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Analytic extension of ultradifferentiable Whitney jets

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Abstract

Let ω be a weight and F be a closed proper subset of $\mathbb{R}^n$. Then for every function f on $\mathbb{R}^n$ belonging to the non quasi-analytic (ω)-class of Beurling (resp. Roumieu) type, there is an element g of the same class which is analytic on $\mathbb{R}^n \setminus F$ and such that $\mathrm{D}^\alpha f(x) = \mathrm{D}^\alpha g(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$.

1. Introduction and statement of the result

In [18], H. Whitney has established that every C^∞-Whitney jet on a closed subset F of $\mathbb{R}^n$ has a C^∞-extension on $\mathbb{R}^n$ which is analytic on $\mathbb{R}^n \setminus F$. Since then several authors have considered the extension problem of jets in different situations; here are references to some of them [3], [4], [5], [6], [7], [9], [10], [11], [12], [16] and [17] — with in [4] and [6], a discussion of the previous literature on the subject. In particular, for the Beurling type, one finds

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a) in [3] (resp. [5] and [12]) conditions on the weight ω (resp. on the sequence $(M_r)_{r\in\mathbb{N}_0}$) under which the restriction map

$$\rho_K: \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(K) \qquad \big(\text{resp. } \rho_K: \mathcal{E}_{(M_r)}(\mathbb{R}^n) \to \mathcal{E}_{(M_r)}(K)\big)$$

is surjective for every compact subset K of $\mathbb{R}^n$,

b) in [6] (resp. [5] and [12]) conditions under which this restriction map has a continuous linear right inverse.

In this paper, we are going to consider this problem in the case of non quasi-analytic ω-Whitney jets of Beurling (resp. Roumieu) type.

In order to make this statement more precise, let us introduce some definitions and notations.

We use the modification introduced by Braun, Meise and Taylor in [2] to Beurling's method in [1] in order to define classes of non quasi-analytic functions. So we consider a *weight* ω, i.e. a function $\omega: [0, +\infty[\to [0, +\infty[$ which is continuous, increasing and verifies the following conditions:

(ω1) there is $l \geq 1$ such that $\omega(2t) \leq l(1 + \omega(t))$ for every $t \geq 0$,

(ω2) $\int_1^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$,

(ω3) $\lim\limits_{t\to+\infty} \frac{\ln(1+t)}{\omega(t)} = 0$,

(ω4) the function $\varphi: [0, +\infty[\to [0, +\infty[, \quad t \mapsto \omega(\mathrm{e}^t)$ is convex.

By the Proposition 1.2(b) of [3], there is then a weight $\sigma \leq \omega$ such that $\sigma(1) = 0$ and $\sigma(t) = \omega(t)$ for large t. As in what follows, the values of $\omega(t)$ are used only for large t, we are going to suppose moreover that we have $\omega(1) = 0$ hence $\varphi(0) = 0$. Then *the Young's conjugate* φ^* of φ is defined as

$$\varphi^*: [0, +\infty[\to [0, +\infty[, \quad y \mapsto \sup_{x\geq 0} (xy - \varphi(x)).$$

It is a convex and increasing function which verifies $\varphi^*(0) = 0$ and $\varphi^*(y)/y$ is an increasing function such that $\lim_{y\to\infty} \varphi^*(y)/y = \infty$.

Moreover the property (ω1) of the definition of the weight ω gives the existence of a constant d_0 such that

$$\varphi(x + 1) \leq d_0(\varphi(x) + 1), \quad \forall x \geq 0,$$

and by the Lemma 1.4 of [2], there is also a constant $y_0 > 0$ such that

$$\varphi^*(y) - y \geq d_0\varphi^*(y/d_0) - d_0, \quad \forall y \geq y_0; \tag{1}$$

of course we may suppose that d_0 is an integer and that $y_0 > d_0$.

We then designate:

a) By $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ the non quasi-analytic ω-class of Beurling type on $\mathbb{R}^n$, i.e. the set of the C^∞-functions f on $\mathbb{R}^n$ such that, for every compact subset K of $\mathbb{R}^n$ and constant $h \geq 1$, one has

$$\sup_{\alpha\in\mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_K \,\mathrm{e}^{-h\varphi^*(|\alpha|/h)} < \infty.$$

b) By $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ the non quasi-analytic ω-class of Roumieu type on $\mathbb{R}^n$, i.e. the set of the C^∞-functions f on $\mathbb{R}^n$ such that, for every compact subset K of $\mathbb{R}^n$, there is a constant $h \geq 1$ such that

$$\sup_{\alpha\in\mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_K \,\mathrm{e}^{-\varphi^*(h|\alpha|)/h} < \infty.$$

In [17], the following results are proved: *let K be a compact subset of $\mathbb{R}^n$.*

(a) *If f is an element of $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$* [resp. $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$], *then the Whitney jet $f|_K$ has an extension belonging to the same space and analytic on $\mathbb{R}^n \setminus K$.*

(b) *If $\mathcal{E}_{(\omega)}(K)$ and $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$* [resp. $\mathcal{E}_{\{\omega\}}(K)$ *and* $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$] *are endowed with their usual topologies and if there is a continuous linear extension map*

$$T\colon \mathcal{E}_{(\omega)}(K) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n) \quad \left[\text{resp. } T\colon \mathcal{E}_{\{\omega\}}(K) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)\right],$$

then there is a continuous linear extension map

$$S\colon \mathcal{E}_{(\omega)}(K) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n) \quad \left[\text{resp. } S\colon \mathcal{E}_{\{\omega\}}(K) \to \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)\right]$$

such that Sf is analytic on $\mathbb{R}^n \setminus K$ for every $f \in \mathcal{E}_{(\omega)}(K)$ [resp. $\mathcal{E}_{\{\omega\}}(K)$].

In [16], similar properties are established for the spaces $\mathcal{E}_{(M_r)}$ and $\mathcal{E}_{\{M_r\}}$.

In this paper, we are going to prove that it is possible to adapt the proof of the result (a) to the case when the compact set K is replaced by a closed subset F of $\mathbb{R}^n$. As the result (b) is concerned, we do not know whether it generalizes to the closed subsets setting. So our main result reads as follows.

Theorem 1.1

For every $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ [resp. $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$] *and every closed subset F of $\mathbb{R}^n$, there is $g \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$* [resp. $g \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$] *such that*

(a) $\mathrm{D}^\alpha f(x) = \mathrm{D}^\alpha g(x)$ *for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$,*

(b) *g is analytic on $\mathbb{R}^n \setminus F$.*

In [14] and [15] respectively, one finds similar results for the spaces $\mathcal{E}_{(M_r)}$ and $\mathcal{E}_{\{M_r\}}$.

2. Proof in the case of the Beurling type

Notations. The proof of the Theorem 1.1 relies very much on the correct value of the numbers λ_r for $r \in \mathbb{N}$. These values cannot be introduced directly and we feel more advisable to set up immediately the different notations that will be used throughout the proof of the Beurling type and that lead to these numbers λ_r.

