Collect. Math. **50**, 1 (1999), 53–72

On reiteration and the behaviour of weak compactness under certain interpolation methods*

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Received March 18, 1997. Revised September 1, 1997

Abstract

This article deals with K- and J-spaces defined by means of polygons. First we establish some reiteration formulae involving the real method, and then we study the behaviour of weakly compact operators. We also show optimality of the weak compactness results.

Introduction

The behaviour of weak compactness under real interpolation has been extensively studied. We refer, for example, to the book of Beauzamy [1] and the papers by

* All three authors have been supported in part by DGICYT (PB94-0252).

Heinrich [9], Maligranda and Quevedo [10] and Mastylo [11]. It turned out that if $0 < \theta < 1$ and $1 < q < \infty$ then a necessary and sufficient condition for $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ to be weakly compact is that $T : A_0 \bigcap A_1 \longrightarrow B_0 + B_1$ is weakly compact. In particular, if just one restriction of T is weakly compact, say $T : A_0 \longrightarrow B_0$, then the interpolated operator is also weakly compact.

In this paper we study how weakly compact operators behave under J- and K-interpolation methods associated to polygons. These methods are similar to the real interpolation method, but they work on N-tuples of Banach spaces ($N \geq 3$) and they incorporate some geometrical elements that play an important role in developing their theory.

In Section 2 we show that if just one restriction of the operator T is weakly compact, then the interpolated operator from a J-space into a K-space also has this property, but in general this is not the case if we consider T acting between two J-spaces or two K-spaces. For these cases we prove that the interpolated operator is weakly compact provided that all but two restrictions of T (located in adjacent vertices of the polygon) are weakly compact. We also show by means of examples that these results are best possible. In other words, the interpolated operator may fail to be weakly compact if we leave out any of our assumptions on restrictions of T or on parameters.

In order to give those examples we require certain relationships between spaces defined by a polygon Π and those defined by a subpolygon $\widetilde{\Pi}$ of Π. These connections are described in Section 1. We also derive there reiteration formulae between methods associated to polygons and the real method. Reiteration results are not connected to weak compactness, but they have interest in their own. They allow to compute J- and K-spaces in certain cases.

1. Some properties of methods defined by polygons

We start by recalling definitions of the J- and K-spaces associated to polygons (see [6]).

Let $\overline{A} = \{A_1, \dots, A_N\}$ be a Banach N-tuple, that is to say, a family of N-Banach spaces A_j all of them embedded in a common linear Hausdorff space. Then we can form their sum $\Sigma(\overline{A}) = A_1 + \cdots + A_N$ and their intersection $\Delta(\overline{A}) = A_1 \bigcap \cdots \bigcap A_N$. These two spaces become Banach spaces when endowed with the norms

$$\|a\|_{\Sigma(\overline{A})} = \inf \left\{ \sum_{j=1}^{N} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j \right\}$$

and

$$\|a\|_{\Delta(\overline{A})} = \max\left\{\|a\|_{A_1}, \dots, \|a\|_{A_N}\right\}$$

respectively.

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. From now on, each space A_j of the N-tuple $\overline{A}$ should be thought of as sitting on the vertex P_j.

Given any two positive numbers t, s, we may equivalently renorm $\Sigma(\overline{A})$ by the K-functional (with respect to the polygon Π)

$$K(t,s;a) = \inf\left\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j\right\}.$$

Similarly, an equivalent norm to $\|\cdot\|_{\Delta(\overline{A})}$ is given by the J-functional

$$J(t,s;a) = \max_{1\le j\le N}\left\{t^{x_j} s^{y_j} \|a\|_{A_j}\right\}.$$

Let $1 \le q \le \infty$ and let (α,β) be any point in the interior of Π $[(\alpha,\beta) \in \operatorname{Int}\Pi]$. We define $\overline{A}_{(\alpha,\beta),q;K}$ to be the space of all elements $a \in \Sigma(\overline{A})$ having a finite norm

$$\|a\|_{(\alpha,\beta),q;K} = \left(\sum_{(m,n)\in\mathbb{Z}^2} \left(2^{-\alpha m-\beta n} K(2^m, 2^n; a)\right)^q\right)^{1/q} \qquad (\text{if } q < \infty)$$

$$\|a\|_{(\alpha,\beta),\infty;K} = \sup_{(m,n)\in\mathbb{Z}^2}\left\{2^{-\alpha m-\beta n} K(2^m, 2^n; a)\right\}.$$

