & 1 \\ & & & & 1 \end{pmatrix}$$

where the index u ranges over some set U_r (the sets U_r do not intersect for distinct r). The index couple $(u, i) = i'$, where $1 \le i \le s(u)$, determines the number of a line or a column in the Jordan form of T. Let $\mathfrak{N} = \{a_v\}$, where $a_v = (a_{1,v}, \ldots, a_{n,v}) \in \mathbb{C}^n$ are the coordinates in the Jordan basis, let the number N satisfy the inequality (3), and let the non-resonance conditions $(2')$ be valid.

Lemma 2

The property (6) *holds if and only if for all* r

$$\text{rang } A_r = \text{card } M_r, \tag{7}$$

where $A_r = \bigl(a_{i',v} : i' \in M_r\bigr)$, $M_r = \bigl\{(u, s(u)) : u \in U_r\bigr\}$, rang *denoting the rank of a matrix and* card, *the cardinality of a set.*

Let $L_l \subset \mathbb{R}^n$ be the linear hull of the set $\mathfrak{N}_l$.

Lemma 3

The conditions (7) *are equivalent to that* $L_{n-1} = \mathbb{R}^n$.

Corollary. *The property* (6) *persists under small perturbations of the set* $\mathfrak{N}$ *and the mapping* T.

To prove Lemmas 2 and 3, firstly one establishes, using the Taylor expansion, that condition (6) for a function F implies this condition for the linear part $dF(0)$ of the mapping F at zero. Thus, the proof is reduced to a problem of linear algebra.

Indeed, for a given point $a_v \in \mathbb{C}^n$, the coordinates of the image $x = T^k a_v$ are linear combinations of terms $\lambda_r^k P(k)$, where $P(k)$ are polynomials of degrees not greater than n. Substitute the expression for $x = T^k a_v$ in the equation $F(x) = 0$ using the Taylor formula $F(x) = Q(x) + O\big(\|x\|^N\big)$, where $Q(x)$ is a polynomial of degree less than N. As the result, one obtains the relation

$$0 = \sum_l \lambda_l^k P_l(k) + O(\rho^k), \quad k \to +\infty, \tag{8}$$

where l ranges over the set of all indices r and multiindices $m = (m_1, \dots, m_n)$ such that $|\lambda^m| \geq \min |\lambda_r| = \nu$, the notation $\lambda_m = \lambda^m$ is introduced, $P_l(\cdot)$ are some polynomials, and the number ρ satisfies the inequalities

$$\max\big\{|\lambda^m| : |\lambda^m| < \nu\big\} < \rho, \qquad \max_r |\lambda_r|^N < \rho, \qquad \rho < \nu.$$

Then $\lambda_r \neq \lambda^m$ due to condition $(2')$ and all the numbers λ_r are distinct. One can assume that all the numbers λ_m in formula (8) are also distinct. Let the leading term of the polynomial $P_l(k)$ be $A_l k^{d_l}$ and the index l runs over p values. Then

$$P_l(K+k) = A_l K^{d_l}\big(1 + o(1)\big) \quad \text{as} \quad K \to +\infty, \quad 0 \leq k < p.$$

Replacing k by $K + k$, $0 \leq k < p$, the equation (8) takes the form

$$\sum_l H_l \lambda_l^k \big(1 + o(1)\big) = g_k, \quad K \to +\infty, \quad \text{where} \quad H_l = A_l K^{d_l} \lambda_l^K, \quad g_k = O(\rho^K).$$

Therefore, $H_l = O(\rho^K)$ for large K because the Vandermonde matrix $\big(\lambda_l^k\big)_{1 \leq l, k+1 \leq p}$ is well-known to be nondegenerate if all the numbers λ_l are distinct. So,

$$A_l = O\Big(\big(\rho/\lambda_l\big)^K K^{-d_l}\Big) \to 0, \quad K \to +\infty$$

and $A_l = 0$, i.e., $P_l \equiv 0$. It remains only to observe that the sum $\sum_r \lambda_r^k P_r(k) \equiv 0$ is obtained by substituting the expressions for x into the linear part $dF(0)$.

Elementary proofs of Lemma 2 in the case of linear F and Lemma 3 that will be omitted here are based on the following Remark 3. The subsequent Remark 5 easily shows that the contraction maps T, such that condition (7) is satisfied for a given set $\mathfrak{N} \neq \{0\}$, are everywhere dense. $\square$

Remark 3. The criterion (7) does not depend on the choice of the Jordan basis. Let V_r be the linear subspace in $\mathbb{C}^n$ of dimension $\operatorname{card} M_r$ that is spanned by the basis vectors with numbers $i' \in M_r$ and let $\pi = \pi_r : \mathbb{C}^n \to V_r$ be the natural projection along the complement subspace $\widetilde{V}_r$ that is spanned by the other basis vectors and that is just the image of the operator $T - \lambda_r \cdot \mathrm{id}$. Obviously, condition (7) is equivalent to the following one: the set $\pi(\mathfrak{N}) \subset V_r$ spans the whole space V_r that can be identified with the factor-space $\mathbb{C}^n/\widetilde{V}_r = \mathbb{C}^n/\mathrm{im}(T - \lambda_r \cdot \mathrm{id}) = \mathrm{coker}(T - \lambda_r \cdot \mathrm{id})$. This provides a completely coordinate-free description of criterion (7).

Remark 4. Basis vectors corresponding to complex-conjugate eigenvalues can be also chosen to be complex-conjugate. Thus, if $\lambda_r \in \mathbb{R}$ then the linear spaces $V_r, \widetilde{V}_r \subset \mathbb{C}^n$ are the complexifications of the corresponding real linear spaces $V_r^{\mathbb{R}}$, $\widetilde{V}_r^{\mathbb{R}}$ (i.e., $V_r = \mathbb{C} \otimes V_r^{\mathbb{R}}$, $\widetilde{V}_r = \mathbb{C} \otimes \widetilde{V}_r^{\mathbb{R}}$). Therefore, the projection π is also the complexification of the natural projection $\pi_r^{\mathbb{R}} : \mathbb{R}^n \to V_r^{\mathbb{R}}$ along $\widetilde{V}_r^{\mathbb{R}}$ and $\pi(\mathfrak{N}) \subset \pi(\mathbb{R}^n) = V_r^{\mathbb{R}}$. The condition (7) is equivalent to the following one: the set $\pi_r^{\mathbb{R}}(\mathfrak{N})$ spans the whole space $V_r^{\mathbb{R}}$. If $\lambda_{r'}$ and $\lambda_{r''}$ are complex-conjugate and distinct then $\widetilde{V}_{r'} = \widetilde{V}_{r',r''} \oplus V_{r''}$, $\widetilde{V}_{r''} = \widetilde{V}_{r',r''} \oplus V_{r'}$, $\widetilde{V}_{r',r''} = \mathbb{C} \otimes \widetilde{V}^{\mathbb{R}}_{r',r''}$, $V_{r',r''} = V_{r'} \oplus V_{r''} = \mathbb{C} \otimes V^{\mathbb{R}}_{r',r''}$ (where $V^{\mathbb{R}}_{r',r''}$, $\widetilde{V}^{\mathbb{R}}_{r',r''}$ are real spaces) and the spaces $V_{r'}$ and $V_{r''}$ are complex-conjugate. Then the projection $\pi_{r',r''} : \mathbb{C}^n \to V_{r',r''}$ along $\widetilde{V}_{r',r''}$ is the complexification of the projection $\pi^{\mathbb{R}}_{r',r''} : \mathbb{R}^n \to V^{\mathbb{R}}_{r',r''}$ along $\widetilde{V}^{\mathbb{R}}_{r',r''}$ and the projection $\pi_{r'} : \mathbb{C}^n \to V_{r'}$ along $\widetilde{V}_{r'}$ decomposes into the product $\widetilde{\pi}_{r'} \circ \pi_{r',r''}$, where $\widetilde{\pi}_{r'} : V_{r',r''} \to V_{r'}$ is the projection along $V_{r''}$.

Consider in more detail the form of criterion (7) in the latter case for the real set $\mathfrak{N}$. Thus, the set $G = \pi^{\mathbb{R}}_{r',r''}(\mathfrak{N}) \subset V^{\mathbb{R}}_{r',r''}$ has to be such that its image under the projection $\widetilde{\pi}_{r'}$ (respectively, $\widetilde{\pi}_{r''}$) $\mathbb{C}$-spans the whole space $V_{r'}$ ($V_{r''}$). The complex-conjugate subspaces $V_{r'}$ and $V_{r''}$ have the trivial intersection only and, consequently, they do not contain real non-zero vectors. It is easily seen that $V_{r'} = \{z + iJ(z) : z \in V^{\mathbb{R}}_{r',r''}\}$ and $V_{r''} = \{z - iJ(z) : z \in V^{\mathbb{R}}_{r',r''}\}$ where J is a linear operator in the real even-dimensional space $V^{\mathbb{R}}_{r',r''}$ such that $J^2 = -\mathrm{id}$. Here, $\widetilde{\pi}_{r'}(2z) = z + iJ(z)$ and $\widetilde{\pi}_{r''}(2z) = z - iJ(z)$ for any $z \in V^{\mathbb{R}}_{r',r''}$. Therefore, condition (7) is equivalent to that $G \cup J(G)$ $\mathbb{R}$-spans the whole space $V^{\mathbb{R}}_{r',r''}$. Concerning the minimal number of elements of G, we note the following fact. For any non-zero vector $z \in V^{\mathbb{R}}_{r',r''}$, the linear space spanned by z and $J(z)$ is two-dimensional and J-invariant. Therefore, G has to contain a subset G' consisting of $\frac{1}{2} \dim_{\mathbb{R}} V^{\mathbb{R}}_{r',r''} = \frac{1}{2} \dim_{\mathbb{C}} V_{r',r''}$ elements and such that $G' \cap J(G') = \emptyset$ and $G' \cup J(G')$ spans the whole $V^{\mathbb{R}}_{r',r''}$.

Remark 5. Mappings T such that the elements of maximal (i.e., the finest) strongly ordered partitions for the set $\{\lambda_i\}$ are of type $\{\gamma\}$, where $\gamma \in \mathbb{R}$, or $\{\alpha \pm i\beta\}$, where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, form an open everywhere dense subset. In this case the condition (7) is equivalent to $\pi_r^{\mathbb{R}}(\mathfrak{N}) \not\subset \{0\}$ or $\pi_{r',r''}^{\mathbb{R}}(\mathfrak{N}) \not\subset \{0\}$, respectively. Thus, for the set $\mathfrak{N}$ it is sufficient (but not necessary) to contain a point that does not belong to the union of a finite number of linear subspaces of codimension 1 or 2.

DEFINITION 8. A set $\mathfrak{P}$ of points on the separatrix $W_i^{\pm}$ is said to be *in general position* if there is a linearization $l_i^{\pm}$ and the conditions (7) are valid for the set $\mathfrak{N} = l_i^{\pm}(\mathfrak{P})$ and the mapping $T = \left(J_i^{\pm}\right)^{\pm 1}$.

This definition generalizes the one used in Theorem 1. Obviously, under non-resonance conditions (2′) for the eigenvalues of $S \mid W_i^{\pm}$, the set $\mathfrak{P} \subset W_i^{\pm}$ is in general position on the separatrix $W_i^{\pm}$ if and only if from conditions $F \in C^N\left(W_i^{\pm}\right)$ and $F \mid \mathfrak{P}_\infty \equiv 0$, where $\mathfrak{P}_\infty = \bigcup_{k=0}^{\infty} S^{\pm k}(\mathfrak{P})$ and the number N was described earlier, it follows that $dF(q_i) = 0$.

5. The main result and a discussion

Theorem 2

Let the following conditions be satisfied:

1) the eigenvalues $\lambda_{i,j}$ and $\mu_{i,j}$ for all i are such that there exist concordant partitions $\xi_i^{\pm} = \left\{\Lambda_{i,s}^{\pm};\ 1 \le s \le p^{\pm}\right\}$ of the sets $\{\lambda_{i,j}\}$ and $\{\mu_{i,j}\}$;
2) there is i^+ (and also i^-) such that the partition $\xi_{i^+}^{+}$ ($\xi_{i^-}^{-}$) is right and the set of homo-(hetero)clinic points r_j lying on the separatrix $W_{i^+}^{+}$ ($W_{i^-}^{-}$), i.e., points such that $i^+ = i^+(j)$ ($i^- = i^-(j)$), is in general position;
3) the diffeomorphism S has class C^N, where $N \ge N_{i^\pm}^{\pm}$ and the numbers $N_i^{\pm}$ are defined in accordance with (4);
4) for any j and $1 \le s < p^{\pm}$ the element $\widetilde{\mathcal{L}}_{i^\pm(j),s}^{\pm}(r_j) \in G_{a_s^\pm}(r_j)$ is in general position with respect to the separatrix $W_{i^\mp(j)}^{\mp}$.

Then property (1) holds. Moreover, in a neighbourhood of any own point the set A' does not lie on a regular C^N-submanifold of positive codimension nor even on the union of a countable set of such submanifolds.

Remark 6. The condition that the element $\widetilde{\mathcal{L}}_{i^\pm(j),p^\pm}^{\pm}(r_j) \in G_{n^\pm}(r_j)$ is in general position with respect to $W_{i^\mp(j)}^{\mp}$ is equivalent to that the homo-(hetero)clinic point $r_j \in W_{i^-(j)}^{-} \cap W_{i^+(j)}^{+}$ is transversal. Therefore, the case $s = p^{\pm}$ is omitted in the statement of the Theorem. The condition 4) of the Theorem together with the transversality condition mentioned above are equivalent to the following one:

$$f_{i^-(j),m}^{-}(r_j) \cap f_{i^+(j),m}^{+}(r_j) = \{0\}, \tag{9}$$

where $f^{\pm}_{i,m}$ are the mappings $f^{\pm}_{m}$ for the hyperbolic point $O = q_i$ and the number m ranges over the set

$$a^-_s \ (1 \le s < p^-), \qquad n^- = a^-_{p^-} = n - a^+_{p^+}, \qquad n - a^+_s \ (p^+ > s \ge 1)\,. \tag{10}$$

To formulate condition 4), one has to assume the validity of condition 1).

Remark 7. Let $N \ge 2$ and m be a number in the set (10). Denote by x^+_i, x^-_i and by $\widetilde{W}^{\pm}_{i,+} = \widetilde{W}^{\pm}_{x^+_i}$, $\widetilde{W}^{\pm}_{i,-} = \widetilde{W}^{\pm}_{x^-_i}$ the hyperbolic points x^+, x^- of C^{N-1}-diffeomorphisms $\mathcal{S}\colon \mathfrak{M}_m \to \mathfrak{M}_m$, $\mathcal{S}\colon \mathfrak{M}_{n-m} \to \mathfrak{M}_{n-m}$ and their separatrices $\widetilde{W}^{\pm}_{x^+}$, $\widetilde{W}^{\pm}_{x^-}$ that correspond to the point $O = q_i$. The condition (9) is equivalent to the following one: the couple of separatrices $\widetilde{W}^{-}_{i^-(j),+}$ and $\widetilde{W}^{+}_{i^+(j),+}$ ($\widetilde{W}^{-}_{i^-(j),-}$ and $\widetilde{W}^{+}_{i^+(j),-}$, respectively) of the hyperbolic points $x^+_{i^-(j)}$ and $x^+_{i^+(j)}$ ($x^-_{i^-(j)}$ and $x^-_{i^+(j)}$) intersect transversally at $\widetilde{r}^{\,+}_j = \big(r_j, f^-_{i^-(j),m}(r_j)\big)$ $\big(\widetilde{r}^{\,-}_j = (r_j, f^+_{i^+(j),m}(r_j))\big)$. So, the hyperbolic points x^+_i (x^-_i) and the trajectories of homo-(hetero)clinic points $\widetilde{r}^{\,+}_j$ ($\widetilde{r}^{\,-}_j$) form a homoclinic structure that projects homeomorphically by the natural projection $\mathfrak{M}_m \to M$ ($\mathfrak{M}_{n-m} \to M$), onto the homoclinic structure formed by the points q_i and the trajectories of the points r_j.

Remark 8. The conditions of the Theorem are satisfied on an open subset in the space of diffeomorphisms, S, of a finite-dimensional manifold M, i.e., they persist under small perturbations of S. These conditions are also satisfied on an open everywhere dense subset in the space of all diffeomorphisms S possessing a homoclinic point. Indeed, a transversal homoclinic point r can be produced by an arbitrarily small perturbation of the mapping S. One can achieve, by additional small perturbations, that: **a)** the eigenvalues λ_j and μ_j at the fixed hyperbolic point satisfy the non-resonance conditions (2) and, moreover, the elements of the maximal strongly ordered partitions are of the form $\{\gamma\}$, where $\gamma \in \mathbb{R}$, or $\{\alpha \pm i\beta\}$, where $\alpha, \beta \in \mathbb{R}$, $\beta \ne 0$, **b)** conditions 2) and 4) of the Theorem are satisfied. One uses here the fact that the Hadamard–Perron theorem on the existence of the separatrices of a hyperbolic fixed point and Sternberg's theorem admit "uniform variants": the separatrices as C^N-manifolds and the linearizations as C^N-mappings depend continuously on the C^N-mappings under consideration. As for the first theorem, the latter result is well-known and follows, for instance, from Lemma 1. As for the second one, this will be explained in Remark 14 below.

The above considerations are valid in the C^N-variants where $N \in \mathbb{N} \cup \{\infty\}$, N being bounded from below by condition 3) of Theorem 2. Moreover, the following two opposite extensions of these results are possible.

i) where the C^N-mapping S is treated in the C^k-topology for arbitrary $1 \leq k \leq N$ provided that it remains uniformly C^N-bounded from above (over each compact set). Indeed, firstly, the uniform C^N-boundedness of the diffeomorphism S guarantees the uniform C^N-boundedness of its separatrices as immersed C^N-manifolds. Thus, an appropriate uniform variant of the Hadamard–Perron theorem is valid. Secondly, an analogous uniform variant of the Sternberg theorem is also valid as will be explained in Remark 14 below.

ii) where the diffeomorphism S is considered in the analytic (C^ω-) category $\mathrm{Diff}^\omega(M)$. Fortunately, if each connected component of the real analytic manifold M has a countable base then the technique developed in [10] (see also [43]) allows us to carry over immediately the corresponding C^∞-results to the analytic case (the only problem is to establish the C^ω-density, see Historical comment 1 below)[9]. For this purpose, one has only to reformulate, in succession, the presence of a transversal homoclinic point and assumptions **a)** and **b)** as specific "Kupka–Smale" or transversality properties. We explain briefly the latter term and discuss the C^ω-topologies. Firstly, the desired assumptions should be expressed as abstract transversality conditions, and, secondly, the following should hold. Recall that all the desired assumptions are considered in succession, one by one, as mentioned above. Then for any S satisfying the "preceding" assumptions, there is an appropriate transversal C^∞-unfolding of S, S_μ, i.e., a perturbation with a compact support not depending on a small (multi)parameter μ such that the family S_μ satisfies the above abstract transversality condition. Note that, usually, a suitable C^∞-unfolding is actually constructed while proving the C^∞-density [10] (see Historical comment 1). The topology τ in $\mathrm{Diff}^\omega(M)$ is defined originally for the case where M is a Euclidean space and then it is directly carried over to the general case of a real analytic manifold M with a countable base (see details in [10, 43]). This transfer is done via the following classic results by H. Grauert [23] (see also [35]): any real analytic manifold M with a countable base can be analytically embedded into a Euclidean space as a closed submanifold and any analytic function on M is extendible to an analytic function on the whole ambient Euclidean space. Any topology in $\mathrm{Diff}^\omega(M)$ that is finer than the C^N-one but coarser than τ is also suitable (cf. [10, 43]). Indeed, the density is guaranteed by that in the topology τ and the openness is guaranteed by that in the C^N-topology. Because of the results described in item **i)**, one may consider topologies in $\mathrm{Diff}^\omega(M)$ that are not finer than τ and not coarser than C^1, provided that the mapping S remains uniformly C^N-bounded from above (over compact sets).

[9] The technique of [10] requires the manifold M to have a countable base. Recall that, anyway, we have to restrict our consideration to a finite number of connected components, to which the homoclinic structure belongs. This guarantees the validity of the assumption.

Returning to our assumptions, we mention that the desired reformulation is an elementary task. As for the existence of a transversal homoclinic point (cf. [10]), one should only utilize local coordinates on M depending smoothly on S, in which one separatrix becomes independent of S; the standard proof of the C^∞-density provides us with a transversal C^∞-unfolding. The task is also easy for assumption **a)** and condition 2) of the Theorem because one has just to avoid a finite number of equalities, each defining, locally, a submanifold, cf. [43]. Finally, as for condition 4) of the Theorem, it is reduced, due to Remark 7, to the assumption on the existence of a homoclinic point. So, we get the assumption discussed before.

Historical comment 1. Usually the proof of the C^∞-density is done using a C^∞-perturbation with a compact support which is not applicable in the analytic case. However, H. W. Broer and F. M. Tangerman [10] (see also references therein) were able to overcome this difficulty using suitable real analytic perturbations and the transversality theory. Firstly, they have shown how given a C^∞-unfolding, S_μ, of an analytic diffeomorphism, to construct a C^∞-close C^ω-unfolding, $S_{\mu,t}$, $t > 0$. The key idea is to use the solutions of the heat equation whose initial conditions are connected with the representatives of the C^∞-unfolding. Let w_μ be a C^∞-unfolding of the zero vector field $w_0 \equiv 0$ with a compact support and let $w_{\mu,t}$, $t \geq 0$, be the solution of the heat equation with the initial condition $w_{\mu,0} = w_\mu$. Then $w_{\mu,t} \to w_\mu$ as $t \to +0$ uniformly in C^k, $k \geq 1$, over compact sets and, for fixed $t > 0$, $w_{\mu,t} \to w_{0,t} \equiv 0$ in C^ω as $\mu \to 0$. The representatives of the desired C^ω-unfolding of S are defined as $S_{\mu,t} = \varphi_{\mu,t} \circ S$ where $\varphi_{\mu,t}$ is the time one flow map of $w_{\mu,t}$ and $t > 0$. Secondly, another basic idea is to represent the desired property as the transversal "Kupka–Smale" property and to use the Thom Transversality Theorem. Therefore, if the unfolding $S_\mu = S_{\mu,0}$ satisfies the transversality condition then there are μ's arbitrarily close to $\mu = 0$ such that $S_{\mu,t}$ possesses the desired property.

J. Palis attached kindly the author's attention to papers [5, 29, 38, 40, 41] where the centralizers of diffeomorphisms satisfying some rather special conditions of the Morse–Smale type were considered. It happens, in particular, that under some conditions the centralizer is trivial, i.e., is generated by the diffeomorphism itself. Incidentally, the constancy of any continuous integral for systems under consideration is evident from the fact that almost every trajectory tends to some basic sets Ω_i being an attractor and a repeller (see below).

Let T be a linear contraction nondegenerate operator. Determine the number N by the spectrum of T in accordance with (3). According to [29], the C^N-centralizer $Z(T)$, i.e., the group of the C^N-diffeomorphisms commuting with T, contains only polynomials of degrees $< N$ (the proof of this fact is almost identical to that for

Sternberg's theorem). Moreover, $Z(T)$ contains only linear mappings if multiplicative non-resonance conditions $(2')$ are satisfied. On the other hand, if the spectrum of T is simple (i.e., there are no multiple eigenvalues) then a linear mapping commutes with T if and only if it is diagonalizable in the same coordinates where T is diagonal. Using this result and Sternberg's theorem, B. Anderson, J. Palis, and J. C. Yoccoz [5, 38, 40, 41] have obtained the following results that we will describe in detail. Let $\mathrm{Diff}(M)$ be the set of the C^∞-diffeomorphisms on a given compact connected boundaryless manifold M, and let $\mathfrak{A}(M)$ be the open subset of the diffeomorphisms satisfying axiom A and the (strong) transversality condition. Recall that $f \in \mathrm{Diff}(M)$ satisfies *axiom A* if the set of non-wandering points, $\Omega(f)$, is hyperbolic and the set of periodic points, $P(f)$, is dense in $\Omega(f)$. Then f is also said to satisfy *the (strong) transversality condition* if for all $x, y \in \Omega(f)$ the incoming ("stable") manifold of x and the outgoing ("unstable") manifold of y are transversal (concerning the invariant manifolds of points of a hyperbolic set, see the beginning of Section 6). Axiom A is obviously satisfied if $\Omega(f)$ is a finite hyperbolic set, i.e., $\Omega(f) = P(f)$ is formed by periodic hyperbolic points. *Morse–Smale diffeomorphisms* are those possessing the latter property and satisfying the strong transversality condition. Finally, let $\mathfrak{A}_1(M)$ be the open subset of $\mathfrak{A}(M)$ formed by diffeomorphisms that exhibit either a sink (periodic attractor) or a source (periodic repeller). Morse–Smale diffeomorphisms form an open subset $MS(M)$ in $\mathfrak{A}_1(M)$. J. Palis and J. C. Yoccoz [40] have proven that the C^∞-centralizer is trivial for C^∞-diffeomorphisms in: **1)** an open and dense subset of $\mathfrak{A}_1(M)$, **2)** an open and dense subset of $\mathfrak{A}(M)$ if $\dim M = 2$ or a residual (Baire second category) subset of $\mathfrak{A}(M)$ if $\dim M \geq 3$. The "localization" argument shows that it suffices to consider the open subset in $\mathfrak{A}(M)$ (respectively, $\mathfrak{A}_1(M)$) formed by the diffeomorphisms with the following properties: **i)** if p is a k-periodic point with $k \leq K$ then both the parts of the spectrum of f^k at p, which lie inside and outside the unit circle, are simple and satisfy $(2')$, **ii)** if p, p' are periodic points of the same period $k \leq K$ then their spectra are distinct unless p, p' belong to the same orbit. Then, firstly, one establishes structurally stable conditions that impose some restrictions on the centralizer and are analogous to those used in Remark 8. Secondly, the following holds due to the fact that these conditions are related to a disposition of some geometrical objects on the separatrices of hyperbolic periodic points as well as to the spectra for the restrictions of the diffeomorphism to these separatrices: arbitrarily small perturbations can lead to the validity of the required conditions (similarly to Remark 8). However, the considerations of [40] are more complicated than ours. In particular, while proving result **1)**, one has to consider some invariant sets or foliations in the basin of attraction of a sink (repulsion of a source), instead of double-asymptotic points. An essential ingredient of the proofs is also the Smale spectral decomposition theorem: if $f \in \mathfrak{A}(M)$ then $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_l$ where each Ω_i, called a *basic set*, is close, f-invariant and topologically transitive. Using the technique [10], J. Rocha [43] carried over the results of [40] to the C^∞-centralizers of C^ω-diffeomorphisms, restricting himself in case **1)** to Morse–Smale diffeomorphisms only. This restriction seems to be caused by the absence of an appropriate analytic perturbation theory for the above-mentioned invariant sets or foliations in the basin of attraction (repulsion). Some further more particular results can be also found in [40, 43].

