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Integration with respect to the canonical spectral measure in sequence spaces

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Abstract

Given a spectral measure P acting in a locally convex space X, there is a subtle connection between the properties of P and its associated space $\mathcal{L}^1(P)$ of P-integrable functions and of the topological properties of the underlying space X and the space $L(X)$ of all continuous linear operators on X (equipped with the strong operator topology). This paper makes a detailed study of the canonical spectral measure P acting in a class of locally convex *sequence spaces* $X \subseteq \mathbb{C}^{\mathbb{N}}$. Special emphasis is placed on developing criteria which guarantee the σ-additivity of P and criteria which allow for an explicit identification of $\mathcal{L}^1(P)$. Moreover, certain desirable features of the integration map $f \mapsto \int f dP$, $f \in \mathcal{L}^1(P)$, are established which are not true for general spectral measures acting in arbitrary locally convex spaces X.

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1. Introduction

Spectral measures in Banach or, more generally, locally convex Hausdorff spaces (briefly, lcHs) are natural extensions of the notion of the resolution of the identity of a normal operator in a Hilbert space. Integration with respect to such operator-valued measures has played an important role in the theory of operator algebras generated by Boolean algebras of projections; see [4], [5], [6], [7], [8], [22], [26], [27], [28] and the references therein, for example. Given a spectral measure P acting in a lcHs X there is a subtle connection between the properties of its associated lcHs $\mathcal{L}^1(P)$ of P-integrable functions and the properties of the underlying space X and the space $L(X)$ of all *continuous* linear operators of X into itself (for the strong operator topology). There are many general results available which provide sufficient conditions for $\mathcal{L}^1(P)$ to be a (complex) lattice, a complete or separable lcHs, etc. and also many examples illustrating the limitations of such general results. Because of the large diversity of possible lcH-spaces X and spectral measures P available it is imperative to be able to decide about two basic questions.

(i) Given a σ-algebra of sets Σ and a multiplicative set function $P : \Sigma \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of *all* linear maps of X into itself, when does P form a genuine spectral measure in $L(X)$? This problem reduces to two basic criteria: one needs to be able to determine that $Px : E \mapsto P(E)x$, for $E \in \Sigma$, is a σ-additive X-valued measure for each $x \in X$, and to be able to verify that each operator $P(E)$ actually belongs to $L(X)$ rather than just belonging to $\mathcal{L}(X)$.

(ii) Associated with each X-valued vector measure Px, for $x \in X$, is its lcHs $\mathcal{L}^1(Px)$ of Px-integrable functions. The general theory of vector measures is well developed and provides a variety of tools which can be applied to determine $\mathcal{L}^1(Px)$ rather concretely for specific examples of X and P. The problem arises in transferring this information to determine the space $\mathcal{L}^1(P)$ associated with P. It is clear that $\mathcal{L}^1(P) \subseteq \bigcap_{x\in X} \mathcal{L}^1(Px)$; the difficulty is to provide sufficient conditions, often of a somewhat delicate topological nature on X and $L(X)$ or on properties of P, which guarantee that this containment is actually an equality.

The aim of this paper is to investigate in depth the questions (i) and (ii) for a particular class of lcH-spaces X and a canonical spectral measure P acting in X. More specifically, X will come from a certain class of sequence spaces (all contained in $\mathbb{C}^{\mathbb{N}}$) with the property that each linear operator $P(E) \in \mathcal{L}(X)$ given by

$$P(E) : x \mapsto \chi_E x, \qquad x \in X,$$

for $E \in 2^{\mathbb{N}}$, is well defined meaning that $\chi_E x$ (defined coordinatewise) is again an element of X whenever $x \in X$ and $E \in 2^{\mathbb{N}}$. The reason for considering this particular setting is three-fold. Firstly, the questions (i) and (ii) are quite tractable, which is surely not the case in the general setting alluded to earlier. Secondly, by varying the lcH-topology to be put on X we are able to exhibit a large variety of spaces X which, even though P is fixed throughout, illustrate many detailed phenomena in relation to question (i). This is meant in the sense that we have quite general positive results and at the same time a wealth of examples illustrating the limitations involved. Thirdly, this canonical spectral measure P turns out to be "concrete enough" so that it is possible to describe the spaces $\mathcal{L}^1(P)$ and $\mathcal{L}^1(Px)$, for each $x \in X$, accurately enough to give an exact answer to question (ii).

The structure of this paper is roughly as follows. Section 2 records some notation and preliminaries needed later. Section 3 is mainly devoted to the question (i). The final section addresses question (ii) and also contains some additional features of the integration map $I_P : \mathcal{L}^1(P) \to L(X)$ which are specific to our setting.

2. Preliminaries

Let Σ be a σ-algebra of subsets of a non-empty set Ω. The space of all $\mathbb{C}$-valued, Σ-simple functions is denoted by $\text{sim}(\Sigma)$. Let Y be a lcHs. A function $m : \Sigma \to Y$ is called a *vector measure* if it is σ-additive. Given y' in the topological dual space Y' of Y, let $\langle m, y' \rangle$ denote the complex measure $E \mapsto \langle m(E), y' \rangle$, $E \in \Sigma$. A $\mathbb{C}$-valued, Σ-measurable function f on Ω is called *m-integrable* if it is $\langle m, y' \rangle$-integrable for each $y' \in Y'$ and if, given any $E \in \Sigma$, there is a (necessarily unique) element $\int_E f dm$ of Y such that $\langle \int_E f dm, y' \rangle = \int_E f d\langle m, y' \rangle$ for each $y' \in Y'$. The linear space of all m-integrable functions is denoted by $\mathcal{L}^1(m)$. Given $E \in \Sigma$ the characteristic function of E is denoted by χ_E. Clearly $\text{sim}(\Sigma) \subseteq \mathcal{L}^1(m)$.

Let X be a lcHs. The linear space $L(X)$ is denoted by $L_s(X)$ when it is equipped with the strong operator topology, that is, the topology of pointwise convergence on X. A vector measure $P : \Sigma \to L_s(X)$ is called a *spectral measure* if it is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$ for all $E, F \in \Sigma$) and if $P(\Omega) = I$, the identity operator on X.

Let $P : \Sigma \to L_s(X)$ be a spectral measure. For each $f \in \mathcal{L}^1(P)$ the operator $\int_\Omega f dP \in L(X)$ is also denoted by $P(f)$. For each $x \in X$, the X-valued set function Px on Σ defined by $Px : E \mapsto P(E)x$, for each $E \in \Sigma$, is σ-additive. Integrability with respect to a spectral measure P is simpler to characterize than for general vector measures (due to the muliplicativity of P).

Lemma 2.1 ([19; Lemma 1.2])

Let X be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. The following statements for a Σ-measurable function $f : \Omega \to \mathbb{C}$ are equivalent.
(i) *The function f is P-integrable.*
(ii) *The function f is $\langle Px, x'\rangle$-integrable for all $x \in X$ and $x' \in X'$, and there exists $T_1 \in L(X)$ such that*

$$\langle T_1 x, x'\rangle = \int_\Omega f d\langle Px, x'\rangle, \qquad x \in X, x' \in X'.$$

(iii) *The function f is Px-integrable for each $x \in X$, and there exists $T_2 \in L(X)$ such that*

$$T_2 x = \int_\Omega f dPx, \qquad x \in X.$$

In this case $T_1 = T_2 = P(f)$ and

$$\int_E f dP = P(f)P(E) = P(E)P(f), \qquad E \in \Sigma.$$

Lemma 2.1 clearly implies the inclusion $\mathcal{L}^1(P) \subseteq \bigcap_{x\in X} \mathcal{L}^1(Px)$. Given a function $f \in \bigcap_{x\in X} \mathcal{L}^1(Px)$ let $P_{[f]} : X \to X$ denote the linear map defined by

$$P_{[f]} : x \mapsto \int_\Omega f dPx, \qquad x \in X.$$

As noted in the Introduction a fundamental question is to decide when the equality

$$\mathcal{L}^1(P) = \bigcap_{x\in X} \mathcal{L}^1(Px) \tag{2.1}$$

holds or, equivalently, when $P_{[f]}$ is continuous on X for *every* $f \in \bigcap_{x\in X} \mathcal{L}^1(Px)$. To discuss this question let us introduce some further terminology. The lcHs X is said to have the *closed graph property* if every closed linear map from X into itself is necessarily continuous. Let $[L_s(X)]_P$ denote the sequential closure in $L_s(X)$ of the linear span of the range $P(\Sigma) = \{P(E) : E \in \Sigma\}$ of P. It is known that $[L_s(X)]_P$ coincides with the sequential closure of the range of the integration map

$$I_P : f \mapsto \int_\Omega f dP, \qquad f \in \mathcal{L}^1(P); \tag{2.2}$$

see the remark prior to Lemma 1.4 in [17]. Consider now the following three conditions:

$(H1)$ X is barrelled.