The J-space $\overline{A}_{(\alpha,\beta),q;J}$ is formed by all those a in $\Sigma(\overline{A})$ which can be represented as

$$a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n} \qquad (\text{convergence in } \Sigma(\overline{A}))$$

with $(u_{m,n}) \subset \Delta(\overline{A})$ and

$$\|a\|_{(\alpha,\beta),q;J} = \left(\sum_{(m,n)\in\mathbb{Z}^2} \left(2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n})\right)^q\right)^{1/q} < \infty$$

(the sum should be replaced by the supremum if $q = \infty$). The norm $\|\cdot\|_{(\alpha,\beta),q;J}$ on $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J} = \inf\left\{\left(\sum_{(m,n)\in\mathbb{Z}^2} \left(2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n})\right)^q\right)^{1/q}\right\}$$

where the infimum is taken over all representations $(u_{m,n})$ as above.

Let us single out some important cases.

Example 1.1: When Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$ and $(\alpha,\beta) \in \operatorname{Int}\Pi$ (i.e., $\alpha > 0$, $\beta > 0$ with $\alpha+\beta < 1$), spaces $\overline{A}_{(\alpha,\beta),q;K}$, $\overline{A}_{(\alpha,\beta),q;J}$ coincide with Sparr spaces $\overline{A}^S_{(1-\alpha-\beta,\alpha,\beta),q;K}$, $\overline{A}^S_{(1-\alpha-\beta,\alpha,\beta),q;J}$, respectively. See [12] (and also [14]).

Example 1.2: If Π is the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ and $0 < \alpha, \beta < 1$, then we obtain spaces studied by Fernandez [8].

Example 1.3: Note that the real interpolation space $(A_0, A_1)_{\theta,q}$ can be described by a similar scheme to the one developed above, but working now in $\mathbb{R}$, with the segment $[0,1]$ taking the role of the polygon Π and $0 < \theta < 1$ being an interior point of $[0,1]$. The space A_0 should be thought of as sitting on 0, while A_1 on 1. In this case

$$(A_0,A_1)_{\theta,q;K} = (A_0,A_1)_{\theta,q;J} = (A_0,A_1)_{\theta,q} \qquad \text{(see [2] and [13])}.$$

In contrast to the theory for couples, K- and J-spaces for N-tuples ($N \geq 3$) do not coincide in general. We only have that

$$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K} \qquad \text{(see [6], Theorem 1.3)}.$$

K- and J-spaces can be equivalently defined using integrals instead of sums but the discrete approach is more convenient for our purposes.

Let now $\widetilde{\Pi} = \overline{P_{j_1} \cdots P_{j_M}}$ be another convex polygon whose vertices all belong to Π. Form the subtuple $\widetilde{A}$ of M spaces $\widetilde{A} = \{A_{j_1}, \ldots, A_{j_M}\}$ by selecting from $\overline{A}$ those spaces sitting on vertices of $\widetilde{\Pi}$. We designate by

$$\widetilde{K}(t,s;\cdot) \quad \text{and} \quad \widetilde{J}(t,s;\cdot)$$

the K- and the J-functionals defined by means of $\widetilde{\Pi}$ over $\Sigma(\widetilde{A})$ and $\Delta(\widetilde{A})$, respectively. For $(\alpha,\beta) \in \operatorname{Int}\widetilde{\Pi}$ and $1 \leq q \leq \infty$, we denote by $\widetilde{A}_{(\alpha,\beta),q;K}$, $\widetilde{A}_{(\alpha,\beta),q;J}$ the interpolation spaces defined by $\widetilde{\Pi}$ over $\widetilde{A}$. The next result follows easily from inequalities

$$K(t,s;a) \leq \widetilde{K}(t,s;a), \quad a \in \Sigma(\widetilde{A})$$
$$\widetilde{J}(t,s;a) \leq J(t,s;a), \quad a \in \Delta(\overline{A})\,.$$

Lemma 1.4

Let Π, $\widetilde{\Pi}$, q, (α,β), $\overline{A}$ and $\widetilde{A}$ as above. Then the following continuous inclusions hold

(a) $$\widetilde{A}_{(\alpha,\beta),q;K} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K}$$

(b) $$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \widetilde{A}_{(\alpha,\beta),q;J}\,.$$