So far we have introduced the weight ω, the function $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ as well as the closed subset F of $\mathbb{R}^n$. Of course we may restrict our attention to the case when the restriction of f to the open subset $\Omega = \mathbb{R}^n \setminus F$ of $\mathbb{R}^n$ is not identically 0. Moreover:

(a) For every integer $m \geq d_0$, let us then denote by $p(m)$ the integer part of m/d_0 and let us set $q_m = \sup\{\mathrm{e}^m, 2^{my_0/d_0}\}$. These numbers will appear in the proof by use of the following remark: as in [17], we note that we have

$$2^r \mathrm{e}^{m\varphi^*(r/m)} \leq q_m \mathrm{e}^{p(m)\varphi^*(r/p(m))}$$

for every r, $m \in \mathbb{N}$ such that $m \geq d_0$: in fact on the one hand, if r verifies $r \leq my_0/d_0$, we certainly have

$$2^r \mathrm{e}^{m\varphi^*(r/m)} \leq 2^{my_0/d_0} \mathrm{e}^{p(m)\varphi^*(r/p(m))}$$

and on the other hand we always have $2^r \mathrm{e}^{m\varphi^*(r/m)} \leq \mathrm{e}^{m/d_0\left(d_0\varphi^*(\frac{rd_0}{m}/d_0)+\frac{rd_0}{m}\right)}$ so for $r > my_0/d_0$ the use of the inequality (1) for $y = rd_0/m$ leads to

$$2^r \mathrm{e}^{m\varphi^*(r/m)} \leq \mathrm{e}^{\frac{m}{d_0}\left(\varphi^*(\frac{rd_0}{m})+d_0\right)} \leq \mathrm{e}^m \mathrm{e}^{p(m)\varphi^*(r/p(m))}.$$

(b) For every $m \in \mathbb{N}$, A_m designates the ball

$$A_m = \{\, x \in \mathbb{R}^n : |x| \leq m + \mathrm{d}(0,\Omega)\}$$

where of course $\mathrm{d}(0,\Omega)$ is the distance of the origin to Ω.

(c) $\{\, K_m : m \in \mathbb{N}\}$ designates a compact cover of Ω such that $K_1 \neq \emptyset$ and $K_m = (K_m)^{\circ -} \subset (K_{m+1})^\circ$, $K_{m+3} \subset A_m$ for every $m \in \mathbb{N}$.

(d) As $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$, we have

$$\|f\|_m = 1 + \sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_{A_m} \mathrm{e}^{-m\varphi^*(|\alpha|/m)} < \infty, \quad \forall m \in \mathbb{N}.$$

(e) The Proposition 1 of [17] provides a sequence $(u_m)_{m\in\mathbb{N}}$ of $\mathrm{C}^\infty(\mathbb{R}^n)$ such that, for every $m \in \mathbb{N}$,

$$\begin{aligned}
&0 \le u_m \le 1,\\
&u_m \equiv 1 \quad \text{on a neighbourhood of } K_{m+2} \setminus (K_{m+1})^\circ,\\
&\operatorname{supp}(u_m) \subset (K_{m+3})^\circ \setminus K_m,\\
&2^{m|\alpha|}\mathrm{e}^{-\varphi^*(|\alpha|)} \|\mathrm{D}^\alpha u_m\|_{\mathbb{R}^n} \le d_{m,1}, \quad \forall \alpha \in \mathbb{N}_0^n,\\
&2^{m|\alpha|}\mathrm{e}^{-(k+1)\varphi^*(\frac{|\alpha|}{k+1})} \|\mathrm{D}^\alpha u_m\|_{\mathbb{R}^n} \le d_{m,k}, \quad \forall \alpha \in \mathbb{N}_0^n, \forall k \in \mathbb{N},
\end{aligned}$$

with, for every m, $k \in \mathbb{N}$,

$$\begin{aligned}
&1 \le d_{m,k} \le d_{m,k+1},\\
&2\,\|f\|_1\,(m+1)d_{m,1}^2 \le d_{m+1,1},\\
&2\,\|f\|_{k+1}\,(m+1)d_{m,k+1}^2 \le d_{m+1,k}\,.
\end{aligned}$$

Now for every $r \in \mathbb{N}$, we set $p_r = d_{r,r}$, $\varepsilon_r = 2^{-rp_{r+2}}$ and

$$\delta_r = \varepsilon_r \left(3nq_r p_r^2 p_{r+1} 2^{r+2+p_{r+2}} \mathrm{e}^{2r\varphi^*(p_{r+2}+1)}\right)^{-1}.$$

(f) Finally, for every $\rho > 0$, we set $\Psi(\rho) = \pi^{-n/2} \int_{|y|\le\rho} \mathrm{e}^{-|y|^2}\,dy$. By the Poisson formula, we have $\Psi(\rho) \uparrow 1$ if $\rho \uparrow +\infty$. So, for every $r \in \mathbb{N}$, there is $\lambda_r > 0$ such that

$$\begin{aligned}
&\|f\|_r\, p_r^2(1 - \Psi(\lambda_r\delta_r)) \le \delta_r,\\
&\pi^{-n/2}\lambda_r^n \mathrm{e}^{-\lambda_r^2 p_r^2}\ \mathrm{mes}(K_{r+3}) \le 1\\
&q_{rd_0} d_{r,rd_0}^2\, \|f\|_{rd_0}\, \pi^{-n/2}\lambda_r^n \mathrm{e}^{-\lambda_r^2 r^{-2}} \mathrm{mes}(K_{r+3}) \le 2^{-r}.
\end{aligned}$$

Now we are all set to start the proof. It consists in the study of the functions G_0, G_1, G_2, $\ldots$ defined on $\mathbb{R}^n$ by $G_0(x) = 0$ and the recursion

$$G_r(x) = \pi^{-n/2}\lambda_r^n \int_{\mathbb{R}^n} u_r(y) \left(f(y) - \sum_{s=0}^{r-1} G_s(y)\right) \mathrm{e}^{-\lambda_r^2|x-y|^2}\,dy, \quad \forall r \in \mathbb{N}\,.$$

In fact, apart the context, these are the functions that H. Whitney considered in [18]. As the functions u_r belong to $\mathrm{C}^\infty(\mathbb{R}^n)$ and have compact support and as the function $\mathrm{e}^{-\lambda_r^2|x-y|^2}$ is the restriction to $\mathbb{R}^n$ of the holomorphic function $\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(w_j-y_j)^2}$ on $\mathbb{C}^n$, the properties of the convolution product tell us directly that the functions G_1, G_2, $\ldots$ are analytic on $\mathbb{R}^n$. In our setting a lot more can be said.

Notations. In order to simplify the notations, we introduce the following shorthand

$$v_r = u_r\left(f - \sum_{s=0}^{r-1} G_s\right), \quad \forall r \in \mathbb{N}\,.$$

Proposition 2.1

The analytic functions $G_0,\ G_1,\ \ldots$ on $\mathbb{R}^n$ are such that, for every integer $m \geq d_0$ and $\alpha \in \mathbb{N}_0^n$, we have

$$\left\| \mathrm{D}^\alpha\Big(f - \sum_{s=0}^{r-1} G_s\Big)\right\|_{A_m} \leq q_m \left\|f\right\|_m d_{r,m} \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))}, \tag{2}$$

$$\left\|\mathrm{D}^\alpha v_r\right\|_{A_m} \leq q_m \left\|f\right\|_m d_{r,m}^2 \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))}, \tag{3}$$

$$\left\|\mathrm{D}^\alpha G_r\right\|_{A_m} \leq q_m \left\|f\right\|_m d_{r,m}^2 \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))}. \tag{4}$$

Proof. **Case** $r = 1$**.** Of course we have

$$\left\|\mathrm{D}^\alpha f\right\|_{A_m} \leq \left\|f\right\|_m \mathrm{e}^{m\varphi^*(|\alpha|/m)} \leq \left\|f\right\|_m d_{1,m} \mathrm{e}^{m\varphi^*(|\alpha|/m)} \tag{5}$$

hence by use of the Leibniz formula

$$\begin{aligned} \left\|\mathrm{D}^\alpha v_1\right\|_{A_m} &\leq \left\|f\right\|_m d_{1,m}^2 \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta} 2^{-|\beta|} \mathrm{e}^{m\varphi^*(|\beta|/m)} \mathrm{e}^{m\varphi^*(|\alpha-\beta|/m)} \\ &\leq \left\|f\right\|_m d_{1,m}^2 (1+2^{-1})^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)}. \end{aligned} \tag{6}$$

Now as $\mathrm{supp}(v_1) \subset \mathrm{supp}(u_1) \subset A_m$ holds for every $m \in \mathbb{N}$, we also have

$$\left\|\mathrm{D}^\alpha G_1\right\|_{A_m} \leq \left\|f\right\|_m d_{1,m}^2 (1+2^{-1})^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)}. \tag{7}$$

Case $r > 1$**.** We proceed by recursion on r. Suppose that, for some integer $r \geq 2$, we have obtained

$$\left\|\mathrm{D}^\alpha G_s\right\|_{A_m} \leq \left\|f\right\|_m d_{s,m}^2 (1 + 2^{-1} + \cdots + 2^{-s})^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)}$$

for every $s \in \{1, \ldots, r-1\}$, $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$.