$(H2)$ X has the closed graph property and the linear map $P_{[f]} \in \mathcal{L}(X)$ is a closed map for each $f \in \bigcap_{x \in X} \mathcal{L}^1(Px)$.

$(H3)$ The lcHs $[L_s(X)]_P$ is sequentially complete.

It was recently shown by J. Bonet [2] that not all barrelled spaces have the closed graph property: he exhibited a class of normed, barrelled spaces on which there exist everywhere defined, closed linear operators which fail to be continuous.

The following result, which is an extension of [6; Proposition 1.2], can be found in [20; Theorem 1].

Lemma 2.2

Let X be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. Then the equality (2.1) *is implied by any one of the conditions* $(H1), (H2)$ *and* $(H3)$.

3. The canonical spectral measure in sequence spaces

We will only be dealing with sequence spaces in the setting of the space $\mathbb{C}^{\mathbb{N}}$ of all $\mathbb{C}$-valued functions on $\mathbb{N}$ (the set of all positive integers). We equip $\mathbb{C}^{\mathbb{N}}$ with the usual product topology so that $\mathbb{C}^{\mathbb{N}}$ becomes a Fréchet space. For more general sequence spaces and various extensions of most of the basic notions that we will consider we refer the reader to [10], for example.

For each $n \in \mathbb{N}$, let e_n denote the characteristic function $\chi_{\{n\}}$. The linear span of the set $\{e_n : n \in \mathbb{N}\}$ is denoted by c_{00}. A linear subspace of $\mathbb{C}^{\mathbb{N}}$ containing c_{00} is called a *sequence space*. Given a sequence space X, its *α-dual* X^{α} is defined to be the space of all $y \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x(n)y(n)| < \infty$ for every $x \in X$, [12]. Clearly $c_{00} \subseteq X^{\alpha}$. Given $f \in \mathbb{C}^{\mathbb{N}}$, define

$$Xf := \{xf : x \in X\},$$

where $(xf)(n) := x(n)f(n)$ for $n \in \mathbb{N}$.

Let $2^{\mathbb{N}}$ denote the σ-algebra of all subsets of $\mathbb{N}$. Then $\mathrm{sim}(2^{\mathbb{N}})$ is a sequence space. A sequence space X is called *monotone* if $X\varphi \subseteq X$ for every $\varphi \in \mathrm{sim}(2^{\mathbb{N}})$.

Example 3.1: Clearly the sequence spaces c_{00}, $\mathrm{sim}(2^{\mathbb{N}})$ and $\mathbb{C}^{\mathbb{N}}$ are monotone. Other frequently used examples of monotone sequence spaces are the space $\ell^p (1 \le p < \infty)$ of all $x \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x(n)|^p < \infty$, the subspace ℓ^{∞} of $\mathbb{C}^{\mathbb{N}}$ consisting of all bounded functions, and the space c_0 of all $x \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_{n\to\infty} x(n) = 0$. The sequence space c consisting of all $x \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_{n\to\infty} x(n)$ exists (in $\mathbb{C}$) is *not* monotone.

A sequence space equipped with a lcH-topology is called a *locally convex sequence space*; briefly, a lcss. The weak and Mackey topologies on a lcss X are denoted by $\sigma(X, X')$ and $\tau(X, X')$, respectively.

Let X be a monotone sequence space. Define the canonical set function $P : 2^{\mathbb{N}} \to \mathcal{L}(X)$ by

$$P(E)x = x\chi_E, \qquad E \in 2^{\mathbb{N}}, x \in X; \tag{3.1}$$

note that the monotone property of X ensures that $P(E)$ exists as an element of $\mathcal{L}(X)$ for each $E \in 2^{\mathbb{N}}$. Moreover, P is multiplicative and satisfies $P(\mathbb{N}) = I$. Let $x \in X$. Recall that $Px : 2^{\mathbb{N}} \to X$ is defined by

$$Px(E) = P(E)x, \qquad E \in 2^{\mathbb{N}}. \tag{3.2}$$

As noted in the Introduction, a fundamental question is to determine when Px becomes σ-additive with respect to a lcH-topology on X. This question is answered in Proposition 3.2 below. But first we require some further notation.

A monotone lcss X is said to be a *weak AK-space* if, given any $x \in X$, the sequence $\{xe_n\}_{n=1}^{\infty}$ is summable to x in X, i.e. $x = \lim_{n\to\infty} \sum_{j=1}^{n} xe_n$ in the topology of X.

Given a lcss X the canonical linear map $J : X' \to \mathbb{C}^{\mathbb{N}}$ is defined by

$$J(x') : n \mapsto \langle e_n, x' \rangle, \qquad n \in \mathbb{N}, \tag{3.3}$$

for each $x \in X'$, where $\langle \cdot, \cdot \rangle$ is the duality of the pair (X, X').

Proposition 3.2

Let X be a monotone lcss. Then the following statements are equivalent.

(i) *For each $x \in X$, the set function $Px : 2^{\mathbb{N}} \to X$ defined by (3.2) is σ-additive.*

(ii) *The lcss X is a weak AK-space.*

(iii) *The lcss $X_{\sigma(X,X')}$ is a weak AK-space.*

(iv) *The range of the canonical map $J : X' \to \mathbb{C}^{\mathbb{N}}$ given by (3.3) is contained in the α-dual X^{α} of X and the identity*

$$\langle x, x' \rangle = \sum_{n=1}^{\infty} J(x')(n)x(n) \tag{3.4}$$

holds, for each $x \in X$ and $x' \in X'$.

Proof. (i)⇒(ii). For each $x \in X$ the statement (i) implies that

$$x = P(\mathbb{N})x = \sum_{n=1}^{\infty} Px(\{n\}) = \sum_{n=1}^{\infty} xe_n.$$

(ii)⇒(iii). This implication is clear.

(iii)⇒(iv). Fix $x' \in X'$. Let $x \in X$ and $E \in 2^{\mathbb{N}}$. By (iii) the sequence $\{x\chi_E e_n\}_{n=1}^{\infty}$ is summable to $x\chi_E$ with respect to the weak topology $\sigma(X, X')$. Since $x' \in (X_{\sigma(X,X')})'$ we have $\sum_{n=1}^{\infty} \langle x\chi_E e_n, x' \rangle = \sum_{n=1}^{\infty} J(x')(n)\chi_E(n)x(n)$ in $\mathbb{C}$. In other words, $\{\langle xe_n, x'\rangle\}_{n=1}^{\infty}$ is unconditionally summable and hence, by a classical result of Riemann, is absolutely summable in $\mathbb{C}$. Accordingly, $J(x') \in X^{\alpha}$ and (3.4) holds by choosing $E = \mathbb{N}$.

(iv)⇒(i). Let $x \in X$. It follows from (iv) that the $\mathbb{C}$-valued set function $\langle Px, x'\rangle$ is σ-additive for each $x' \in X'$. The Orlicz-Pettis theorem, [9; p.308], then implies (i). □

Definition 3.3. A monotone lcss X has the $2^{\mathbb{N}}$-*summability property* if the canonical set function $P : 2^{\mathbb{N}} \to \mathcal{L}(X)$ given by (3.1) is an $L_s(X)$-valued spectral measure.

Let $1 \le p < \infty$. Then the sequence space ℓ^p equipped with the usual norm $||x||_p = (\sum_{n=1}^{\infty} |x(n)|^p)^{1/p}$, for $x \in \ell^p$, has the $2^{\mathbb{N}}$-summability property. The same is true of c_0 when equipped with the norm $||x||_{\infty} = \sup_n |x(n)|$, for $x \in c_0$. However, the space ℓ^{∞} does not have the $2^{\mathbb{N}}$-summability property with respect to the norm $||x||_{\infty} = \sup_n |x(n)|$, for $x \in \ell^{\infty}$.

A direct consequence of Definition 3.3 and Proposition 3.2 is the following result.

Corollary 3.4

A monotone lcss X has the $2^{\mathbb{N}}$-summability property if and only if X is a weak AK-space and the inclusion

$$P(2^{\mathbb{N}}) = \left\{P(E) : E \in 2^{\mathbb{N}}\right\} \subseteq L(X) \tag{3.5}$$

holds.