Using some special arguments, in addition to the localization principle for $K = 1$ and Sternberg's theorem, J. Palis and J. C. Yoccoz [41] have proven that for an open and dense subset of Anosov C^∞-diffeomorphisms of the torus $\mathbb{T}^n$, the C^∞-centralizer is also trivial. The results and ideas of [40] generalize those of [5, 38]. B. Anderson [5] have proven that the C^∞-centralizer is trivial for an open and dense subset of $MS(M)$. On the other hand, J. Palis [38] stated that the C^∞-centralizer is C^0-discrete for an open and dense subset of $\mathfrak{A}(M)$. The first step in arguments outlined in [38] is as follows. If h is a homeomorphism near the identity on M and $h \circ f = f \circ h$ then $h \equiv \mathrm{id}$ on each basic set Ω_i. This has motivated item a) of our Proposition 4. The considerations of [5, 38] are easily carried over to the C^∞-centralizers of C^ω-diffeomorphisms.

Note that the case of infinite smoothness was discussed in [5, 38, 40, 41]. Nevertheless, it is easily seen that all the statements remain valid if one considers the C^N-centralizer for diffeomorphisms in a small C^N-neighbourhood of a diffeomorphism $f \in C^N$, provided that the number N exceeds numbers (4) corresponding to the spectra of the restrictions of the mapping f to the separatrices of periodic points that are used in the proof. Moreover, one can consider C^N-diffeomorphisms in the C^k-topology, $1 \le k \le N$, provided that they remain uniformly C^N-bounded from above. Note in this connection that the statement of [5] on the C^3-openness is wrong because of the reason explained in Remark 14 below.

Problem 1. To carry over all the results of [40, 41] to the C^∞-centralizers of C^ω-diffeomorphisms (as it was pointed out, the paper [43] contains particular results in this direction).

Remark 9. It often happens in applications that the mapping S on M is the *first return map (Poincaré mapping)* of some flow v. Obviously, an integral F_S of the mapping S and that F_v of the flow v are related by the formula $F_S = F_v \mid M$ and exist or do not exist simultaneously (under the proper assumptions about the smoothness).

We proceed to the problem of the existence of an analytic symmetry group for the mapping S and flow v. Recall that the flow v is a one-parameter group of diffeomorphisms, $\{v^t : t \in \mathbb{R}\}$, generated by a vector field that we denote again as v. ¿From now on we will consider only analytic vector fields v in contrast to mappings S that should be only smooth. The analyticity of the flow v will be an essential ingredient of the proofs. Our technique does not allow us to weaken this assumption.

Proposition 3

Under assumptions (1) *or if the conditions of Theorem* 1 *are satisfied*

a) the mapping S has no one-parameter nontrivial symmetry group,
b) any one-parameter symmetry group of the flow v is generated (modulo a linear change of a parameter) by this flow.

The definition of the centralizer for a flow is entirely analogous to that for a diffeomorphism. So, the centralizer $Z(v)$ consists of diffeomorphisms f, for which the flow v is a one-parameter symmetry group. The concept of the triviality of the centralizer remains also valid (now the group (v) is just the flow v). However, we should correct the definition of the discreteness of the centralizer.

DEFINITION 9. The centralizer $Z(v)$ of a flow v is *discrete* if for any $f \in Z(v)$ there is a neighbourhood $\mathcal{V}$ of f in the space of diffeomorphisms such that $Z(v) \cap \mathcal{V} \subset f \circ (v) \equiv (v) \circ f = \left\{ f \circ v^t \equiv v^t \circ f : t \in \mathbb{R} \right\}$.

Proposition 3 is a direct corollary of the following.

Proposition 4

Under the conditions of Proposition 3

a) the analytic centralizer $Z(S)$ of the mapping, or
b) the analytic centralizer $Z(v)$ of the flow
is discrete in the compact–open (weak C^0-) topology.

Proof of Propositions 3 and 4. First of all, while proving Proposition 4, it suffices to establish the discreteness of the centralizer at the identity diffeomorphism id. This means that there is a neighbourhood $\mathcal{V}$ of id such that **a)** $Z(S) \cap \mathcal{V} = \{\mathrm{id}\}$ or **b)** $Z(v) \cap \mathcal{V} = (v) \cap \mathcal{V}$. Second, items b) concerning the flow v can be immediately derived from the corresponding items a) concerning the first return map due to the fact that the proofs of items a) are based on the presence of S-invariant compact topologically transitive key sets C (the direct proofs for items b) can be also obtained by a simple modification of those for items a)). We explain this in detail. Let $\widetilde{M}$ be the manifold carrying the flow v. In a small neighbourhood of $M \subset \widetilde{M}$, V, the natural projection $\pi\colon V \to M$ along the trajectories of the flow v is well-defined. We desire the inclusion $\tilde{f}(M) \subset V$ for all the diffeomorphisms $\tilde{f}$ on $\widetilde{M}$ that are close to id. This is true only if M is compact. Fortunately, in the sequel, we are able to replace M by a region in M, M', which has the compact closure in M, contains the set C and is a DKP of the latter. Instead of $\widetilde{M}$ we will consider the manifold $\widetilde{M'} \subset \widetilde{M}$ filled with the trajectories crossing $M(= M')$ at least once.

Suppose now that $Z(S)$ is discrete at id. If $\tilde{f} \in Z(v)$ is close enough to id then $\tilde{f}(M) \subset V$ and the map $f = \pi \circ \tilde{f} \mid M$ belongs to $Z(S)$ and is close to id. Thus, $f \equiv \mathrm{id}$ and $\tilde{f}(r) = v^{\lambda(r)}(r)$ for every $r \in M$ where $\lambda\colon M \to \mathbb{R}$ is some continuous function. This formula and the function λ are then extendible onto the whole manifold $\widetilde{M'} \subset \widetilde{M}$. Here, λ happens to be a first integral of the flow v. Next, the set $\widetilde{C}$, filled with the trajectories passing through the points of C, is a v-invariant topologically transitive key set. Hence, $\lambda \mid \widetilde{C} = \lambda_0 \equiv \mathrm{const}$. Finally, analytic mappings $\tilde{f}$ and v^{λ_0} coincide over the key set $\widetilde{C}$ and are therefore identical everywhere. The discreteness of $Z(v)$ at id has been proven.

Now it suffices to prove item a) of Proposition 4. Nevertheless, because of simplicity, we present a reduction of item b) to item a) in Proposition 3 and a direct proof of item a). If a vector field $\tilde{u}$ generates a one-parameter symmetry group of the flow v then the infinitesimal variant of the above consideration yields the following conclusion. There is a one-parameter (local) symmetry group of S generated by a vector field u defined by the formula $u(r) = \pi_r\big(\tilde{u}(r)\big)$, $r \in M$, where $\pi_r : T_r\widetilde{M} \to T_r M$ is the projection along the vector $v(r)$ of the flow. So, if u vanishes on M then $\tilde{u}(r) = \lambda(r)v(r)$ for every $r \in M$ where λ is a continuous function. The rest of the proof that $\tilde{u} \equiv \lambda_0 v$ everywhere with some constant λ_0 is analogous to that presented above.

Now we prove item a) of Proposition 3. In the set A' of quasi-random motions, the set of periodic hyperbolic points, P, of the mapping S is everywhere dense. The set P is a key set. Recall that under the conditions of Theorem 1, the set B is also a key set. It is easily seen that a vector field generating a symmetry group vanishes upon P or B and, thus, is identical to zero.

Finally, we have to prove item a) of Proposition 4. If $f \in Z(S)$ is close enough to id, which will be assumed in the sequel, then $f \mid C \equiv \mathrm{id}$. This implies the desired result $f \equiv \mathrm{id}$ due to the key property of C. The proof is very simple in the case $C = B$ (under the conditions of Theorem 1). The symmetry f permutes hyperbolic fixed points. So, $f(O^{\pm}) = O^{\pm}$ and f preserves the separatrices $W^{\pm}$. Next, f preserves also the isolated double-asymptotic point[10] r which implies $f \mid B \equiv \mathrm{id}$. Now let implication (1) hold. Recall that the compact set A' is the maximal S-invariant set in an open region. Therefore, $f(A') \subseteq A'$. Applying the inverse symmetry, one sees that $f(A') \supseteq A'$. So, f preserves A'. It is well-known that the restriction of S to the hyperbolic set A' is *expansive* (see, for instance, [4, 37]), i.e., there exists an expansive constant $\delta > 0$ such that for any distinct points $x, y \in A'$ the distance between $S^m(x)$ and $S^m(y)$ is greater than δ for some $m \in \mathbb{Z}$. (In our case, this is a direct corollary of the expansiveness of the corresponding TMC.) Therefore, $f \mid A' \equiv \mathrm{id}$. (We note that the arguments used here for the set A' are also applicable for the set B.) This completes the proof. $\square$

Remark 10. All the results remain valid if one considers meromorphic integrals instead of analytic ones. Indeed, at first, assume that a first integral has the form $F = f/g$ where f and g are analytic functions, $g \not\equiv 0$. Then $g \neq 0$ over an open set, U. Recall that there exists an S-invariant topologically transitive key set C. Then $U \cap C$ is a key set. Indeed, $h \mid U \cap C \equiv 0 \Rightarrow hg \mid C \equiv 0 \Rightarrow hg \equiv 0 \Rightarrow h \mid U \equiv 0 \Rightarrow h \equiv 0$ for any function h analytic in a connected DKP of C. The function F is analytic at the points of $U \cap C$ and the latter set is S-invariant and topologically transitive. One can assume that $F \mid U \cap C \equiv 0$. Then $f \mid U \cap C \equiv 0$ and, consequently, $F \equiv 0$ via $f \equiv 0$. A meromorphic function is defined as one locally represented as the quotient of complex holomorphic (or real analytic) functions. The problem of such a global representation is called the *Poincaré problem*. It has been positively solved for the functions over the polycylinder domains in $\mathbb{C}^n$ with simply connected cofactors in $\mathbb{C}$ [20, 44], and even for those over an arbitrary domain in $\mathbb{C}^n$ [28]. It is easy to choose a C^1-chart that covers a small vicinity of the closure of the DKP of B while local coordinates range over a rectangular parallelepiped $I = I_1 \times \cdots \times I_n$ where I_ν are closed intervals in $\mathbb{R}$. Then, C^1-approximating the coordinate functions by analytic ones (for instance, with the use of the Weierstrass Approximation Theorem [35]), one gets analytic local coordinates ranging over $\operatorname{int} I$ and covering the DKP. Consider the meromorphic first integral, F, as being expressed via these coordinates. Then F is a function meromorphic in the desired polycylinder domain $D = D_1 \times \cdots \times D_n$ where $D_\nu = \{z : \mathrm{Re} z \in \operatorname{int} I_\nu, |\mathrm{Im} z| < \varepsilon_\nu\} \subset \mathbb{C}$ provided that $\varepsilon_\nu > 0$ were chosen to be sufficiently small. As for the set A', one does not need to cover the whole A' by a chart. Fortunately, it suffices to consider, instead of A' itself, the intersection of A' and a neighbourhood of any of its points. Other considerations remain unchanged.

Analogously, Proposition 3 remains valid if one considers meromorphic vector fields that generate *local symmetries*, i.e., local phase flows commuting with the mapping S or the flow v.

[10] We mean that the double-asymptotic point is isolated in the topology of the immersed submanifolds $W^\pm$.

Note that, in the case of an analytic diffeomorphism S, every nontrivial single-valued first integral or vector field generating a local symmetry, has to be not meromorphic at one of the points $O^{\pm}$, if the conditions of Theorem 1 are satisfied, or at each point of the set A' of quasi-random motions, if assumption (1) is met. This is a direct consequence of the following two facts. Firstly, the images of any neighbourhood of $\{O^{+}, O^{-}\}$ under iterations of S cover a DKP of B, and, secondly, the images of a neighbourhood of any of the points of A' cover the whole A'.

6. Proof of main Theorem 2. Nonautonomous linearization on stable or unstable manifolds

6.1. Two basic Lemmas and proof of the main Theorem

It is well-known that the set A described by the Alekseev theorem is a hyperbolic set and for each point $r \in A$, are defined the *incoming ("stable")* W_r^{+} and the *outgoing ("unstable")* W_r^{-} local C^N-manifolds that depend (in the C^N-norm) continuously on r. They are quite analogous to the separatrices of a hyperbolic periodic point, and coincide with them if the point r is periodic (see, for example, [2, 27, 32, 34, 37] and a correction in [19]). These results can be proven via a nonautonomous variant of the construction described in the proof of Lemma 1 (cf. [2, 32, 34] and see Remark 16 below).

Let $\mathfrak{A}_s \to A$ be the bundle of s-dimensional tangent subspaces over A (the restriction of the bundle $\mathfrak{M}_s \to M$). For any finite number set K, let K^t, $|K|$, and $\ln K$ be the sets whose elements are obtained from the elements of K via the corresponding mathematical operations (raising to the power t and evaluating moduli and logarithms). Finally, let $[K] = \left[\min_{\lambda\in K}|\lambda|, \max_{\lambda\in K}|\lambda|\right]$ be the minimal closed interval containing $|K|$.

Lemma 4

Let a homoclinic structure generating, due to the Alekseev theorem, a set A satisfy the conditions 1), 4) *of Theorem* 2[11] *and let m be any element of the set* (10). *Then, over the set A, the sections g_m^{+} and g_m^{-} corresponding to all the hyperbolic points $O = q_i$ are extendible naturally to invariant continuous sections $u_m^{+} = (id, h_m^{+})\colon A \to \mathfrak{A}_{n-m}$ and $u_m^{-} = (id, h_m^{-})\colon A \to \mathfrak{A}_m$ that possess the same properties. More precisely speaking:*

[11] A is not assumed to be a set of quasi-random motions, i.e., the graph Γ may be disconnected.

a) $h_m^{\pm} = f_{i,m}^{\pm}$ *over* $W_i^{\pm} \cap A$,
b) $h_m^+(r) \oplus h_m^-(r) = T_r M$, $h_{n^-}^{\pm}(r) = T_r W_r^{\pm}$ *for any* $r \in A$,
c) for $z^+ \in \mathfrak{A}_m$ *or* $z^- \in \mathfrak{A}_{n-m}$ *the point* $\mathcal{S}^t(z^{\pm})$ *tends to the set* $u_m^{\mp}(A)$ *as* $t \to \pm\infty$ *(i.e., the limit set of the corresponding semitrajectory lies in* $u_m^{\mp}(A)$*) if and only if* $z^{\pm} = (r, \sigma_r)$, *where* $\sigma_r \cap h_m^{\pm}(r) = \{0\}$. *Moreover,* $\mathcal{S}^t(z^{\pm})$ *approaches* $u_m^{\mp}(A)$ *exponentially rapidly, so that the distance from the point* $\mathcal{S}^t(z^{\pm})$ *to the set* $u_m^{\mp}(A)$ *does not exceed* $\mathrm{const} \cdot \kappa^{\pm t}$, *where the number* $0 < \kappa < 1$ *does not depend on the choice of* $r \in A$.

Furthermore:

d) if $0 < m_1 < \cdots < m_p < n$, *where* $p = p^+ + p^- - 1$, *are all the elements of the set* (10), *then there exist invariant sections* (id, φ_s) *of the bundles of* $(m_s - m_{s-1})$*-dimensional subspaces, where* $1 \le s \le p+1$, $m_0 = 0$, *and* $m_{p+1} = n$, *such that*

$$h_{m_s}^- = \varphi_1 \oplus \cdots \oplus \varphi_s, \qquad h_{m_s}^+ = \varphi_{s+1} \oplus \cdots \oplus \varphi_{p+1}, \qquad \varphi_s = h_{m_s}^- \cap h_{m_{s+1}}^+ ,$$

e) under the conditions of item d), *denote by* $\varphi_s^-(r) = \varphi_s(r)$, $1 \le s \le p^-$, *and* $\varphi_s^+(r) = \varphi_{p+2-s}(r)$, $1 \le s \le p^+$, *the subspaces whose direct sums are* $T_r W_r^-$ *and* $T_r W_r^+$, *respectively. Let* $r \in A$ *be* T*-periodic. Obviously, the eigenvalues of the mappings* $S^T \mid W_r^{\pm}$ *at the fixed point* r *decompose into partitions* $\xi^{\pm} = \{\Lambda_s^{\pm} : 1 \le s \le p^{\pm}\}$ *whose elements correspond to the invariant subspaces* $\varphi_s^{\pm}(r)$. *Then the partitions* $\xi^{\pm}$ *are concordant with* $\xi_i^{\pm}$. *Moreover, the distances between different sets* $\ln|\Lambda_s^{\pm}|^{1/T} \subset \mathbb{R}$ *(i.e., ones corresponding to different subscripts or superscripts) are bounded from below by a positive constant, not depending on the choice of* $r \in A$. *Actually, these distances are* $\ge -\ln\kappa$, *where* κ *satisfies the condition of item* c) *for all* m *under consideration.*

Note that the sections (id, φ_s) separate the Lyapunov exponents for the trajectories in A. A simplified version of this result concerning the periodic points is given in item e) of the Lemma. Some relevant results in a more general aspect will be discussed in Appendix A. Analogously to Lemma 1, this Lemma has a clear geometrical sense and its simple proof will be omitted here. Let us mention that this proof can be performed in a manner analogous to the one used below in Remarks 16 and 19. So, "negative" $\{S^t(r), t \le 0\}$ and "positive" $\{S^t(r), t \ge 0\}$ "semitrajectories" of the point r are considered to build the tangent subspaces $h_m^-(r)$ and $h_m^+(r)$, respectively. This geometrical construction of the bundles $h_m^-(r)$ ($h_m^+(r)$, respectively) can also be interpreted in somewhat different terms: the mapping $\mathcal{S}: \mathfrak{A}_m \to \mathfrak{A}_m$ ($\mathcal{S}^{-1}: \mathfrak{A}_{n-m} \to \mathfrak{A}_{n-m}$) induces an automorphism of the space of continuous sections $u = (\mathrm{id}, h): A \to \mathfrak{A}_m$ ($u = (\mathrm{id}, h): A \to \mathfrak{A}_{n-m}$) which possesses an attracting point—the required section u_m^- (u_m^+) (cf., for example, [27, points (2.11)–(2.13)]).

An elementary proof of item c) will be given in Appendix B. □

Remark 11. If a diffeomorphism S is of class C^2 then $u_m^+(A)$ ($u_m^-(A)$, respectively) is an invariant set of the diffeomorphism $\mathcal{S}\colon \mathfrak{M}_m \to \mathfrak{M}_m$ ($\mathcal{S}\colon \mathfrak{M}_{n-m} \to \mathfrak{M}_{n-m}$, respectively) that exists due to Alekseev's theorem near the homoclinic structure described above in Remark 7 and that covers the homoclinic structure of the diffeomorphism S. If $S \notin C^2$ then the points x_i^+ (x_i^-) and the trajectories of the points $\widetilde{r}_j^+$ ($\widetilde{r}_j^-$) form also an analog of homoclinic structure in the non-smooth case and the set $u_m^+(A)$ ($u_m^-(A)$) is again a maximal $\mathcal{S}$-invariant set in some neighbourhood of this "homoclinic structure".

Remark 12. Let the conditions of Lemmas 1 (for one of the two superscripts $\pm$) and 4 be valid for numbers m belonging to a collection $\mathbf{m} = (m_1, \dots, m_p)$, where $0 < m_1 < \dots < m_p < n$. Denote $n - \mathbf{m} = (n - m_p, \dots, n - m_1)$. Then one can pass in the formulations of the mentioned Lemmas from the mappings of bundles $\mathfrak{M}_m$ or $\mathfrak{M}_{n-m}$ to the mappings of tangent bundles $\mathfrak{M}_{\mathbf{m}}$ or $\mathfrak{M}_{n-\mathbf{m}}$ of flags $\mathcal{F}_{\mathbf{s}}$ of type $\mathbf{s} = \mathbf{m}$ or $\mathbf{s} = n - \mathbf{m}$, i.e., $\mathfrak{M}_{\mathbf{s}} = \bigcup\limits_{r \in M} \mathcal{F}_{\mathbf{s}}(r)$, where $\mathbf{s} = (s_1, \dots, s_p)$ and $\mathcal{F}_{\mathbf{s}}(r) = \bigl\{d_{s_1} \subset \dots \subset d_{s_p} : d_{s_1} \in G_{s_1}(r), \dots, d_{s_p} \in G_{s_p}(r)\bigr\}$. Here nested sequences of spaces

$$\begin{aligned} f_{\mathbf{m}}^-(r) = {}& \bigl\{f_{m_1}^-(r) \subset \dots \subset f_{m_p}^-(r)\bigr\}, \quad f_{\mathbf{m}}^+(r) = \bigl\{f_{m_p}^+(r) \subset \dots \subset f_{m_1}^+(r)\bigr\}, \\ h_{\mathbf{m}}^-(r) = {}& \bigl\{h_{m_1}^-(r) \subset \dots \subset h_{m_p}^-(r)\bigr\}, \quad h_{\mathbf{m}}^+(r) = \bigl\{h_{m_p}^+(r) \subset \dots \subset h_{m_1}^+(r)\bigr\} \end{aligned}$$

can be treated as the images of sections $g_{\mathbf{m}}^\pm = (\mathrm{id}, f_{\mathbf{m}}^\pm)$ and $u_{\mathbf{m}}^\pm = (\mathrm{id}, h_{\mathbf{m}}^\pm)$. Remarks 2, 7 and 11 remain valid with the only change that now

$$\begin{aligned} \widetilde{W}^+_{x^+_{\mathbf{m}}} = \bigl\{(r, \sigma_r) \in \mathfrak{M}_{\mathbf{m}} \ :\ & \sigma_r = (\sigma_{r,m_1} \subset \dots \subset \sigma_{r,m_p}) \,, \\ & (r, \sigma_{r,m_i}) \in \widetilde{W}^+_{x^+_{m_i}} \quad \text{for all} \quad i\bigr\}, \\ \widetilde{W}^-_{x^-_{\mathbf{m}}} = \bigl\{(r, \sigma_r) \in \mathfrak{M}_{n-\mathbf{m}} \ :\ & \sigma_r = (\sigma_{r,n-m_p} \subset \dots \subset \sigma_{r,n-m_1}) \,, \\ & (r, \sigma_{r,n-m_i}) \in \widetilde{W}^-_{x^-_{m_i}} \quad \text{for all} \quad i\bigr\}, \end{aligned}$$

where the subscripts m_i and $\mathbf{m}$ of $x^\pm$ are used to point the corresponding number m or collection $\mathbf{m} = (m_1, \dots, m_p)$.

Lemma 5

Let the assumptions of Theorem 2 be satisfied and $V_{i^\pm}$ be a given small neighbourhood of the point $q_{i^\pm}$. Furthermore, let a number $\delta > 0$ be small enough. If *the trajectory of a periodic point $r \in A'$ spends, on the average, a relative part of the whole time that does not exceed δ out of the neighbourhood $V_{i^\pm}$ and, moreover, $S^m(r) \in V_{i^\pm}$ for $|m| < (1-\delta)T/2$, where T is the period of r, then on $W_r^\pm$ the mapping S^T possesses a linearization C^N-close to the linearization of $S \mid W_{i^\pm}^\pm$. Moreover, the partition $\xi^\pm = \{\Lambda_s^\pm : 1 \le s \le p^\pm\}$ of the eigenvalues of the mapping $S^T \mid W_r^\pm$, which was introduced in item* e) *of Lemma 4, satisfies the following condition: the sets $\left|\Lambda_s^\pm\right|^{1/T}$ are contained in small vicinities of $\left[\Lambda_{i^\pm,s}^\pm\right]$, respectively. In particular, the partition $\xi^\pm$ is right concordant with $\xi_{i^\pm}^\pm$. Besides, if the partitions $\xi_i^\pm$ are right concordant then it is sufficient to require the domain V to be small and the periodic point $r \in A'$ to be close to $q_{i^\pm}$.*

Remark 13. The assumption that $A = A'$ is a set of quasi-random motions is not used while proving Lemma 5. However, all the periodic trajectories from A which pass near the point q_i and do not coincide with it belong to the set of quasi-random motions (only due to this fact the set of quasi-random motions is dealt with in the statement of Theorem 2). The latter corresponds to the maximal connected branched subgraph Γ' of the graph Γ. This subgraph is formed just by all the closed paths passing through the vertex i.