Firstly we get

$$\begin{aligned} \left\| \mathrm{D}^\alpha\Big(f - \sum_{s=0}^{r-1} G_s\Big)\right\|_{A_m} &\leq \left\|\mathrm{D}^\alpha f\right\|_{A_m} + \sum_{s=1}^{r-1} \left\|\mathrm{D}^\alpha G_s\right\|_{A_m} \\ &\leq r \left\|f\right\|_m d_{r-1,m}^2 (1 + 2^{-1} + \cdots + 2^{-r+1})^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)} \end{aligned}$$

hence

$$\Big\|\mathrm{D}^\alpha\Big(f-\sum_{s=0}^{r-1}G_s\Big)\Big\|_{A_1}\le\frac{1}{2}\,d_{r,1}(1+2^{-1}+\cdots+2^{-r+1})^{|\alpha|}\mathrm{e}^{\varphi^*(|\alpha|)}\tag{8}$$

and, for every $m\in\mathbb{N}$,

$$\Big\|\mathrm{D}^\alpha\Big(f-\sum_{s=0}^{r-1}G_s\Big)\Big\|_{A_{m+1}}\le\frac{1}{2}\,d_{r,m}(1+2^{-1}+\cdots+2^{-r+1})^{|\alpha|}\mathrm{e}^{(m+1)\varphi^*(|\alpha|/(m+1))}.\tag{9}$$

Next by use of the Leibniz formula we have

$$\|\mathrm{D}^\alpha v_r\|_{A_1}\le\tfrac{1}{2}\,d_{r,1}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{\varphi^*(|\alpha|)}\tag{10}$$

and, in the same way, for every $m\in\mathbb{N}$,

$$\|\mathrm{D}^\alpha v_r\|_{A_{m+1}}\le\frac{1}{2}\,d_{r,m}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{(m+1)\varphi^*(|\alpha|/(m+1))}.\tag{11}$$

Finally:

a) For $m\ge r$, we have $\mathrm{supp}(v_r)\subset A_r\subset A_m$ hence

$$\begin{aligned}\|\mathrm{D}^\alpha G_r\|_{A_m}&\le\ \|\mathrm{D}^\alpha v_r\|_{A_m}\\&\le\ \left\{\begin{matrix}\tfrac{1}{2}\,d_{r,1}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{\varphi^*(|\alpha|)} & \text{if } m=1\\ \tfrac{1}{2}\,d_{r,m}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{m\varphi^*(|\alpha|/m)} & \text{if } m\ge 2\end{matrix}\right\}.\end{aligned}$$

b) For $1\le m<r$ and every $x\in A_m$, we get

$$\begin{aligned}|\mathrm{D}^\alpha G_r(x)|&\le\ \pi^{-n/2}\lambda_r^n\left(\int_{A_{m+1}}+\int_{K_{r+3}\setminus A_{m+1}}\right)|\mathrm{D}^\alpha v_r(y)|\,\mathrm{e}^{-\lambda_r^2|x-y|^2}\,dy\\&\le\ \|\mathrm{D}^\alpha v_r\|_{A_{m+1}}+\|\mathrm{D}^\alpha v_r\|_{A_r}\,\pi^{-n/2}\lambda_r^n\mathrm{e}^{-\lambda_r^2}\ \mathrm{mes}(K_{r+3})\end{aligned}$$

hence, by use of the formula (11) and of one condition imposed on λ_r,

$$\begin{aligned}&\|\mathrm{D}^\alpha G_r\|_{A_m}\\&\quad\le\ \frac{1}{2}\,d_{r,m}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{(m+1)\varphi^*(|\alpha|/(m+1))}\\&\qquad+\frac{1}{2}\,p_r^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{r\varphi^*(|\alpha|/r)}\pi^{-n/2}\lambda_r^n\mathrm{e}^{-\lambda_r^2}\ \mathrm{mes}(K_{r+3})\\&\quad\le\ \frac{1}{2}\,d_{r,m}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{m\varphi^*(|\alpha|/m)}\\&\qquad+\frac{1}{2}\,(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{m\varphi^*(|\alpha|/m)}\\&\quad\le\ d_{r,m}^2(1+2^{-1}+\cdots+2^{-r})^{|\alpha|}\mathrm{e}^{m\varphi^*(|\alpha|/m)}.\end{aligned}$$

So the recursion is complete and, for every m, $r \in \mathbb{N}$,
a) the inequalities (5), (8) and (9) give

$$\left\| \mathrm{D}^{\alpha}\Big(f - \sum_{s=0}^{r-1} G_s\Big) \right\|_{A_m} \leq \|f\|_m \, d_{r,m} 2^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)},$$

b) the inequalities (6), (10) and (11) give

$$\|\mathrm{D}^{\alpha} v_r\|_{A_m} \leq \|f\|_m \, d_{r,m}^2 2^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)},$$

c) the inequality (7) and the recursion give

$$\|\mathrm{D}^{\alpha} G_r\|_{A_m} \leq \|f\|_m \, d_{r,m}^2 2^{|\alpha|} \mathrm{e}^{m\varphi^*(|\alpha|/m)}.$$

Hence the conclusion by use of the inequality we have obtained in part (a) of the Notations. □

Lemma 2.2

For every integer $m \geq d_0$ and x, $y \in A_m$, $\alpha \in \mathbb{N}_0^n$, we have

$$|\mathrm{D}^{\alpha} v_r(x) - \mathrm{D}^{\alpha} v_r(y)| \leq n \, |x - y| \, q_m \, \|f\|_m \, d_{r,m}^2 \mathrm{e}^{p(m)\varphi^*((|\alpha|+1)/p(m))}.$$

Proof. For every $j \in \{1, \ldots, n\}$, let us designate by ϵ_j the j-th unit vector of $\mathbb{R}^n$. Then comes

$$|\mathrm{D}^{\alpha} v_r(x) - \mathrm{D}^{\alpha} v_r(y)| \leq \sum_{j=1}^{n} |x_j - y_j| \, \left\| \mathrm{D}^{\alpha+\epsilon_j} v_r \right\|_{A_m}.$$

Hence the conclusion by use of the formula (3) of the Proposition 2.1. □

Lemma 2.3

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $d_0 \leq m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^{\alpha} G_r - \mathrm{D}^{\alpha} v_r\|_{A_m} \leq \|f\|_{m+1} \, \varepsilon_r (p_{r+1} 2^{r+2+p_{r+2}} \mathrm{e}^{r\varphi^*(p_{r+2}+1)})^{-1}.$$

Proof. For every $x \in A_m$, we get

$$|\mathrm{D}^{\alpha} G_r(x) - \mathrm{D}^{\alpha} v_r(x)| \leq J_1 + J_2$$

with successively

$$\begin{aligned}J_1 = {} & \pi^{-n/2}\lambda_r^n \int_{|x-y|\geq\delta_r} \big(|\mathrm{D}^\alpha v_r(x)| + |\mathrm{D}^\alpha v_r(y)|\big)\mathrm{e}^{-\lambda_r^2|x-y|^2}\,dy \\ \leq {} & 2q_r\,\|f\|_r\,p_r^2\mathrm{e}^{p(r)\varphi^*(|\alpha|/p(r))}\big(1-\Psi(\lambda_r\delta_r)\big) \\ \leq {} & 2q_r\delta_r\mathrm{e}^{p(r)\varphi^*(|\alpha|/p(r))}\end{aligned}$$

and, by Lemma 2.2, as $\{\, y : |x-y| \leq \delta_r\} \subset A_{m+1}$,

$$\begin{aligned}J_2 = {} & \pi^{-n/2}\lambda_r^n \int_{|x-y|\leq\delta_r} |\mathrm{D}^\alpha v_r(y) - \mathrm{D}^\alpha v_r(x)|\,\mathrm{e}^{-\lambda_r^2|x-y|^2}\,dy \\ \leq {} & n\delta_r q_{m+1}\,\|f\|_{m+1}\,d_{r,m+1}^2\mathrm{e}^{p(m+1)\varphi^*((|\alpha|+1)/p(m+1))} \\ \leq {} & n\delta_r q_r p_r^2\,\|f\|_{m+1}\,\mathrm{e}^{p(m)\varphi^*((|\alpha|+1)/p(m))}.\end{aligned}$$