If (α, β) lies on some diagonal of Π, then we can compare $\overline{A}_{(\alpha,\beta),q;K}$ and $\overline{A}_{(\alpha,\beta),q;J}$ with spaces obtained by using the real method for couples. Let $\mathcal{D}$ be the set of all couples $\{i, k\}$ such that (α, β) belongs to the diagonal joining P_i and P_k. For $\{i, k\} \in \mathcal{D}$ let $0 < \theta_{i,k} < 1$ be the (unique) number such that

$$(\alpha, \beta) = (1 - \theta_{i,k})P_i + \theta_{i,k}P_k.$$

Theorem 1.5

The following inclusions hold

$$(i) \qquad \sum_{\{i,k\}\in\mathcal{D}} (A_i, A_k)_{\theta_{i,k},\infty} \hookrightarrow \overline{A}_{(\alpha,\beta),\infty;K}$$

$$(ii) \qquad \overline{A}_{(\alpha,\beta),1;J} \hookrightarrow \bigcap_{\{i,k\}\in\mathcal{D}} (A_i, A_k)_{\theta_{i,k},1}.$$

Proof. Assume that $\{1, 3\} \in \mathcal{D}$, let $\theta = \theta_{1,3}$ and for $\lambda > 0$ put

$$\widehat{K}(\lambda; a) = \inf\{\|a_1\|_{A_1} + \lambda\|a_3\|_{A_3} : a = a_1 + a_3, a_1 \in A_1, a_3 \in A_3\},$$

$$\widehat{J}(\lambda; a) = \max\{\|a\|_{A_1}, \lambda\|a\|_{A_3}\}, \quad a \in A_1 \bigcap A_3.$$

Given $(m, n) \in \mathbb{Z}^2$, write

$$(1) \qquad \begin{cases} w = m(x_3 - x_1) + n(y_3 - y_1) \\ v = m(x_3 - x_1) - n(y_3 - y_1) \end{cases}$$

where $P_1 = (x_1, y_1)$ and $P_3 = (x_3, y_3)$. Since

$$(\alpha, \beta) = (1 - \theta)(x_1, y_1) + \theta(x_3, y_3),$$

we have

$$\begin{aligned} &2^{-\alpha m-\beta n}K(2^m, 2^n; a) \\ &\le 2^{m(x_1-\alpha)+n(y_1-\beta)}\widehat{K}(2^{m(x_3-x_1)+n(y_3-y_1)}; a) \le 2\, 2^{-\theta[w]}\widehat{K}(2^{[w]}; a). \end{aligned}$$

Here $[w]$ stands for the integer part of w, that is, the largest integer which is less than or equal to w. Hence

$$\|a\|_{(\alpha,\beta),\infty;K} \le 2\|a\|_{(A_1,A_3)_{\theta,\infty}}$$

and embedding (i) follows.

To check the other inclusion let $a \in \overline{A}_{(\alpha,\beta),1;J}$ and given any $\varepsilon > 0$ find a representation $a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$ with

$$\sum_{(m,n)\in\mathbb{Z}^2} 2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}) \leq (1+\varepsilon)\|a\|_{(\alpha,\beta),1;J}.$$

For $(m,n) \in \mathbb{Z}^2$, let $(w,v) \in \mathbb{R}^2$ be the numbers defined by (1). The relevant inequality says now

$$2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}) \geq 2^{-\theta} 2^{-\theta[w]} \widehat{J}(2^{[w]}; u_{m,n}).$$

We can get a representation $a = \sum_{r\in\mathbb{Z}} d_r$ of a in $(A_1, A_3)_{\theta,1}$ by putting

$$d_r = \sum_{(m,n)\in I_r} u_{m,n}$$

where

$$I_r = \{(m,n) \in \mathbb{Z}^2 : [w] = r\}.$$

Then

$$\begin{aligned} \|a\|_{(A_1,A_3)_{\theta,1}} &\leq \sum_{r\in\mathbb{Z}} 2^{-\theta r} \hat{J}(2^r; d_r) \leq \sum_{r\in\mathbb{Z}} \sum_{(m,n)\in I_r} 2^{-\theta r} \hat{J}(2^r; u_{m,n}) \\ &\leq 2^{\theta} \sum_{(m,n)\in\mathbb{Z}^2} 2^{-m\alpha-n\beta} J(2^m, 2^n; u_{m,n}) \leq 2^{\theta}(1+\varepsilon)\|a\|_{(\alpha,\beta),1;J} \end{aligned}$$

This gives (ii) and completes the proof. □

Combining Theorem 1.5 with the fact that $(A_j, A_j)_{\frac{1}{2},1} = (A_j, A_j)_{\frac{1}{2},\infty} = A_j$ we get