Proof of Theorem 2. If a point $r \in A'$ is close to $q_{i^\pm}$ then on $W_r^\pm$ there exist homo-(hetero)clinic points close to those on $W_{i^\pm}^\pm$. Next, trajectories of all the points r described in Lemma 5 are everywhere dense in A'. Therefore, the required result concerning the validity of condition (1) follows easily from Lemmas 3 and 5. Moreover, the set A' in a neighbourhood U of any of its points does not lie on a regular C^N-submanifold K of positive codimension. So, to prove the second part of the statement of the Theorem, one can exploit an iterative procedure of the diagonal process-type. If $\{K_i\}$ is a sequence of C^N-submanifolds such that $\bigcup_i K_i \supset A' \cap U$ then there exists a sequence $\{U_i\}$ of open sets, for which $U_i \cap A' \neq \emptyset$ and $\overline{U}_{i+1} \subset U_i \setminus K_i$. Then for any sequence of points $r_i \in A' \cap U_i$ there is a limit point $r_\infty \in A' \setminus \bigcup_{i=1}^\infty K_i$ which completes the proof. $\square$

6.2. Proof of Lemma 5. Nonautonomous linearization

The key idea in the proof of Lemma 5 is the following one. Sternberg's proof [48] can be slightly rephrased in such a way that the linearization of a contraction mapping $T: U \to U$, $T(0) = 0$, of a neighbourhood $U \subset \mathbb{R}^n$ of the origin (under conditions (2) for the spectrum at the origin which are sharper than conditions (2′) considered in [48]) will be sought as a fixed point of a hyperbolic endomorphism $\mathcal{D}_T$

in the Banach space of C^N-mappings f. The endomorphism $\mathcal{D}_T$ is defined so that the following diagram is commutative:

$$\begin{array}{ccc} & \xrightarrow{\;\;T\;\;} & \\ \mathcal{D}_T(f+\mathrm{id})\Big\downarrow & & \Big\downarrow f+\mathrm{id} \\ & \xrightarrow[\;\;J\;\;]{} & \end{array}$$

where $T(0)=0$, $J=dT(0)$, $f\colon U\to\mathbb{R}^n$, and $f(0)=0$, $df(0)=0$. Then the operator $\mathcal{D}_T$ transforms functions defined in the image of the mapping T into functions defined in the pre-image, and $l=\mathrm{id}+f$ is the required linearization if $\mathcal{D}_T l=l$. In order to deal with the linear space of C^N-functions f whose 1-jets vanish at the origin, it is convenient to represent the required f as a fixed point of the mapping $(\cdot)\mapsto\overline{\mathcal{D}}_T(\cdot)=\mathcal{D}_T(\cdot)+d_T$, where $d_T=\mathcal{D}_T\mathrm{id}-\mathrm{id}=J^{-1}\circ T-\mathrm{id}$. Obviously, $\overline{\mathcal{D}}_{T_1\circ T_2}=\overline{\mathcal{D}}_{T_2}\circ\overline{\mathcal{D}}_{T_1}$, $\mathcal{D}_{T_1\circ T_2}=\mathcal{D}_{T_2}\circ\mathcal{D}_{T_1}$, i.e., the correspondences $T\mapsto\overline{\mathcal{D}}_T$, $\mathcal{D}_T$ are *contravariant functors*. The important Proposition 5, which utilizes the above ideas, will be prefaced by the following

Remark 14. Now we explain the proof of the Sternberg theorem under conditions $(2')$ and establish its local uniform variant as follows. The linearizing C^N-transformation depends continuously on a C^N-diffeomorphism $T\colon U\to\mathbb{R}^n$ if all the mappings are treated in the C^k-topology for arbitrary $1\le k\le N$ and T remains uniformly C^N-bounded from above. Here $N\ge 2$ is determined by inequality (3). Actually, we adopt the original proof [48]. A fixed point of the contractive map T depends continuously on $T\in C^1$. Hence, without loss of generality, one can consider perturbations of T that leave the fixed point to be located at the origin. There is a norm in $\mathbb{R}^n$ that induces an operator norm $\|\cdot\|$ such that $\|J\|<1$ and $\mu=\big\|J^{-1}\big\|\|J\|^N<1$ where $J=dT(0)\colon\mathbb{R}^n\to\mathbb{R}^n$. Let U be a small enough ball in this norm centered at the origin and let G_* be the space of C^N-functions $f\colon U\to\mathbb{R}^n$ with zero $(N-1)$-jets at the origin. Equip G_* with the norm $\|\cdot\|_*$ being the exact upper bound for the norm of the N-th differential at all the points of U. Here the N-th differential is considered as a polylinear mapping $\underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{N}\to\mathbb{R}^n$ symmetrical in each cofactor [15]. Then $\mathcal{D}_T$ is well-defined and the key fact is that $\big\|\mathcal{D}_T\mid G_*\big\|_*\le\tilde{\mu}$, the operator norm being induced by the norm $\|\cdot\|_*$ in G_*, and $\tilde{\mu}$ being close to μ if the neighbourhood U is chosen to be sufficiently small. This is easily derived from the expressions for the monomials constituting the N-th differential and the estimates $\big\|d^if\big\|=O(\varepsilon)\|f\|_*$, $0\le i<N$, and $\big\|dT(x)-J\big\|=O(\varepsilon)\|T\|_{C^2}$ where d^i denotes the i-th differential, $x\in U$, and ε is the radius of the ball U. Then $|\tilde{\mu}-\mu|\le\varepsilon\cdot\big\|J^{-1}\big\|P_N\big(\|T\|_{C^N}\big)$ for small $\varepsilon>0$ where P_N is some polynomial depending on N only. (We notice, however, that the linear operators L and $\mathcal{D}_T$ are not close as will be explained in Remark 15.) So, the operator $\mathcal{D}_T \mid G_*$ is contractive and its Lipschitz constant can be chosen to be uniform and < 1 under C^1-small perturbations of T provided that T remains uniformly C^N-bounded. There exists a C^N-transformation from the old coordinates $x \in U \subset \mathbb{R}^n$ to the new ones $y = R(x)$ such that $R(0) = 0$, $dR(0) = \mathrm{id}$ and in the coordinates y, the $(N-1)$-jet of the mapping T at the origin, $j_0^{N-1}T$, will coincide with the $(N-1)$-jet of the differential $J = dT(0)$, i.e., the mapping T will take the form $T' = R \circ T \circ R^{-1}$ such that $T' - J \in G_*$. This corresponds to performing $(N-2)$ steps of the formal normalization procedure and this is possible due to the absence of resonances. Here, for any $k < N$, the k-jet of the normalizing transformation R at zero, $j_0^k R$, is determined only by the k-jet $j_0^k T$ and depends continuously on the latter. This fact remains valid for $k = N$ if one performs an additional step of the formal normalization procedure so that the N-jet of T' at the origin will coincide with the N-jet of $J = dT(0)$. If $k = N$ or $k = N-1$ then one can put $R(x)$ to be a polynomial transformation identified with $j_0^k R$. If $k < N-1$ then one can choose uniformly C^N-bounded mapping R to be arbitrarily close to $j_0^k R$ in the C^k-norm and to have the required $(N-1)$-jet, $j_0^{N-1}R$. Then in the new coordinates y, the desired linearization is sought for as $l' = \mathrm{id} + f$ where f is a fixed point of the contractive map $\overline{\mathcal{D}}_{T'} \mid G_*$. In the original coordinates, the linearization is written as $l = l' \circ R$. One can present another version of the above proof where the preliminary transformation of coordinates is not performed. Now $R\colon U \to \mathbb{R}^n$ is a map such that $R(0) = 0$, $dR(0) = \mathrm{id}$, and $h = \mathcal{D}_T R - R \in G_*$. The desired linearization is sought for as $l = R + f$ where f is a fixed point of the contractive map $(\cdot) \mapsto \mathcal{D}_T(\cdot) + h$ in the space G_*.

Therefore, it suffices to prove the following result where all the mappings are considered in the C^k-topology. Let $g_T \in G_*$ remain uniformly C^N-bounded and depend continuously on $T \in C^N$. Denote $\widetilde{\mathcal{D}}_T(\cdot) \equiv \mathcal{D}_T(\cdot) + g_T$. The unique fixed point $f \in G_*$ of the mapping $\widetilde{\mathcal{D}}_T \mid G_*$ depends continuously on T. In fact, the desired fixed point can be expressed by the formula

$$f = \sum_{j=0}^{+\infty} \mathcal{D}_T^j g_T.$$

The series is uniformly convergent in G_*. This fact is due to the uniformity of the Lipschitz constant < 1 for $\mathcal{D}_T \mid G_*$ and the uniform C^N-boundedness of g_T. Consequently, the series is also uniformly convergent in the C^k-norm and the desired result follows immediately from here because of the continuous dependence of each term on T in the C^k-topology.

B. Anderson [5] claimed that the linearization as a C^{k-1}-mapping depends continuously on the diffeomorphism T as a C^k-mapping, without assuming T to be uniformly C^N-bounded in a neighbourhood of the fixed point. The following two examples show that the latter statement is not correct and our result is the best possible. Let $N_0 + 1$ be the minimal N satisfying (3). We construct C^∞-mappings $T\colon \mathbb{R}^n \to \mathbb{R}^n$ that are arbitrarily C^{N_0}-close to a given non-resonant linear mapping $J = dT(0)$ while the corresponding linearizations are not C^0-close in any neighbourhood of the origin. The second example is essentially one-dimensional $(n = 1)$ where $N_0 = 1$. The first example treats an arbitrary N_0 and can be easily generalized to any dimension ≥ 2. Actually, the first example is some modification of the second one.

Example 2: Let $J(x, y) = (\lambda x, \lambda^K y)$, where $N_0 < K < N_0 + 1$, and let $U \subset \mathbb{R}^n$ be a bounded neighbourhood of the origin such that $\overline{J(U)} \subset U$. Introduce a bump C^∞-function $\theta\colon \mathbb{R}^n \to \mathbb{R}$ such that $\theta \equiv 1$ over U and supp$\theta \subset J^{-1}(U)$. Define $f_r(z) = \theta(z/r)x^{N_0}$, where $z = (x, y)$ and $r > 0$. Then the j-th derivatives of f_r are $O(r^{N_0 - j})$. So, the map

$$T\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda x \\ \lambda^K y + c f_r(x, y) \end{pmatrix}$$

is $O(c)$-close to J in C^{N_0} provided that r remains bounded. On the other hand, the linearization of T and that of J are not C^0-close in any given neighbourhood of the origin if r is sufficiently small. Indeed, the desired linearization of T takes the following form in the neighbourhood U of the origin:

$$l(x, y) = \big(x, y - cg(x)\big), \quad \text{where} \quad g(x) = x^{N_0}/(\lambda^{N_0} - \lambda^K),$$

and the invariant curve $C : y = cg(x)$ represents the x-axis of the linearizing coordinates. Let (x_0, y_0) be one of the points of $C \cap \partial(rU)$ where the domain rU is obtained from U by applying the homothety with coefficient r and center at the origin. Then $y_0 = cg(x_0)$ and $x_0 \neq 0$, $x_0 = O(r)$ as $r \to 0$. Let $m > 0$ be an integer such that the point $(x_m, y_m) = T^{-m}(x_0, y_0)$ has the x-coordinate, x_m, of the order of unit. Then

$$|y_m| = |y_0||x_m/x_0|^K = \frac{c}{\lambda^{N_0} - \lambda^K}|x_m|^K|x_0|^{N_0 - K} \to \infty$$

as $r \to 0$ provided that $c \neq 0$ is not changed. Here we have used the fact that all the points (x_k, y_k), $k > 0$, are located in the complement to $J^{-1}(rU)$ where $T \equiv J$. The point (x_m, y_m) lies on the x-axis of the linearizing coordinates. This implies the desired result and shows, moreover, that the x-axis of the linearizing coordinates of T and that of J diverge.

EXAMPLE 3: Now $n = 1$, $J(x) = \lambda x$ and $\theta\colon \mathbb{R} \to \mathbb{R}$, $f_r(x) = \theta(x/r)x$ will be defined as above. The map $T(x) = \lambda x + c\bigl(x - f_r(x, y)\bigr)$ is $O(c)$-close to J in C^1 over any given bounded neighbourhood of the origin. It is easily seen that for small r the linearization of T and that of J are not C^0-close in the neighbourhood of the origin.

We proceed to the following important

Proposition 5

Let the following "concordance" and "uniformity" conditions be satisfied for C^N-mappings $T_i\colon U \to \mathbb{R}^n$ of a domain $U \subset \mathbb{R}^n$ that have the common fixed point $0 \in U$:

i) the sets of eigenvalues for the operators $J_i = dT_i(0)\colon \mathbb{R}^n \to \mathbb{R}^n$ possess the right concordant partitions $\xi_i = \bigl\{\Lambda_{i,s} : 1 \le s \le p\bigr\}$, and, moreover, invariant subspaces M_s corresponding to the elements $\Lambda_{i,s}$ of the partitions ξ_i are coincided. The number N is defined by the inequalities of type (3) via the eigenvalues of the operators J_i;

ii) moreover, some norm in $\mathbb{R}^n$ and its restrictions to M_s induce operator norms $\|\cdot\|$ such that $\bigl\|J_i\bigr\| < 1$ and the partition into classes $\Bigl\{\bigl\|J_{i,s}\bigr\|, \bigl\|J_{i,s}^{-1}\bigr\|^{-1}\Bigr\}$, $J_{i,s} = J_i \mid M_s$, that correspond to right concordant partitions ξ_i, are also right concordant with ξ_i. These conditions must be persistent under δ-perturbations of the operators $J_{i,s}$, where $\delta > 0$ is some number;

iii) the C^N-norms of T_i and T_i^{-1} are uniformly bounded.

Then in a small enough ball (in the sense of the norm from condition ii)*) U, centered at the origin $0 \in \mathbb{R}^n$, all the mappings T_i are contractive. Besides, there exist a number $0 < \mu < 1$, a direct sum decomposition $E = E^+ \oplus E^-$ of the space E of C^N-functions $f\colon U \to \mathbb{R}^n$ whose 1-jets vanish at the origin, and a norm in E (that is equivalent to the original one) such that the following statements hold. Each linear operator $L_i = \mathcal{D}_{J_i}$ is hyperbolic with separatrices E^+ and E^- and, moreover, $\Bigl\|\bigl(L_i \mid E^\pm\bigr)^{\pm 1}\Bigr\| \le \mu$ for the corresponding operator norms. The functional spaces $E^\pm = E^\pm(U)$ do not depend on the choice of U in the sense that for $U' \subset U''$, restricting the functions $f \in E^\pm(U'')$ onto U' generates the natural projections of $E^\pm(U'')$ onto $E^\pm(U')$. Obviously, to the decomposition of E, there correspond the block forms of the linear operators $\mathcal{D}_{T_i}$:*

$$\mathcal{D}_{T_i} = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},$$

where $A_i\colon E^+ \to E^+$, $B_i\colon E^- \to E^+$, $C_i\colon E^+ \to E^-$, and $D_i\colon E^- \to E^-$ are linear operators. Next, given $\varepsilon > 0$, the small enough neighbourhood U of the origin and the norm in E can be chosen in such a manner that in addition the following condition will hold: for all i the norms of the block components satisfy estimates $\|A_i\| \le \mu + \varepsilon$, $\|B_i\| \le \varepsilon$, $\|C_i\| \le \varepsilon$, and $\bigl\|D_i^{-1}\bigr\| \le \mu + \varepsilon$, and the norms of the vectors d_{T_i} do not exceed unit.

Corollary

If the subscript i ranges over the set $\mathbb{N}$ then, under the conditions of Proposition 5, there exists a unique sequence of uniformly C^N-bounded mappings l_i possessing at the origin a tangency of (at least) first order with the identity map id and such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{T_{-1}} & & \xrightarrow{T_0} & & \xrightarrow{T_1} & & \xrightarrow{T_2} & \cdots \\
 & & {\scriptstyle l_{-1}}\big\downarrow & & {\scriptstyle l_0}\big\downarrow & & {\scriptstyle l_1}\big\downarrow & & \\
\cdots & \xrightarrow[J_{-1}]{} & & \xrightarrow[J_0]{} & & \xrightarrow[J_1]{} & & \xrightarrow[J_2]{} & \cdots
\end{array} \tag{11}$$

Historical comment 2. Recently the author found out that, in essence, this construction had been considered by Y. Yomdin [49] a few years ago. He called the mappings l_i *"nonautonomous linearization"*. The simplicity of the spectra assumed by Y. Yomdin is not essential. On the other hand, he considered a slightly more general case where the sequences $\{T_i\}$, $\{J_i\}$, and $\{l_i\}$ grow not faster than exponentially. This generalized situation can also be easily described in terms of itinerary schemes as it will be done in Remark 16. Y. Yomdin dealt also with some special situation where all the elements of the sequence $\{J_i\}$ coincide and possess a simple spectrum that lies in the Siegel domain, and, moreover, all the mappings are analytic (we treat a situation where the spectra belong to the Poincaré domain).

Remark 15. In Proposition 5, the linear operators $\mathcal{D}_{T_i}$ and $L_i = \mathcal{D}_{J_i}$ are not close, although the norms of their block components admit the close upper estimates. Indeed, let $T\colon U \to U$ be a C^N-mapping possessing a nondegenerate fixed point and let J be the corresponding linear part of T. Assume that a mapping $T'\colon U \to U$ is C^N-close to T. Then, using the chain rule, one can represent the N-th differential of $(\mathcal{D}_{T'} - \mathcal{D}_T)(f)$ at a point $x \in U$ as a sum of monomials such that the norm of one of them,

$$J^{-1}\Big(d^N f \mid_{T'(x)} - d^N f \mid_{T(x)}\Big)\Big(\underbrace{dT \mid_x, \cdots, dT \mid_x}_{N}\Big),$$

can be estimated from above only in terms of the continuity modulus of $d^N f$, while the norms of the other monomials admit a suitable upper estimate $\|T' - T\|_{C^N} \cdot \|J^{-1}\|^2 P_N\Big(\|T\|_{C^N}\Big) \cdot \|f\|_{C^N}$ where P_N is a polynomial depending on N only. Therefore, Proposition 5 is unimprovable in the above-mentioned sense if one imposes no restrictions on the continuity modulus of the N-th differential of f. Due to the same reason, the operator $\mathcal{D}_T$ does not depend continuously on $T \in C^N$ although $\mathcal{D}_T f$ does for each $f \in E$.

Remark 16. The previous Corollary is easily established by standard means of hyperbolic theory (the finite-dimensional case with invertible mappings S_i is usually considered. We deal here with the *infinite-dimensional Banach case with non-invertible* S_i). We sketch the underlying geometrical ideas that are very natural and can be found in many works (see also a relevant discussion in [6] and [27]). These ideas are "nonautonomous" and a many-dimensional variant of the above-described J. Hadamard's "graph transform" approach. In essence, we will construct a mathematical object that was called an *itinerary scheme* by V. M. Alekseev [2, Part 1]. Different names were attached to the elements of the below-defined spaces Σ^-, Σ^+. For example, they can be called *unstable* and *stable slices*. Let $R > 0$ be large enough and $D = D^+ \times D^-$, where $D^\pm$ is an R-ball in $E^\pm$ (centered at zero). Consider spaces $\Sigma^\pm$ of continuous mappings $f^\pm: D^\pm \to D^\mp$ whose Lipschitz constants are $\le \rho^\pm$, where the positive numbers ρ^- and ρ^+ are such that $\rho^-\rho^+ < 1$. We will identify functions $f^\pm$ and their graphs. Let $S = S_i = \overline{\mathcal{D}}_{T_i}$. Then the correspondences $\alpha \mapsto S^{\mp 1}(\alpha) \cap D$, for $\alpha \in \Sigma^\pm$, will determine well-defined contracting (in the usual C^0-norm) mappings $\mathfrak{S}_i^\pm = \mathfrak{S}^\pm: \Sigma^\pm \to \Sigma^\pm$ (although the mappings S_i are non-invertible) if small enough $\varepsilon > 0$ was chosen depending on μ and $\rho^\pm$. The Lipschitz constant $\lambda < 1$ of these contracting mappings and a lower bound for the radius R depend only on μ, $\rho^\pm$, and ε. Thus, to (11) there corresponds a double-infinite chain

$$\cdots \xleftarrow{\overline{\mathcal{D}}_{T_{-1}}} \xleftarrow{\overline{\mathcal{D}}_{T_0}} \xleftarrow{\overline{\mathcal{D}}_{T_1}} \xleftarrow{\overline{\mathcal{D}}_{T_2}} \cdots . \tag{12}$$

Then the sets $\mathfrak{S}_{i+1}^- \circ \mathfrak{S}_{i+2}^- \circ \cdots \circ \mathfrak{S}_m^-(\Sigma^-)$ and $\mathfrak{S}_i^+ \circ \mathfrak{S}_{i-1}^+ \circ \cdots \circ \mathfrak{S}_{-m}^+(\Sigma^+)$ shrink, respectively, exponentially fast to some elements $\alpha_i^- \in \Sigma^-$ and $\alpha_i^+ \in \Sigma^+$ as $m \to +\infty$. The intersection $\alpha_i^- \cap \alpha_i^+$ consists of a single point due to the inequality $\rho^-\rho^+ < 1$ and coincides with the mapping $l_i - \mathrm{id}$. Due to the affine character of the mappings S, one can consider in our case only linear mappings $f^\pm$, i.e., the affine stable and unstable slices.

We notice that the mapping $\mathfrak{S}^+$ of the stable slices is well-defined and contractive although S may be non-invertible. A simple construction of $\mathfrak{S}^+$, that is based on the invertibility of the restrictions of the mapping S onto unstable slices, can be found in [27, Proof of Theorem 5.1; 25; 33] and will be shortly described here. A stable slice $\alpha \in \Sigma^+$ will be transformed into the set $\mathfrak{S}^+(\alpha) = S^{-1}(\alpha) \cap D \in \Sigma^+$ which is the graph of a function $g: D^+ \to D^-$ to be constructed as follows. The mapping S transfers an unstable slice $\beta_\xi = \{\xi\} \times D^- \in \Sigma^-$, $\xi \in D^+$, into its image, whose piece which is located inside $D = D^+ \times D^-$ is an unstable slice $\widetilde{\beta_\xi} = \mathfrak{S}^-(\beta_\xi)$. Then $\big(\xi, g(\xi)\big) \in D^+ \times D^-$ is the pre-image of a unique point of intersection, $\alpha \cap \widetilde{\beta_\xi}$, under the invertible map $S \mid \beta'_\xi: \beta'_\xi \to \widetilde{\beta_\xi}$, where $\beta'_\xi = S^{-1}(\widetilde{\beta_\xi}) \cap \beta_\xi$.

We return to the problem of constructing the topological equivalence between TMC and the restriction of the diffeomorphism S to an invariant set A. The descriptive exposition of the relevant ideas can be also found in the lectures [32]. The basic property of the diffeomorphism S allowing one to build the itinerary scheme, i.e., to choose the proper graph Γ and the subsets D_i and $l_{i,j}$, is *hyperbolicity*, or more precisely, the possibility to decompose the tangent space at any point x in some subset of M into the direct sum of two (many-dimensional) maybe non-invariant[12] subspaces such that the expansion occurs along one of them (E_x^-) and the contraction takes place along the other (E_x^+) (in an appropriate metric on M). The sets D_i are chosen as "rectangles" $D_i^+ \times D_i^-$ whose sides $D_i^\pm$ are parallel to these subspaces and are closed balls or, more precisely, are nearly closed balls (in the mentioned metric). More rigorously speaking, using a chart, a small vicinity U of the point x of the manifold M can be identified with a neighbourhood of zero in the tangent space T_xM. This makes possible to construct the required set D_i in U. Denote by $\Sigma_i^\pm$ the corresponding spaces $\Sigma^\pm$ for the rectangle $D_i^+ \times D_i^-$. Next, only "good" intersections [32] $l_{i,j}$, where the rectangles $S(D_i)$ and D_j cross, must be considered[13]. In this case the corresponding maps $\mathfrak{S}^-(l_{i,j})\colon \Sigma_i^- \to \Sigma_j^-$ and $\mathfrak{S}^+(l_{i,j})\colon \Sigma_j^+ \to \Sigma_i^+$ are well-defined and contractive under a proper choice of ρ^+ and ρ^-. The precise formulations and strict estimates for these geometrically visual results can be found in [2, Part 1]and will be reproduced in the continuation (Part II) of the present paper. Thus, every sequence $\omega \in \Omega^\Pi$ determines stable $W_r^+ \in \Sigma_{\omega_0}^+$ and unstable $W_r^- \in \Sigma_{\omega_0}^-$ slices by the above-described way: the images of the compositions

$$\Sigma_{\omega_{-m}}^- \to \Sigma_{\omega_{-m+1}}^- \to \cdots \to \Sigma_{\omega_0}^- \quad \text{and} \quad \Sigma_{\omega_m}^+ \to \Sigma_{\omega_{m-1}}^+ \to \cdots \to \Sigma_{\omega_0}^+$$

shrink exponentially fast to W_r^- and W_r^+, respectively, as $m \to +\infty$. The required point $\psi(\omega) = r$ is a single point of intersection $W_r^- \cap W_r^+$. Therefore, the Markov property is valid. Note that the slice $W_r^\pm$ is determined by the sequence $[\omega_n : \pm n \geq 0]$ infinite in one direction. The mapping ψ constructed is indeed continuous because any change of far elements of the sequence ω leads only to a small perturbation of the manifolds $W_r^\pm$ and the mappings $\kappa_i\colon \Sigma_i^+ \times \Sigma_i^- \to D_i$, defined as $\kappa_i(\alpha^+, \alpha^-) = \alpha^+ \cap \alpha^-$, are continuous (in fact, Lipschitz). Moreover, the mapping ψ is easily seen to be Hölder if the space Ω is equipped with the conventional distance function

$$\rho\Big([\omega_n'], [\omega_n'']\Big) = \sum_{n=-\infty}^{+\infty} \frac{\delta_{\omega_n', \omega_n''}}{a^{|n|}} ,$$

where $a > 1$ is an arbitrary constant and $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

12 We keep in mind the invariance of the bundles formed by these subspaces with respect to the action of the tangent mapping $TS{:}TM \to TM$.

13 One could consider "good" connected components of the intersection. In this case, the multigraph Γ should be dealt with where a few edges may possess the common origins and ends. However, the equivalent *dual* graph can be considered whose vertices and edges are identified with the edges and vertices of the multigraph Γ, respectively. We emphasize that this representation was used in [2] .