A lcss X is called a K-*space* if

$$P(\{n\}) \in L(X), \qquad n \in \mathbb{N}. \tag{3.6}$$

This is equivalent to $P(E) \in L(X)$ for each finite and cofinite subset E of $\mathbb{N}$ which, in turn, is equivalent to continuity of the natural injection from X into the Fréchet space $\mathbb{C}^{\mathbb{N}}$. A lcss X is called an AK-*space* if it is both a K-space and a weak AK-space. Since $(\mathbb{C}^{\mathbb{N}})' = c_{00}$ we see that a weak AK-space is an AK-space if and only if $X' \supseteq c_{00}$, where X' is regarded as a linear subspace of $\mathbb{C}^{\mathbb{N}}$; see (3.3) and Proposition 3.2. Clearly a lcss with the $2^{\mathbb{N}}$-summability property is an AK-space. The following example shows that there exist monotone lc-sequence spaces (even a Banach space) which are neither K-spaces nor weak AK-spaces.

EXAMPLE 3.5: Let the monotone sequence space ℓ^1 be equipped with its usual norm $||\cdot||_1$. Then there exists a discontinuous linear functional $x^* : \ell^1 \to \mathbb{C}$ which vanishes on c_{00}. To see this, let Y be an algebraic complement of c_{00} in ℓ^1 so that ℓ^1 is the algebraic direct sum of c_{00} and Y. Let $\pi : \ell^1 \to Y$ denote the associated projection. Equip the subspace Y of ℓ^1 with the relative topology. Then π is not continuous because Y is not closed in ℓ^1. Therefore π also fails to be continuous with respect to the topologies $\sigma(\ell^1, \ell^\infty)$ and $\sigma(Y, Y')$ on ℓ^1 and Y, respectively. In other words, there is $y' \in Y'$ for which $x^* = y' \circ \pi$ is not continuous on $\ell^1_{\sigma(\ell^1,\ell^\infty)}$. Hence x^* also fails to be continuous for the norm $||\cdot||_1$.

Given $x \in \ell^1$ we have $x = (x(1) - \langle x, x^* \rangle)e_1 + [\langle x, x^*\rangle e_1 + (x - e_1 x)]$. Let X be the vector space ℓ^1 equipped with the norm $||\cdot||_X$ defined by

$$||x||_X = |x(1) - \langle x, x^*\rangle| + ||xe_1 - x||_1, \qquad x \in X.$$

Then X is complete with respect to $||\cdot||_X$ because X is isomorphic to the topological direct sum of ℓ^1 and the linear span of $\{e_1\}$.

Choose any $x \in X$ for which $\langle x, x^*\rangle \neq 0$. Since x^* vanishes on c_{00} it follows that

$$\Big\|x - \sum_{k=1}^{n} xe_k\Big\|_X = |\langle x, x^*\rangle| + \Big\|x - \sum_{k=1}^{n} xe_k\Big\|_1 \to |\langle x, x^*\rangle|, \qquad n \to \infty.$$

This shows that X is not a weak AK-space.

To see that X is not a K-space it suffices to verify that $P(\{1\}) \notin L(X)$. Since x^* is not continuous on ℓ^1 we can choose a sequence $\{y_n\}_{n=1}^\infty$ in ℓ^1 such that $||y_n||_1 \to 0$, yet $n \mapsto \langle y_n, x^*\rangle$, for $n \in \mathbb{N}$, does not converge to 0 as $n \to \infty$. Define a sequence $\{x_n\}_{n=1}^\infty$ in X by

$$x_n = \langle y_n - e_1, x^*\rangle e_1 + (y_n - e_1 y_n), \qquad n \in \mathbb{N}.$$

Then the sequence of complex numbers

$$x_n(1) = \langle y_n - e_1, x^*\rangle = \langle y_n, x^*\rangle, \qquad n \in \mathbb{N},$$

fails to converge to 0 in $\mathbb{C}$, but $||x_n||_X \to 0$ as $n \to \infty$ because $||x_n||_X = ||y_n - e_1 y_n||_1 \leq ||y_n||_1$ for each $n \in \mathbb{N}$. Since $P(\{1\})x_n = x_n(1)e_1$ for each $n \in \mathbb{N}$, the linear map $P(\{1\})$ is not continuous on X.

There also exist monotone lc-sequence spaces (even normed ones) which are K-spaces but not weak AK-spaces.

EXAMPLE 3.6: (i) $X = \ell^\infty$ with norm $||\cdot||_\infty$ is a Banach space which is a K-space but not a weak AK-space.

(ii) Let $X = \ell^1$ as a vector space and let x^* be as in Example 3.5. Define a norm on X by

$$||x|| = |\langle x, x^*\rangle| + ||x||_1, \qquad x \in X.$$

Then X is not complete. For, if it were, then the identity map from ℓ^1 onto X would be continuous by the open mapping theorem and so x^* would be continuous on ℓ^1 for the norm $||\cdot||_1$ (which is not the case).

Since x^* vanishes on c_{00} we see that (3.6) holds, that is, X is a K-space. However, X is not a weak AK-space since

$$\lim_{n\to\infty} \Big\langle \sum_{k=1}^{n} x e_k, x^* \Big\rangle = 0 \neq \langle x, x^*\rangle, \qquad x \in X \setminus (x^*)^{-1}(\{0\}).$$

We now turn our attention to finding sufficient conditions which guarantee that (3.5) holds. A seminorm q on a sequence space X is called *absolutely monotone* if $q(x) \le q(y)$ for all $x, y \in X$ satisfying $|x| \le |y|$, [10; p.64], where by definition $|x|$ has co-ordinates $|x(n)|$, $n \in \mathbb{N}$, for each $x \in \mathbb{C}^{\mathbb{N}}$. For instance, the usual norm in each space ℓ^p, $1 \le p \le \infty$, is absolutely monotone.

Lemma 3.7

Let X be a monotone lcss whose topology is defined by a fundamental set of absolutely monotone seminorms. Then the inclusion (3.5) *necessarily holds.*

Proof. This is a consequence of the fact that

$$q\big(P(E)x\big) = q(x\chi_E) \le q(x), \qquad x \in X, E \in 2^{\mathbb{N}},$$

whenever q is an absolutely monotone seminorm on X. □

It is clear that any lcss with the properties assumed in Lemma 3.7 is a K-space. Example 3.6 (i) shows that it does not follow that such a space is an AK-space.

Lemma 3.7 allows us to exhibit a large class of lc-sequence spaces which have the $2^{\mathbb{N}}$-summability property.

EXAMPLE 3.8: Let X be a monotone sequence space and let Y be a linear subspace of X^α containing c_{00}. Given $y \in Y$, the seminorm r_y on X defined by

$$r_y(x) = \sum_{n=1}^{\infty} \left|(x(n)y(n)\right|, \qquad x \in X,$$

is absolutely monotone. The lcH-topology on X generated by the family of seminorms $\{r_y : y \in Y\}$ is denoted by $|\sigma|(X, Y)$. Proposition 3.2 implies that $X_{|\sigma|(X,Y)}$ is a weak AK-space. It then follows from Corollary 3.4 and Lemma 3.7 that $X_{|\sigma|(X,Y)}$ has the $2^{\mathbb{N}}$-summability property. See also the notion of normal topology as given in [12].

The topology $\sigma(X, X')$ on a monotone lcss X does *not* satisfy the assumption of Lemma 3.7 unless $X' = c_{00}$. So, to ensure the $2^{\mathbb{N}}$-summability property in spaces equipped with their weak topology we require other criteria; see Proposition 3.10 below. First we require a technical result.

Lemma 3.9

Let X be a monotone sequence space and let Y be a linear subspace of X^α such that $c_{00} \subseteq Y$. Then the following statements hold.

(i) *The lcss $X_{\sigma(X,Y)}$ is an AK-space.*

(ii) *The lcss $X_{\sigma(X,Y)}$ has the $2^{\mathbb{N}}$-summability property if and only if Y is monotone.*

Proof. Statement (i) is clear.

Statement (ii) follows because

$$\langle P(E)x, y\rangle = \sum_{n=1}^{\infty} y(n)\chi_E(n)x(n) = \sum_{n=1}^{\infty}(y\chi_E)(n)x(n)\,, \qquad x \in X,$$

for $E \in \Sigma$ and $y \in Y$, and because Y is monotone if and only if $Y\chi_E \subseteq Y$ for all $E \in \Sigma$. □

Via Lemma 3.9 it is easy to exhibit AK-spaces without the $2^{\mathbb{N}}$-summability property. For instance, $X = \ell^1_{\sigma(\ell^1,c)}$ provides such an example because $Y = c$ is not monotone.

The fact that the weak topology $\sigma(X, X')$ on a lcHs X is compatible with the duality (X, X') leads to the following result.