Corollary 1.6

Let $\Pi = \overline{P_1 \cdots P_{2N}}$ be a regular polygon with $2N$ vertices, let (α,β) be the center of Π and let $\{A_1, \ldots, A_N\}$ be an N-tuple. Consider the $2N$-tuple obtained by sitting each space A_j on the vertex P_j and on its symmetrical vertex P_{j+N}. Then we have

(a) $$(A_1, \ldots, A_N, A_1, \ldots, A_N)_{(\alpha,\beta),\infty;K} = \sum_{j=1}^{N} A_j$$

(b) $$(A_1, \ldots, A_N, A_1, \ldots, A_N)_{(\alpha,\beta),1;J} = \bigcap_{j=1}^{N} A_j.$$

The corollary extends a result of Cwikel and Janson for the unit square [7], Example 1.25.

Next we establish certain reiteration formulae between methods associated to polygons and the real method.

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$ where $x_j \geq 0$, $y_j \geq 0$. Let (A_0, A_1) be any Banach couple, let $0 < \theta, \eta < 1$ and assume that

$$\theta_j = \theta x_j + \eta y_j \leq 1, \quad j = 1, \dots, N.$$

Consider the N-tuple $\overline{B} = \{B_1, \dots, B_N\}$ formed by

$$B_j = \begin{cases} (A_0, A_1)_{\theta_j, s_j} & \text{if} \quad 0 < \theta_j < 1 \\ A_0 & \text{if} \quad \theta_j = 0 \\ A_1 & \text{if} \quad \theta_j = 1 \end{cases}$$

where $1 \leq s_j \leq \infty$.

Theorem 1.7

If $(\alpha, \beta) \in$ *Int* Π *lies on some diagonal of* Π*, say*

$$(\alpha, \beta) = (1 - \delta) P_1 + \delta P_3 \quad (0 < \delta < 1)$$

and $\theta_1 \neq \theta_3$*, then it holds*

$$(i) \qquad \overline{B}_{(\alpha,\beta),1;J} = (A_0, A_1)_{\alpha\theta + \beta\eta, 1}$$

$$(ii) \qquad \overline{B}_{(\alpha,\beta),\infty;K} = (A_0, A_1)_{\alpha\theta + \beta\eta, \infty}\,.$$

Proof. We start with the J-method. By Theorem 1.5/(ii) and the reiteration theorem for the real method, we obtain

$$\overline{B}_{(\alpha,\beta),1;J} \hookrightarrow (B_1, B_3)_{\delta,1} = \big((A_0, A_1)_{\theta_1, s_1}, (A_0, A_1)_{\theta_3, s_3}\big)_{\delta,1} = (A_0, A_1)_{\alpha\theta + \beta\eta, 1}.$$

Let us see the converse inclusion. We claim that

(2) There exists a constant $C > 0$ such that for any $a \in A_0 \bigcap A_1$ we have

$$\|a\|_{(\alpha,\beta),1;J} \leq C \|a\|_{A_0}^{1 - \alpha\theta - \beta\eta} \|a\|_{A_1}^{\alpha\theta + \beta\eta}.$$

Indeed, by [4], Theorem 1.3, there exists $C_1 > 0$ such that

$$\|a\|_{(\alpha,\beta),1;J} \leq C_1 \max \Big\{ \|a\|_{B_i}^{c_i} \|a\|_{B_k}^{c_k} \|a\|_{B_r}^{c_r} : \{i, k, r\} \in \mathcal{P} \Big\}.$$

Here $\mathcal{P}$ is the collection of all those triples $\{i,k,r\}$ such that (α,β) belongs to the triangle with vertices P_i, P_k, P_r, and (c_i, c_k, c_r) are the barycentric coordinates of (α,β) with respect to P_i, P_k, P_r. On the other hand, using that

$$\|a\|_{B_j} \le C_2 \|a\|_{A_0}^{1-\theta_j} \|a\|_{A_1}^{\theta_j},$$

we get

$$\begin{aligned}
\|a\|_{B_i}^{c_i} \|a\|_{B_k}^{c_k} \|a\|_{B_r}^{c_r} &\le C_3 \|a\|_{A_0}^{(1-\theta_i)c_i+(1-\theta_k)c_k+(1-\theta_r)c_r} \|a\|_{A_1}^{\theta_i c_i+\theta_k c_k+\theta_r c_r} \\
&= C_3 \|a\|_{A_0}^{1-\theta_i c_i-\theta_k c_k-\theta_r c_r} \|a\|_{A_1}^{\theta_i c_i+\theta_k c_k+\theta_r c_r} \\
&= C_3 \|a\|_{A_0}^{1-\alpha\theta-\beta\eta} \|a\|_{A_1}^{\alpha\theta+\beta\eta}
\end{aligned}$$

and (2) follows.