We mention in this connection the following facts. Firstly, the stable, W_r^+, and unstable, W_r^-, manifolds of a point r of the hyperbolic set A can be constructed in this way (see [32]) on the basis of the corresponding itinerary scheme and "positive" $\{S^t(r), t \geq 0\}$ and "negative" $\{S^t(r), t \leq 0\}$ "semitrajectories" of the point $r \in A$. Secondly, the proof of Alekseev's theorem can be carried out via the construction of an itinerary scheme that deals with (nonlinear) diffeomorphisms S. It is closely related to the C^1-variant of the well-known λ-lemma [39; 37, Lemma 6.1]. Moreover, as it was explained, the construction of the mapping $\mathfrak{S}^+$ described above remains valid if one considers an itinerary scheme in the nonlinear Banach non-invertible case. In essence, the itinerary scheme was constructed in [25, 33] to establish an analog of Alekseev's theorem for a homoclinic structure of a smooth, possibly non-invertible map of a Banach manifold (the simplest case of a single homoclinic trajectory was discussed in [25, 33], but the result obtained is immediately transferable to the general case). Here one has to require the differential of the mapping to be uniformly continuous in a small vicinity of the trajectories of hyperbolic points q_i, otherwise the C^1-variant of the λ-lemma does not hold. The necessity of this condition was pointed out in [25] (the C^2-case was considered in [33]). The special features of the problem (the infinite dimension and non-invertibility) require also some accuracy in stating the conditions concerning the points r_j (see [25, 33]).

Remark 17. Let us consider the sequence of mappings $S_i\colon D_i \to E_{i+1}$,

$$\cdots \xrightarrow{S_{-1}} \xrightarrow{S_0} \xrightarrow{S_1} \cdots$$

(in contrast to (12) the order is direct), and rectangles $D_i = D_i^+ \times D_i^- \subset E_i$ which determine the itinerary scheme with some positive maximal "slopes" ρ^+, ρ^- of stable and unstable slices and the corresponding Lipschitz constant $0 < \lambda = \lambda(\rho^+, \rho^-) < 1$. Then the stable $\alpha_i^+ \in \Sigma_i^+$ and unstable $\alpha_i^- \in \Sigma_i^-$ slices, defined as above by the itinerary scheme, and their intersection, $r_i = \alpha_i^+ \cap \alpha_i^-$, will depend continuously on the sequence of mappings S_j. To attach the precise meaning to the last statement, one should consider $[S_i : i \in \mathbb{Z}]$ as an element of the corresponding Tychonoff product (over $i \in \mathbb{Z}$) of the spaces V_i, where V_i is the space of mappings S_i for which $S_i(D_i)$ and D_{i+1} cross and define the contractive maps $\mathfrak{S}_i^-\colon \Sigma_i^- \to \Sigma_{i+1}^-$ and $\mathfrak{S}_i^+\colon \Sigma_{i+1}^+ \to \Sigma_i^+$ with Lipschitz constant λ. Equip each V_i with the usual C^0-norm and recall that $\Sigma_i^\pm$ are equipped with the analogous norms. Then, firstly, $\mathfrak{S}_i^- \beta^- \in \Sigma_{i+1}^-$ ($\mathfrak{S}_i^+ \beta^+ \in \Sigma_i^+$, respectively) depends continuously on the map $S_i \in V_i$ and the slice $\beta^- \in \Sigma_i^-$ ($\beta^+ \in \Sigma_{i+1}^+$). Next, recall that, secondly, the C^0-diameter of $\mathfrak{S}_{i-1}^- \circ \cdots \circ \mathfrak{S}_{-m}^-(\Sigma_{-m}^-)$ ($\mathfrak{S}_i^+ \circ \cdots \circ \mathfrak{S}_{m-1}^+(\Sigma_m^+)$, respectively) tends uniformly to zero as $m \to +\infty$ for all the mappings $S_j \in V_j$, and, thirdly, the intersection map $\kappa_i\colon \Sigma_i^+ \times \Sigma_i^- \to D_i$ is continuous. It is easily seen from these results that $\alpha_i^\pm$ and $r_i = \alpha_i^+ \cap \alpha_i^-$ depend continuously on $[S_i]$. (Moreover, the mapping κ_i and the dependence of $\mathfrak{S}_i^\pm \beta^\pm$ on $S_i \in V_i$ are Lipschitz with some constants determined by ρ^+ and ρ^- only. Therefore, the dependencies of $\alpha_i^\pm$ and r_i on $S_j \in V_j$ are also Lipschitz with constant $C\lambda^{|i-j|}$ where $C = C(\rho^+, \rho^-) > 0$.) For further purposes, we notice that the analogous results can be easily proven if the spaces $\Sigma_i^\pm$ and V_i are equipped with the pointwise convergence topologies. We emphasize that these topologies are finer than the C^0-topologies if $E^\pm$ or E are infinite-dimensional, respectively. This is a direct consequence of the analogous result that in the space of linear operators $E^\pm \to \mathbb{R}$, whose norms do not exceed a given positive constant, the topology defined by the operator norm (the strong topology) is coarser than that of the pointwise convergence (the weak topology).

Proof of Proposition 5. The existence of the required ball U follows immediately from conditions ii) and iii). The operators L_i are hyperbolic and the separatrices E^+, E^- for all of them coincide if condition i) is satisfied. Indeed, let Φ be the finite set of all couples (s, m) such that $|m| \geq 2$ and $|\lambda_s| < |\lambda^m|$ in inequality (2), where $\{\lambda_j\}$ is the spectrum of J_i. Given a multiindex $m = (m_1, \ldots, m_n)$, introduce $\widetilde{m} = (\widetilde{m}_1, \ldots, \widetilde{m}_p)$, where $\widetilde{m}_t = \sum_{\lambda_j \in \Lambda_{i,t}} m_j \geq 0$, and denote $(s, \widetilde{m}) = \Psi(s, m)$ and $\widetilde{\Phi} = \Psi(\Phi)$. Then the correspondence Ψ does not depend on the number i and $\Phi = \Psi^{-1}(\widetilde{\Phi})$ due to condition i). Obviously, $|\widetilde{m}| = |m|$. Furthermore, let π_s be the projection of $\mathbb{R}^n$ onto M_s along $\bigoplus_{l \neq s} M_l$, and let $G_{s,\widetilde{m}}$ be the linear space of the polynomials of the form $P\bigl(\pi_1(\cdot), \ldots, \pi_p(\cdot)\bigr)$, where $P\colon M_1 \times \cdots \times M_p \to M_s$ is a homogeneous M_s-valued polynomial in p variables that has degree $\widetilde{m}_t$ in the t-th argument. Let

$$F = \bigoplus_{(s,\widetilde{m}) \in \widetilde{\Phi}} G_{s,\widetilde{m}}, \qquad H = \bigoplus_{\substack{(s,\widetilde{m}) \notin \widetilde{\Phi} \\ |\widetilde{m}| < N}} G_{s,\widetilde{m}} .$$

Then the outgoing separatrix E^- is F and the incoming separatrix E^+ is the space of functions that have at the origin $0 \in \mathbb{R}^n$ $(N-1)$-jets lying in H, i.e., $E^+ = H \oplus G_*$, where G_* is the space of functions with zero $(N-1)$-jets at the origin. Indeed, in the spaces $G_* \subset E$ and $G_{s,\widetilde{m}}$ invariant under L_i, the following norms $\|\cdot\|_*$ and $\|\cdot\|_{s,\widetilde{m}}$ will be induced. The norm $\|\cdot\|_*$ was defined above as the exact upper bound for the norm of the N-th differential, and $\|\cdot\|_{s,\widetilde{m}}$ is the norm in the space of polynomials P that are considered as polylinear mappings $P\colon M_1^{\widetilde{m}_1} \times \cdots \times M_p^{\widetilde{m}_p} \to M_s$ symmetrical in each cofactor $M_t^{\widetilde{m}_t}$ (the case $p = 1$ being discussed in [15] but the results being transferable immediately to the case of arbitrary $p \geq 1$). The corresponding operator norms (denoted by the same symbols) satisfy inequalities

$$\left\| \left(L_i \mid G_{s,\widetilde{m}} \right)^{-1} \right\|_{s,\widetilde{m}} \leq \mu \quad \text{for} \quad (s, \widetilde{m}) \in \widetilde{\Phi},$$
$$\left\| L_i \mid G_{s,\widetilde{m}} \right\|_{s,\widetilde{m}} \leq \mu \quad \text{for} \quad (s, \widetilde{m}) \notin \widetilde{\Phi}, \quad |\widetilde{m}| < N, \qquad \text{and}$$
$$\left\| L_i \mid G_* \right\|_* \leq \mu$$

for some $0 < \mu < 1$. Recall that the inequality for $\| \cdot \|_*$ was the basic one in [48]. In our case

$$\mu = \max \left\{ \max_{(s,\widetilde{m}) \in \widetilde{\Phi}} \| J_{i,s} \| \prod_{j=1}^{p} \| J_{i,j}^{-1} \|^{\widetilde{m}_j}; \right.$$
$$\left. \max_{(s,\widetilde{m}) \notin \widetilde{\Phi}} \| J_{i,s}^{-1} \| \prod_{j=1}^{p} \| J_{i,j} \|^{\widetilde{m}_j} \right\}. \tag{13}$$

It follows from condition ii) that the number μ can be chosen independently of i. Define a norm in

$$E = E^+ \oplus E^- = \bigoplus_{\substack{(s,\widetilde{m}) \\ |\widetilde{m}| < N}} G_{s,\widetilde{m}} \oplus G_*$$

as the direct sum (or maximum) of the norms in $G_{s,\widetilde{m}}$, G_*. Then $\left\| \left(L_i \mid E^{\pm} \right)^{\pm 1} \right\| \leq \mu$. Due to condition iii), using a homothetic transformation with center at the origin one can make the difference between T_i and J_i arbitrarily small in the C^N-norm if the neighbourhood U was chosen to be small enough (with a size of order ε). Then the operators $\mathcal{D}_{T_i}$ and $L_i = \mathcal{D}_{J_i}$ satisfy the required conditions and the norms of the vectors d_{T_i} are close to zero. However, another representation of this idea is preferable: the norm in E is built of the norms in $G_{s,\widetilde{m}}$ and G_* with corresponding weight coefficients $\kappa_{\widetilde{m}}$ and κ_* that decrease quickly as $|\widetilde{m}|$ rises (one supposes $|\widetilde{m}| = N$ for G_*),

$$\left\| \bigoplus_{\substack{(s,\widetilde{m}) \\ |\widetilde{m}| < N}} u_{s,\widetilde{m}} \oplus u_* \right\| = \kappa_* \| u_* \|_* + \sum_{\substack{(s,\widetilde{m}) \\ |\widetilde{m}| < N}} \kappa_{\widetilde{m}} \| u_{s,\widetilde{m}} \|_{s,\widetilde{m}}$$

(the sums in the right-hand side can be replaced by the maxima). The possibility of such a choice follows from the two auxiliary results. Firstly, the operator $\mathcal{D}_T$ has triangular form: if $G_t = \bigoplus_{\substack{s,\widetilde{m} \\ |\widetilde{m}|=t}} G_{s,\widetilde{m}}$ for $2 \le t < N$, $G_N = G_*$, and $\mathcal{D}_T\big(\oplus_t u_t\big) = \oplus_t v_t$, where $u_t, v_t \in G_t$, then

$$
\begin{aligned}
v_t &= \mathcal{D}_J u_t + \Theta_t\Big(\bigoplus_{t'<t} u_{t'}\Big), \quad t < N, \\
v_N &= \mathcal{D}_T u_N + \Theta_N\Big(\bigoplus_{t'<N} u_{t'}\Big)
\end{aligned}
\tag{14}
$$

(the subscript i is omitted for brevity). Secondly, the required estimate $\|\mathcal{D}_T \mid G_*\|_* \le \mu + \varepsilon/2$ is provided by the arguments of Remark 14 if the neighbourhood U is chosen to be of a size of order ε. Proposition 5 is proven. □

Remark 18. As a required norm in E, one can take the C^N-norm induced by the norm

$$
\|x_1 \oplus \cdots \oplus x_p\|^* = C\varepsilon^{-1} \sum_{s=1}^{p} \|x_s\|_s \tag{15}
$$

in $\mathbb{R}^n$, where $x_s \in M_s$, $\|\cdot\|_s$ are the restrictions of the original norm in $\mathbb{R}^n$ (which is used in condition ii)) to M_s, and $C > 0$ is some constant. (Recall that the t-th derivative of a mapping $\mathbb{R}^n \to \mathbb{R}^n$ is a polylinear symmetrical mapping $\big(\mathbb{R}^n\big)^t \to \mathbb{R}^n$ [15, 37] which allows one to define the C^N-norm in a coordinate-free way [37].) This result follows immediately from some ideas of the above proof (that concern the use of the homothetic transformation) and from the following facts. Firstly, for each polylinear mapping $u\colon \big(\mathbb{R}^n\big)^t \to \mathbb{R}^n$, the function $\|u(\cdot,\ldots,\cdot)\|_{(+)}$ attains its maximum over the product B^t of unit balls $B \subset \mathbb{R}^n$ of the norm

$$
\|x_1 \oplus \cdots \oplus x_p\|_{(+)} = \sum_{s=1}^{p} \|x_s\|_s
$$

at some point of the set

$$
\mathcal{B}^t = \bigcup_{s_1,\ldots,s_t} \Big(\mathcal{B}_{s_1} \times \cdots \times \mathcal{B}_{s_t}\Big),
$$

where $\mathcal{B} = \bigcup_{s=1}^{p} \mathcal{B}_s$ is the union of "ribs"

$$
\mathcal{B}_s = \big\{x = x_1 \oplus \cdots \oplus x_p :\ x_i \in M_i,\ x_j = 0 \ \text{ for } \ j \ne s,\ \|x_s\|_s = 1\big\}
$$

of the ball B (because B^t is the convex hull of $\mathcal{B}^t$). Secondly, each homogeneous polynomial $v = \bigoplus_{\substack{s,\widetilde{m}\\ |\widetilde{m}|=t}} u_{s,\widetilde{m}} \in G_t$ of degree t if considered as a polylinear symmetrical mapping $v\colon \big(\mathbb{R}^n\big)^t \to \mathbb{R}^n$ is reduced over $\mathcal{B}^t$ to one of the mappings $v_{\widetilde{m}} = \bigoplus_s u_{s,\widetilde{m}}$. More precisely speaking, $v = v_{\widetilde{m}}$ over $\mathcal{B}_{s_1} \times \cdots \times \mathcal{B}_{s_t}$, where $\widetilde{m}_i$ are the quantities of numbers $j\,(1 \le j \le t)$ such that $s_j = i$. Thirdly, the mapping $\mathcal{D}_J\colon G_t \to G_t$ in the space of polylinear mappings has the form

$$\mathcal{D}_J u(\cdot,\ldots,\cdot) = J^{-1} u\big(J(\cdot),\ldots,J(\cdot)\big)$$

and induces operators φ^- and φ^+ in the spaces of polylinear mappings

$$M_{s_1} \times \cdots \times M_{s_t} \to \bigoplus_{s\colon (s,\widetilde{m})\in\Phi} M_s \subset \mathbb{R}^n \quad \text{and}$$
$$M_{s_1} \times \cdots \times M_{s_t} \to \bigoplus_{s\colon (s,\widetilde{m})\notin\Phi} M_s \subset \mathbb{R}^n,$$

where $\big\|(\varphi^-)^{-1}\big\| \le \mu$ and $\|\varphi^+\| \le \mu$ for the corresponding operator norms, the number μ is defined by formula (13) for $t \ge 2$. Fourthly (this statement is also related to the homothetic transformation), the norm (15) induces norms in the spaces of polylinear mappings such that $\big\|d^t u_*\big\| \le \mathrm{const}\, C\varepsilon \big\|d^N u_*\big\|$ for any $u_* \in G_*$ and $t < N$, where d^t denotes the t-th differential. The proofs of the last two inequalities for operator norms and the inequality for norms of differentials are quite elementary and will be omitted. Note, however, that the norms of the polylinear mappings

$$v_{\widetilde{m}} = \bigoplus_s u_{s,\widetilde{m}}\colon M_{s_1} \times \cdots \times M_{s_t} \to \bigoplus_s M_s$$

are not expressed by any definite formula via the norms of the mappings

$$u_{s,\widetilde{m}}\colon M_{s_1} \times \cdots \times M_{s_t} \to M_s.$$

Thus, there is no definite expression for the norm $\|v\| = \max \|v_{\widetilde{m}}\|$ of homogeneous polynomial $v \in G_t$ in terms of the norms $\|u_{s,\widetilde{m}}\|_{s,\widetilde{m}}$. *A fortiori*, there is no such expression for the C^N-norm under consideration. Obviously, one can take in the right-hand side of (15) a slightly more general expression $C\varepsilon^{-1} \sum_{s=1}^{p} \alpha_s \|x_s\|_s$, where $\alpha_s > 1$.

Now we will explain briefly, omitting technical details, the basic ideas needed to prove Lemma 5. Some of these ideas will be partially revised in the sequel. First of all, assume the partitions $\xi_i^\pm$ to be right concordant. Then the invariant sections $(\mathrm{id}, \varphi_s^\pm)$ allow one to define local coordinates on $W_r^\pm$, $r \in A'$ such that for the mappings $T = S^{\pm m} \mid W_r^\pm \colon W_r^\pm \to W_{S^{\pm m}(r)}^\pm$ with large m the conditions of Proposition 5 will be satisfied (with the common invariant subspaces identified with $\varphi_s^\pm$). This enables one to construct the required linearizations. Incidentally, one can take $m = 1$ here via the use of a simple generalization of a result [11, 12] which will be discussed in Appendix A. However, if the partitions $\xi_i^\pm$ are not right concordant then these mappings, in the local coordinates under consideration, possess the common invariant subspaces M_s and the corresponding partitions $\xi^\pm$ of their eigenvalues are concordant but maybe not right. To a periodic point $r \in A'$ of period T, there corresponds a periodic chain of mappings

$$W_r^\pm \xrightarrow{S^{\pm 1}} W_{S^{\pm 1}(r)}^\pm \xrightarrow{S^{\pm 1}} \cdots \xrightarrow{S^{\pm 1}} W_{S^{\pm T}(r)}^\pm \tag{16}$$

with identified ends. In turn, to this chain there corresponds a periodic diagram (11) and an associated periodic chain

$$\xleftarrow{\overline{\mathcal{D}}_{S^{\pm 1}}} \xleftarrow{\overline{\mathcal{D}}_{S^{\pm 1}}} \cdots \xleftarrow{\overline{\mathcal{D}}_{S^{\pm 1}}} \tag{17}$$

whose endspaces are also identified and coincide with the space E of the C^N-functions over $W_r^\pm$. In a piece of relative length δ in the middle of diagram (16), there are placed "bad" mappings, for which the partitions $\xi^\pm$ of the eigenvalues do not satisfy the required condition of right concordance with $\xi_{i^\pm}^\pm$. These mappings correspond to the points of the trajectory $\{S^m(r)\}$ that are placed out of a small vicinity of $q_{i^\pm}$. However, if $\delta > 0$ is small enough then the composition of mappings (17) possesses a unique fixed point that will coincide with the required linearization (under the subtraction of the identity mapping). Moreover, the constructed linearization on $W_r^\pm$ tends to that on $W_{i^\pm}^\pm$ as $r \to q_{i^\pm}$, i.e., $T \to +\infty$, and this fact is a corollary of a continuous and (exponentially fast) decreasing dependence of the built stable and unstable slices on the mappings of the Banach space. This property is quite typical for situations described by itinerary schemes (see Remark 17). To explain this briefly, let us attribute to "bad" mappings those which occupy the part of relative length 2δ in the middle of diagram (16). Then all the other mappings tend C^N-uniformly to $S^{\pm 1} \mid W_{i^\pm}^\pm$ as $T \to +\infty$ and their influence on the linearization will vanish. Here, one uses Remark 17 and the fact that the map $\overline{\mathcal{D}}_T$, being treated in the pointwise convergence topology, depends continuously on $T \in C^N$ (see Remark 15). Next, if $\delta > 0$ is small enough then will also vanish the influence of "bad" mappings placed at a distance of order T from both ends of the diagram.

Now we will explain in detail for W_r^+ how to construct the linearizations and to establish their proximity to the linearization over $W_{i^+}^+$. For $u \in A'$ close to q_{i^+}, local C^N-coordinates, $y \in \mathbb{R}^{n^+}$, on W_u^+ in a neighbourhood of q_{i^+} will be defined that depend continuously (in the C^N-norm) on u. One can set the coordinates of each one of these points $u \in W_u^+$ to be $y = 0$ and suppose that to the tangent subspaces $\varphi_s(u), m_s > n^-$, there corresponds the same unique $(m_s - m_{s-1})$-dimensional subspace in $\mathbb{R}^{n^+}$ for each s. Let positive δ_1, δ_2 be small enough and $\delta = \delta_1\delta_2$, $p = \big[(1-\delta)T/2\big]$, $p_1 = \big[(1-\delta_1)T/2\big]$. The period T of a point $r \in A$ occurring in the condition of Lemma 5 tends to infinity as the vicinity V_{i^+} shrinks to the point q_{i^+}. The mapping $S^T \mid W_r^+$ can be represented as the composition of mappings

$$W_r^+ \xrightarrow{S} W_{S(r)}^+ \xrightarrow{S} \cdots \xrightarrow{S} W_{S^{p_1}(r)}^+ \xrightarrow{S^{T-2p_1}} W_{S^{-p_1}(r)}^+ \xrightarrow{S} \cdots \xrightarrow{S} W_r^+ \tag{18}$$

and the manifolds $W_{S^t(r)}^+$ $(|t| \le p_1)$ can be identified near the point q_{i^+} with a neighbourhood of the origin in $\mathbb{R}^{n^+}$ by means of coordinates $y \in \mathbb{R}^{n^+}$. Then all the conditions i)–iii) of Proposition 5 happen to be satisfied for the mappings S considered in (18). We introduce $\rho^\pm > 0$ and choose a small $\varepsilon > 0$ and an appropriate neighbourhood U of the origin $0 \in \mathbb{R}^{n^+}$ in the statement of Proposition 5. Due to Remark 16, the mappings $\overline{\mathcal{D}}_S$ that correspond to diffeomorphisms S in the chain (18) will define the contracting mappings of the spaces Σ^+ and Σ^- with Lipschitz constant $\lambda < 1$. We shall prove that this condition remains also valid for $\overline{\mathcal{D}}_{S^{T-2p_1}}$ if one chooses small enough $\varepsilon > 0$ and large enough $R > 0$ depending on r. Moreover, the norm of $d_{S^{T-2p_1}}$ admits an upper exponential estimate C^{T-2p_1}, where $C > 1$ does not depend on r. The mapping S^{T-2p_1} is the composition of

$$W_{S^{p_1}(r)}^+ \xrightarrow{S} \cdots \xrightarrow{S} W_{S^p(r)}^+ \xrightarrow{S^{T-2p}} W_{S^{-p}(r)}^+ \xrightarrow{S} \cdots \xrightarrow{S} W_{S^{-p_1}(r)}^+ .$$

Due to the smallness assumption for $\delta_2 > 0$, the mappings S^{T-2p_1} for all r satisfy also condition i), but not conditions ii)–iii).

Some trouble is the fact that, generally speaking, there exists no neighbourhood U of the origin that is invariant for all the mappings (18). Therefore, at first we sharpen the definition of the mappings $\overline{\mathcal{D}}_S$ and $\overline{\mathcal{D}}_{S^{T-2p_1}}$. Let U be a small ball centered at the origin $0 \in \mathbb{R}^{n^+}$ with respect to the norm from condition ii) for the mappings S. If $\delta_2 > 0$ is small enough then the mappings $S^t \mid W_r^+$ $(0 < t \le T)$ cast U onto domains $U^{(t)}$, where $U^{(t)} \subset U$ under the identification of $W_{S^t(r)}^+$ and $\mathbb{R}^{n^+}$ for $t \le p$ or $t \ge T - p$. For other values of t, the sizes of the domains $U^{(t)} \subset W_{S^t(r)}^+$

will be also small enough and upper estimates for their diameters do not depend on the choice of $r \in A$ and are proportional to the diameter of U. On each separatrix $W^+_{S^t(r)} = W^+_{S^{t-T}(r)}$ $(0 \leq t < T)$, we confine our consideration to the domain $U^{(t)}$. Note now that the estimates of Proposition 5 remain valid if $\overline{\mathcal{D}}_T$ is considered as an operator from the space of the C^N-functions on U'' to the space of the C^N-functions on U', where $T(U') \subset U''$, $0 \in U'$, $U'' \subset U$.

To analyze the mapping S^{T-2p} we observe that, since the set A is of dimension zero (totally disconnected), the following holds true. The above-discussed C^N-coordinates $y \in \mathbb{R}^{n^+}$ in a neighbourhood of $u \in A$ on W^+_u which depend continuously on u can be introduced for all $u \in A$. Thus, the mappings $S\colon U^{(t)} \to U^{(t+1)}$ for all $0 \leq t < T$ can be considered as mappings of vicinities of the origin in $\mathbb{R}^{n^+}$ into analogous vicinities. (Incidentally, such the representation is suitable to prove that all the domains $U^{(t)}$ are small enough.) Here, for the operator $L = \mathcal{D}_J$, where J is the differential of S at the origin, the subspaces $E^\pm$ are invariant. Moreover, for the operators $\bigl(L \mid E^-\bigr)^{-1}$ and $L \mid E^+$ and for the vector d_S, their norms are bounded from above by a constant that does not depend on r. The desired properties of the operator $\mathcal{D}_{S^{T-2p_1}}$ and the estimate for the norm of the vector $d_{S^{T-2p_1}}$ follow from the facts described and the functorial nature of the correspondence $T \mapsto \overline{\mathcal{D}}_T$. Note that in the case $E^- = \{0\}$ (the operators $\mathcal{D}_S$ are contractive) there is no necessity of the choice of $\varepsilon > 0$ and U depending on $r \in A$.