Hence the conclusion by the evaluation of δ_r since we have

$$J_1 + J_2 \leq 3nq_r\delta_r p_r^2\,\|f\|_{m+1}\,\mathrm{e}^{p(m)\varphi^*((|\alpha|+1)/p(m))}. \ \square$$

Lemma 2.4

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $d_0 \leq m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^\alpha G_{r+1}\|_{K_{r+2}\cap A_m} \leq \|f\|_{m+1}\,\varepsilon_r 2^{-(r+1)}.$$

Proof. As the function u_r is identically 1 on a neighbourhood of the set $K_{r+2} \setminus (K_{r+1})^\circ$, the Lemma 2.3 leads directly to the following auxiliary inequality $(*)$

$$\begin{aligned}&\Big\|D^\alpha\Big(f - \sum_{s=1}^{r} G_s\Big)\Big\|_{(K_{r+2}\setminus(K_{r+1})^\circ)\cap A_m} \\ &\quad = \|\mathrm{D}^\alpha G_r - \mathrm{D}^\alpha v_r\|_{(K_{r+2}\setminus(K_{r+1})^\circ)\cap A_m} \\ &\quad \leq \|f\|_{m+1}\,\varepsilon_r(p_{r+1}2^{r+2+p_{r+2}}\mathrm{e}^{r\varphi^*(p_{r+2}+1)})^{-1}.\end{aligned}$$

Therefore by use of the Leibniz formula we get

$$\begin{aligned}&\|\mathrm{D}^\alpha v_{r+1}\|_{(K_{r+2}\setminus(K_{r+1})^\circ)\cap A_m} \\ &\quad \leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}d_{r+1,1}2^{-(r+1)|\beta|}\mathrm{e}^{\varphi^*(|\beta|)}\,\|f\|_{m+1}\,\varepsilon_r(p_{r+1}2^{r+2+p_{r+2}}\mathrm{e}^{r\varphi^*(p_{r+2}+1)})^{-1} \\ &\quad \leq \|f\|_{m+1}\,\varepsilon_r 2^{-(r+2)}.\end{aligned}$$

Now as u_{r+1} vanishes identically on K_{r+1}, these last inequalities are indeed valid on $K_{r+2} \cap A_m$. Therefore the Lemma 2.3 leads to

$$\begin{aligned}
&\|\mathrm{D}^\alpha G_{r+1}\|_{K_{r+2}\cap A_m} \\
&\leq \|\mathrm{D}^\alpha G_{r+1} - \mathrm{D}^\alpha v_{r+1}\|_{A_m} + \|\mathrm{D}^\alpha v_{r+1}\|_{K_{r+2}\cap A_m} \\
&\leq \|f\|_{m+1}\, \varepsilon_{r+1}(p_{r+2} 2^{r+3+p_{r+3}} \mathrm{e}^{(r+1)\varphi^*(p_{r+3}+1)})^{-1} + \|f\|_{m+1}\, \varepsilon_r 2^{-(r+2)} \\
&\leq \|f\|_{m+1}\, 2^{-(r+1)}. \ \Box
\end{aligned}$$

Proposition 2.5

For every compact subset K of Ω and $\alpha \in \mathbb{N}_0^n$, the series $\sum_{r=1}^{\infty} \|\mathrm{D}^\alpha G_r\|_K$ converges.

Therefore the series $G = \sum_{r=1}^{\infty} G_r$ defines a C^∞-function on Ω and can be differentiated term by term.

Proof. Indeed if we choose integers m, $r \in \mathbb{N}$ such that $d_0 \leq m < r$, $|\alpha| \leq p_{r+2}$ and $K \subset K_r \cap A_m$, the Lemma 2.4 leads to

$$\|\mathrm{D}^\alpha G_{r+p}\|_K \leq \|\mathrm{D}^\alpha G_{r+p}\|_{K_{r+p+1}\cap A_m} \leq \|f\|_{m+1}\, \varepsilon_{r+p-1} 2^{-(r+p)}$$

for every $p \in \mathbb{N}$. Hence the conclusion. $\Box$

Lemma 2.6

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $d_0 \leq m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^\alpha G - \mathrm{D}^\alpha f\|_{A_m \cap \Omega \setminus K_{r+1}} \leq \|f\|_{m+1}\, \varepsilon_r.$$

Proof. Let x be any element of $A_m \cap \Omega \setminus K_{r+1}$. Then we designate by q the first positive integer such that $x \in K_{r+q}$; of course we have $q \geq 2$. So, on the one hand, the Lemma 2.4 leads to

$$|\mathrm{D}^\alpha G_{r+s}(x)| \leq \|f\|_{m+1}\, \varepsilon_{r+s-1} 2^{-(r+s)}$$

for every integer $s \geq q-1$. On the other hand, the auxiliary inequality $(*)$ that appears at the beginning of the proof of the Lemma 2.4 leads to

$$\left|\mathrm{D}^\alpha \left(f(x) - \sum_{s=1}^{r+q-2} G_s(x) \right)\right| \leq \|f\|_{m+1}\, \varepsilon_{r+q-2} 2^{-(r+q)}.$$

So we get

$$\begin{aligned}|\mathrm{D}^\alpha G(x) - \mathrm{D}^\alpha f(x)| \leq\ & \|f\|_{m+1}\, \varepsilon_{r+q-2} 2^{-(r+q)} \\ & + \sum_{s=q-1}^{\infty} \|f\|_{m+1}\, \varepsilon_{r+s-1} 2^{-(r+s)} \\ \leq\ & \|f\|_{m+1}\, \varepsilon_r. \ \square\end{aligned}$$

Proposition 2.7

The function g defined on $\mathbb{R}^n$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ G(x) & \text{if } x \in \Omega = \mathbb{R}^n \setminus F \end{cases}$$

belongs to $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ and is such that $\mathrm{D}^\alpha g(x) = \mathrm{D}^\alpha f(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$.

Proof. By the Proposition 2.5, the Lemma 2.6 and a classical argument, we know already that g belongs to $\mathrm{C}^\infty(\mathbb{R}^n)$ and is such that $\mathrm{D}^\alpha g(x) = \mathrm{D}^\alpha f(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$.

So to conclude, we just need to establish that g belongs to $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$. As f belongs to $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$, we just need to concentrate our attention to the restriction G of g to Ω.

For every integer m such that $p(m) \geq d_0$, we are going to prove that, if $q(m)$ designates the integer part of $p(m)/d_0$, we have

$$\sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha G\|_{A_m \cap \Omega}\, \mathrm{e}^{-q(m)\varphi^*(|\alpha|/q(m))} < \infty.$$

The conclusion then follows at once.