Proposition 3.10

Let X be a monotone lcss which is an AK-space and, via (3.3) *and Proposition* 3.2(iv)*, regard X' as a subspace of X^α. Then the following statements hold.*

(i) *If X has the $2^{\mathbb{N}}$-summability property, then so does $X_{\sigma(X,X')}$.*

(ii) *The following conditions are equivalent:*

(a) X' is monotone.

(b) $X_{\sigma(X,X')}$ has the $2^{\mathbb{N}}$-summability property.

(c) $X_{\tau(X,X')}$ has the $2^{\mathbb{N}}$-summability property.

(iii) *If X has the $2^{\mathbb{N}}$-summability property, then X' is monotone.*

(iv) *For each $E \in 2^{\mathbb{N}}$, suppose that*

$$\left\{ \sum_{k=1}^{n} P\big(E \cap \{k\}\big) : n \in \mathbb{N} \right\} \tag{3.7}$$

is an equicontinuous subset of $L(X)$. Then X has the $2^{\mathbb{N}}$-summability property.

(v) *If X is barrelled, then X has the $2^{\mathbb{N}}$-summability property.*

(vi) *If X has the closed graph property, then X has the $2^{\mathbb{N}}$-summability property.*

Proof. Statement (i) follows from the fact that $L(X) \subseteq L(X_{\sigma(X,X')})$.

(ii) Since $c_{00} \subseteq X'$ (see the comments after Corollary 3.4) the equivalence of (a) and (b) follows from Lemma 3.9 (ii). The equivalence of (b)$\Leftrightarrow$ (c) follows from Proposition 3.2 applied to $X_{\tau(X,X')}$ and the fact that $L(X_{\sigma(X,X')}) = L(X_{\tau(X,X')})$; see [9; Corollary 8.6.5].

(iii) Apply (i) and (ii).

(iv) Fix $E \in 2^{\mathbb{N}}$. Let q be a continuous seminorm on X. By assumption there is a continuous seminorm r on X such that

$$q\left(\sum_{k=1}^{n} Px(E\cap\{k\})\right) = q\left(\left[\sum_{k=1}^{n} P(E\cap\{k\})\right] x\right) \le r(x), \qquad x \in X,$$

for each $n \in \mathbb{N}$. The σ-additivity of Px, for each $x \in X$, implies that

$$\begin{aligned} q(P(E)x) = {}& q(Px(E)) = q\left(\lim_{n\to\infty} \sum_{k=1}^{n} Px(E\cap\{k\})\right) \\ = {}& \lim_{n\to\infty} q\left(\sum_{k=1}^{n} Px(E\cap\{k\})\right) \le r(x)\,. \end{aligned}$$

Hence, $P(E) \in L(X)$. Since $E \in 2^{\mathbb{N}}$ is arbitrary it follows from Corollary 3.4 that X has the $2^{\mathbb{N}}$-summability property.

(v) Given $x \in X$ and $E \in 2^{\mathbb{N}}$ we have

$$\sum_{k=1}^{n} P(E \cap \{k\})x = Px(E \cap \{1, 2, \ldots, n\}) \in Px(2^{\mathbb{N}}), \qquad n \in \mathbb{N}. \tag{3.8}$$

Moreover, the range $Px(2^{\mathbb{N}})$ of the X-valued measure Px is a bounded subset of X, [11; II, Lemma 1.2]. Hence, (3.8) implies that the set (3.7) is bounded in $L_s(X)$ and so (3.7) is actually equicontinuous because X is barrelled, [13; (2),§39.3]. Then (iv) gives the desired conclusion.

(vi) Let $E \in 2^{\mathbb{N}}$. Consider a net $\{x_\alpha\}$ in X with the property that the nets $\{x_\alpha\}$ and $\{P(E)x_\alpha\}$ are convergent to elements x and y in X, respectively. Since the topology on X is stronger than that induced by $\mathbb{C}^{\mathbb{N}}$ it follows that $x_\alpha \to x$ pointwise on $\mathbb{N}$ and hence, also $x_\alpha \chi_E \to x\chi_E$ pointwise on $\mathbb{N}$. Since also $x_\alpha \chi_E = P(E)x_\alpha \to y$ pointwise on $\mathbb{N}$, we conclude that $y = x\chi_E = P(E)x$. Thus $P(E) \in \mathcal{L}(X)$ is a closed map and so $P(E) \in L(X)$ by the assumption on X. The required conclusion then follows from Corollary 3.4. □

A lcHs is called a *Mackey space* if its given topology is equal to its Mackey topology $\tau(X, X')$. Every quasi-barrelled space is a Mackey space, [9; p.222]. Accordingly, Proposition 3.10(ii) implies that a quasi-barrelled AK-space X has the $2^{\mathbb{N}}$-summability property if and only if X' is monotone. Note that the class of quasi-barrelled spaces is quite extensive: it includes all bornological spaces and hence, all metrizable spaces, [12; (1), (4) in §28.1].

Question. Does there exist a monotone lcss which is a quasi-barrelled AK-space but fails to have the $2^{\mathbb{N}}$-summability property?

EXAMPLE 3.11: The following spaces X have the $2^{\mathbb{N}}$-summability property, either by Example 3.8 or by Proposition 3.10.

(i) A monotone sequence space X equipped with the topology $\sigma(X, c_{00}) = |\sigma|(X, c_{00})$.

(ii) The space $X = c_{00}$ equipped with lc-direct sum topology as a subspace of the direct sum of countably many copies of $\mathbb{C}$, or the topology $\sigma(c_{00}, Y)$ for any monotone sequence space Y containing c_{00}.

(iii) A monotone linear subspace X of ℓ^p, $1 \le p < \infty$, equipped with either the $||\cdot||_p$-norm topology, or one of the topologies $\sigma(X, Y)$, $|\sigma|(X, Y)$, $\tau(X, Y)$ where Y is any monotone linear subspace of ℓ^q (with $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$) containing c_{00}.

(iv) A monotone linear subspace X of ℓ^∞, equipped with one of the topologies $\sigma(X,Y)$, $|\sigma|(X,Y)$ or $\tau(X,Y)$, where Y is any monotone linear subspace of ℓ^1 containing c_{00}.

It is known that the range of any purely atomic spectral measure is always a sequentially closed subset of $L_s(X)$, [25; Theorem]. For the canonical spectral measure P of this paper, which is surely purely atomic, a stronger result is true.

Proposition 3.12

Let X be a monotone lcss with the $2^{\mathbb{N}}$-summability property. Then the range of the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(X)$ defined by (3.1) is actually a complete subset of $L_s(X)$. In particular, it is also a closed subset of $L_s(X)$.

Proof. Let $\{P(E_\alpha)\}_{\alpha\in A}$ be a Cauchy net in $L_s(X)$. Fix $n \in \mathbb{N}$. Since $\{P(E_\alpha)e_n\}_{\alpha\in A}$ is a Cauchy net in the (1-dimensional) complete subspace of X spanned by e_n it has a limit of the form $a_n e_n$ where the complex number $a_n \in \{0,1\}$. Let $E = \{n \in \mathbb{N} : a_n = 1\}$. Then

$$\lim_\alpha \chi_{E_\alpha}(n) = \chi_E(n), \qquad n \in \mathbb{N}. \tag{3.9}$$

The claim is that $\{P(E_\alpha)\}_{\alpha\in A}$ converges to $P(E)$ in $L_s(X)$. To see this fix $x \in X$. Let $\epsilon > 0$ and let q be a continuous seminorm on X. For each $n \in \mathbb{N}$ let $F_n = \mathbb{N} \setminus \{1,\dots,n\}$. Defining

$$q(Px) : F \mapsto \sup\big\{q(Px(G)) : G \subseteq F\big\}, \qquad F \in 2^{\mathbb{N}},$$

we have that $\lim_{n\to\infty} q(Px)(F_n) = 0$; see Lemmas 1.1 and 1.2 in Chapter II of [11]. Since

$$q\big(Px(E_\alpha \cap F_n)\big) + q\big(Px(E \cap F_n)\big) \le 2q(Px)(F_n)$$

for each $\alpha \in A$ and $n \in \mathbb{N}$, there is an integer $K \ge 2$ such that

$$\sup\Big\{q\big(Px\big(E_\alpha \cap F_K\big)\big) + q\big(Px(E \cap F_K)\big) : \alpha \in A\Big\} < \epsilon. \tag{3.10}$$