The next result follows from the above-found properties of the mappings $\overline{\mathcal{D}}_S$ and $\overline{\mathcal{D}}_{S^{T-2p_1}}$ pertaining to diffeomorphisms forming the chain (18): an itinerary scheme is defined for the closed chain obtained via identification of the ends of the associated diagram

$$\xleftarrow{\overline{\mathcal{D}}_S} \dots \xleftarrow{\overline{\mathcal{D}}_S} \xleftarrow{\overline{\mathcal{D}}_{S^{T-2p_1}}} \xleftarrow{\overline{\mathcal{D}}_S} \dots \xleftarrow{\overline{\mathcal{D}}_S} \ . \tag{19}$$

If δ_1 is small enough then $C^{T-2p_1}\lambda^{p_1} \to 0$ as $T \to +\infty$ (this condition is required to "paralyze" the influence of the "bad" mapping $\overline{\mathcal{D}}_{S^{T-2p_1}}$ in the chain (19) upon its ends) and if $r \to q_{i^+}$ then $T \to +\infty$. Therefore, the following holds true. Let E be the space of the C^N-functions on W^+_r, to which the endpoints of diagram (19) correspond. Then the itinerary scheme determines affine subspaces $\mathcal{W}^\pm$ of E that are invariant under the composition of mappings (19) and are close to those for $\overline{\mathcal{D}}_{S|W^+_{i^+}}$ if r is close to q_{i^+}. The intersection $\mathcal{W}^+ \cap \mathcal{W}^-$ is just the desired linearization (with the identity mapping subtracted) and the required result would follow immediately from here if the number $\varepsilon > 0$ and, consequently, the ball U in $\mathbb{R}^{n^+}$ and the norm in E were also chosen independently of r. One can overcome this difficulty via a slight modification of the proof proposed. The idea is to use the triangle form (14) of the operators $\mathcal{D}$. In this case, instead of the chain (19) of mappings $\overline{\mathcal{D}}_T$, one considers the corresponding chains for subspaces G_t, $2 \le t \le N$, and the quantity Θ_t, that depends on the dynamics in the subspaces $G_{t'}$ with less indices $t' < t$, can be interpreted as a supplement to the G_t-component of the vector d_T. The above constructions can be repeated almost literally in sequence for increasing indices t (Θ_t is determined by a point of the space $\bigoplus_{t'<t} G_{t'}$ that was found at the previous steps and corresponds to the linearization sought for) and decreasing numbers $\delta_1 = \delta_1(t) > 0$. For each $t > 2$, the quantity $\delta_1(t)/\delta_1(t-1)$ should be chosen small enough in order to "paralyze" the influence of "bad" mappings, as was done above with respect to $\mathcal{D}_{S^{T-2p_1}}$. The quantity δ_2 can be chosen independently of t. Now, there is no problem of the choice of $\varepsilon > 0$ and U depending on r because the role of the operator $\mathcal{D}_T$ for $t < N$ is played by $\mathcal{D}_J$ and for $t = N$, by the contractive operator $\mathcal{D}_T \mid G_N$ whose unstable separatrix is $E^- = \{0\}$. As a result, one establishes the desired proximity of the linearizations in the C^N-norm not depending on the choice of $r \in A$. At last, the proof is elementary for the case where the partitions ξ_i^+ are right concordant. It is based on the fact that $S^T \mid W_r^+$ can be represented as the composition of mappings satisfying the conditions i)–iii) with uniform estimates that do not depend on $r \in A$. The statement of Lemma 5 that concerns the properties of the moduli for the eigenvalues of the mapping $S^T \mid W_r^+$ will also follow from the above analysis. So, the proof of Lemma 5 has been completed. $\square$

We will discuss briefly in the following two Remarks the construction of the nonautonomous linearization.

Remark 19. The conditions i)–iii) of Proposition 5 and the construction of the nonautonomous linearization described above are stable under small perturbations of the mappings T_i. Precisely speaking, the uniform estimates of conditions i)–iii) for T_i and J_i will determine a number $\Delta > 0$ such that the following statement holds. If each C^N-mapping $T_i'\colon U \to \mathbb{R}^n$ is Δ-close to T_i in the C^1-norm and its C^N-norm admits the previous upper bound then for the sequence T_i':

a) there exist unique sequences of points $r_i' \in U$ and spaces $M_{i,s}' \in G_{\dim M_s}(r_i')$ ($1 \le s \le p$) close, respectively, to zero $r_i = 0 \in U$ and $M_s \subset T_0U$ such that they satisfy the "invariance" conditions $T_i'(r_i') = r_{i+1}'$, $dT_i'\big(M_{i,s}'\big) = M_{i+1,s}'$;
b) there exists a unique sequence of mappings l_i' such that diagram (11) commutes, where the mappings T_i and l_i are replaced by T_i' and l_i', the conditions $dl_i(0) = \mathrm{id}$ are replaced by $dl_i'(r_i') = \mathrm{id}$, and all the mappings l_i' are uniformly bounded in the C^N-norm.

Let U_i be the definition domain of the mappings $T_i, T_i'\colon U_i \to U_{i+1}$ and $l_i, l_i'\colon U_i \to \mathbb{R}^n$. Embedding $U_i \subset \mathbb{R}^n$ induces the coordinates in U_i that will be referred to as *"natural"*. According to item a), there exist C^N-coordinates uniformly C^N-close in domains U_i to the "natural" ones and such that the mappings T'_i in these coordinates will satisfy the conditions of Proposition 5 with uniform estimates close to the "unperturbed" ones (i.e., those for T_i in "natural" coordinates). The item b) follows immediately from both the latter fact and the general construction of an itinerary scheme for the mappings $\overline{\mathcal{D}}_{T_i}$ which is stable under small perturbations of the uniform estimates. The item a) is easily proven via an analogous geometrical construction: the desired sequence r'_i is determined by the itinerary scheme for the mappings T'_i and spaces $M'_{i,1}\oplus\cdots\oplus M'_{i,s}$ and $M'_{i,s+1}\oplus\cdots\oplus M'_{i,p}$ $(1 \le s < p)$ are built on the basis of negative $\bigl[r'_j,\, j \le i\bigr]$ and positive $\bigl[r'_j,\, j \ge i\bigr]$ "semitrajectories"of the point r'_i by the method used earlier in Remark 16 to construct stable and unstable slices α_i^+ and α_i^-. (For the itinerary scheme determining the sequence r'_i, the unstable subspace degenerates into the trivial one and the set Σ^+ of stable slices consists of a single element, the set Σ^- of unstable slices coincides with the set of the points of U_i. Therefore, the point r'_i is determined by the sequence $\bigl[T'_j,\, j \le i\bigr]$ infinite to the left.) Moreover, the sequences of points r'_i, subspaces $M'_{i,s}$, and C^N-mappings l'_i treated in the C^k-topology, $1 \le k \le N$, depend continuously on the sequence of C^N-mappings T'_i treated in the C^k-topology. To attach the precise meaning to the last statement, one should consider $\bigl[r'_i\bigr]$, $\bigl[M'_{i,s}\bigr]$, $\bigl[l'_i\bigr]$, and $\bigl[T'_i\bigr]$ as elements of the corresponding Tychonoff products (over $i \in \mathbb{Z}$) of the spaces U_i, $\mathfrak{U}_{i,s}$, $C^N\bigl(U_i, \mathbb{R}^n\bigr)$, and B_i, where $\mathfrak{U}_{i,s} \to U_i$ is the tangent bundle of the Grassmannian manifolds of the linear spaces of dimension $\dim M_s$ and B_i is the space of C^N-mappings $U_i \to U_{i+1}$ that are Δ-close to T_i in the C^1-norm.

This result is proven by a nonautonomous version of the arguments of Remark 14. Note that in the case $k = N$, it is a direct consequence of Remark 17 and the fact that the mapping $\overline{\mathcal{D}}_T$, being treated in the pointwise convergence topology, depends continuously on $T \in C^N$ (this argument was used in the proof of Lemma 5). In the general case $1 \le k \le N$, it suffices to assume that $r'_i = 0$ for all i (small parallel translations should be used) because r'_i depends continuously on T_j. Next, the k-jets of the nonautonomous linearizations l'_i at $r'_i = 0$, $j_0^k l'_i$, are determined only by the k-jets $j_0^k T'_j$ and depend continuously on the latters. Indeed, $j_0^k l'_i$ are determined by an itinerary scheme obtained from the original one by considering, instead of the spaces of all the functions $f\colon U_i \to \mathbb{R}^n$ with zero 1-jets at the origin, only the spaces of the k-jets for these functions at the origin (see Appendix B). We will perform the C^N-transformation from the old coordinates $x \in U_i \subset \mathbb{R}^n$ to the new ones $y = R_i(x)$ such that all the R_i are uniformly C^N-bounded, are arbitrarily close to $j_0^k l'_i$ in the C^k-norm, and have the desired N-jets, $j_0^N l'_i$. Then in the new coordinates y, the mappings T'_i will take the form $\widetilde{T}'_i = L_i \circ J_i \circ L_{i-1}^{-1}$ where $L_i = R_i \circ l_i'^{-1}$, and the desired nonautonomous linearizations are sought for as $l_i' = \mathrm{id} + f_i$ where $f_i \in G_*$ are determined by the sequence of the contractive mappings $\overline{\mathcal{D}}_{\widetilde{T}_i'} \mid G_*$. Then

$$f_i = \sum_{j=1}^{+\infty} \mathcal{D}_{T_{i+1}} \circ \cdots \circ \mathcal{D}_{T_{i+j}} g_{T_{i+j}}$$

and the arguments of Remark 14 provide the desired result. Note that this expression in terms of the uniformly convergent series remains valid under the conditions of Lemma 5 although the assumptions of Proposition 5 are not necessarily met. Therefore, under the conditions of Lemma 5, the C^N-linearization of S^T on $W_r^\pm$ depends continuously on $S \in C^N$ if all the mappings are treated in the C^k-topology and S remains uniformly C^N-bounded.

Remark 20. **a)** Let the partitions $\xi_i^\pm$ be right concordant and the neighbourhood V of the homoclinic structure be small enough. Then, in accordance with the Corollary of Proposition 5, for each point $r \in A$ a nonautonomous C^N-linearization $W_r^\pm \to \mathbb{R}^{n^\pm}$ can be naturally introduced. This mapping plays the role of linearization, coincides with it if the point r is periodic, and depends continuously on r. (Note that the graph Γ can be disconnected and, thus, set A is not necessarily a set of quasi-random motions.)

b) The local manifolds $W_r^\pm$ are well-known to be extendible to the global manifolds by means of iterations of the mapping S [37]. Obviously, the commutativity condition for the diagram (11) enables one to extend uniquely the nonautonomous linearization onto the global manifolds $W_r^\pm$.

c) The nonautonomous linearization l_i admits an invariant coordinate-free description. Indeed, consider a sequence $\{T_i\}$ that satisfies the conditions of Proposition 5. Let, as above, $U_i = U$ be the definition domain of the mapping $T_i \colon U_i \to U_{i+1}$. There exists a unique sequence $r_i = 0 \in U$ such that $T_i(r_i) = r_{i+1}$. The nonautonomous linearization is given by the mappings $l_i \colon U_i \to \mathbb{R}^n$ such that $l_i(r_i) = 0$, $dl_i(r_i) = \mathrm{id} \mid \mathbb{R}^n$. Using the "natural" coordinates in $U_i \subset \mathbb{R}^n$, one can identify the spaces $\mathbb{R}^n$ containing the images $\mathrm{im} l_i$ with the tangent spaces $T_{r_i} U_i$. Then the nonautonomous linearization is just the sequence of C^N-mappings $l_i \colon U_i \to T_{r_i} U_i$ uniformly C^N-bounded and such that $l_i(r_i) = 0 \in T_{r_i} U_i$, $dl_i(r_i) = \mathrm{id} \mid T_{r_i} U_i \equiv \mathrm{id} \colon T_{r_i} U_i \supset$ (the space $T_r U$ is equipped with the natural linear structure and, therefore, for each $\xi \in T_r U_i$ there is the canonical identification $T_r U_i \equiv T_\xi \big(T_r U_i\big)$ used here for $\xi = (r_i, 0)$, $0 \in T_{r_i} U_i$) and diagram (11) commutes, where $J_i = dT_i(r_i) \colon T_{r_i} U_i \to T_{r_{i+1}} U_{i+1}$. All the constructions are analogous in the case of the nonautonomous linearization on the separatrices $W_r^\pm, r \in A$. Here the dimension $n^\pm = \dim W_r^\pm$, a trajectory of the mapping $S^{\pm 1}$ in A, some vicinity of r_i in $W_{r_i}^{\pm}$, and the restriction $S^{\pm 1} \mid U_i$ play the roles of n, $\big[r_i : i \in \mathbb{Z}\big]$, U_i, and T_i, respectively. A completely coordinate-free representation is in some sense impossible because of the following two reasons. Firstly, in appropriate coordinates on U_i the mappings T_i are supposed to satisfy the conditions of Proposition 5, and, secondly, the uniform boundedness of l_i is required in the corresponding C^N-norm that will be called an *"original"* norm. Let $g_i \colon U_i \to \mathbb{R}^n$ be C^N-mappings introducing new coordinates in U_i. If all g_i and g_i^{-1} are uniformly bounded in the original C^N-norm then the uniform boundedness of l_i in the original C^N-norm is equivalent to that in C^N-norms induced by coordinates $y = g_i(u), u \in U_i$. In particular, in the case of the nonautonomous linearization on the separatrices $W_r^{\pm}$, due to the compactness of A, one can set on $W_r^{\pm}$ arbitrary C^N-coordinates depending continuously on $r \in A$. Thus, in this situation a completely coordinate-free description of nonautonomous linearization is possible because conditions i)–iii) of Proposition 5 are guaranteed by the assumption that the partitions $\xi_i^{\pm}$ are right concordant. Note in conclusion the following fact. If $l_i \colon \mathbb{R}^n \supset U \to \mathbb{R}^n = T_0 U$ is an arbitrary mapping written in the natural coordinates then in the coordinates $y = g_i(u), u \in U_i$, where $g_i(0) = 0$, this mapping possesses the form $\mathcal{D}_{g_i^{-1}} l_i$.

Problem 2. It would be interesting to investigate in detail the above mathematical object — nonautonomous linearization $\{l_i\}$, and its relation to the sequence $\big\{T_i\big\}$.

7. Modifications of condition 2) of main Theorem 2

Let the partition $\xi_i^{\pm}$ be right, and let $N \geq N_i^{\pm}$. It is convenient to use on $W_i^{\pm}$ coordinates $y \in \mathbb{R}^{n^{\pm}}$ defined by the diffeomorphism $l_i^{\pm}$ and such that the mapping S becomes linear in these coordinates: $y \mapsto J_i^{\pm} y$. To each element $\Lambda_{i,s}^{\pm}$ of the partition $\xi_i^{\pm}$, there corresponds a $c_s^{\pm}$-dimensional invariant subspace $M_s^{\pm} \subset \mathbb{R}^{n^{\pm}}$ of the operator $J_i^{\pm}$ that can be chosen independently of the number i. Moreover, $\bigoplus\limits_{s=1}^{p^{\pm}} M_s^{\pm} = \mathbb{R}^{n^{\pm}}$. The spaces $L_{i,s}^{\pm} = \bigoplus\limits_{t \leq s} M_t^{\pm}$ were used above.

Let $T = \big(J_i^{\pm}\big)^{\pm 1}$ and U be a small vicinity of the origin $0 \in \mathbb{R}^{n^{\pm}}\{y\}$ identified with the point q_i, $T(U) \subset U$. Denote the subset Φ of the set of couples (s, m) and the linear subspaces $F = E^{-}$ and H in the space of polynomials (which were introduced in the proof of Proposition 5) as $\Phi^{\pm}$, $F^{\pm}$, and $H^{\pm}$, respectively, for the map T under consideration. Consider the mapping $\mathcal{D}_T$ defined on the functions $f \colon U \to \mathbb{R}^{n^{\pm}}$ with 1-jets vanishing at the origin. The role of the outgoing separatrix for the hyperbolic fixed point $f \equiv 0$ is played by $F^{\pm}$, and the role of the incoming separatrix is played by the space of functions whose $(N-1)$-jets lie in $H^{\pm}$. The mapping $g = \mathrm{id} + \Delta l$ realizes a diffeomorphism $\mathbb{R}^{n^{\pm}} \to \mathbb{R}^{n^{\pm}}$ for every $\Delta l \in F^{\pm}$ because its inverse g^{-1} is also of the form $\mathrm{id} + \widetilde{\Delta l}$, $\widetilde{\Delta l} \in F^{\pm}$. Indeed, the mappings g of such form are exactly all the mappings having, at the origin, a tangency (at least) of first order with the identity map id, and such that the following diagram commutes:

$$\begin{tikzcd}
{} \arrow[r, "T"] \arrow[d, "g"'] & {} \arrow[r, "T"] \arrow[d] & {} \arrow[r, "T"] \arrow[d] & \cdots \\
{} \arrow[r, "T=J"'] & {} \arrow[r, "T=J"'] & {} \arrow[r, "T=J"'] & \cdots
\end{tikzcd}$$

where all the mappings are uniformly bounded in the C^N-norm. The mapping g^{-1} also satisfies this condition because it is possible to invert the direction of the vertical arrows, replacing simultaneously the corresponding mappings by their inverses. One establishes, analogously, the more general result: the mappings $\mathrm{id}+F^{\pm}$ form a group with respect to the composition operation. If the mapping g has, at the origin, a tangency of first order with the identity map, then the k-jet of g^{-1} at the origin is a polynomial in the j-jets of g at the origin, where $j \leq k$. Therefore, the mapping of the set $\mathrm{id} + F^{\pm}$ into itself that links a function and its inverse is polynomial.

Let the conditions 1) and 4) of Theorem 2 be satisfied and $r = r_j$ be a point such that $i = i^{\pm}(j), i_1 = i^{\mp}(j)$, let the partitions $\xi_i^{\pm}, \xi_{i_1}^{\pm}$ be right concordant, and $N \geq N_i^{\pm} = N_{i_1}^{\pm}$. In particular, it is sufficient to consider the homoclinic structure formed by the trajectory of a single double-asymptotic point $r = r_j$ and to require the partitions $\xi_i^{\pm}, \xi_{i_1}^{\pm}$ to be right concordant, the appropriate inequality on number N to be true, and condition 4) of Theorem 2 to be satisfied, under the assumption that each partition $\xi_i^{\mp}, \xi_{i_1}^{\mp}$ contains one element, i.e., $p^{\mp} = 1$. Then, due to Remark 20, in a neighbourhood, U_r, of the point r on $W_i^{\pm}$, there is a well-defined nonautonomous linearization $l^{\pm}\colon U_r \to \mathbb{R}^{n^{\pm}}$ associated with the trajectory of r. Moreover,

$$g^{\pm}(\cdot) = \left(l^{\pm}\right)^{-1}(\cdot) = y(r) + (\cdot) + \Delta g^{\pm}(\cdot) \tag{20}$$

in coordinates y on U_r, where $\Delta g^{\pm} \in F^{\pm}$, $y(r) = l_i^{\pm}(r)$ are the coordinates of the point r, and the domain U_r is extendible to the whole separatrix $W_i^{\pm}$ because of the invertibility of $g^{\pm}$ and due to Remark 20.

Now some generalizations of condition 2) of the main Theorem 2, which are based on this nonautonomous linearization corresponding to a double-asymptotic point, will be given.

Let conditions 1) and 4) of the main Theorem 2 be satisfied and the concordant partitions $\xi_i^{\pm}$ for all i be refinements of concordant partitions $\widetilde{\xi}_i^{\pm}$: $\xi_i^{\pm} \succeq \widetilde{\xi}_i^{\pm}$ (the case $\xi_i^{\pm} = \widetilde{\xi}_i^{\pm}$ is possible). In accordance with Lemma 4, to the elements of the partition $\xi_i^{\pm} = \left\{\Lambda_{i,s}^{\pm} : 1 \le s \le p^{\pm}\right\}$ there correspond invariant subspaces that were naturally denoted as $\varphi_s^{\pm}$ $(1 \le s \le p^{\pm})$.

Lemma 6

Let V_j be a given small neighbourhood of the closure of the trajectory of the point r_j and $r \in A'$ be a periodic point placed near r_j. Let the partitions $\widetilde{\xi}_i^{\pm}$ and $\widetilde{\xi}_{i_1}^{\pm}$ be right concordant, where $i = i^{\pm}(j), i_1 = i^{\mp}(j)$, and the mapping S be of class C^N, where $N \ge N^{\pm} = N_i^{\pm} = N_{i_1}^{\pm}$. Suppose that a number $\delta > 0$ is small enough and that the trajectory of the periodic point r spends, on the average, a relative part of the whole time that does not exceed δ out of the neighbourhood V_j and, moreover, $S^m(r) \in V_j$ for $|m| < (1-\delta)T/2$, where T is the period of r. Then on $W_r^{\pm}$ the mapping $S^{\pm T}$ possesses a linearization C^N-close to the nonautonomous linearization $l^{\pm}\colon W_i^{\pm} \supset U_r \to \mathbb{R}^{n^{\pm}}$ associated with the trajectory of r_j. Moreover, the eigenvalues of the mapping $S^{\pm T} \mid W_r^{\pm}$ have a partition $\xi^{\pm}$ concordant with $\xi_i^{\pm}$ and a partition $\widetilde{\xi}^{\pm}$ right concordant with $\widetilde{\xi}_i^{\pm}, \widetilde{\xi}_{i_1}^{\pm}$. The eigensubspaces that correspond to the elements $\Lambda_s^{\pm}$ of the partition $\xi^{\pm}$ are $\varphi_s^{\pm}(r)$.

This Lemma differs from Lemma 5 by considering the linearization on the separatrices of a periodic point placed near a homo-(hetero)clinic point instead of one near a fixed point.

Definition 10. A set $\mathfrak{P} \subset W_i^{\pm}$ will be said to be *in general position with respect to a point* $r \in A \cap W_i^{\pm}$, asymptotic to q_i, if there is a nonautonomous linearization $l^{\pm}$ on a vicinity U_r of the point r in $W_i^{\pm}$ and the following conditions are valid for the set $\mathfrak{N} = l^{\pm}(\mathfrak{P})$: the images of $\mathfrak{N}$ under the projections into eigensubspaces $\varphi_s^{\pm}(r) = \dot{l}^{\pm} \big|_r \left(\varphi_s^{\pm}(r)\right)$ $\left(\bigoplus\limits_s \varphi_s^{\pm}(r) = T_r W_i^{\pm}\right)$ span these subspaces.

Remark 21. The coordinates of the points of the projection of $\mathfrak{N}$ into $\varphi_s^{\pm}(r)$ are polynomials of degrees not greater than $N^{\pm} - 1$ in the coordinates of the points of the set $l_i^{\pm}(\mathfrak{P})$. The coefficients of these polynomials are determined in terms of $\Delta g^{\pm}$, $y(r) = l_i^{\pm}(r)$, and $\varphi_s^{\pm}(r)$ (if the tangent spaces $T_r W_i^{\pm}$ at all the points $r \in W_i^{\pm}$ are identified by using the linearization $l_i^{\pm}$).

Remark 22. If there is a set of homo-(hetero)clinic points on $W_i^{\pm}$ which is in general position then the following holds. For each $r_j \in W_i^{\pm}$ such that the partitions $\xi_i^{\pm}$ and $\xi_{i^{\mp}(j)}^{\pm}$ are right concordant, there exists also a set of homo-(hetero)clinic points on $W_i^{\pm}$ in general position with respect to r_j. The proof follows from the fact that, due to the Corollary of Lemma 3, this condition is satisfied for a point of the trajectory of r_j which is close to q_i because the corresponding nonautonomous linearization $l^{\pm}$ is close to $l_i^{\pm}$. Therefore, the new formulation of condition 2), that will be written in Proposition 6, is more general than the preceding one if the condition of right concordance mentioned here holds.

Proposition 6

Condition 2) *of Theorem* 2 *can be replaced by the following one for one or both cases marked by the superscripts* $\pm$, *respectively. There exist* i^+ (i^-, *respectively) and* j *such that* $i^+(j) = i^+$, $i^-(j) = i_1^+$ ($i^-(j) = i^-$, $i^+(j) = i_1^-$), *there exist right concordant partitions* $\widetilde{\xi}_{i^+}^+$ *and* $\widetilde{\xi}_{i_1^+}^+$ ($\widetilde{\xi}_{i^-}^-$ *and* $\widetilde{\xi}_{i_1^-}^-$) *such that* $\widetilde{\xi}_{i^+}^+ \preceq \xi_{i^+}^+$ ($\widetilde{\xi}_{i^-}^- \preceq \xi_{i^-}^-$) *(and, as a consequence,* $\widetilde{\xi}_{i_1^+}^+ \preceq \xi_{i_1^+}^+$ ($\widetilde{\xi}_{i_1^-}^- \preceq \xi_{i_1^-}^-$)), *and the set of double-asymptotic points on* $W_{i^+}^+$ ($W_{i^-}^-$) *is in general position with respect to* r_j.

Proof. It is based on using Lemma 6 instead of Lemma 5. □

Remark 23. One can obtain concordant partitions $\widetilde{\xi}_i^{\pm}$ for all i via roughing partitions $\xi_i^{\pm}$. In contrast to the earlier situation, where the linearizations on separatrices $W_{i^{\pm}}^{\pm}$ were dealt with, in the present case one has also to consider finer partitions $\xi_i^{\pm}$. This is because of the necessity to define the generality of position of a set $\mathfrak{P}$ with respect to an asymptotic point r_j via using the corresponding invariant subspaces $\varphi_s^{\pm}(r_j)$. Indeed, if one uses the partitions $\widetilde{\xi}_i^{\pm}$ in Definition 10 instead of the partitions $\xi_i^{\pm}$, then the set $\mathfrak{P}$ can break the general position condition.

Both variants of condition 2) of Theorem 2 admit also the following generalization. Let $I^{\pm}$ be a set of indices i, for which there exist right concordant partitions $\widetilde{\xi}_i^{\pm}$ such that $\widetilde{\xi}_i^{\pm} \preceq \xi_i^{\pm}$.