For this purpose, let us first consider a multi-index $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq p_{m+2}$. Two cases are possible:

a) *The point x belongs to $A_m \cap K_{m+2}$.* In this case, the formula (4) of the Proposition 2.1 gives the existence of a constant $k_m > 0$ such that

$$\sum_{s=1}^{m+1} \|\mathrm{D}^\alpha G_s\|_{A_m} \leq k_m \|f\|_m\, \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))}.$$

Then the Proposition 2.5 and the Lemma 2.4 successively give

$$\begin{aligned}|\mathrm{D}^\alpha G(x)| \leq & \sum_{s=1}^{m+1} |\mathrm{D}^\alpha G_s(x)| + \sum_{s=m+2}^{\infty} |\mathrm{D}^\alpha G_s(x)| \\ \leq & \; k_m \|f\|_m \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))} + \sum_{s=m+2}^{\infty} \|f\|_{m+1} \varepsilon_{s-1} 2^{-s} \\ \leq & \; \|f\|_{m+1} (k_m+1)\mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))}.\end{aligned}$$

b) *The point x belongs to $(\Omega \cap A_m) \setminus K_{m+2}$.* In this case, the Lemma 2.6 gives

$$\begin{aligned}|\mathrm{D}^\alpha G(x)| \leq & \; |\mathrm{D}^\alpha G(x) - \mathrm{D}^\alpha f(x)| + |\mathrm{D}^\alpha f(x)| \\ \leq & \; \|f\|_{m+1} \varepsilon_{m+1} + \|f\|_m \mathrm{e}^{m\varphi^*(|\alpha|/m)} \quad \leq \quad 2\|f\|_{m+1} \mathrm{e}^{m\varphi^*(|\alpha|/m)}.\end{aligned}$$

Let us now consider a multi-index $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > p_{m+2}$. Then we first introduce the first positive integer r such that $p_{r+1} < |\alpha| \leq p_{r+2}$; of course, we have $r > m$. Once more two cases are possible:

a') *The point x belongs to $A_m \cap K_{r+1}$.* On the one hand, the Lemma 2.4 and the value of ε_{r+s-1} lead to

$$\sum_{s=1}^{\infty} |\mathrm{D}^\alpha G_{r+s}(x)| \leq \sum_{s=1}^{\infty} \|f\|_{m+1} \varepsilon_{r+s-1} 2^{-(r+s)} \leq \|f\|_{m+1} \varepsilon_r 2^{-r}.$$

On the other hand, by use the formula (4) of the Proposition 2.1, we also have

$$\begin{aligned}\sum_{s=1}^{r} |\mathrm{D}^\alpha G_s(x)| \leq & \sum_{s=1}^{r} q_m \|f\|_m d_{s,m}^2 \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))} \\ \leq & \; r q_m \|f\|_m d_{r,m}^2 \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))} \\ \underset{(*)}{\leq} & \; q_m 2^{|\alpha|} \mathrm{e}^{p(m)\varphi^*(|\alpha|/p(m))} \quad \underset{(**)}{\leq} \quad q_m^2 \mathrm{e}^{q(m)\varphi^*(|\alpha|/q(m))}\end{aligned}$$

(to get the inequality $(*)$, we note that $r\|f\|_m d_{r,m}^2 \leq d_{r+1,m} \leq p_{r+1} \leq 2^{p_{r+1}} \leq 2^{|\alpha|}$; to get the inequality $(**)$, we use an evaluation made in the part (a) of the Notations). These two informations put together give

$$|\mathrm{D}^\alpha G(x)| \leq (\|f\|_{m+1} + q_m^2)\mathrm{e}^{q(m)\varphi^*(|\alpha|/q(m))}.$$

b') *The point x belongs to $(\Omega \cap A_m) \setminus K_{r+1}$.* Then the Lemma 2.6 leads to

$$\begin{aligned}|\mathrm{D}^\alpha G(x)| \leq & \; |\mathrm{D}^\alpha f(x)| + |\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha G(x)| \\ \leq & \; \|f\|_m \mathrm{e}^{m\varphi^*(|\alpha|/m)} + \|f\|_{m+1} \varepsilon_r \quad \leq \quad 2\|f\|_{m+1} \mathrm{e}^{m\varphi^*(|\alpha|/m)}.\end{aligned}$$

Therefore we finally have

$$\|\mathrm{D}^\alpha G\|_{\Omega \cap A_m} \leq C_m \mathrm{e}^{q(m)\varphi^*(|\alpha|/q(m))}, \quad \forall \alpha \in \mathbb{N}_0^n$$

for $C_m = \max\{\|f\|_{m+1}(k_m+1), 2\|f\|_{m+1}, \|f\|_{m+1} + q_m^2\}$ and the proof is complete. □

Proposition 2.8

The function G has a holomorphic extension on the following open subset $\Omega^* = \{u + iv : u \in \Omega, v \in \mathbb{R}^n, |v| < \mathrm{d}(u, \partial\Omega)\}$ of $\mathbb{C}^n$. Therefore g is analytic on Ω.

Proof. It is clear that Ω^* is an open subset of $\mathbb{C}^n$.

Let H be any compact subset of Ω^*. Then as in [16], one establishes immediately by contradiction the existence of an integer $r_0 \in \mathbb{N}$ such that

$$\delta^2 = \inf \left\{ \sum_{j=1}^{n} \left((u_j - y_j)^2 - v_j^2 \right) : u, v, y \in \mathbb{R}^n; u + iv \in H; y \notin K_{r_0} \right\} > 0\,.$$

So for every integer $r > \max\{r_0, \delta^{-1}\}$ and every point $w = u + iv$ of H with u, $v \in \mathbb{R}^n$, the formula (3) of the Proposition 2.1 leads to

$$\begin{aligned} |G_r(w)| &= \left| \pi^{-n/2} \lambda_r^n \int_{\mathbb{R}^n} v_r(y) \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (w_j - y_j)^2}\, dy \right| \\ &\le \pi^{-n/2} \lambda_r^n \, \|v_r\|_{A_{rd_0}} \int_{K_{r+3} \setminus K_r} \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n ((u_j - y_j)^2 - v_j^2)}\, dy \\ &\le q_{rd_0} \, \|f\|_{rd_0} \, d^2_{r, rd_0} \pi^{-n/2} \lambda_r^n \mathrm{e}^{-\lambda_r^2 r^{-2}} \ \mathrm{mes}(K_{r+3}) \underset{(*)}{\le} 2^{-r} \end{aligned}$$

(the inequality $(*)$ comes from the last requirement we imposed on λ_r).

Therefore the series $\sum_{r=1}^{\infty} G_r$ converges absolutely and uniformly on H. So it represents a holomorphic function on Ω^* since each of the functions $G_r(w)$ is holomorphic on $\mathbb{C}^n$. Hence the conclusion. □

Proof of the Theorem 1.1 in the case of the Beurling type. The main result is now a direct consequence of the Propositions 2.7 and 2.8. □

3. Proof in the case of the Roumieu type

The pattern of the proof of this case is very much comparable to the one relative to the Beurling case. Therefore we are going to indicate the intermediate results and to only mention the differences.

Notations. Let us first set up the different notations that will be used throughout the proof and that lead, up to a requirement that will appear inside the proof of the Proposition 3.2, to the definition of the numbers λ_r.

So far we have introduced the weight ω, the function $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ as well as the closed subset F of $\mathbb{R}^n$. Of course we may restrict our attention to the case when the restriction of f to the open subset $\Omega = \mathbb{R}^n \setminus F$ of $\mathbb{R}^n$ is not identically 0. Moreover:

(a) We set $q'_m = \max\{e^{1/m}, 2^{y_0/(md_0)}\}$ for every $m \in \mathbb{N}$ and remark as in [17] that the inequality
$$2^r e^{\varphi^*(mr)/m} \le q'_m e^{\varphi^*(mrd_0)/(md_0)}$$
holds for every r, $m \in \mathbb{N}$: in fact if r verifies $r \le y_0/(md_0)$, we certainly have
$$2^r e^{\varphi^*(mr)/m} \le 2^{y_0/(md_0)} e^{\varphi^*(md_0 r)/(md_0)},$$
and if $r > y_0/(md_0)$, we successively have
$$\begin{aligned} 2^r e^{\varphi^*(mr)/m} &\le e^{1/(md_0)(d_0\varphi^*(md_0 r/d_0)+md_0 r)} \\ &\le e^{1/(md_0)(\varphi^*(md_0 r)+d_0)} \quad = \quad e^{1/m} e^{\varphi^*(md_0 r)/(md_0)}. \end{aligned}$$

(b) For every $r \in \mathbb{R}$, we set $A_r = \{\, x \in \mathbb{R}^n : |x| \le r + \mathrm{d}(0,\Omega)\}$.