Since $\chi_{G\setminus H} = \chi_G \chi_{\mathbb{N}\setminus H}$ for any $G, H \in 2^{\mathbb{N}}$ it follows from (3.9) that

$$\lim_\alpha \chi_{E_\alpha\setminus F_K}(n) = \chi_{E\setminus F_K}(n), \qquad n \in \mathbb{N}. \tag{3.11}$$

Moreover, the sets $E\setminus F_K$ and $E_\alpha \setminus F_K$ are contained in the finite set $\{1,\dots,(K-1)\}$, for each $\alpha \in A$. Accordingly, (3.11) and the inequality

$$q\big(Px(E_\alpha \setminus F_K) - Px(E \setminus F_K)\big) \le \sum_{j=1}^{K-1} |x(j)| q(e_j) |\chi_{E_\alpha\setminus F_K}(j) - \chi_{E\setminus F_K}(j)|,$$

which is valid for each $\alpha \in A$, imply that there exists $\alpha_0 \in A$ for which

$$q\big(Px(E_\alpha \setminus F_K) - Px(E \setminus F_K)\big) < \epsilon, \qquad \alpha \geq \alpha_0. \tag{3.12}$$

Since the identity

$$Px(E_\alpha) - Px(E) = Px(E_\alpha \cap F_K) + \big[Px(E_\alpha \setminus F_K) - Px(E \setminus F_K)\big] + \big[-Px(E \cap F_K)\big]$$

is valid for each $\alpha \in A$, it follows from this identity, the triangle inequality for q, and the inequalities (3.10) and (3.12) that $q(Px(E_\alpha) - Px(E)) < 2\epsilon$ whenever $\alpha \geq \alpha_0$. This establishes that $P(E_\alpha)x \to P(E)x$ in X. Since $x \in X$ is arbitrary it follows that $P(E_\alpha) \to P(E)$ in $L_s(X)$. □

4. The space of P-integrable functions

Unless stated otherwise, throughout this section X is a monotone lcss which has the $2^{\mathbb{N}}$-summability property. Hence, the canonical set function P on $2^{\mathbb{N}}$ defined by

$$P(E)x = \chi_E x, \qquad E \in 2^{\mathbb{N}}, x \in X, \tag{4.1}$$

is an $L_s(X)$-valued spectral measure. By applying (3.3) and Proposition 3.2 to the weak AK-space X we may assume that $c_{00} \subseteq X' \subseteq \mathbb{C}^{\mathbb{N}}$ and $\langle x, x' \rangle = \sum_{n=1}^{\infty} x(n)x'(n)$, for all $x \in X$ and $x' \in X'$. For each $x \in X$ recall that $Px : 2^{\mathbb{N}} \to X$ is the σ-additive measure given by $Px : E \mapsto P(E)x$, for each $E \in 2^{\mathbb{N}}$. Finally, let $\delta_n : 2^{\mathbb{N}} \to \mathbb{C}$ denote the Dirac point measure at each $n \in \mathbb{N}$.

Lemma 4.1

Let $x \in X$. Then

$$\mathcal{L}^1(Px) = \{f \in \mathbb{C}^{\mathbb{N}} : xf \in X\} \tag{4.2}$$

and, for each $f \in \mathcal{L}^1(Px)$, we have

$$\int_E f\,dPx = \chi_E xf = P(E)xf, \qquad E \in 2^{\mathbb{N}}. \tag{4.3}$$

Consequently,

$$\bigcap_{x \in X} \mathcal{L}^1(Px) = \left\{ f \in \mathbb{C}^{\mathbb{N}} : Xf \subseteq X \right\}. \tag{4.4}$$

Proof. For all $x \in X$ and $x' \in X'$ we have $\langle Px, x' \rangle = \sum_{n=1}^{\infty} x(n)x'(n)\delta_n$, which implies both (4.2) and (4.3) since, for each $\varphi \in L^1(\langle Px, x' \rangle)$, we have

$$\int_E \varphi\, d\langle Px, x' \rangle = \sum_{n=1}^{\infty} x(n)x'(n)\chi_E(n)\varphi(n), \qquad E \in 2^{\mathbb{N}}.$$

The identity (4.4) then follows from (4.2). □

Given a function $f : \mathbb{N} \to \mathbb{C}$ satisfying $Xf \subseteq X$ we define a linear map $M_f : X \to X$ by $M_f : x \mapsto xf$, for each $x \in X$. By using the assumption that the topology on X is stronger than that induced by $\mathbb{C}^{\mathbb{N}}$ it can be shown that M_f is a closed linear map, along the same lines that $P(E)$ was shown to be closed in the proof of Proposition 3.10(vi). Moreover, given any $f \in \bigcap_{x\in X} \mathcal{L}^1(Px)$, it follows from Lemma 4.1 that the operator $P_{[f]} \in \mathcal{L}(X)$ defined by $x \mapsto \int_{\mathbb{N}} f dPx$, for each $x \in X$, is precisely the closed linear map M_f. Accordingly, for our canonical spectral measure P the condition $(H2)$ may be reformulated as:

$(H2)^*$ X has the closed graph property.

We now turn our attention to the space $\mathcal{L}^1(P)$. The following result is immediate from Lemma 4.1.

Proposition 4.2

A function $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{L}^1(P)$ if and only if $Xf \subseteq X$ and $M_f \in L(X)$. In this case $\int_{\mathbb{N}} f dP = M_f$.

In view of the inclusion $L(X) \subseteq L(X_{\sigma(X,X')})$, let $\Lambda : L_s(X) \to L_s(X_{\sigma(X,X')})$ be the natural injection. Since Λ is continuous, the set function $\Lambda \circ P : 2^{\mathbb{N}} \to L_s(X_{\sigma(X,X')})$ is a spectral measure satisfying $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\Lambda\circ P)$. Moreover, Lemma 4.1 implies that

$$\mathcal{L}^1\big((\Lambda \circ P)x\big) = \big\{f \in \mathbb{C}^{\mathbb{N}} : xf \in X\big\} = \mathcal{L}^1(Px), \tag{4.5}$$

for each $x \in X$. Thus

$$\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\Lambda \circ P) \subseteq \bigcap_{x\in X} \mathcal{L}^1(Px). \tag{4.6}$$

By Lemma 2.2 any one of the conditions $(H1)$, $(H2)^*$ and $(H3)$ ensures that

$$\mathcal{L}^1(P) = \bigcap_{x\in X} \mathcal{L}^1(Px). \tag{4.7}$$

On the other hand, (4.7) implies the identity $\mathcal{L}^1(\Lambda \circ P) = \bigcap_{x\in X} \mathcal{L}^1((\Lambda \circ P)x)$. Accordingly, whenever X has the property that the inclusion

$$\mathcal{L}^1(\Lambda \circ P) \subseteq \bigcap_{x\in X} \mathcal{L}^1\big((\Lambda \circ P)x\big) \tag{4.8}$$

is strict, then also the inclusion

$$\mathcal{L}^1(P) \subseteq \bigcap_{x\in X} \mathcal{L}^1(Px) \tag{4.9}$$

is strict. It turns out that the space $\mathcal{L}^1(\Lambda \circ P)$ can be easily described.

Proposition 4.3

A function $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{L}^1(\Lambda \circ P)$ if and only if

$$Xf \subseteq X \quad \text{and} \quad X'f \subseteq X'. \tag{4.10}$$

This proposition is a direct consequence of the following result.

Lemma 4.4

The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ satisfying $Xf \subseteq X$ are equivalent:

(i) $M_f \in L(X_{\sigma(X,X')})$.
(ii) $M_f \in L(X_{\tau(X,X')})$.
(iii) $X'f \subseteq X'$.

Proof. The equivalence of (i) and (iii) follows from the identity

$$\langle M_f x, x' \rangle = \sum_{n=1}^{\infty} x(n)x'(n)f(n), \qquad x \in X, x' \in X'.$$

The equivalence (i)⇔(ii) is clear from $L(X_{\sigma(X,X')}) = L(X_{\tau(X,X')})$. □

Corollary 4.5

The following statements hold.

(i) *If $f \in \mathcal{L}^1(P)$, then* (4.10) *is satisfied.*
(ii) *The identity*

$$\mathcal{L}^1(P) = \mathcal{L}^1(\Lambda \circ P) \tag{4.11}$$

holds if and only if every function $f : \mathbb{N} \to \mathbb{C}$ which satisfies (4.10) *is P-integrable.*
(iii) *If X is quasi-barrelled, then* (4.11) *holds.*
(iv) *If any one of* $(H1)$, $(H2)^*$ *and* $(H3)$ *holds, then every function $f : \mathbb{N} \to \mathbb{C}$ satisfying $Xf \subseteq X$ also satisfies $X'f \subseteq X'$.*

Proof. To establish (i) and (ii) we apply (4.6) and Proposition 4.3. Statement (iii) follows from the fact that X is a Mackey space. Statement (iv) is a consequence of statement (i) and (4.4), after noting that (4.7) holds by Lemma 2.2. □

An example of a monotone lcss X with the $2^{\mathbb{N}}$-summability property for which strict inclusion holds in (4.9) has been given in [20; Example 7]. An extensive collection of additional examples is now presented (cf. Examples 4.6 and 4.7).