DEFINITION 11. Let us say that a set $\mathfrak{P} \subset W_{I^{\pm}}^{\pm} = \bigcup\limits_{i \in I^{\pm}} W_i^{\pm}$ is *in general position* if for each $s = 1, \ldots, p^{\pm}$ either

a) there is $i = i_s^{\pm} \in I^{\pm}$ such that conditions (7) are satisfied for the set $\mathfrak{N}_s$ being the projection of $\mathfrak{N} = l_i^{\pm}\big(\mathfrak{P} \cap W_i^{\pm}\big)$ into $L_{i,s}^{\pm} = \varphi_s^{\pm}(q_i)$ along the subspaces $L_{i,s'}^{\pm} = \varphi_{s'}^{\pm}(q_i)$ ($s' \neq s$) and for the mapping $T_s = \big(J_i^{\pm}\big)^{\pm 1} \Big| \varphi_s^{\pm}(q_i)$, or

b) there is $j = j_s^{\pm}$ such that $i^+ = i^+(j), i^- = i^-(j) \in I^{\pm}$ and, therefore, a nonautonomous linearization $l^{\pm}$ on a vicinity $U_{r_j} \subset W_{i^{\pm}}^{\pm}$ of the point r_j, which is associated with the trajectory of r_j, is well-defined; moreover, the image of $\mathfrak{N} = l^{\pm}\big(\mathfrak{P} \cap W_{i^{\pm}}^{\pm}\big)$ under the projection into $\varphi_s^{\pm}(r_j)$ along $\bigoplus\limits_{s' \neq s} \varphi_{s'}^{\pm}(r_j)$ spans the whole space $\varphi_s^{\pm}(r_j)$.

Proposition 7

The condition 2) *of Theorem* 2 *admits the following generalization:* $I^{\pm} \neq \emptyset$ *and the set of homo-(hetero)clinic points lying on* $W_{I^{\pm}}^{\pm}$ *is in general position.*

Proof. It is based on the consideration of everywhere dense sets, $P^{\pm}$, in A' formed by periodic trajectories that spend most of the time near the points $q_i, i \in I^{\pm}$ and pass close to all the points $q_{i_s^{\pm}}$ and $r_{j_s^{\pm}}$ described above. We require in addition that the set $P^{\pm}$ contains only points of the trajectories γ possessing the following property: for every $s = 1, \dots, p^{\pm}$ there exists $r_s \in \gamma$ such that $S^m(r_s) \in V_k$ for $|m| < g_s T/2$, where $k = i_s^{\pm}$ or $k = j_s^{\pm}$ in accordance to the cases a) and b). Here T is the period of the trajectory γ and $g_s > 0$ are some given constants such that $g_1 + \dots + g_{p^{\pm}} < 1$ (for brevity we drop the superscripts $\pm$ of r_s and g_s). The last inequality guarantees the existence of the desired trajectories γ and their density in A'. Then the eigenvalues of $S^T \mid W_{r_s}^{\pm}$ at the point r_s satisfy the non-resonance condition and the mapping S^T on the separatrix $W_{r_s}^{\pm}$ has a linearization C^N-close to $l_{i_s^{\pm}}^{\pm}$ or $l^{\pm}$, where $l^{\pm}$ is the nonautonomous linearization associated with the trajectory of the point $r_{j_s^{\pm}}$. The proof of this fact is based on the ideas of the proof of Lemma 5. Introduce the number $p_1 = \big[(1-\delta_1) g_s T/2\big]$, where $0 < \delta_1 < 1$, and consider the chain of mappings (18) and the corresponding chain of functional endomorphisms (19). If the trajectory of a point r spends a relative part of the whole time that does not exceed $\delta > 0$ out of small vicinities V_i of the points $q_i, i \in I^{\pm}$, then the norm of the vector $d_{S^{T-2p_1}}$ is easily shown to admit the upper exponential estimate const $\cdot\, C^{\delta T}$. If δ was chosen small enough then $C^{\delta T} \lambda^{p_1} \to 0$ as $T \to +\infty$. Obviously, one can take δ such that $\delta < \delta_0 \min\limits_{1 \le s \le p^{\pm}} g_s$, where the small number $\delta_0 > 0$ is chosen independently of g_s in order to satisfy the inequality $C^{2\delta_0} \lambda^{1-\delta_1} < 1$. If $\delta_0 > 0$ is small enough then the non-resonance conditions for the eigenvalues of the composition of mappings (18) are also valid. The rest of the proof is quite analogous to the preceding one. Next, linearizations on the separatrices $W_r^{\pm}$ of points r belonging to the same trajectory γ are conjugated by powers S^p of the mapping S and by their differentials $\dot{S}^p \mid_r$. Thus, there is a set of homo-(hetero)clinic points on $W_r^{\pm}$ which is in general position in the sense of Definition 8. The proof is finished as in Section 6. $\square$

8. The case of a diffeomorphism close to a direct product

We shall formulate now some general statements that allows one, in particular, to transfer results concerning the non-existence of an integral and symmetry group to systems of an arbitrary finite number of weakly interacting particles in an external time-periodical force field. Note that the direct product of topological Markov chains $T \mid \Omega^{\Pi_t}$ defined by the graphs $\Gamma_t\,(t \in \mathfrak{t})$ is also a TMC, $T \mid \Omega^{\Pi}$, such that the corresponding graph Γ is the direct product of the graphs Γ_t. Elements of the matrices $\Pi_t = \left(\pi^t_{a_t,b_t}\right)$ and $\Pi = (\pi_{a,b})$ determining edges of the graphs Γ_t and Γ are connected by the relations $\pi_{a,b} = \prod_t \pi^t_{a_t,b_t}$ for $a = [a_t : t \in \mathfrak{t}]$, $b = [b_t : t \in \mathfrak{t}]$. Consider diffeomorphisms $S_t\colon M_t \to M_t\,(t = 1, \ldots, k)$ of manifolds M_t that possess homoclinic structures $\mathcal{H}_t$ satisfying conditions of Theorem 2 (for $t \in \mathfrak{t}$) or Theorem 1 (for $t \notin \mathfrak{t}$), where $\mathfrak{t}$ is a given subset of $\{1; \ldots; k\}$. Define a direct product $S = S_1 \times \cdots \times S_k\colon M \to M$ of the mappings S_t, where $M = M_1 \times \cdots \times M_k$. Then the direct product $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_k$ is the homoclinic structure for S. It is easily seen, by choosing a neighbourhood U of the set $\mathcal{H}$ in the form of the direct product $U = U_1 \times \cdots \times U_k$, that analogous decompositions take place for the TMC and its graph $\Gamma = \prod_t \Gamma_t$ which exist due to the V. M. Alekseev theorem, and for the set A of points whose trajectories lie in U. Let $C = A' \times B \subset A$, where $A' = \prod_{t\in\mathfrak{t}} A'_t$ and $B = \prod_{t\notin\mathfrak{t}} B_t$, and A'_t and B_t denote the above-introduced sets A' and B for the diffeomorphisms S_t.

Proposition 8

The diffeomorphism S has no analytical first integral or one-parameter symmetry group. Moreover, its analytic centralizer is discrete in the compact–open topology.

Proof. The direct product of key sets is easily seen to be also a key set whose DKP is the product of the DKP's of the original sets. Therefore, it is sufficient to show that a continuous first integral is constant over C. The α- and ω-limit sets for the trajectories in C lie, respectively, in the sets $A' \times \left\{O^-\right\}$ and $A' \times \left\{O^+\right\}$, where $O^{\pm} = \left[O^{\pm}_t : t \notin \mathfrak{t}\right]$, and $O^{\pm}_t$ denote points $O^{\pm}$ for the maps S_t, $t \notin \mathfrak{t}$. Thus, we can restrict ourselves to the case $\mathfrak{t} = \{1; \ldots; k\}$. Due to the fact that the graph Γ_t contains a circuit formed by one edge and corresponding to a fixed point, the related TMC $T \mid \Omega^{\Pi_t}$ is *primitive* [4], i.e., for some N every two vertices can be joined by a path of length exactly equal to N. The direct product of indecomposable primitive TMC has also these properties. The desired result related to first integrals and one-parameter symmetry groups follows from here because there is a trajectory everywhere dense in A'. Moreover, the hyperbolic set C is expansive which guarantees the discreteness of the centralizer (see the proof of Proposition 4). □

Let $\mathfrak{t} = \{1; \ldots; k\}$, i.e., the conditions of Theorem 2 are valid for all the diffeomorphisms S_t. The mapping S will have a homoclinic structure $\mathcal{H}' \subset \mathcal{H}$ satisfying the conditions of Theorem 2 if simple additional assumptions hold. However, it is more convenient to remove non-essential conditions by considering a homoclinic structure $\mathcal{H}' \subset A'$ which does not belong, generally speaking, to $\mathcal{H}$. Note that one can also point out additional restrictions under which the required homoclinic structure will exist in a neighbourhood of a given one $\mathcal{H}_0 \subset \mathcal{H}$. All the hyperbolic fixed points for S are $q_i = \bigl[q_{t,i_t} : 1 \le t \le k\bigr]$ and they possess eigenvalue sets $\bigl\{\lambda_{t,i_t,l_t}, \mu_{t,i_t,l_t} : 1 \le t \le k\bigr\}$, with multiindices $i = (i_1, \ldots, i_k)$ where q_{t,i_t} and $\bigl\{\lambda_{t,i_t,l_t}, \mu_{t,i_t,l_t}\bigr\}$ are hyperbolic fixed points and their eigenvalues for S_t. Let $\mathbf{I}^{\pm} = I_1^{\pm} \times \cdots \times I_k^{\pm}$, where $I_t^{\pm}$ is the set $I^{\pm}$ for S_t.

Proposition 9

For some $i^+ \in \mathbf{I}^+$ ($i^- \in \mathbf{I}^-$, respectively) let the eigenvalues of the mapping S at q_{i^+} (q_{i^-}), whose moduli are less (greater) than unit, satisfy the non-resonance conditions (2) *and $S \in C^N$, where $N \ge N^{\pm}_{i^{\pm}}$, $N^{\pm}_i$ are determined by* (4). *Then there exists a homoclinic structure $\mathcal{H}' \subset A'$ satisfying the conditions of Theorem 2. This result remains valid in the more general situation of the presence of non-empty subsets $\widetilde{\mathbf{I}}^{\pm} \subset \mathbf{I}^{\pm}$ possessing the following property. For any couple (s, m), where the multiindex $m = (m_1, \ldots, m_{n^{\pm}})$ is such that $\sum_j m_j \ge 2$, $m_j \ge 0$ and $1 \le s \le n^{\pm}$, there is $i^{\pm} \in \widetilde{\mathbf{I}}^{\pm}$ for which the eigenvalues of the mapping S at q_{i^+} (or at q_{i^-}), whose moduli are less (greater) than the unit, satisfy the non-resonance condition* (2) *for the given (s, m). Here $S \in C^N$ must be supposed with $N \ge \min\limits_{i^{\pm} \in \widetilde{\mathbf{I}}^{\pm}} N^{\pm}_{i^{\pm}}$.*

Proof. One should take $\delta > 0$ and $\varepsilon > 0$ small enough and consider the set $\mathbf{P}^{\pm}$ of periodic trajectories that spend near all the points q_i, $i \in \mathbf{I}^{\pm}$ at least a relative part $1-\delta$ of the whole time and near $q_{i^{\pm}}$, at least a relative part $(1-\varepsilon)(1-\delta)$ of the whole time. Under the projections $\pi_t \colon M \to M_t$, to these trajectories there correspond the quite analogous trajectories for S_t. Moreover, an additional restriction must be imposed on the trajectories forming the set $\mathbf{P}^{\pm}$: $\pi_t\bigl(\mathbf{P}^{\pm}\bigr) = P^{\pm}_t$, where $P^{\pm}_t$ is the set $P^{\pm}$ for the diffeomorphism S_t, $P^{\pm}$ having been constructed in the proof of Proposition 7. Precisely speaking, constants $g_{t,s}$ $\bigl(1 \le s \le p^{\pm}_t\bigr)$ for each of the sets $P^{\pm}_t$ are given in such a manner that the following is valid: $g_{t,s_t} > (1-\varepsilon)(1-\delta)$ for the indices s_t satisfying the condition $i^{\pm}_t = i^{\pm}_{t,s_t}$, where $i^{\pm} = \bigl[i^{\pm}_t : 1 \le t \le k\bigr]$, and $p^{\pm}_t$, $g_{t,s}$, and $i^{\pm}_{t,s}$ are the numbers $p^{\pm}$, g_s, and $i^{\pm}_s$ for the mapping S_t. If the number $i^{\pm}_t$ does not occur among $i^{\pm}_{t,s}$ then one should put $i^{\pm}_{t,p^{\pm}_t+1} = i^{\pm}_t$ and increase the number $p^{\pm}_t$ by one unit. The above-supposed additional condition of “delay” near $q_{i^\pm}$ guarantees the validity of the non-resonance conditions for the eigenvalues. Moreover, the linearizations on the separatrices of periodic points of the mapping S are the direct products of the corresponding linearizations for the mappings S_t. Let $r^\pm \in \mathbf{P}^\pm$, T be a common period for both points $r^\pm$. It is seen from the proof of Theorem 2 that for each t the mapping S_t^T has a homoclinic structure $\mathcal{H}'_t$ that is constituted by trajectories double-asymptotic to hyperbolic fixed points $\pi_t(r^\pm)$ and satisfies the conditions of Theorem 2. It follows easily from Lemma 3 that the homoclinic structure $\mathcal{H}' = \mathcal{H}'_1 \times \cdots \times \mathcal{H}'_k$ also satisfies the conditions of Theorem 2.

Next, if the sharpened variant of the statement of Proposition is satisfied then there exist numbers $f_i^\pm \geq 0, i \in \mathbf{I}^\pm$ such that $\sum\limits_{i\in\mathbf{I}^\pm} f_i^\pm = 1$ and the sets

$$\left\{\lambda_{t,l_t} \equiv \prod_{i=(i_1,\dots,i_k)\in\mathbf{I}^+} \lambda_{t,i_t,l_t}^{f_i^+}\right\} \quad \text{and} \quad \left\{\mu_{t,l_t} \equiv \prod_{i=(i_1,\dots,i_k)\in\mathbf{I}^-} \mu_{t,i_t,l_t}^{f_i^-}\right\}$$

satisfy the non-resonance conditions (2) for all (s, m). The sets of all such $\{f_i^\pm\}$ constitute the complements to a finite collections of hyperplanes. Therefore, one can choose $f_{i^\pm}^\pm$ close to the unit for a subscript $i^\pm \in \widetilde{\mathbf{I}}^\pm$, on which the number $N_{i^\pm}^\pm$ attains its minimum. Then the numbers N^+ and N^-, determined by the sets $\{\lambda_{t,l_t}\}$ and $\{\mu_{t,l_t}\}$ according to (4), will coincide with $N_{i^+}^+$ and $N_{i^-}^-$, respectively. One can suppose that $f_i^\pm > 0$ for all $i \in \mathbf{I}^\pm$. Now the proof of Proposition follows without any change if one considers the sets $\mathbf{P}^\pm$ of periodic trajectories satisfying the following conditions that we shall describe in detail. Let $Q_t = \{q_{t,i_t}, r_{t,j_t}\}$ be the set of the hyperbolic and double-asymptotic points for the diffeomorphism S_t and let $Q = Q_1 \times \cdots \times Q_k$ be the analogous set for S. Choose any subset $\widetilde{Q}^\pm = \{w_\nu^\pm : 1 \leq \nu \leq h^\pm\} \subset Q$ containing $\{q_i : i \in \mathbf{I}^\pm\}$ and such that $\pi_t(\widetilde{Q}^\pm)$ coincides with the set of all the points $q_i, i \in I_t^\pm$ and $r_{j_s^\pm}$ for the mapping S_t. Let $V_\nu^\pm$ be a small enough neighbourhood of the closure of the trajectory of the point $w_\nu^\pm \in M$. Choose a small $\varepsilon > 0$ and fix $g_\nu = f_i^\pm(1-\varepsilon)$ for $w_\nu^\pm = q_i$ and, later, take small enough numbers g_ν depending on ε for the other indices ν in order to satisfy $\sum\limits_{\nu=1}^{h^\pm} g_\nu < 1$. Next, let $\delta > 0$ be chosen small enough depending on g_ν: $\delta < \delta_0 \min\limits_{1\leq\nu\leq h^\pm} g_\nu$, where the small $\delta_0 > 0$ does not depend on ε, $f_i^\pm$, and g_ν. The desired set $\mathbf{P}^\pm$ is formed by periodic trajectories γ such that every γ spends a relative part of the whole time that does not exceed $\delta > 0$ out of the domains $V_\nu^\pm$, and for each ν there is a point $r_\nu \in \gamma$ such that $S^m(r_\nu) \in V_\nu^\pm$ if $|m| < g_\nu T/2$, T being the period of γ. Then $\pi_t(\mathbf{P}^\pm) \subseteq P_t^\pm$ (the set of numbers $i_{t,s}^\pm$ is extended to the whole set $I_t^\pm$) and the non-resonance conditions for the eigenvalues of the mapping

$S^T \mid W_r^{\pm}, r \in \gamma$ at the point r (the multiplicators of the trajectory γ lying at one side of the unit circle) are satisfied because ε was chosen small enough depending on $f_i^{\pm}$. Moreover, the corresponding number $N^{\pm}$ defined by formula (4) will coincide with $N_{i^{\pm}}^{\pm}$. $\square$

Proposition 10

Let $S\colon M \to M$ be a diffeomorphism satisfying the conditions of Theorem 2 and A' be the set of quasi-random motions. If $R \subset M$ is a closed S-invariant set that does not contain the whole A' (this is valid if in a neighbourhood of some point of A', R is the union of a finite collection of regular C^N-submanifolds of positive codimensions), then there exist a vicinity, U, of the set R and a number $\delta > 0$ such that the following property holds true. Any diffeomorphism, S', that is δ-close to S outside the domain U in the space of C^N-functions will possess a homoclinic structure satisfying the conditions of Theorem 2.

Proof. The sets $P^{\pm}$ of periodic points, which are everywhere dense in A', have been used in the proof of Theorem 2. There is an element of $P^{\pm} \setminus R$ near an arbitrary point of the set $A' \setminus R$. Let $r^{\pm} \in P^{\pm} \setminus R$ and T be the common period of both points $r^{\pm}$. Then the mapping S^T has a homoclinic structure $\mathcal{H}$ that is constituted by trajectories double-asymptotic to the hyperbolic fixed points $r^{\pm}$ and satisfies the conditions of Theorem 2. Obviously, the set $\mathcal{H}' = \mathcal{H} \cup S(\mathcal{H}) \cup \cdots \cup S^{T-1}(\mathcal{H})$ is S-invariant. Moreover, $\mathcal{H}' \cap R = \emptyset$ and it is enough to choose a vicinity U not intersecting some neighbourhood of the set $\mathcal{H}'$ and, later, a small enough $\delta > 0$. The desired homoclinic structure is a perturbation of $\mathcal{H}'$. $\square$

Remark 24. Let A'' be the maximal S-invariant subset of A' out of some small neighbourhood of R. Then $A'' \neq \emptyset$ is a hyperbolic set, and under the perturbation S' of S described in Proposition 10 this set is deformed into the S'-invariant hyperbolic set A''' [37] possessing property (1).

Remark 25. Propositions 9 and 10 can be useful in studies of a system of many weakly interacting particles in an external time-periodic force field. In this case R is the set of collision trajectories, the mapping S corresponds to the absence of interaction between the particles. Here $S = S_1 \times \cdots \times S_k$, where S_i is the first return map (Poincaré map) along the temporal period for the i-th particle. If the dynamical properties of all the particles are identical then all the mappings S_i coincide, and the validity of the conditions of Theorem 2 for one mapping S_i implies the validity of the conditions of Propositions 9 and 10. The situation under consideration, where any particle does not affect the external time-periodic field, can appear as a result of the transition to the *"restricted"* setting of the problem upon a common nonsingular level M_J of *a priori* first integrals J of the "general" problem. Then S is identified with some first return map for the flow on M_J. The perturbation of the system, which is the transition from the "restricted" setting of the problem to the "general" one, leads only to a small perturbation of the flow on M_J and, hence, to the persistence of the non-integrability conditions[14]. We will describe generally the situation discussed here. Let the motion equations depend smoothly on a (multi-dimensional) "perturbation parameter" ε and possess a collection of first integrals J that depend also smoothly on ε. Furthermore, suppose that for $\varepsilon = 0$, which corresponds to the "restricted" setting of the problem, the following conditions are satisfied:

1) The influence of the variables corresponding to the *infinitesimal particles* vanishes on the *field variables*. This means that, firstly, the finite-dimensional phase space, M, of the continuous dynamical system (flow) $\{v_t\}$ is the total space of a locally-trivial bundle $\pi\colon M \to Y$ with fibers $M_y = {}_1X_y \times \cdots \times {}_kX_y, y \in Y$ and base Y and, secondly, the dynamical system $\{v_t\}$ on M "covers" some dynamical system $\{u_t\}$ on Y (i.e., the flows $v\colon M \times \mathbb{R} \to M$ and $u\colon Y \times \mathbb{R} \to Y$ make the diagram

$$\begin{array}{ccc} M & \xrightarrow{v_t} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[u_t]{} & Y \end{array}$$

to be commutative for each $t \in \mathbb{R}$. One says also that the flow $\{v_t\}$ is *fibered* or that the system $\{v_t\}$ is an *extension* of the system $\{u_t\}$ and the system $\{u_t\}$ is a *factor* of the system $\{v_t\}$). Here Y is the space of the field variables and ${}_iX_y$ is the space of variables corresponding to the i-th infinitesimal particle for a given $y \in Y$. Moreover, since a mutual influence of the particles vanishes also, there are flows $\{{}_iv_t\}$ on bundles ${}_iM \to Y$ with fibers ${}_iX_y, y \in Y$ and base Y such that

$$v_{(y),t}({}_1x_y \times \cdots \times {}_kx_y) = {}_1v_{(y),t}({}_1x_y) \times \cdots \times {}_kv_{(y),t}({}_kx_y),$$

where the notations

$$v_t(y, x_y) = \big(u_t(y), v_{(y),t}(x_y)\big) \quad \text{and} \quad {}_iv_t(y, {}_ix_y) = \big(u_t(y), {}_iv_{(y),t}({}_ix_y)\big)$$

[14] Such approach will be used in a forthcoming part of this paper to prove the non-integrability of the planar problem of more than three point vortices in an ideal incompressible liquid and the planar problem of more than two bodies attracting by the Newton law.

for $y \in Y$, $x_y \in M_y$, $v_{(y),t}(x_y) \in M_{u_t(y)}$, and ${}_ix_y \in {}_iX_y$, ${}_iv_{(y),t}({}_ix_y) \in {}_iX_{u_t(y)}$ have been introduced. In other words, the flow $\{v_t\}$ covers the flow $\{{}_iv_t\}$ under the natural projection $M \to {}_iM$. The flow $\{{}_iv_t\}$ corresponds to the dynamics of the i-th infinitesimal particle. If the particles are identical then the corresponding bundles ${}_iM \to Y$ and the flows $\{{}_iv_t\}$ will coincide. Note that the bundle $M \to Y$ (not necessarily a vector one) could be called the *Whitney sum* of the bundles ${}_iM \to Y$.

2) The dependence of the integrals J vanishes on the point of the fiber M_y. Thus, $J = \Phi \circ \pi$, where Φ is the collection of integrals for the flow $\{u_t\}$ on the base Y. Moreover, the considered common level of the integrals Φ on Y is supposed to have a nonsingular connected component which is a closed trajectory[15] γ_J of the flow $\{u_t\}$. Then the level, M_J, of the integrals J has the nonsingular component $\pi^{-1}(\gamma_J) \subset M$. To the point $y \in \gamma_J$, there correspond the first return maps $S\colon M_y \to M_y$ and $S_i\colon {}_iX_y \to {}_iX_y$ according to formulas $S(x_y) = v_{(y),T}(x_y)$ and $S_i({}_ix_y) = {}_iv_{(y),T}({}_ix_y)$, where T is the period of the trajectory γ_J.

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Appendix A. Exponential separation, cocycles, and the construction of the adapted metric

Let $\pi\colon X \to B$ be a finite-dimensional real or complex vector bundle with compact base B and let $f\colon B \to B$, $F\colon X \to X$ be some homeomorphisms. Recall some definitions (see also Remark 25). Homeomorphisms f and F generate dynamical systems with discrete time $n \in \mathbb{Z}$, i.e., cascades $\{f^n\}$ and $\{F^n\}$. The cascade $\{F^n\}$ is called an *extension* of $\{f^n\}$ if F^n "covers" f^n, i.e., $\pi \circ F = f \circ \pi$. The extension $\{F^n\}$ is said to be *linear* if the restriction of F to any fiber $X(b) = \pi^{-1}(b)$, $b \in B$, is linear. For simplicity, we consider linear extensions of cascades. However, all the results are easily carried over to the case of linear extensions of flows. The presentation will follow, in general, [12] (where the proofs are given for the case of flows). A *norm* in the vector bundle $X \to B$ is a continuous function $X \to \mathbb{R}$ that is a norm on each fiber $X(b)$. A continuous function $\|\cdot\|_*\colon X \to \mathbb{R}$ is called a *quasinorm* if the following conditions are satisfied: **1)** $\|x\|_* \geq 0$ for any $x \in X$ and

[15] In particular, the number of *a priori* integrals is one unit less than the dimension of the field variables space Y.

$\|x\|_* = 0$ if and only if x is the zero element of a fiber, **2)** $\|kx\|_* = |k| \cdot \|x\|_*$ for any element $x \in X$ and real number k. In contrast to a norm, one does not require here the validity of the triangle inequality $\|x + y\|_* \leq \|x\|_* + \|y\|_*$, $\pi(x) = \pi(y)$. Due to the compactness of the base B and the finite-dimensionality of the bundle, any two quasinorms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in the sense that inequalities $c^{-1}\|x\|_2 \leq \|x\|_1 \leq c\|x\|_2$, $x \in X$, hold for some $c > 0$.