(c) $\{\, K_m : m \in \mathbb{N}\}$ is the same regular compact cover of Ω.

(d) As $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$, there is a strictly increasing sequence $(n_m)_{m\in\mathbb{N}}$ of positive integers such that
$$\sup_{\alpha\in\mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_{A_{m+1}} e^{-\varphi^*(n_m|\alpha|)/n_m} \le n_m, \quad \forall m \in \mathbb{N}\,.$$

(e) Proposition 1 of [17] provides a sequence $(u_r)_{r\in\mathbb{N}}$ of $\mathrm{C}^\infty(\mathbb{R}^n)$ such that, for every $r \in \mathbb{N}$,
$$\begin{aligned} &0 \le u_r \le 1, \\ &u_r \equiv 1 \quad \text{on a neighbourhood of } K_{r+2} \setminus (K_{r+1})^\circ, \\ &\mathrm{supp}(u_r) \subset (K_{r+3})^\circ \setminus K_r, \\ &\sup_{\alpha\in\mathbb{N}_0^n} 2^{r|\alpha|} e^{-m\varphi^*(|\alpha|/m)} \|\mathrm{D}^\alpha u_r\|_{\mathbb{R}^n} \le d_{r,m} \end{aligned}$$
where, for every r, $m \in \mathbb{N}$, $d_{r,m}$ is a positive integer such that
$$\begin{aligned} &d_{r,m} \le d_{r+1,m}, \\ &d_{r,m} \le d_{r,m+1}, \\ &(r+1)n_m d_{r,n_m}^2 \le \frac{1}{3}\, d_{r+1,m}\,. \end{aligned}$$

(f) For every m, $r \in \mathbb{N}$, $\{(A_{m+2^{-r+1}})^\circ, \mathbb{R}^n \setminus A_{m+2^{-r+1/2}}\}$ is an open cover of $\mathbb{R}^n$. So, by use of the Proposition 1 of [17] again and of convolution products, there is a C^∞-partition of unity $\{\varphi_{1,m,r}, \varphi_{2,m,r}\}$ of $\mathbb{R}^n$ such that

$$\operatorname{supp}(\varphi_{1,m,r}) \subset (A_{m+2^{-r+1}})^\circ, \quad \operatorname{supp}(\varphi_{2,m,r}) \subset \mathbb{R}^n \setminus A_{m+2^{-r+1/2}}$$

and

$$\sup_{\alpha \in \mathbb{N}_0^n} 2^{r|\alpha|} e^{-l\varphi^*(|\alpha|/l)} \|D^\alpha \varphi_{j,m,r}\|_{\mathbb{R}^n} < \infty$$

for every $j \in \{1,2\}$ and $l \in \mathbb{N}$. We then set $v_{m,r} = u_r\varphi_{1,m,r}$ and $w_{m,r} = u_r\varphi_{2,m,r}$; of course we get $u_r = v_{m,r} + w_{m,r}$ with

$$\operatorname{supp}(v_{m,r}) \subset K_{r+3} \cap (A_{m+2^{-r+1}})^\circ \quad \text{and} \quad \operatorname{supp}(w_{m,r}) \subset K_{r+3} \setminus A_{m+2^{-r+1/2}}.$$

Moreover we may suppose, up to a modification of the numbers $d_{r,m}$, that we have

$$\sup_{\alpha \in \mathbb{N}_0^n} 2^{r|\alpha|} e^{-l\varphi^*(|\alpha|/l)} \|D^\alpha w_{m,r}\|_{\mathbb{R}^n} \leq d_{r,m,l}$$

for some positive numbers $d_{r,m,l}$ verifying $d_{r,m,m} = d_{r,m}$.

(g) For every $r \in \mathbb{N}$, we set $p_r = d_{r,n_r}$, $\varepsilon_r = 2^{-rp_{r+2}}$ and

$$\delta_r = \varepsilon_r \left(3nq_r p_r^2 p_{r+1} 2^{r+2+p_{r+2}} e^{2\varphi^*(n_r d_0(p_{r+2}+1))/(n_r d_0)}\right)^{-1}.$$

(h) For every $r \in \mathbb{N}$, there is then $\lambda'_r > 0$ such that

$$\begin{aligned} n_r p_r^2\big(1 - \Psi(\lambda\delta_r)\big) &\leq \delta_r, \\ \pi^{-n/2} q_r n_r d_{r,n_r}^2 \lambda^n e^{-\lambda^2 r^{-2}} \operatorname{mes}(K_{r+3}) &\leq 2^{-r} \end{aligned}$$

for every $\lambda \geq \lambda'_r$. This will allow us to fix the value of numbers $\lambda_r \geq \lambda'_r$ satisfying one more requirement inside the proof of the Proposition 3.2.

Remark. The existence of a function such as u_1 in the case $n = 1$ allows to get an evaluation of how fast the sequence $(e^{\varphi^*(r)})_{r \in \mathbb{N}_0}$ grows to $+\infty$.

Lemma 3.1

For every $C > 0$, the sequence $(r!C^r e^{-\varphi^*(r)})_{r \in \mathbb{N}_0}$ converges to 0.

Proof. In order to simplify the notations, let us set $M_r = e^{\varphi^*(r)}$ for every $r \in \mathbb{N}_0$. As φ^* is a convex and increasing function on $[0, +\infty[$ such that $\varphi^*(0) = 0$, we certainly get $M_0 = 1$ as well as $M_r \geq 1$ and $M_r^2 \leq M_{r-1}/M_{r+1}$ for every $r \in \mathbb{N}$. Moreover we have $\sum_{r=1}^{\infty} M_{r-1}/M_r < \infty$: as, in the case $n = 1$, the non-zero function u_1 belongs to $C^\infty(\mathbb{R})$, has compact support and verifies

$$\|D^r u_1\|_{\mathbb{R}} \leq d_{1,1} 2^{-r} M_r, \quad \forall r \in \mathbb{N}_0,$$

the class $\mathcal{E}_{(M_r)}(\mathbb{R})$ is non quasi-analytic, hence the conclusion by the Denjoy-Carleman-Mandelbrojt theorem.

Now we conclude as in the Lemma 1.1 of [15]: the series $\sum_{r=1}^{\infty} M_{r-1}/M_r$ is converging and the sequence $(M_{r-1}/M_r)_{r\in\mathbb{N}}$ decreases to 0 therefore it is well known that the sequence $(rM_{r-1}/M_r)_{r\in\mathbb{N}}$ tends to 0. Hence, by the ratio test, for every $C > 0$, the series $\sum_{r=1}^{\infty} r!C^r/M_r$ converges. Hence the conclusion. □

Now we are all set to start the proof. It consists in the study of the same functions G_0, G_1, G_2, ... and to simplify the notations, we introduce the same shorthand v_r.

Proposition 3.2

The analytic functions G_1, G_2, ... on $\mathbb{R}^n$ are such that, for every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we have

$$\left\| D^\alpha \Big(f - \sum_{s=0}^{r-1} G_s \Big) \right\|_{A_m} \leq q_m n_m d_{r,n_m} e^{\varphi^*(n_m d_0 |\alpha|)/(n_m d_0)}, \tag{12}$$

$$\|D^\alpha v_r\|_{A_m} \leq q_m n_m d_{r,n_m}^2 e^{\varphi^*(n_m d_0 |\alpha|)/(n_m d_0)}, \tag{13}$$

$$\|D^\alpha G_r\|_{A_m} \leq q_m n_m d_{r,n_m}^2 e^{\varphi^*(n_m d_0 |\alpha|)/(n_m d_0)}. \tag{14}$$

Proof. First of all we are going to prove by recursion that

$$\|D^\alpha G_r\|_{A_{m+2^{-r}}} \leq n_m d_{r,n_m}^2 (1 + 2^{-1} + \ldots + 2^{-r})^{|\alpha|} e^{\varphi^*(n_m |\alpha|)/n_m}$$

holds for every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$.