EXAMPLE 4.6: Fix $1 \leq p \leq \infty$. Let $X = c_{00}$ be equipped with the norm topology induced from the Banach space ℓ^p, in which case X is quasi-barrelled. Applying Proposition 4.3 and Corollary 4.5 (iii) we see that $\mathcal{L}^1(P) = \mathcal{L}^1(\Lambda \circ P) = \ell^\infty$ since $X' = \ell^q$ (where $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$) and because a function $f : \mathbb{N} \to \mathbb{C}$ satisfies $\ell^q f \subseteq \ell^q$ if and only if $f \in \ell^\infty$. On the other hand it is clear that $\mathbb{C}^{\mathbb{N}} = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

EXAMPLE 4.7: Let Y be a monotone sequence space. Let Z be a monotone linear subspace of Y^α such that $c_{00} \subseteq Z$. Then Lemma 3.9 implies that $X = Y_{\sigma(Y,Z)}$ is a monotone lcss with the $2^{\mathbb{N}}$-summability property. Moreover, Proposition 4.3 then implies that

$$\mathcal{L}^1(P) = \{f \in \mathbb{C}^{\mathbb{N}} : Yf \subseteq Y \text{ and } Zf \subseteq Z\}. \tag{4.12}$$

On the other hand, Lemma 4.1 yields

$$\bigcap_{x \in X} \mathcal{L}^1(Px) = \{f \in \mathbb{C}^{\mathbb{N}} : Yf \subseteq Y\}. \tag{4.13}$$

So, if there exists a function $f : \mathbb{N} \to \mathbb{C}$ satisfying $Yf \subseteq Y$ but $Zf \not\subseteq Z$, then the inclusion (4.9) is strict. The descriptions given by (4.12) and (4.13) allow us to exhibit an array of "curious examples".

(i) Let $Y = \ell^1$ and Z be either c_{00}, or c_0, or ℓ^p (for any $1 \leq p \leq \infty$). Then $\mathcal{L}^1(P) = \ell^\infty = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(ii) Let $Y = \ell^1$ and $Z = \mathrm{sim}(2^{\mathbb{N}})$. Then $\mathcal{L}^1(P) = \mathrm{sim}(2^{\mathbb{N}})$, but $\ell^\infty = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(iii) Let $Y = \mathrm{sim}(2^{\mathbb{N}})$ and Z be any monotone subspace of $\mathbb{C}^{\mathbb{N}}$ satisfying $c_{00} \subseteq Z \subseteq \ell^1$. Then $\mathcal{L}^1(P) = \mathrm{sim}(2^{\mathbb{N}}) = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(iv) Let $Y = c_{00}$ and $Z = c_{00}$. Then $\mathcal{L}^1(P) = \mathbb{C}^{\mathbb{N}} = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(v) Let $Y = c_{00}$ and $Z = \mathrm{sim}(2^{\mathbb{N}})$. Then $\mathcal{L}^1(P) = \mathrm{sim}(2^{\mathbb{N}})$, but $\mathbb{C}^{\mathbb{N}} = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(vi) Let $Y = c_{00}$ and $Z = c_0$ or ℓ^p (for any $1 \leq p \leq \infty$). Then $\mathcal{L}^1(P) = \ell^\infty$, but $\mathbb{C}^{\mathbb{N}} = \bigcap_{x \in X} \mathcal{L}^1(Px)$.

(vii) Let $E(1)$ denote the set of all odd integers in $\mathbb{N}$ and $E(2) = \mathbb{N} \setminus E(1)$. Let

$$Y = \{y \in \mathbb{C}^{\mathbb{N}} : y\chi_{E(1)} \in \ell^1, y\chi_{E(2)} \in c_{00}\}$$

and

$$Z = \{z \in \mathbb{C}^{\mathbb{N}} : z\chi_{E(1)} \in \mathrm{sim}(2^{\mathbb{N}}), z\chi_{E(2)} \in c_{00}\}.$$

Then $\mathcal{L}^1(P) = \{\varphi \in \mathbb{C}^{\mathbb{N}} : \varphi\chi_{E(1)} \in \mathrm{sim}(2^{\mathbb{N}})\}$ but $\bigcap_{x\in X} \mathcal{L}^1(Px) = \{\varphi \in \mathbb{C}^{\mathbb{N}} : \varphi\chi_{E(1)} \in \ell^\infty\}$.

(viii) Let $E(1) = \{3n-2 : n \in \mathbb{N}\}, E(2) = \{3n-1 : n \in \mathbb{N}\}$ and $E(3) = \{3n : n \in \mathbb{N}\}$. Let

$$Y = \{y \in \mathbb{C}^{\mathbb{N}} : y\chi_{E(1)} \in \ell^1, y\chi_{E(2)} \in \mathrm{sim}(2^{\mathbb{N}}), y\chi_{E(3)} \in c_{00}\}$$

and

$$Z = \{z \in \mathbb{C}^{\mathbb{N}} : z\chi_{E(1)} \in \mathrm{sim}(2^{\mathbb{N}}), z\chi_{E(2)} \in \Delta, z\chi_{E(3)} \in c_{00}\},$$

where Δ is either c_{00} or ℓ^1. Then $\mathcal{L}^1(P) = \{\varphi \in \mathbb{C}^{\mathbb{N}} : \varphi\chi_{E(1)\cup E(2)} \in \mathrm{sim}(2^{\mathbb{N}})\}$, but $\bigcap_{x\in X} \mathcal{L}^1(Px) = \{\varphi \in \mathbb{C}^{\mathbb{N}} : \varphi\chi_{E(1)} \in \ell^\infty, \varphi\chi_{E(2)} \in \mathrm{sim}(2^{\mathbb{N}})\}$.
And so on!

Either, if X is quasi-barrelled or if X is equipped with its weak topology, then (4.11) holds; the latter case is obvious and the former is statement (iii) of Corollary 4.5. The following example provides another sufficient condition.

EXAMPLE 4.8: Let Y be a monotone sequence space and Z be a linear subspace of Y^α containing c_{00}. Let $X = Y_{|\sigma|(Y,Z)}$; for the definition see Example 3.8. Then every function $f : \mathbb{N} \to \mathbb{C}$ satisfying (4.10) is P-integrable because the dual space X' is the ideal generated by Z in the vector lattice $\mathbb{C}^{\mathbb{N}}$, [1; p.128]. Hence, (4.11) holds by Corollary 4.5 (ii).

The condition $(H3)$ always guarantees that

$$\ell^\infty \subseteq \mathcal{L}^1(P); \tag{4.14}$$

see [17; Proposition 2.2 (i)]. Example 4.7 shows that this is surely not always the case. In order to discuss some implications of (4.14) and earlier results in this section we recall that a sequence space Y is called *solid* if every function $f \in \mathbb{C}^{\mathbb{N}}$ satisfying $|f| \le |y|$ for some $y \in Y$ necessarily belongs to Y itself. In particular, if $y \in Y$, then also $|y| \in Y$. Of course, $|y|$ is the function $n \mapsto |y(n)|$, for each $n \in \mathbb{N}$. A simple example of a sequence space which fails to be solid is c. It is clear that Y is solid if and only if $Yf \subseteq Y$ for each $f \in \ell^\infty$. Moreover, if Y is solid, then it is also monotone.

Proposition 4.9

The following statements hold for a lcss X.

(i) *X is solid if and only if $\ell^\infty \subseteq \bigcap_{x\in X} \mathcal{L}^1(Px)$.*
(ii) *X and X' are both solid if and only if $\ell^\infty \subseteq \mathcal{L}^1(\Lambda \circ P)$.*
(iii) *If $\ell^\infty \subseteq \mathcal{L}^1(P)$, then both X and X' are solid.*
(iv) *The condition $(H3)$ implies that both X and X' are solid.*

Proof. Statements (i) and (ii) follow from Lemma 4.1 and Proposition 4.3, respectively. Statement (ii) and (4.6) imply (iii). Finally, (iv) is a consequence of (iii) because $(H3)$ implies (4.14). □

We note that statement (iii) of Proposition 4.9 need not hold if the assumption $\ell^\infty \subseteq \mathcal{L}^1(P)$ is replaced by the weaker requirement that

$$\ell^\infty \cap \left(\bigcap_{x \in X} \mathcal{L}^1(Px) \right) \subseteq \mathcal{L}^1(P). \tag{4.15}$$

Indeed, the space X in (iii) of Example 4.7 satisfies $\bigcap_{x \in X} \mathcal{L}^1(Px) = \mathrm{sim}(2^{\mathbb{N}}) = \mathcal{L}^1(P)$ but, X is not solid.