Let now $a \in (A_0, A_1)_{\alpha\theta+\beta\eta,1}$, and $\varepsilon > 0$. Put $\widehat{J}(\lambda; a) = \max\{\|a\|_{A_0}, \lambda\|a\|_{A_1}\}$ and find a $\widehat{J}$-representation $a = \sum_{r\in\mathbb{Z}} v_r$ with

$$\sum_{r\in\mathbb{Z}} 2^{-r(\alpha\theta+\beta\eta)} \widehat{J}(2^r; v_r) < (1+\varepsilon)\|a\|_{(A_0,A_1)_{\alpha\theta+\beta\eta,1}}.$$

We have

$$\begin{aligned}
\|a\|_{(\alpha,\beta),1;J} &\le \sum_{r\in\mathbb{Z}} \|v_r\|_{(\alpha,\beta),1;J} \le C \sum_{r\in\mathbb{Z}} \|v_r\|_{A_0}^{1-\alpha\theta-\beta\eta} \|v_r\|_{A_1}^{\alpha\theta+\beta\eta} \\
&\le C \sum_{r\in\mathbb{Z}} 2^{-r(\alpha\theta+\beta\eta)} \widehat{J}(2^r; v_r) < (1+\varepsilon)\|a\|_{(A_0,A_1)_{\alpha\theta+\beta\eta,1}}.
\end{aligned}$$

This proves (i).

Next we pass to the K-space. One inclusion follows easily from Theorem (1.5)/(i) and the reiteration theorem for the real method. Namely,

$$(A_0, A_1)_{\alpha\theta+\beta\eta,\infty} = (B_1, B_3)_{\delta,\infty} \hookrightarrow \overline{B}_{(\alpha,\beta),\infty;K}.$$

To establish the reverse inclusion, put

$$\widehat{K}(\lambda; a) = \inf\left\{\|a_0\|_{A_0} + \lambda\|a_1\|_{A_1} : a = a_0 + a_1\right\}$$

and denote by F_r the space $A_0 + A_1$ normed by $\widehat{K}(2^r; \cdot)$. Definition of B_j's yields that the operator

$$\begin{aligned}
T:\quad B_j \quad &\longrightarrow \quad \ell_\infty\left(2^{-r\theta_j} F_r\right) \\
a \quad &\longrightarrow \quad Ta = (\dots, a, a, a, \dots)
\end{aligned}$$

is bounded. Interpolating we have that

$$T : \overline{B}_{(\alpha,\beta),\infty;K} \longrightarrow \big(\ell_\infty(2^{-r\theta_1}F_r), \dots, \ell_\infty(2^{-r\theta_N}F_r)\big)_{(\alpha,\beta),\infty;K}$$

is also bounded. But, taking into account that

$$\begin{aligned}&\sup_{t>0,s>0}\left[\min_{1\le j\le N}\{t^{x_j-\alpha}s^{y_j-\beta}2^{-r\theta_j}\}\right]\\ &= \min\{2^{-r(\theta_i c_i+\theta_k c_k+\theta_r c_r)} : \{i,k,r\}\in\mathcal{P}\} = 2^{-r(\alpha\theta+\beta\eta)}\end{aligned}$$

and arguing as in [5], Theorem 2.3, one can check that

$$\big(\ell_\infty(2^{-r\theta_1}F_r), \dots, \ell_\infty(2^{-r\theta_N}F_r)\big)_{(\alpha,\beta),\infty;K} \hookrightarrow \ell_\infty\big(2^{-r(\alpha\theta+\beta\eta)}F_r\big).$$

Hence, there is a constant $C > 0$ such that for all $a \in \overline{B}_{(\alpha,\beta),\infty;K}$ we have

$$\|a\|_{(A_0,A_1)_{\alpha\theta+\beta\eta,\infty}} = \|Ta\|_{\ell_\infty\left(2^{-r(\alpha\theta+\beta\eta)}F_r\right)} \le C\|a\|_{(\alpha,\beta),\infty;K}.$$

The proof is complete. □

We close this section with an application of Theorem 1.7 to interpolation of ℓ_p-spaces.