A continuous mapping $G\colon B \times \mathbb{Z} \to \mathbb{R}_+$ is called a *multiplicative cocycle* of the cascade $\{f^n\}$ if $G(b, n+m) = G(b, n)G\big(f^n(b), m\big)$ for any $b \in B$ and integers n, m. A cocycle is uniquely determined by setting an arbitrary continuous everywhere positive function $g(\cdot) = G(\cdot, 1)$.

One gives convenient two-sided estimates for quasinorms $\big\|F^n(x)\big\|$ in terms of cocycles. Indeed, let

$$\begin{aligned} \Phi(x) &= \frac{\big\|F(x)\big\|}{\|x\|}, \\ g^-(b) &= \inf\big\{\Phi(x) \,:\, \|x\| = 1,\, \pi(x) = b\big\},. \\ g^+(b) &= \sup\big\{\Phi(x) \,:\, \|x\| = 1,\, \pi(x) = b\big\} \end{aligned} \tag{A1}$$

It follows from the uniform continuity of Φ over the compact set $\{x \in X \,:\, \|x\| = 1\}$ that the functions $g^\pm\colon B \to \mathbb{R}$ are also continuous. The functions $g^\pm$ determine cocycles $G^\pm$. Then

$$G^-(b, n) \leq \frac{\big\|F^n(x)\big\|}{\|x\|} \leq G^+(b, n), \quad n \geq 0, \quad x \in X(b).$$

Let $X = X_1 \oplus X_2$ be the decomposition of the bundle X into the Whitney sum of F-invariant subbundles. The subbundles X_1, X_2 (up to their permutation) are said to be *exponentially separated* if there are numbers $d > 0$ and $\lambda > 1$ such that

$$\frac{\big\|F^n(x_2)\big\|}{\|x_2\|} : \frac{\big\|F^n(x_1)\big\|}{\|x_1\|} \geq d\lambda^n \tag{$A2$}$$

for any $n \geq 0$ and $x_1 \in X_1$, $x_2 \in X_2$, $\pi(x_1) = \pi(x_2)$. This means that the vectors in the fibers of X_2 are "stretched" under iterations of F faster, than those in the fibers of X_1. In this case we will write $X_1 \prec X_2$. This definition does not depend on the choice of the norm or quasinorm $\|\cdot\|$ because this choice affects d only.

Proposition A1

Let $X = X_1 \oplus \cdots \oplus X_p$ be the decomposition of the bundle X into the direct sum of F-invariant subbundles. The following two properties are equivalent:

1) any two vector subbundles X_i and X_j $(i \neq j)$ are exponentially separated;
2) the subbundles X_s, $1 \leq s \leq p$, can be renumbered in such a way that for any s, $1 \leq s < p$, subbundles

$$Y_s^- = X_1 \oplus \cdots \oplus X_s, \quad Y_s^+ = X_{s+1} \oplus \cdots \oplus X_p$$

are exponentially separated, so that $Y_s^- \prec Y_s^+$.

Moreover, estimates (A2) *for pairs of subbundles X_s, X_{s+1} and Y_s^-, Y_s^+ will be satisfied with the same λ's (but, maybe, with different d's).*

Proof. This result is rather simple and well-known. The relation $\prec$ is an *order relation* on the set $\{X_s\}$ and determines the required reordering of the elements of this set.

Remark. Under the conditions of Lemma 4, the TS-invariant bundles $\bigcup\limits_{r \in A} \varphi_s(r) \to A$ happen to be exponentially separated. Moreover, the corresponding estimates (A2) are satisfied with the constant $\lambda = \kappa^{-1}$.

Proposition A2

Let the conditions of Proposition A1 *be met so that the pairs of subbundles Y_s^-, Y_s^+ satisfy inequalities* (A2) *with some constants $\lambda = \lambda_s > 1$. Furthermore, let numbers μ_s lie in the intervals $(1, \lambda_s)$. Then there is a norm $\|\cdot\|$ in X such that for all the pairs Y_s^-, Y_s^+ inequalities* (A2) *are valid with constants $d = 1$ and $\lambda = \mu_s$, respectively. In particular, the same holds for the pairs X_s, X_{s+1}. The desired norm $\|\cdot\|$ can be chosen to be Euclidean (in the real case) or Hermitian (in the complex case).*

So, by slightly decreasing λ_s, one is able to put all the constants d's to be equal to 1.

Lemma A1

Let the conditions of Proposition A2 *be satisfied. Then there are cocycles $G_s^{\pm}$, $1 \leq s \leq p$, such that the following two properties hold:*

a) $g_{s+1}^-(b) : g_s^+(b) \geq \mu_s$, $1 \leq s < p$, $b \in B$, where $g_s^{\pm}(\cdot) = G_s^{\pm}(\cdot, 1)$.
b) $g_s^-(b) \leq g_s^+(b)$ and $C^{-1} G_s^-(b,n) \leq \|F^n(x)\|/\|x\| \leq C G_s^+(b,n)$, $n \geq 0$, $x \in X_s(b) \equiv X(b) \cap X_s$, $b \in B$, where constant $C > 0$ depends on the choice of a quasinorm $\|\cdot\|$.

Proof. Set $\|x\|_* = \left(\prod_{n=0}^{N-1} \|F^n(x)\|_1\right)^{1/N}$, $\|\cdot\|_1$ being some (quasi)norm and N, a sufficiently large positive integer. Obviously, $\|\cdot\|_*$ is a quasinorm. However, $\|\cdot\|_*$ can be not a norm even if $\|\cdot\|_1$ is a norm (an appropriate example is easily constructed). Let Φ_*, g_s^-, and g_s^+ be the functions Φ, g^-, and g^+ for the invariant bundles X_s which are determined by formulas (A1) via the constructed quasinorm $\|\cdot\|_*$. The functions $g_s^{\pm}$ determine the required cocycles $G_s^{\pm}$. The item b) of the Lemma is obviously satisfied. In order to prove item a), we recall that

$$\frac{\|F^n(x_{s+1})\|_1}{\|x_{s+1}\|_1} : \frac{\|F^n(x_s)\|_1}{\|x_s\|_1} \geq d\lambda_s^n, n \geq 0,\ x_s \in X_s,\ x_{s+1} \in X_{s+1},\ \pi(x_s) = \pi(x_{s+1}),$$

where $d > 0$. Therefore

$$\Phi_*(x_{s+1}) : \Phi_*(x_s) = \left(\frac{\|F^N(x_{s+1})\|_1}{\|x_{s+1}\|_1} : \frac{\|F^N(x_s)\|_1}{\|x_s\|_1}\right)^{1/N} \geq d^{1/N}\lambda_s \geq \mu_s$$

if $N > 0$ is chosen to be large enough. The item a) of the Lemma follows immediately from this inequality.

Lemma A2

Let $G^{\pm}$ be cocycles such that $g^-(b) \leq g^+(b)$, $b \in B$, where $g^{\pm}(\cdot) = G^{\pm}(\cdot, 1)$, and

$$C^{-1}G^-(b,n) \leq \frac{\|F^n(x)\|}{\|x\|} \leq CG^+(b,n), \quad n \geq 0,\ x \in X(b),$$

$C > 0$ being some constant and $\|\cdot\|$, a quasinorm in X. Let $\nu > 1$. Then there exists a norm $\|\cdot\|_0$ in X such that

$$\nu^{-n}G^-(b,n) \leq \frac{\|F^n(x)\|_0}{\|x\|_0} \leq \nu^n G^+(b,n), \quad n \geq 0,\ x \in X(b).$$

The desired norm $\|\cdot\|_0$ can be chosen to be Euclidean (in the real case) or Hermitian (in the complex case).

Proof. Without loss of generality, one can assume that $\|\cdot\|$ is a norm. Set

$$\|x\|_0 = \sum_{n\geq 0} \|F^n(x)\| \big(G^+(b,n)\big)^{-1}\nu^{-n} + \sum_{n<0} \|F^n(x)\| \big(G^-(b,n)\big)^{-1}\nu^{n}.$$

Obviously, these two series converge uniformly and $\|\cdot\|_0$ is a norm. Then it is easily seen that

$$\nu^{-1}g^-(b) \leq \frac{\|F(x)\|_0}{\|x\|_0} \leq \nu g^+(b), \quad x \in X(b),$$

which implies that the norm $\|\cdot\|_0$ is the desired one. If the norm $\|\cdot\|$ was chosen to be Euclidean or Hermitian then the norm $\|\cdot\|_0$ possesses the same property.

Proof of Proposition A2. It follows easily from Lemmas A1 and A2. Apply Lemma A1, using some numbers μ'_s instead of numbers μ_s. Then we apply Lemma A2 to each invariant subbundle X_s, assuming that $G^\pm = G^\pm_s$, $g^\pm = g^\pm_s$, $\nu = \nu_s$. As a result, some norms $\|\cdot\|_s$ in subbundles X_s will be obtained. Finally, we require inequalities $\mu'_s/(\nu_s\nu_{s+1}) \geq \mu_s$ and construct the desired norm $\|\cdot\|$ in $X = X_1 \oplus \cdots \oplus X_p$ from the norms $\|\cdot\|_s$ in X_s. For instance, one can set

$$\|x_1 \oplus \cdots \oplus x_p\| = \left(\sum_{s=1}^{p} \|x_s\|_s^2\right)^{1/2}, \quad x_s \in X_s,\ \pi(x_s) = b.$$

Historical comment. Proposition A2 for the case $p = 2$ was proven by I. U. Bronshtein [11; 12, point 6.30]. The generalization of his proof to the case $p > 2$ requires some accuracy. For this purpose, we state Lemma A2 on the existence of a norm satisfying the two-sided estimates, which generalizes the well-known result on the existence of the *adapted* or *Lyapunov* norm. The Lyapunov norm is the norm satisfying the one-sided estimate with $g^- \equiv \text{const} > 1$ or $g^+ \equiv \text{const} < 1$. Note that any cocycle G and linear extension $\{F^n\}$ determine a linear extension $\{F^n_G\}$ via the formula $F^n_G(x) = G\big(n, \pi(x)\big)F^n(x)$. The possibility of constructing a norm satisfying the one-sided estimate follows immediately from the existence of the Lyapunov norm for the corresponding linear extension $\{F^n_G\}$. This idea was utilized in [12, 11] for proving Proposition A2 in the case $p = 2$. The case of the two-sided estimates with $g^- = \text{const}$, $g^+ = \text{const}$ is considered in [27, point 2.8–2.9]. The corresponding proof generalizes the well-known construction of the Lyapunov norm. In turn, our proof is an immediate generalization of the latter proof. The proof of Lemma A1 is reproduced from [12].

Appendix B. Geometrical digression: induced mappings of k-jet bundles

Assume that the assumptions described at the beginning of Section 7 are met, i.e., the conditions 1) and 4) of Theorem 2 are satisfied and $r = r_j$ is a point such that $i = i^\pm(j), i_1 = i^\mp(j)$, the partitions $\xi^\pm_i, \xi^\pm_{i_1}$ are right concordant, and $N \geq N^\pm_i = N^\pm_{i_1}$. Recall that, due to Remark 20, in a neighbourhood, U_r, of the point r on $W^\pm_i$, there is a well-defined nonautonomous linearization $l^\pm\colon U_r \to \mathbb{R}^{n^\pm}$ associated with the trajectory of r and formula (20) holds in the linearizing coordinates y defined by the diffeomorphism $l^\pm_i$.

Now we draw our attention to some *invariant*, i.e., *coordinate-free*, geometrical objects.

The element $\Delta g^{\pm}$ of the finite-dimensional space $F^{\pm}$ can be found by virtue of the itinerary scheme that determines the nonautonomous linearization $l^{\pm}$. This itinerary scheme corresponds to the trajectory $\{S^t(r_j)\}$ of the point r_j and to the manifolds $W^{\pm}_{S^t(r_j)}$. One can reduce this itinerary scheme to the finite-dimensional and invertible case if one considers, instead of the spaces of all the functions $f\colon W^{\pm}_{S^t(r_j)} \to T_{S^t(r_j)} W^{\pm}_{S^t(r_j)}$ such that $f\bigl(S^t(r_j)\bigr) = 0$, $df\bigl(S^t(r_j)\bigr) = \mathrm{id}$, only the spaces of the k-jets for these functions at the points $S^t(r_j)$, where $k \le N$. The corresponding geometric constructions will be discussed below. It is convenient to use the invariant coordinate-free language for this purpose. All the constructions are easily reformulated into the coordinate language and can be used in concrete applications.

Let $T = S^{\pm 1} \mid W^{\pm}_i$ and let U be a vicinity of q_i in $W^{\pm}_i$ such that $T(U) \subset U$. The linear operator $\mathcal{D}_T$ acts in the space of functions $l\colon U \to T_{q_i} W^{\pm}_i$. The restriction of $\mathcal{D}_T$ onto the $\mathcal{D}_T$-invariant affine subset $Z^{\pm}_i$ of the C^N-functions l, such that $l(q_i) = 0 \in T_{q_i} W^{\pm}_i$ and $dl(q_i) = \mathrm{id}$, has a hyperbolic fixed point — the linearization $l^{\pm}_i$. To prove this it is sufficient to consider the parametrization of U by the linearizing coordinates. Moreover, the linearizing coordinates determine some affine parametrization $F^{\pm} \to \mathcal{W}^{\mp,\pm}_i$ of the outgoing separatrix $\mathcal{W}^{\mp,\pm}_i$ for the hyperbolic point $l^{\pm}_i$ of the mapping $\mathcal{D}_T \mid Z^{\pm}_i$ by associating the map $\mathrm{id} + \widetilde{\Delta l}$ with the element $\widetilde{\Delta l} \in F^{\pm}$. Analogously, there is an affine parametrization of the incoming separatrix $\mathcal{W}^{\pm,\pm}_i$ by the space of functions whose $(N^{\pm} - 1)$-jets lie in $H^{\pm}$.

DEFINITION B1. A class of equivalence of C^k-manifolds with respect to the equivalence relation “tangency of order k at the point r” will be called the *k-jet of an m-dimensional (sub-)manifold at the point r.*

The difference from the usual definition for the k-jet of a mapping (see, for example, [9, 22]) consists in what distinguishes a manifold from a map: a particular parametrization is ignored in the definitions for a manifold and a jet of a manifold, and these definitions have a coordinate-free nature. An equivalent definition for a manifold’s jet is contained in [22, Chapter 7, §3].

Consider *pairs* (W, f) formed by a C^k-manifold $W \subset M$ and a C^k-mapping $f\colon W \to N$, where N is some manifold.

DEFINITION B2. We say that the pairs (W, f) and (W', f') *possess a tangency of order k at the point $r \in M$* if the following conditions are valid: **1)** the manifolds W and W' of the same dimension, m, possess a tangency of order k at the point r, i.e., there is a mapping $i\colon W \to M$ that has at the point r a tangency of order k with the embedding $W \hookrightarrow M$ and transforms a neighbourhood of the point r in W onto a neighbourhood of this point in W', and **2)** the mappings f and $f' \circ i\colon W \to N$ have a tangency of order k at the point r.

The tangency of pairs at a given point $r \in M$ is an equivalence relation.

Definition B3. A class of equivalence of pairs (W, f) with respect to the equivalence relation "tangency of order k at the point r" will be called the *k-jet of a pair at the point r.*

Analogously to the conventional notations for k-jets of mappings, we denote by $J^k_m(r)$ the space of k-jets of m-dimensional manifolds at the point $r \in M$, and by $\mathfrak{M}^k_m = \bigcup_{r\in M} J^k_m(r)$, the bundle of k-jets of m-dimensional manifolds (at all the points $r \in M$. The sets $J^k_m(r)$ and $\mathfrak{M}^k_m$ possess, respectively, the natural structures of an analytical manifold and a manifold whose class of smoothness is k units less than that of M. Obviously, $J^1_m(r) = G_m(r)$ and $\mathfrak{M}^1_m = \mathfrak{M}_m$, and there exists a natural projection $J^{k+1}_m(r) \to J^k_m(r)$. Analogous objects can be defined for k-jets of pairs (W, f), and there exist their natural projections onto the corresponding objects for k-jets of manifolds W. However, we need only to consider k-jets of pairs lying in some regular submanifolds in the manifold of all the k-jets. Setting $N = TM$ and $k \geq 1$, we denote by $\widehat{J}^k_m(r)$ the analytical manifold of the k-jets of pairs (W, f) at the point r such that

$$\begin{aligned} \dim W = {}& m, \quad f(W) \subset T_rW, \\ f(r) = {}& 0 \in T_rW, \quad df(r) = \mathrm{id} \mid T_rW. \end{aligned} \tag{B1}$$

Obviously, $\widehat{\mathfrak{M}}^k_m = \bigcup_{r\in M} \widehat{J}^k_m(r)$ is a manifold whose class of smoothness is equal to that for $\mathfrak{M}^k_m$. The diffeomorphism $S\colon M \to M$ of class C^N, where $N \geq k$, induces natural analytical mappings $J^k_m(r) \to J^k_m\big(S(r)\big)$ forming a fiber mapping $\mathcal{S}\colon \mathfrak{M}^k_m \to \mathfrak{M}^k_m$ of class C^{N-k}. Each mapping $J^k_m(r) \to J^k_m\big(S(r)\big)$ is determined by the k-jet of the mapping S at the point r. Therefore, by some misuse of language, we denote it, as well as the k-jet, by $j^k_r S$. Obviously, $j^1_r S = dS\mid_r = \dot{S}\mid_r$. Furthermore, by associating to the pair (W, f), which satisfies condition (B1), another pair $\big(S(W), \mathcal{D}_{(S|W)^{-1}} f\big)$, which also satisfies condition (B1), one can introduce natural analytical mappings $\hat{j}^k_r S\colon \widehat{J}^k_m(r) \to \widehat{J}^k_m\big(S(r)\big)$ (each of them being determined by the k-jet of S at the point r) that form the fiber mapping $\widehat{\mathcal{S}}\colon \widehat{\mathfrak{M}}^k_m \to \widehat{\mathfrak{M}}^k_m$ of class C^{N-k}.

Remark B1. Lemma 1 and items a), b), and c) of Lemma 4, with their proofs as well as Remarks 2, 7, and 11, can be immediately generalized to the case where one considers, instead of the bundles $\mathfrak{M}_m^{\pm}$, $\mathfrak{M}_{n-m}^{\pm}$, $\mathfrak{M}_m$, $\mathfrak{M}_{n-m}$, $\mathfrak{A}_m$, and $\mathfrak{A}_{n-m}$ of 1-jets of manifolds, the quite analogous bundles $\mathfrak{M}_m^{k,\pm}$, $\mathfrak{M}_{n-m}^{k,\pm}$, $\mathfrak{M}_m^k$, $\mathfrak{M}_{n-m}^k$, $\mathfrak{A}_m^k$, and $\mathfrak{A}_{n-m}^k$ of k-jets. However, the following restriction concerning the numbers m and k must be imposed on the spectrum $(\gamma_1, \dots, \gamma_n)$ at each hyperbolic point $O = q_i$ under consideration. The sections $g_m^{k,-} = (\mathrm{id}, f_m^{k,-})$ and $u_m^{k,-} = (\mathrm{id}, h_m^{k,-})$ ($g_m^{k,+} = (\mathrm{id}, f_m^{k,+})$ and $u_m^{k,+} = (\mathrm{id}, h_m^{k,+})$, respectively) analogous to $g_m^- = g_m^{1,-}$ and $u_m^- = u_m^{1,-}$ ($g_m^+ = g_m^{1,+}$ and $u_m^+ = u_m^{1,+}$) exist and are unique if the linear operator $\dot{S}\mid_O$ ($\dot{S}^{-1}\mid_O$) in $T_O M$ induces a mapping in the space of k-jets of the m-dimensional ($(n-m)$-dimensional) manifolds $W \subset T_O M$ at the point $0 \in T_O M$ such that the k-jet of the linear subspace α_m^+ (α_m^-) is a *sink*, i.e., a fixed point whose spectrum lies inside the unit circle. This means exactly that

$$|\gamma_j||\gamma_{j'}|^{-s} < 1 \qquad \left(\ |\gamma_j|^s|\gamma_{j'}|^{-1} < 1\ \right)$$

$$\text{for all} \quad j \le m < j',\ 1 \le s \le k$$

(cf. [27, Theorem 5.1]). In particular, one can always assume that $m = n^-$ or $k = 1$. The following modifications must be performed in the statements of Lemmas 1 and 4 and in Remarks 2, 7, and 11:

1) to read C^{N-k} everywhere instead of C^{N-1}; the corresponding mapping $\mathcal{S}$: $\mathfrak{M}_m^k \to \mathfrak{M}_m^k$ ($\mathcal{S}\colon \mathfrak{M}_{n-m}^k \to \mathfrak{M}_{n-m}^k$) is a diffeomorphism if $N \ge k+1$;

2) to take some k-jet $\alpha_m^{k,\mp}$ such that $\pi(\alpha_m^{k,\mp}) = \alpha_m^{1,\mp} \equiv \alpha_m^{\mp}$ under the natural projection $\pi\colon J_s^k(r) \to J_s^1(r) = G_s(r), r = O$, instead of $\alpha_m^{\mp}$ everywhere except for item 2f) of Lemma 1; the k-jet $\alpha_m^{k,-}$ ($\alpha_m^{k,+}$) is the unique attracting point of the diffeomorphism $j_O^k S$ ($j_O^k S^{-1}$) in the space $J_m^k(O)$ ($J_{n-m}^k(O)$). (If in some coordinate system in a neighbourhood of the point $O \in M$, the mapping S has at O the k-jet coinciding with the k-jet of $\dot{S}\mid_O$, then in this coordinate system $\alpha_m^{k,\mp}$ is the k-jet of the linear subspace $\alpha_m^{\mp}$.);

3) to talk in item 2f) of Lemma 1 about C^k-linearizations and to assume that $\alpha_m^{\mp}$ and $\dot{l}$ are the k-jet of the linear subspace $\alpha_m^{\mp} \subset T_O M$ at the point $0 \in T_O M$ and the k-jet of the mapping l; the latter is interpreted as the mapping of the spaces of k-jets of manifolds;

4) to replace conditions $\sigma_r \cap f_m^{\mp}(r) = \{0\}$ in item 2e) of Lemma 1 and in Remark 2, and conditions $\sigma_r \cap h_m^{\mp}(r) = \{0\}$ in item c) of Lemma 4 by the conditions $\pi(\sigma_r^k) \cap f_m^{\mp}(r) = \{0\}$ and $\pi(\sigma_r^k) \cap h_m^{\mp}(r) = \{0\}$, where $z^{k,+} = (r, \sigma_r^k) \in \mathfrak{M}_m^k$ or $z^{k,-} = (r, \sigma_r^k) \in \mathfrak{M}_{n-m}^k$. In accordance with this fact, the first formula in item b) of Lemma 4 and condition (9), that were stated above for $k = 1$, preserve their

form, but one has to take into account that $\pi\big(f_m^{k,\pm}(r)\big) = f_m^{1,\pm}(r) \equiv f_m^{\pm}(r)$ and $\pi\big(h_m^{k,\pm}(r)\big) = h_m^{1,\pm}(r) \equiv h_m^{\pm}(r)$. Moreover, obviously, $f_m^{k+1,\pm}(r)$ and $h_m^{k+1,\pm}(r)$ are transformed into $f_m^{k,\pm}(r)$ and $h_m^{k,\pm}(r)$, respectively, under the natural projections $J_s^{k+1}(r) \to J_s^k(r)$. The conditions that $f_{m_1}^{k,+}(r) \subset f_{m_2}^{k,+}(r)$, $f_{m_1}^{k,-}(r) \supset f_{m_2}^{k,-}(r)$, $h_{m_1}^{k,+}(r) \subset h_{m_2}^{k,+}(r)$, and $h_{m_1}^{k,-}(r) \supset h_{m_2}^{k,-}(r)$ for $m_1 > m_2$ remain valid if the embedding $\beta_1 \subset \beta_2$ of a jet $\beta_1 \in J_{s_1}^k(r)$ into a jet $\beta_2 \in J_{s_2}^k(r)$ is naturally interpreted as the existence of manifolds $W_1 \subset W_2$ possessing k-jets β_1 and β_2;

5) to consider k-jets of manifolds $W^{\pm}$ and $W_r^{\pm}$ instead of their tangent spaces in the first formula of item 2c) of Lemma 1 and in the second formula of item b) of Lemma 4. Note that, in accordance with items 2c) and 2d) of Lemma 1, $\alpha_{n^-}^{k,\mp}$ is the k-jet of $W^{\pm}$ at the point O.