Case $r = 1$**.** For every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we clearly have

$$\|D^\alpha f\|_{A_{m+2^{-1}}} \leq \|D^\alpha f\|_{A_{m+1}} \leq n_m e^{\varphi^*(n_m |\alpha|)/n_m} \tag{15}$$

and, as $\operatorname{supp}(v_1) \subset \operatorname{supp}(u_1) \subset K_{1+3} \subset A_1 \subset A_{m+1}$,

$$\begin{aligned} \|D^\alpha v_1\|_{\mathbb{R}^n} \le\ & n_m d_{1,n_m} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} 2^{-|\beta|} e^{n_m\varphi^*(|\beta|/n_m)} e^{\varphi^*(n_m|\alpha-\beta|)/n_m} \\ \underset{(*)}{\le}\ & n_m d_{1,n_m} (1+2^{-1})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m} \end{aligned} \tag{16}$$

(the inequality $(*)$ comes from $e^{n_m\varphi^*(|\beta|/n_m)} \le e^{\varphi^*(|\beta|)} \le e^{\varphi^*(n_m|\beta|)/n_m}$) hence

$$\|D^\alpha G_1\|_{\mathbb{R}^n} \le n_m d_{1,n_m} (1+2^{-1})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m}, \tag{17}$$

i.e. the case $r = 1$ is certainly established.

Case $r > 1$. Suppose that, for some integer $r \ge 2$ and every $s \in \{1, \dots, r-1\}$, $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we have obtained

$$\|D^\alpha G_s\|_{A_{m+2^{-s}}} \le n_m d_{s,n_m}^2 (1 + 2^{-1} + \dots + 2^{-s})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m}.$$

Then, for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we certainly have

$$\begin{aligned} & \Big\| D^\alpha \Big(f - \sum_{s=0}^{r-1} G_s \Big) \Big\|_{A_{m+2^{-r+1}}} \\ \le\ & \|D^\alpha f\|_{A_{m+1}} + \sum_{s=1}^{r-1} \|D^\alpha G_s\|_{A_{m+2^{-s}}} \\ \le\ & r n_m d_{r-1,n_m}^2 (1 + 2^{-1} + \dots + 2^{-r+1})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m} \\ \le\ & \frac{1}{3} d_{r,n_m} (1 + 2^{-1} + \dots + 2^{-r+1})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m}. \end{aligned} \tag{18}$$

Therefore for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, if we designate by k any of the functions u_r or $w_{m,r}$, the Leibniz formula leads to

$$\begin{aligned} & \Big\| D^\alpha \Big(k \Big(f - \sum_{s=0}^{r-1} G_s \Big) \Big) \Big\|_{A_{m+2^{-r+1}}} \\ \le\ & \frac{1}{3} d_{r,n_m}^2 (1 + 2^{-1} + \dots + 2^{-r})^{|\alpha|} e^{\varphi^*(n_m|\alpha|)/n_m}. \end{aligned} \tag{19}$$

Now for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we evaluate $\|\mathrm{D}^\alpha G_r\|_{A_{m+2^{-r}}}$. Let us consider any point $x \in A_{m+2^{-r}}$. As $\mathrm{supp}(u_r) \subset K_{r+3}$, we get

$$\begin{aligned}\mathrm{D}^\alpha G_r(x) = {} & \pi^{-n/2}\lambda_r^n \int_{K_{r+3}\cap A_{m+2^{-r+1}}} \mathrm{D}^\alpha v_r(y)\, \mathrm{e}^{-\lambda_r^2|x-y|^2}\, dy \\ & + \pi^{-n/2}\lambda_r^n \int_{K_{r+3}\setminus A_{m+2^{-r+1}}} \mathrm{D}^\alpha v_r(y)\, \mathrm{e}^{-\lambda_r^2|x-y|^2}\, dy\end{aligned}$$

where the inequality (19) implies that the absolute value of the first term of the second member is

$$\leq \frac{1}{3}\, d_{r,n_m}^2 \left(1 + 2^{-1} + \cdots + 2^{-r}\right)^{|\alpha|} \mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m}. \tag{20}$$

To evaluate the second term, we remark first that, on $K_{r+3}\setminus A_{m+2^{-r+1}}$, the functions u_r and $w_{m,r}$ coincide. So it is equal to

$$\begin{aligned}& -\pi^{-n/2}\lambda_r^n \int_{K_{r+3}\cap A_{m+2^{-r+1}}\setminus A_{m+2^{-r}}} \mathrm{D}^\alpha \left(w_{m,r}(y)\Big(f(y) - \sum_{s=0}^{r-1} G_s(y)\Big)\right) \mathrm{e}^{-\lambda_r^2|x-y|^2} dy \\ & + \pi^{-n/2}\lambda_r^n \int_{K_{r+3}\setminus A_{m+2^{-r}}} \mathrm{D}^\alpha \left(w_{m,r}(y)\Big(f(y) - \sum_{s=0}^{r-1} G_s(y)\Big)\right) \mathrm{e}^{-\lambda_r^2|x-y|^2} dy\end{aligned}$$

where again the inequality (19) implies that the absolute value of the first term is

$$\leq \frac{1}{3}\, d_{r,n_m}^2 \left(1 + 2^{-1} + \cdots + 2^{-r}\right)^{|\alpha|} \mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m}. \tag{21}$$

To evaluate the second term, as $w_{m,r}$ has its support contained in the set $K_{r+3} \setminus A_{m+2^{-r+1/2}}$, we may as well integrate on $\mathbb{R}^n$. So, by use of a standard property of the convolution product, we find that it is equal to

$$\pi^{-n/2}\lambda_r^n \int_{\mathbb{R}^n} w_{m,r}(y)\Big(f(y) - \sum_{s=0}^{r-1} G_s(y)\Big)\, \mathrm{D}_x^\alpha \mathrm{e}^{-\lambda_r^2|x-y|^2}\, dy. \tag{22}$$

To evaluate the value of this last integral (22), we first set

$$b_r = \pi^{-n/2} \left\| f - \sum_{s=0}^{r-1} G_s \right\|_{K_{r+3}} \geq \pi^{-n/2} \left\| w_{m,r}\Big(f - \sum_{s=0}^{r-1} G_s\Big)\right\|_{\mathbb{R}^n}.$$

Next we note that for $a = 2^{1/4} - 1 > 0$, the compact subsets $A_{m+2^{-r}}$ and

$$B_{m,r} = \Big\{\, w \in \mathbb{C}^n : |\Re w| \le \mathrm{d}(0,\Omega) + m + 2^{-r+1/4}, |\Im w| \le 2^{-r}a \Big\}$$

of $\mathbb{C}^n$ are of course such that

$$\big\{\, w \in \mathbb{C}^n : \mathrm{d}(w, A_{m+2^{-r}}) \le 2^{-r}a \big\} \subset B_{m,r}.$$

Moreover for every $w \in B_{m,r}$ and $y \in \mathrm{supp}(w_{m,r}) \subset K_{r+3} \setminus A_{m+2^{-r+1/2}}$, we have

$$\begin{aligned} \left| \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (w_j - y_j)^2} \right| &= \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n ((\Re w_j - y_j)^2 - (\Im w_j)^2)} \\ &\le \mathrm{e}^{-\lambda_r^2 2^{-2r}((2^{1/2}-2^{1/4})^2 - a^2)} \quad \le \quad \mathrm{e}^{-2^{-2r}a^2\lambda_r^2(\sqrt{2}-1)}. \end{aligned}$$