As noted in Section 2, the space $[L_s(X)]_P$ is the sequential closure in $L_s(X)$ of the range of the integration map I_P. Since a set $E \in 2^{\mathbb{N}}$ satisfying $\langle Pe_n, e_n \rangle(E) = 0$ for all $n \in \mathbb{N}$ must be empty it follows that the measure P is *countably determined* in the sense of [18; p.33]. Accordingly, [18; Proposition 2.6] implies that $I_P(\mathcal{L}^1(P))$ is sequentially closed, from which it follows that $[L_s(X)]_P = I_P(\mathcal{L}^1(P))$. More is true; it is shown in Proposition 4.10 to follow that $I_P(\mathcal{L}^1(P))$ is actually closed in $L_s(X)$. Consequently, for our particular spectral measure P the condition $(H3)$ turns out to be equivalent to the condition:

- $(H3)^*$ The range of the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is sequentially complete.

Proposition 4.10

The range of the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ *is closed.*

Proof. Let T belong to the closure of $I_P(\mathcal{L}^1(P))$, in which case there exists a net $\{f_\alpha\}_{\alpha \in A}$ in $\mathcal{L}^1(P)$ such that

$$Tx = \lim_\alpha I_P(f_\alpha)x = \lim_\alpha x f_\alpha, \qquad x \in X, \tag{4.16}$$

in the topology of X. Since $c_{00} \subseteq X'$ we have that

$$\langle Te_n, e_n \rangle = \lim_\alpha f_\alpha(n), \qquad n \in \mathbb{N}. \tag{4.17}$$

Let $f : \mathbb{N} \to \mathbb{C}$ be the function defined by the left-hand-side of (4.17). By (4.16) and (4.17) we see that

$$(Tx)(n) = \lim_\alpha (xf_\alpha)(n) = x(n)\lim_\alpha f_\alpha(n) = (xf)(n), \qquad n \in \mathbb{N},$$

for each $x \in X$. Thus $M_f = T \in L(X)$ and so $f \in \mathcal{L}^1(P)$ with $T = P(f) \in I_P(\mathcal{L}^1(P))$. □

In view of the condition $(H3)^*$ we point out that the range of I_P is not always sequentially complete.

EXAMPLE 4.11: Fix $1 \le p \le \infty$. Let $X = \ell^p$ be equipped with the topology $\sigma(\ell^p, c_{00})$. Then Proposition 4.3 implies that $\mathcal{L}^1(P) = \ell^\infty$. Fix any $f \in \mathbb{C}^{\mathbb{N}} \setminus \ell^\infty$. Then the $2^{\mathbb{N}}$-simple functions $f_n = f\chi_{\{1,\dots,n\}}$ belong to $\mathcal{L}^1(P)$ for each $n \in \mathbb{N}$ and $\{P(f_n)\}_{n=1}^\infty$ is a Cauchy sequence in $L_s(X)$ because $X' = c_{00}$. However, $\{P(f_n)\}_{n=1}^\infty$ has no limit in $L_s(X)$; if so, then the limit would have to be M_f which is impossible as $Xf \not\subseteq X$.

In the previous example the fact that X itself is not sequentially complete is not the only reason why $I_P(\mathcal{L}^1(P))$ fails to be sequentially complete.

EXAMPLE 4.12: Consider the following three spaces.

(a) $X = c_0$ equipped with the topology $\sigma(c_0, \ell^1)$.
(b) $X = \ell^1$ equipped with the topology $\sigma(\ell^1, \ell^\infty)$.
(c) $X = \ell^\infty$ equipped with its weak-star topology $\sigma(\ell^\infty, \ell^1)$.

In each case $I_P(\mathcal{L}^1(P))$ is topologically isomorphic to the quasicomplete lcHs $\ell^\infty_{\sigma(\ell^\infty,\ell^1)}$. Whereas the space X in (b) is sequentially complete and that in (c) is quasicomplete, the space X of (a) is not sequentially complete.

Our canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(X)$ is called *equicontinuous* if its range is an equicontinuous subset of $L(X)$. If X is quasi-barrelled, then P is always equicontinuous, [19; Proposition 2.5]. Or, if the given topology on X is specified by a fundamental set of absolutely monotone seminorms, then P is also equicontinuous; see the proof of Lemma 3.7. In each of (a)–(c) in Example 4.12 the measure P is *not* equicontinuous, [16; Proposition 4]. However, when P does happen to be equicontinuous, then the condition $(H3)^*$ implies a rather desirable property of P.

Proposition 4.13

Suppose that the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(X)$ is equicontinuous and that $I_P(\mathcal{L}^1(P))$ is sequentially complete in $L_s(X)$, i.e. $(H3)^*$ is satisfied. Then $I_P(\mathcal{L}^1(P))$ is actually a complete subspace of $L_s(X)$.

Proof. Let $\mathcal{N}(X)$ denote the class of all continuous seminorms on X. For each $q \in \mathcal{N}(X)$ and $x \in X$ define a seminorm q_x on $\mathcal{L}^1(P)$ by

$$q_x(f) = \sup\left\{ q\left(\int_E f\,dPx\right) : E \in 2^{\mathbb{N}} \right\}, \qquad f \in \mathcal{L}^1(P).$$

If we equip $\mathcal{L}^1(P)$ with the lcH-topology determined by the seminorms $\{q_x : q \in \mathcal{N}(X), x \in X\}$ then the equicontinuity of P implies that I_P is a topological isomorphism of $\mathcal{L}^1(P)$ onto its range (with the relative topology from $L_s(X)$), [21; Lemma 1.11].

Since P is absolutely continuous with respect to the σ-finite measure $\lambda = \sum_{n=1}^{\infty} \delta_n$, it follows that P is a *closed measure* in the sense of I. Kluvánek, [11; IV, Theorem 7.3]. In other words, the subset $\{\chi_E : E \in 2^{\mathbb{N}}\}$ of $\mathcal{L}^1(P)$ is a complete uniform space [11; p.71]. Then [24; Theorem 2] implies that $\mathcal{L}^1(P)$ is complete and hence, the range of the topological isomorphism I_P is also complete. □

As a consequence of Propositions 4.10 and 4.13 we have the following result.

Corollary 4.14

If the lcHs $L_s(X)$ is sequentially complete and our canonical spectral measure P is equicontinuous, then $I_P(\mathcal{L}^1(P))$ is a complete subspace of $L_s(X)$.

Of course, the quasicompleteness of $L_s(X)$ always implies its sequential completeness (but not conversely). For instance, if the lcss X is a Fréchet space, then $L_s(X)$ is always quasicomplete; this is a special case of the criterion which states that if X is a Mackey space, then the lcHs $L_s(X)$ is quasicomplete if and only if X is both barrelled and quasicomplete, [23; Corollary 1.1]. An example of a quasicomplete lcss X for which $L_s(X)$ is sequentially complete but not quasicomplete is given in [23; Example 5]. However, even the stronger condition of X being complete need not imply the sequential completeness of $L_s(X)$ in general, [19; Example 3.10]. We conclude with an example for which our canonical spectral measure P is equicontinuous and both the lcss X and the space $I_P(\mathcal{L}^1(P))$ are complete, yet $L_s(X)$ is not even sequentially complete!

Example 4.15: Let c_0 be equipped with the usual norm $||\cdot||_\infty$. Let $X = \ell^1$ be equipped with the lcH-topology $c(\ell^1, c_0)$ of uniform convergence on the compact subsets of c_0, regarding X as the dual space of c_0. Then $X' = c_0$. The monotone lcss X is complete but $L_s(X)$ is not even sequentially complete, [19; Example 3.10].

First X will be shown to have the $2^{\mathbb{N}}$-summability property. Let $E \in 2^{\mathbb{N}}$. Then the linear operator $P(E) : X \to X$ given by $P(E)x = \chi_E x$ for each $x \in X$ is the dual operator of $Q(E) \in L(c_0)$ defined by $Q(E)y = \chi_E y$ for each $y \in c_0$. In other words,

$$\langle y, P(E)x\rangle = \langle Q(E)y, x\rangle, \qquad y \in c_0, x \in X.$$

Since $Q(E)$ maps each compact subset of c_0 to a relatively compact subset of c_0, the operator $P(E)$ belongs to $L(X)$. Clearly the lcss X is a weak AK-space and so

Corollary 3.4 ensures that X has the $2^{\mathbb{N}}$-summability property. Hence, $P : 2^{\mathbb{N}} \to L_s(X)$ is a spectral measure.