In conclusion, we shall explain briefly the proof of items 1), 2e) of Lemma 1 and item c) of Lemma 4. The behaviour of the mapping $\mathcal{S}\colon \mathfrak{M}_m^k \to \mathfrak{M}_m^k$ ($\mathcal{S}^{-1}\colon \mathfrak{M}_{n-m}^k \to \mathfrak{M}_{n-m}^k$, respectively) near the point $x^{k,+} = (O, \alpha_m^{k,+})$ ($x^{k,-} = (O, \alpha_m^{k,-})$) is characterized by the expansion along the separatrix W^- (W^+) and the contraction along the separatrix W^+ (W^-) and fiber $J_m^k(O)$ ($J_{n-m}^k(O)$). Therefore, the outgoing manifold $\widetilde{W}_{x^{k,+}}^-$ ($\widetilde{W}_{x^{k,-}}^+$) of the point $x^{k,+}$ ($x^{k,-}$) happens to be the image of the section $g_m^{k,-}\colon W^- \to \mathfrak{M}_m^k$ ($g_m^{k,+}\colon W^+ \to \mathfrak{M}_{n-m}^k$). Next, the domain of attraction of the sink α_m^+ (α_m^-) of the diffeomorphism $F = \dot{S}\,|_O = j_O^1 S$ ($F = \dot{S}^{-1}\,|_O = j_O^1 S^{-1}$) on the manifold $N = G_m(O) = J_m^1(O)$ ($N = G_{n-m}(O) = J_{n-m}^1(O)$) is the set $P = \big\{\beta \in N : \alpha \cap \beta = \{0\}$, where $\alpha = \alpha_m^-$ ($\alpha = \alpha_m^+$, respectively) and β are considered as linear subspaces in $T_O M\big\}$. Moreover, the invariant set $N \setminus P = \partial P$ of the mapping F repels the neighboring points in the sense that

$$\rho(\beta, \partial P) < \Delta \Rightarrow \rho\big(F(\beta), \partial P\big) > C\rho(\beta, \partial P), \quad \text{where} \quad \beta \in P \tag{B2}$$

for some distance ρ on N and some numbers $\Delta > 0$ and $C > 1$. It is convenient to take as $\rho(\alpha, \beta)$ the angle between the subspaces $\alpha, \beta \subset T_O M$ in the Riemannian metric used in the proof of Lemma 1. Define also the distance function ρ on neighboring fibers $G_m(r)$ ($G_{n-m}(r)$) using this metric. Due to the fact that the section g_m^- (g_m^+) is continuous and $\mathcal{S}$-invariant and $f_m^-(O) = \alpha_m^+$ ($f_m^+(O) = \alpha_m^-$), the following holds true: (B2) remains also valid for all neighboring fibers if one uses $f_m^-(r)$ ($f_m^+(r)$) instead of α_m^+ (α_m^-) in the definition of P. From here there follow items 1) and 2e) of Lemma 1 and, thus, the formula of the incoming manifold $\widetilde{W}_{x^+}^+$ ($\widetilde{W}_{x^-}^-$) for the case $k = 1$. The general case $k \geq 1$ follows from the particular one, $k = 1$, due to the following fact: for k-jets $\beta \in J_m^k(O)$ ($\beta \in J_{n-m}^k(O)$), such that $\pi(\beta) \in N$ is close to α_m^+ (α_m^-), the mapping $j_O^k S\colon J_m^k(O) \to J_m^k(O)$ ($j_O^k S^{-1}\colon J_{n-m}^k(O) \to J_{n-m}^k(O)$) happens to be contractive and this property is possessed by the mappings $j_r^k S\colon J_m^k(r) \to J_m^k\big(S(r)\big)$ ($j_r^k S^{-1}\colon J_{n-m}^k(r) \to J_{n-m}^k\big(S^{-1}(r)\big)$)

of the neighboring fibers. The item c) of Lemma 4 is proven in an analogous way. In particular, formula (B2) turns out to be true if one carries out the following replacements: firstly, using $h^{\pm}_m(r)$ instead of $\alpha^{\mp}_m$ or $f^{\pm}_m(r)$ in the definition of P; secondly, determining a Riemannian norm in T_rM depending continuously on $r \in A$ and such that the spaces $h^{+}_m(r)$ and $h^{-}_m(r)$ are orthogonal with respect to it; thirdly, taking some power S^p, $p > 0$, instead of S in the definition of F. If one uses a Lyapunov metric then the third point is unnecessary, i.e., $p = 1$. The proof is analogous in other details.

Remark B2. Denote the nonautonomous linearization on $W^{\pm}_r$, $r \in A$, by $l^{\pm}_r\colon W^{\pm}_r \to T_rW^{\pm}_r$. The k-jets of the pairs $\bigl(W^{\pm}_i, l^{\pm}_i\bigr)$ and $\bigl(W^{\pm}_r, l^{\pm}_r\bigr)$ admit a description analogous to that considered in the previous Remark. However, there appears a series of distinctions here in the sense that the incoming and outgoing manifolds of the fixed points are organized in another manner. For a hyperbolic point $O = q_i$ of the C^N-diffeomorphism $S\colon M \to M$, let the partition $\xi^- = \xi^-_i$ ($\xi^+ = \xi^+_i$, respectively) be right, $N \ge N^- = N^-_i$ ($N \ge N^+ = N^+_i$), $W^{\pm} = W^{\pm}_i$, $l^- = l^-_i$ ($l^+ = l^+_i$), $Z^- = Z^-_i$ ($Z^+ = Z^+_i$), and $\mathcal{W}^{+,-} = \mathcal{W}^{+,-}_i$, $\mathcal{W}^{-,-} = \mathcal{W}^{-,-}_i$ ($\mathcal{W}^{-,+} = \mathcal{W}^{-,+}_i$, $\mathcal{W}^{+,+} = \mathcal{W}^{+,+}_i$). Then Remark B1 for $m = n^-$ is transferable, with some modifications, to the case where one considers, instead of mappings $\mathcal{S}$ of the bundles, $\mathfrak{M}^{k,-}_m$ and $\mathfrak{M}^k_m$ ($\mathfrak{M}^{k,+}_{n-m}$ and $\mathfrak{M}^k_{n-m}$), of the k-jets of manifolds W, the quite analogous mappings $\widehat{\mathcal{S}}$ of the bundles, $\widehat{\mathfrak{M}}^{k,-}_{n^-}$ and $\widehat{\mathfrak{M}}^k_{n^-}$ ($\widehat{\mathfrak{M}}^{k,+}_{n^+}$ and $\widehat{\mathfrak{M}}^k_{n^+}$), of the k-jets of pairs (W, χ).

There exist the bundles

$$\begin{array}{c} \widehat{J}^k_{n^\mp}(r) \xrightarrow{Z^{k,\mp}} J^k_{n^\mp}(r) \\ \widehat{\mathfrak{M}}^k_{n^\mp} \xrightarrow{Z^{k,\mp}} \mathfrak{M}^k_{n^\mp} \xrightarrow{J^k_{n^\mp}} M\,, \end{array}$$

for which the mappings $\widehat{\mathcal{S}}$ and $\mathcal{S}$ are fibered. Here the roles of the fibers $J^k_{n^\mp}$ and $Z^{k,\mp}$ are played, respectively, by the space of the k-jets of $n^\mp$-dimensional manifolds at a given point and by the affine space of the k-jets of pairs $(W, \chi) \in \widehat{J}^k_{n^\mp}$ at a given point with a given k-jet of the manifold W at this point. The dynamics of the restriction $j^k_O S^{\pm 1}$ of the mapping $\mathcal{S}^{\pm 1}$ to the invariant fiber $J^k_{n^\mp}(O)$ lying over the fixed point O of the base M is very simply constructed. This leads to a simple representation of the incoming $\widetilde{W}^{\pm}_{x^{k,\pm}}$ and outgoing $\widetilde{W}^{\mp}_{x^{k,\pm}}$ manifolds of the point $x^{k,\pm} = (O, \alpha^{k,\pm}_{n^-})$ (see above). It is more difficult to provide a description of the incoming $\widetilde{W}^{\pm}_{\widehat{x}^{k,\pm}}$ and outgoing $\widetilde{W}^{\mp}_{\widehat{x}^{k,\pm}}$ manifolds of a fixed point $\widehat{x}^{k,\pm} = \bigl(O, \widehat{\alpha}^{k,\pm}_{n^-}\bigr)$ of the mapping $\widehat{\mathcal{S}}^{\pm 1}$. Over the fixed point $x^{k,\pm} \in \mathfrak{M}^k_{n^\mp}$, there lies the invariant fiber $Z^{k,\mp}(x^{k,\pm})$ which is formed exactly by k-jets of functions $l \in Z^{\mp}$ defined on the manifold $W^{\mp}$ having the given k-jet $\alpha^{k,\mp}_{n^-} \in J^k_{n^\mp}(O)$. If the partition $\xi^{\mp}$ is right then the restriction $\hat{j}^k_O\bigl(S \mid W^\mp\bigr)^{\pm 1}$ of the mapping $\widehat{\mathcal{S}}^{\pm 1}$ to the invariant affine subspace $Z^{k,\mp}(x^{k,\pm})$ is affine hyperbolic. This restriction has a fixed point $\widehat{\alpha}^{k,\pm}_{n^-}$ and also invariant affine subspaces — incoming $\mathcal{W}^{k,\mp,\mp}$ and outgoing $\mathcal{W}^{k,\pm,\mp}$ separatrices. The correspondence j^k_O that associates to a function $l\colon W^\mp \to T_O W^\mp$ its k-jet $j^k_O l$ will realize a mapping of $\mathcal{W}^{\pm,\mp}$ onto $\mathcal{W}^{k,\pm,\mp}$. The latter is an isomorphism if $k \geq N^\mp$. Here $\widehat{\alpha}^{k,\pm} = \widehat{\alpha}^{k,\pm}_{n^-}$ is the k-jet $j^k_O l^\mp$ of the linearization $l^\mp$ on the separatrix $W^\mp$ or, more precisely speaking, the k-jet of the pair $(W^\mp, l^\mp)$. (If in some coordinate system in a neighbourhood of the point $O \in M$, the mapping S has at O the k-jet coinciding with the k-jet of $\dot{S} \mid_O$ then in this coordinate system $\widehat{\alpha}^{k,\pm}$ is the k-jet of the pair formed by the linear subspace $\alpha^\pm$ and the embedding $\alpha^\pm \hookrightarrow T_O M$.) Analogously, $j^k_O\bigl(\mathcal{W}^{\mp,\mp}\bigr) = \mathcal{W}^{k,\mp,\mp}$. Obviously, $\widehat{\alpha}^{k,\pm}$, $\mathcal{W}^{k,\pm,\mp}$, and $\mathcal{W}^{k,\mp,\mp}$ are transformed onto the analogous objects with lower values of k under the natural projections.

So, the behaviour of the mapping $\widehat{\mathcal{S}}^{\pm 1}$ near the fixed point $\widehat{x}^{k,\pm}$ is characterized by the expansion along $W^\mp$ and $\mathcal{W}^{k,\pm,\mp}$ and the contraction along $W^\pm$, $J^k_{n^\mp}(O)$, and $\mathcal{W}^{k,\mp,\mp}$. Therefore, the outgoing manifold $\widetilde{W}^\mp_{\widehat{x}^{k,\pm}}$ of the fixed point $\widehat{x}^{k,\pm}$ of the mapping $\widehat{\mathcal{S}}^{\pm 1}$ happens now to be the total space of an affine bundle $\widetilde{W}^\mp_{\widehat{x}^{k,\pm}} \to \widetilde{W}^\mp_{x^{k,\pm}}$ of class C^{N-k} with fiber $\mathcal{W}^{k,\pm,\mp}$. Thus, $\widetilde{W}^\mp_{\widehat{x}^{k,\pm}}$ will be the image of a mapping $\widehat{g}^{k,\mp} = (\mathrm{id}, \widehat{f}^{k,\mp})\colon W^\mp \times \mathcal{W}^{k,\pm,\mp} \to \widehat{\mathfrak{M}}^{k,\mp}_{n^\mp}$ covering the mapping $g^{k,\mp}_{n^-}\colon W^\mp \to \mathfrak{M}^{k,\mp}_{n^\mp}$ that associates the k-jet of $W^\mp$ at r to a point $r \in W^\mp$. It is easily verified that for $k \geq N^\mp$ the mapping $\widehat{f}^{k,\mp}$ can be determined by the following formula which is analogous to item 2f) of Lemma 1 and should replace the latter: $\widehat{f}^{k,\mp}(r, l^k) \in \widehat{J}_{n^\mp}(r)$ is the k-jet of the pair $\bigl(W^\mp, \chi_{r,l}\bigr)$ at the point $r \in W^\mp$, where

$$\begin{aligned}\chi_{r,l}(\cdot) &= \bigl(\dot{l} \mid_r\bigr)^{-1}\bigl(l(\cdot) - l(r)\bigr)\\ &= \mathcal{D}_{l(\cdot)-l(r)}\bigl(\mathrm{id} \mid T_O W^\mp\bigr)\colon W^\mp \to T_r W^\mp\end{aligned}$$

and the mapping $l \in \mathcal{W}^{\pm,\mp}$ is such that $j^k_O l = l^k \in \mathcal{W}^{k,\pm,\mp}$. Indeed, the formula

$$\hat{j}^k_r S\left(j^k_r \chi_{r,l}\right) = j^k_{S(r)} \chi_{S(r), \mathcal{D}_{S^{-1}|W^\mp} l}$$

holds true for k-jets $j^k_r \chi_{r,l}$ of functions $\chi_{r,l}$ at the points $r \in W^\mp$ and the desired result follows immediately from here. In turn, the proof of this formula follows immediately from the facts that, in accordance with Remark 20, the following propositions are valid. Firstly, in the coordinates on $W^\mp$ defined by the mapping $l(\cdot) - l(r)\colon W^\mp \to T_O W^\mp$, the mapping $\chi_{r,l}$ will become the identity mapping $\mathrm{id} \mid T_O W^\mp$. Secondly, the mapping $S \mid W^\mp$ and its differential $\dot{S} \mid_O$ transform these coordinates into those defined by the mapping $h(\cdot) = \mathcal{D}_{S^{-1}|W^\mp}\bigl(l(\cdot) - l(r)\bigr) = \bigl(g(\cdot) - g(r)\bigr)\colon W^\mp \to T_O W^\mp$, where $g = \mathcal{D}_{S^{-1}|W^{\mp}} l \colon W^{\mp} \to T_O W^{\mp}$. Thirdly, the expression in the left-hand side of the formula under consideration is the k-jet at the point $S(r)$ of the mapping $W^{\mp} \to T_{S(r)} W^{\mp}$ which becomes the identity mapping id $|\ T_O W^{\mp}$ in the coordinates h. The constructed mapping $\widehat{f}^{k,\mp}(r,\cdot)$ realizes an isomorphism of $\mathcal{W}^{k,\pm,\mp}$ onto the fiber over the point $r \in W^{\mp}$. However, this map is polynomial but not affine. Using coordinates $y(\cdot) = l^{\mp}(\cdot)$ on $W^{\mp}$ introduced by a given nonautonomous linearization $l^{\mp} \in \mathcal{W}^{\pm,\mp}$, one can also determine the mapping $g^{\mp} = \chi_{r,l}^{-1}$ to be of the form $g^{\pm}(\cdot) = y(r) + l(\cdot)$ (see formula (20)), where $l(\cdot) = (\cdot) + \Delta g^{\pm}$. In this case the mapping $\widehat{f}^{k,\mp}(r,\cdot)$ is also polynomial. However, such a definition uses explicitly some nonautonomous linearization $l^{\mp}$ and depends on it. It is natural to take $l^{\mp}$ being the linearization.

Next, the incoming manifold $\widetilde{W}^{\pm}_{\widehat{x}^{k,\pm}}$ of the point $\widehat{x}^{k,\pm}$ is now the total space of an affine bundle $\widetilde{W}^{\pm}_{\widehat{x}^{k,\pm}} \to \widetilde{W}^{\pm}_{x^{k,\pm}}$ of class C^{N-k} with fiber $\mathcal{W}^{k,\mp,\mp}$, i.e., $\widetilde{W}^{\pm}_{\widehat{x}^{k,\pm}}$ will be the image of some C^{N-k}-mapping

$$\bigl(\pi, \widehat{v}^{k,\pm}\bigr) \colon \widetilde{W}^{\pm}_{x^{k,\pm}} \times \mathcal{W}^{k,\mp,\mp} \to \widetilde{W}^{\pm}_{\widehat{x}^{k,\pm}},$$

where π is the projection onto the first factor. A detailed description of this bundle is a separate question which will be formulated below as a Problem. Note only that an affine fiber $w^{k,\mp,\mp}(z) \subset Z^{k,\mp}(z)$, where $z \in \widetilde{W}^{\pm}_{x^{k,\pm}}$, is a stable slice determined by the itinerary scheme for the sequence of mappings

$$Z^{k,\mp}(z_t) \xrightarrow{\hat{j}^k_{y_t} S^{\pm 1}} Z^{k,\mp}(z_{t+1}), \quad t \geq 0 \tag{B3}$$

infinite to one side. Here $z_t = \mathcal{S}^{\pm t}(z) \to x^{k,\pm}$, $t \to +\infty$ and $y_t \in W^{\pm}$ is the image of z_t under the projection $\mathfrak{M}^{k,\pm}_{n^{\mp}} \to W^{\pm}$. In an analogous way, an affine fiber $w^{k,\pm,\mp}(z) \subset Z^{k,\mp}(z)$, $z \in \widetilde{W}^{\mp}_{x^{k,\pm}}$, of the bundle $\widetilde{W}^{\mp}_{\widehat{x}^{k,\pm}} \to \widetilde{W}^{\mp}_{x^{k,\pm}}$ described above is an unstable slice determined by the itinerary scheme for the infinite to one side sequence of mappings (B3) with the only difference that now $t \leq 0$ and $z_t \to x^{k,\pm}$ as $t \to -\infty$ and $y_t \in W^{\mp}$ is the image of z_t under the projection $\mathfrak{M}^{k,\mp}_{n^{\mp}} \to W^{\mp}$.

Let now the partitions ξ_i^- (ξ_i^+) be right concordant, the diffeomorphism S be of class C^N, where $N \geq N_i^-$ ($N \geq N_i^+$), and condition 4) of Theorem 2 be valid. Then nonautonomous linearizations l_r^- (l_r^+) on the separatrices W_r^- (W_r^+), $r \in A$, are defined. Moreover, the pairs $\bigl(W_r^-, l_r^-\bigr)$ ($\bigl(W_r^+, l_r^+\bigr)$, respectively) satisfy condition (B1) and, consequently, they have k-jets $\hat{l}_r^{k,-} \in \widehat{J}^k_{n^-}(r)$ ($\hat{l}_r^{k,+} \in \widehat{J}^k_{n^+}(r)$). For each point $O = q_i$ we denote by $f^{k,\pm}_{i,m}$ and $\widetilde{W}^{k,\pm}_{i,+}$, $\widehat{W}^{k,\pm}_{i,+}$ ($\widetilde{W}^{k,\pm}_{i,-}$, $\widehat{W}^{k,\pm}_{i,-}$), respectively, the mapping $f^{k,\pm}_m$ and the incoming and outgoing manifolds $\widetilde{W}^{\pm}_{x^{k,+}}$ and $\widetilde{W}^{\pm}_{\widehat{x}^{k,+}}$ ($\widetilde{W}^{\pm}_{x^{k,-}}$ and $\widetilde{W}^{\pm}_{\widehat{x}^{k,-}}$) for the fixed points $x^{k,+} = \bigl(O, \alpha^{k,+}_{n^-}\bigr)$ and

$\widehat{x}^{k,+}$ ($x^{k,-} = \big(O, \alpha^{k,-}_{n^-}\big)$ and $\widehat{x}^{k,-}$) of the mappings $\mathcal{S}$ and $\widehat{\mathcal{S}}$. Then the manifolds $\widehat{W}^{k,-}_{i^-(j),+}$ and $\widehat{W}^{k,+}_{i^+(j),+}$ ($\widehat{W}^{k,-}_{i^-(j),-}$ and $\widehat{W}^{k,+}_{i^+(j),-}$, respectively) intersect at a point $\widehat{r}^{k,+}_j = \big(\widetilde{r}^{k,+}_j, \chi\big) = \hat{l}^{k,-}_{r_j}$ ($\widehat{r}^{k,-}_j = \big(\widetilde{r}^{k,-}_j, \chi\big) = \hat{l}^{k,+}_{r_j}$) lying over the point $\widetilde{r}^{k,+}_j = \big(r_j, f^{k,-}_{i^-(j),n^-}(r_j)\big)$ ($\widetilde{r}^{k,-}_j = \big(r_j, f^{k,+}_{i^+(j),n^-}(r_j)\big)$) of intersection of the manifolds $\widetilde{W}^{k,-}_{i^-(j),+}$ and $\widetilde{W}^{k,+}_{i^+(j),+}$ ($\widetilde{W}^{k,-}_{i^-(j),-}$ and $\widetilde{W}^{k,+}_{i^+(j),-}$). The k-jet $\chi = j^k_{r_j} l^-_{r_j} \in Z^{k,-}\big(\widetilde{r}^{k,+}_j\big)$ ($\chi = j^k_{r_j} l^+_{r_j} \in Z^{k,+}\big(\widetilde{r}^{k,-}_j\big)$) is determined by the itinerary scheme for the sequence of mappings

$$\hat{j}^k_{y_t} S: \; Z^{k,-}(z_t) \to Z^{k,-}(z_{t+1})$$
$$\Big(\; \hat{j}^k_{y_t} S^{-1}: \; Z^{k,+}(z_t) \to Z^{k,+}(z_{t+1}) \; \Big), \quad t \in \mathbb{Z}$$

infinite in both the directions. Here $z_t = \mathcal{S}^t\big(\widetilde{r}^{k,+}_j\big)$ ($z_t = \mathcal{S}^{-t}\big(\widetilde{r}^{k,-}_j\big)$) and $y_t = S^t(r_j)$ ($y_t = S^{-t}(r_j)$) is the projection of z_t. Moreover, χ happens to be the point of transversal intersection of the stable $w^{k,-,-}_{i^-(j)}(z_0)$ ($w^{k,+,+}_{i^+(j)}(z_0)$, respectively) and the unstable $w^{k,+,-}_{i^+(j)}(z_0)$ ($w^{k,-,+}_{i^-(j)}(z_0)$, respectively) slices lying in the fiber $Z^{k,-}(z_0)$ ($Z^{k,+}(z_0)$). Here $w^{k,\pm,-}_i(z)$ ($w^{k,\pm,+}_i(z)$) denotes the affine space $w^{k,\pm,-}(z)$ ($w^{k,\pm,+}(z)$) corresponding to the hyperbolic point $O = q_i$, i.e., the fiber of the manifold $\widehat{W}^{k,\pm}_{i,+}$ ($\widehat{W}^{k,\pm}_{i,-}$) that lies over $z \in \widetilde{W}^{k,\pm}_{i,+}$ ($z \in \widetilde{W}^{k,\pm}_{i,-}$). Therefore, in the case $N \geq k+1$ the point $\widehat{r}^{k,+}_j$ ($\widehat{r}^{k,-}_j$) happens to be the point of transversal intersection of the separatrices $\widehat{W}^{k,-}_{i^-(j),+}$ and $\widehat{W}^{k,+}_{i^+(j),+}$ ($\widehat{W}^{k,-}_{i^-(j),-}$ and $\widehat{W}^{k,+}_{i^+(j),-}$). Thus, although a satisfactory description of the manifolds $\widehat{W}^{k,+}_{i,+}$ ($\widehat{W}^{k,-}_{i,-}$) has not been obtained, one can state the following result. Under the conditions indicated above, items a)–c) of Lemma 4 with suitable modifications as well as Remarks 11 and B1 are transferable to the case where instead of bundles $\mathfrak{M}^k_{n^\mp}$ and $\mathfrak{A}^k_{n^\mp}$, the analogous bundles $\widehat{\mathfrak{M}}^k_{n^\mp}$ and $\widehat{\mathfrak{A}}^k_{n^\mp}$ of the k-jets of pairs are considered. We mention without any detail that items a) and c) have to be modified if the space $\mathcal{W}^{k,\pm,\mp}$ is non-trivial (i.e., does not reduce to a single point), and in the second formula of item b) instead of the tangent spaces $T_r W^\mp_r$ or, more generally, the k-jets of manifolds $W^\mp_r$ at the points $r \in A$, the k-jets $\hat{l}^{k,\mp}_r$ of the pairs $\big(W^\mp_r, l^\mp_r\big)$ appear. The triviality of $\mathcal{W}^{k,\pm,\mp}$ is equivalent to that $\Phi^\mp = \emptyset$, i.e., $N^\mp_0 = 1$, where $\Phi^\mp$ is the set Φ and $N^\mp_0$ is the number N_0 for the mapping $T = S^{\pm 1} \mid W^\pm_i$.

Problem. To carry out a more detailed investigation of the structure of the separatrix $\widetilde{W}^\pm_{\widehat{x}^{k,\pm}}$. In particular, what is the construction for a piece of the separatrix that lies over a neighbourhood of the boundary $\partial\widetilde{W}^\pm_{x^{k,\pm}}$? Note that to answer these questions, one requires first of all to use a more detailed description (see [45, 46]) of the dynamics of the mapping $\dot{S} \mid_O : G_{n^\mp}(O) \to G_{n^\mp}(O)$. An answer is easily given in the case of the skew product: if $\pi\colon M \to W^{\mp}$ is some C^N-retraction such that the diagram

$$\begin{tikzcd} M \arrow[r, "S"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ W^{\mp} \arrow[r, "S|W^{\mp}"'] & W^{\mp} \end{tikzcd}$$

commutes, i.e., S "covers" $S \mid W^{\mp}$, then the mapping $\widehat{v}^{k,\pm}$ can be defined by the following formula: $\widehat{v}^{k,\pm}(z, l^k)$ is the k-jet of the pair $\big(W, \chi_{r,l}\big)$ at the point $r \in W^{\pm}$ lying under $z \in \widetilde{W}^{\pm}_{x^{k,\pm}}$, the manifold W has k-jet z at r, the mapping $\chi_{r,l} = \mathcal{D}_{\pi|W} l$, and the mapping $l \in \mathcal{W}^{\mp,\mp}$ is such that $j^k_O l = l^k \in \mathcal{W}^{k,\mp,\mp}$. According to the theorem of Chen [26, Chapter 9], for any natural n and k and real $0 < \rho_1 < \rho_2 < 1$ there is N such that the following statement holds. If the eigenvalues λ_i of the n-dimensional C^N-diffeomorphism S at the hyperbolic fixed point O satisfy the inequalities $\rho_1 < \min\big\{|\lambda_i|, |\lambda_i|^{-1}\big\} < \rho_2$ and admit no resonances $(2')$ up to order N (i.e., for $2 \le |m| \le N$) then S will be C^k-equivalent (i.e., conjugated by a C^k-diffeomorphism) to its linear part, i.e., it becomes linear in some local C^k-coordinates. Let $\pi\colon M \to W^{\mp}$ be the C^k-retraction which is the projection along $W^{\pm}$ in these coordinates. Then the formula determining $\widehat{v}^{k,\pm}$ remains valid.

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