So, by use of the Cauchy inequality, we get

$$\left| \mathrm{D}_x^\alpha \mathrm{e}^{-\lambda_r^2 |x-y|^2} \right| \le \alpha!\, (2^r n a^{-1})^{|\alpha|} \mathrm{e}^{-2^{-2r}a^2\lambda_r^2(\sqrt{2}-1)}$$

for every $y \in \mathrm{supp}(w_{m,r})$. Taking all these informations into account, we get that the absolute value of the expression (22) is

$$\le b_r \mathrm{mes}(K_{r+3}) \lambda_r^n \mathrm{e}^{-2^{-2r}a^2\lambda_r^2(\sqrt{2}-1)} \alpha! (2^r n a^{-1})^{|\alpha|}. \tag{23}$$

At this stage, we note that, by the Lemma 3.1, there is a constant $C_r > 0$ such that

$$j!\,(2^r n a^{-1})^j \le C_r \mathrm{e}^{\varphi^*(j)}, \quad \forall j \in \mathbb{N}\,.$$

This allows us to formulate our last requirement fixing $\lambda_r \ge \lambda'_r$: we choose λ_r large enough so that

$$C_r b_r \mathrm{e}^{-2^{-2r}a^2\lambda_r^2(\sqrt{2}-1)} \lambda_r^n \mathrm{mes}(K_{r+3}) \le \frac{1}{3}.$$

Then with the help of the evaluations (20), (21) and (23), we finally get

$$|\mathrm{D}^\alpha G_r(x)| \le d_{r,n_m}^2 (1 + 2^{-1} + \cdots + 2^{-r})^{|\alpha|} \mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m},$$

i.e. the recursion is complete.

It is now a direct matter to check that the formulae (15) and (18), (resp. (16) and (19); (17) and the recursion formula) lead directly to the formulae (2) (resp. (13); (14)) respectively. □

Lemma 3.3

For every m, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and x, $y \in A_m$, we have

$$|\mathrm{D}^\alpha v_r(x) - \mathrm{D}^\alpha v_r(y)| \le n\,|x-y|\, q_m n_m d_{r,n_m}^2 \mathrm{e}^{\varphi^*(n_m d_0(|\alpha|+1))/(n_m d_0)}.$$

Proof. Using (13) instead of (3), the proof of the Lemma 2.2 applies. □

Lemma 3.4

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^\alpha G_r - \mathrm{D}^\alpha v_r\|_{A_m} \leq n_{m+1}\varepsilon_r \left(p_{r+1} 2^{r+2+p_{r+2}} \mathrm{e}^{\varphi^*(n_r d_0 (p_{r+2}+1))/(n_r d_0)}\right)^{-1}.$$

Proof. Following the lines of the proof of the Lemma 2.3 leads to

$$\begin{aligned} J_1 &\leq 2q_r n_r p_r^2 \mathrm{e}^{\varphi^*(n_r d_0|\alpha|)/(n_r d_0)}\big(1-\Psi(\lambda_r\delta_r)\big) \quad \leq \quad 2q_r\delta_r \mathrm{e}^{\varphi^*(n_r d_0|\alpha|)/(n_r d_0)}, \\ J_2 &\leq n\delta_r q_{m+1} n_{m+1} d_{r,n_{m+1}}^2 \mathrm{e}^{\varphi^*(n_{m+1}d_0(|\alpha|+1))/(n_{m+1}d_0)} \end{aligned}$$

hence the conclusion by use of the evaluation of δ_r. □

Lemma 3.5

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^\alpha G_{r+1}\|_{A_m \cap K_{r+2}} \leq n_{m+1}\varepsilon_r 2^{-(r+1)}.$$

Proof. Using Lemma 3.4 in the proof of the Lemma 2.4 applies. □

Proposition 3.6

For every compact subset K of Ω and $\alpha \in \mathbb{N}_0^n$, the series $\sum_{r=1}^\infty \|\mathrm{D}^\alpha G_r\|_K$ converges.

Therefore the series $G = \sum_{r=1}^\infty G_r$ defines a C^∞-function on Ω and can be differentiated term by term.

Proof. Using Lemma 3.5 in the proof the Proposition 2.5 applies. □

Lemma 3.7

For every m, $r \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $m < r$ and $|\alpha| \leq p_{r+2}$, we have

$$\|\mathrm{D}^\alpha G - \mathrm{D}^\alpha f\|_{A_m \cap \Omega \setminus K_{r+1}} \leq n_{m+1}\varepsilon_r.$$

Proof. Using Lemma 3.5 in the proof of the Lemma 2.6 applies. □

Proposition 3.8

The function g defined on $\mathbb{R}^n$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ G(x) & \text{if } x \in \Omega = \mathbb{R}^n \setminus F \end{cases}$$

belongs to $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ and is such that $\mathrm{D}^\alpha g(x) = \mathrm{D}^\alpha f(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$.

Proof. By the Proposition 3.6, the Lemma 3.7 and a classical argument, we know already that g belongs to $\mathrm{C}^\infty(\mathbb{R}^n)$ and is such that $\mathrm{D}^\alpha g(x) = \mathrm{D}^\alpha f(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in F$.

To establish that g belongs to $\mathcal{E}_{\{M_r\}}(\mathbb{R}^n)$, we just need to prove that, for every $m \in \mathbb{N}$, there is a constant $C_m > 0$ such that

$$\|\mathrm{D}^\alpha g\|_{\Omega\cap A_m} \leq C_m \mathrm{e}^{\varphi^*(n_m d_0^2|\alpha|)/(n_m d_0^2)}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

To get this we just have to follow the steps of the proof of the Proposition 2.7, making the obvious modifications. Indeed for $m \in \mathbb{N}$ fixed, this leads to the following evaluations. For a multi-index $\alpha \in \mathbb{N}_0^n$ verifying $|\alpha| \leq p_{m+2}$, we get the existence of a constant $k_m > 0$ such that

$$\begin{aligned} \|\mathrm{D}^\alpha G\|_{A_m\cap K_{m+2}} &\leq k_m \mathrm{e}^{\varphi^*(n_m d_0|\alpha|)/(n_m d_0)}, \\ \|\mathrm{D}^\alpha G\|_{\Omega\cap A_m\setminus K_{m+2}} &\leq 2n_{m+1}\mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m}. \end{aligned}$$

For a multi-index $\alpha \in \mathbb{N}_0^n$ verifying $|\alpha| > p_{m+2}$, we introduce r as the first positive integer such that $p_{r+1} < |\alpha| \leq p_{r+2}$. This leads to

$$\begin{aligned} \|\mathrm{D}^\alpha G\|_{A_m\cap K_{r+1}} &\leq (n_{m+1} + q_m^2)\mathrm{e}^{\varphi^*(n_m d_0^2|\alpha|)/(n_m d_0^2)}, \\ \|\mathrm{D}^\alpha G\|_{\Omega\cap A_m\setminus K_{r+1}} &\leq n_m \mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m} + n_{m+1}\varepsilon_r \quad \leq \quad 2n_{m+1}\mathrm{e}^{\varphi^*(n_m|\alpha|)/n_m}. \end{aligned}$$

Therefore the constant $C_m = \sup\{k_m, 2n_{m+1}, n_{m+1} + q_m^2\}$ suits our goal. □

Proposition 3.9

The function G has a holomorphic extension on the following open subset $\Omega^* = \{\, u + iv : u \in \Omega, v \in \mathbb{R}^n, |v| < \mathrm{d}(u, \partial\Omega)\}$ of $\mathbb{C}^n$. Therefore g is analytic on Ω.

Proof. One has just to reproduce the proof of the Proposition 2.8 making the obvious modifications. □

Proof of the Theorem 1.1 in the case of the Roumieu type. The main result is now a direct consequence of the Propositions 3.8 and 3.9. □

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