Secondly we claim that P is equicontinuous. Let $\hat{c}_0$ be the space of all sequences with real entries which converge to zero, equipped with the norm $||\cdot||_\infty$. Let K be a compact subset of $\hat{c}_0$. By [15; Theorem 2.1.12] the set K has a supremum, say y (in the usual order of the (real) Banach lattice $\hat{c}_0$), and so K is contained in the order interval $[-y, y]$. Hence, the *solid hull* $\tilde{K}$, of K, also satisfies $\tilde{K} \subseteq [-y, y]$. Since $\hat{c}_0$ is *discrete* the order interval is compact, [1; Corollary 21.13] and we conclude that $\tilde{K}$ is relatively compact. By the usual complexification argument it follows that c_0 also has the property that $\tilde{K}$ is relatively compact whenever K is a compact subset of c_0. In particular, $Q(2^{\mathbb{N}})(K) = \bigcup\{Q(E)(K) : E \in 2^{\mathbb{N}}\}$ is relatively compact in c_0 whenever $K \subseteq c_0$ is compact and hence, its closure $\overline{Q(2^{\mathbb{N}})(K)}$ is compact. This implies that P is equicontinuous since

$$\sup\left\{|\langle y, P(E)x\rangle| : y \in K, E \in 2^{\mathbb{N}}\right\} \le \sup\left\{|\langle z, x\rangle| : z \in \overline{Q(2^{\mathbb{N}})(K)}\right\},$$

for each $x \in X$. Alternatively, one can use the fact that a subset A of c_0 is relatively compact whenever $\lim_k \sup_{x\in A} \|(0, \ldots, x_{k+1}, x_{k+2}, \ldots)\| = 0$, [3].

Thirdly we claim that $\mathcal{L}^1(P) = \ell^\infty$. In fact, it follows from Proposition 4.2 that every $f \in \mathcal{L}^1(P)$ has to satisfy $\ell^1 f \subseteq \ell^1$, which implies that $f \in \ell^\infty$. Conversely let $f \in \ell^\infty$. From the fact that the set Kf is compact in c_0 whenever $K \subseteq c_0$ is compact, it follows that $M_f \in L(X)$. So, again by Proposition 4.2 we obtain $f \in \mathcal{L}^1(P)$ and $P(f) = M_f$.

In order to be able to apply Proposition 4.13 we need to verify that $I_P(\mathcal{L}^1(P))$ is sequentially complete in $L_s(X)$. To this end, let $\{f_n\}_{n=1}^\infty$ be a sequence in $\ell^\infty = \mathcal{L}^1(P)$ such that $\{P(f_n)\}_{n=1}^\infty$ is Cauchy in $L_s(X)$. Then, given $x \in X$, the sequence $\{xf_n\}_{n=1}^\infty$ is norm bounded in ℓ^1 because it is bounded with respect to the topology $\sigma(X, X') = \sigma(\ell^1, c_0)$, i.e. the weak-star topology on ℓ^1. By the uniform boundedness principle the sequence $\{M_{f_n}\}_{n=1}^\infty$ is bounded with respect to the operator norm $||\cdot||_u$ on $L(\ell^1)$ when ℓ^1 is equipped with its norm topology $||\cdot||_1$. Since $||M_{f_n}||_u = ||f_n||_\infty$ for each $n \in \mathbb{N}$ the sequence $\{f_n\}_{n=1}^\infty$ satisfies $\beta = \sup\{||f_n||_\infty : n \in \mathbb{N}\} < \infty$. Furthermore, $\{f_n\}_{n=1}^\infty$ converges pointwise on $\mathbb{N}$ to some function $f : \mathbb{N} \to \mathbb{C}$ because $c(\ell^1, c_0)$ is stronger than the coordinatewise convergence topology induced by $\mathbb{C}^{\mathbb{N}}$. Since $\beta < \infty$ the function f belongs to ℓ^∞. To see that $P(f_n) \to P(f)$ in $L_s(X)$ let $K \subseteq c_0$ be compact. Fix $x \in X$ and let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$|\langle y, xf_n - xf_m\rangle| \le \epsilon, \qquad m, n \ge N,$$

for each $y \in K$. Hence, $\sup\{|\langle y, xf_n - xf\rangle| : y \in K\} \leq \epsilon$ for all $n \geq N$, which implies that $P(f_n) \to P(f)$ in $L_s(X)$. This establishes that $I_P(\mathcal{L}^1(P))$ is sequentially complete in $L_s(X)$.

So, we can now apply Proposition 4.13 to conclude that $I_P(\mathcal{L}^1(P))$ is actually a complete subspace of $L_s(X)$.

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References

1. C.D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York, 1978.
2. J. Bonet, Closed linear maps from a barrelled normed space into itself need not be continuous, *Bull. Austral. Math. Soc.* **57** (1998), 177–179.
3. J. Diestel, *Sequences and series in Banach spaces*, Springer, New York, 1984.
4. P.G. Dodds and B. de Pagter, Orthomorphisms and Boolean algebras of projections, *Math. Z.* **187** (1984), 361–381.
5. P.G. Dodds and W.J. Ricker, Spectral measures and the Bade reflexivity theorem, *J. Funct. Anal.* **61** (1985), 136–163.
6. P.G. Dodds, B. de Pagter and W.J. Ricker, Reflexivity and order properties of scalar-type spectral operators in locally convex spaces, *Trans. Amer. Math. Soc.* **293** (1986), 355–380.
7. P.G. Dodds and B. de Pagter, Algebras of unbounded scalar-type spectral operators, *Pacific J. Math.* **130** (1987), 41–74.
8. N. Dunford and J.T. Schwartz, *Linear Operators III: Spectral operators*, Wiley-Interscience, New York, 1971.
9. H. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, 1981.
10. P.K. Kamthan and Manjul Gupta, *Sequence spaces and series*, Lecture Notes in Pure and Applied Math. **65**, Marcel Dekker, New York, 1981.
11. I. Kluvánek and G. Knowles, *Vector measures and control systems*, North Holland, Amsterdam, 1976.
12. G. Köthe, *Topological vector spaces I*, Springer-Verlag, Heildelberg, 1969.
13. G. Köthe, *Topological vector spaces II*, Springer-Verlag, Heildelberg, 1979.
14. D.R. Lewis, Integration with respect to vector measures, *Pacific J. Math.* **33** (1970), 157–165.
15. P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, New York, 1991.
16. S. Okada and W.J. Ricker, Spectral measures which fail to be equicontinuous, *Period. Math. Hungar.* **28** (1994), 55–61.
17. S. Okada and W.J. Ricker, Vector measures and integration in non-complete spaces, *Arch. Math.* **63** (1994), 344–353.

18. S. Okada and W.J. Ricker, The range of the integration map of a vector measure, *Arch. Math.* **64** (1995), 512–522.
19. S. Okada and W.J. Ricker, Continuous extensions of spectral measures, *Colloq. Math.* **71** (1996), 115–132.
20. S. Okada and W.J. Ricker, Spectral measures and automatic continuity, *Bull. Belgian Math. Soc.* **3** (1996), 267–279.
21. S. Okada and W.J. Ricker, Boolean algebras of projections and ranges of spectral measures, *Dissertationes Math.* **365** (1997), 33pp.
22. W.J. Ricker, On Boolean algebras of projections and scalar-type spectral operators, *Proc. Amer. Math. Soc.* **87** (1983), 73–77.
23. W.J. Ricker, Remarks on completeness in spaces of linear operators, *Bull. Austral. Math. Soc.* **34** (1986), 25–35.
24. W.J. Ricker, Completeness of the L^1-space of closed vector measures, *Proc. Edinburgh Math. Soc.* **33** (1990), 71–78.
25. W.J. Ricker, The sequential closedness of σ-complete Boolean algebras of projections, *J. Math. Anal. Appl.* **208** (1997), 364–371.
26. W.J. Ricker and H.H. Schaefer, The uniformly closed algebra generated by a complete Boolean algebra of projections, *Math. Z.* **201** (1989), 429–439.
27. H.H. Schaefer, Spectral measures in locally convex algebras, *Acta Math.* **107** (1962), 125–173.
28. B. Walsh, Structure of spectral measures on locally convex spaces, *Trans. Amer. Math. Soc.* **120** (1965), 295–326.