Example 1.8: Take Π equal to the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$, let $1 < p < \infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and put $\theta=\frac{1}{p'}$, $\eta=\frac{1}{p}$. Then

$$\theta_1 = 0,\ \theta_2=\theta,\ \theta_3=\eta,\ \theta_4=\theta+\eta=1.$$

The choice

$$(A_0,A_1)=(\ell_1,\ell_\infty)\,,\quad s_1=p,\ s_2=p'$$

gives then the 4-tuple

$$\overline{B}=\{\ell_1,\ell_p,\ell_{p'},\ell_\infty\}$$

and so for $0<\alpha<1$ we obtain the following Lorentz sequence spaces by interpolation of $\overline{B}$

$$(\ell_1,\ell_p,\ell_{p'},\ell_\infty)_{(\alpha,\alpha),1;J} = \ell_{1/(1-\alpha),1}$$

$$(\ell_1,\ell_p,\ell_{p'},\ell_\infty)_{(\alpha,\alpha),\infty;K} = \ell_{1/(1-\alpha),\infty}.$$

2. Weak compactness and interpolation

Given two Banach N-tuples $\overline{A} = \{A_1, \dots, A_N\}$ and $\overline{B} = \{B_1, \dots, B_N\}$, we write $T : \overline{A} \longrightarrow \overline{B}$ to mean that T is a linear operator from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j gives a bounded operator from A_j into B_j. It is not hard to verify that if $T : \overline{A} \longrightarrow \overline{B}$ then the restriction of T to $\overline{A}_{(\alpha,\beta),q;K}$ defines a bounded operator $T : \overline{A}_{(\alpha,\beta),q;K} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$. For J-spaces, we have that $T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \overline{B}_{(\alpha,\beta),q;J}$ is also bounded.

In this section we investigate the behaviour of weak compactness under the K- and J-method. Recall that a linear operator $T : A \longrightarrow B$ between two Banach spaces is said to be weakly compact if it transforms the closed unit ball of A onto a relatively weakly compact subset of B. For example, the identity operator of a reflexive Banach space is weakly compact.

As we said at the Introduction, for the case of the real method, it is known that weak compactness in just one restriction of the operator T is enough to assure that the interpolated operator is weakly compact. However, in our context of N-tuples ($N \geq 3$) this is not true in general as we show next by means of examples based on results of Section 1.

Example 2.1: Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with N vertices ($N \geq 4$). Let P_i, P_k, P_r be three vertices of Π and let (α, β) be any point in the interior of the triangle $\overline{P_i, P_k, P_r}$. Consider the N-tuple $\overline{A} = \{A_1, \dots, A_N\}$ where

$$A_j = \begin{cases} \ell_\infty & \text{for } j = i, k, r \\ \ell_2 & \text{otherwise,} \end{cases}$$

take $B = \ell_\infty$ and choose T as the identity operator $T(\xi_n) = (\xi_n)$. Since ℓ_2 is a reflexive space while ℓ_∞ is not reflexive, all restrictions $T : A_j \longrightarrow B$ are weakly compact except for those to A_i, A_k, and A_r. However, for any $1 \leq q \leq \infty$, the interpolated operator $T : \overline{A}_{(\alpha,\beta),q:K} \longrightarrow B$ fails to be weakly compact because, by Lemma 1.4, we have $\overline{A}_{(\alpha,\beta),q:K} = \ell_\infty$.

Example 2.2: Let Π, (α, β) and q as in the previous example. Take the N-tuple $\overline{B} = \{B_1, \dots, B_N\}$ where

$$B_j = \begin{cases} \ell_1 & \text{for } j = i, k, r \\ \ell_2 & \text{otherwise,} \end{cases}$$

take $A = \ell_1$ and choose again T as the identity operator. It follows from Lemma 1.4 that $\overline{B}_{(\alpha,\beta),q;J} = \ell_1$. Hence $T : A \longrightarrow \overline{B}_{(\alpha,\beta),q;J}$ is not weakly compact although all but three restrictions of T are weakly compact.

These two examples refer to the especial case when one of the N-tuples reduces to a single Banach space. Next we establish positive results in that situation. Arguments are similar to those in the instance of Banach couples (see [9]).

Given any Banach space E, we denote by U_E the closed unit ball of E, and for $D \subset E$ we write $\ell_1(D)$ to mean the collection of all absolutely summable families of scalars with D as index set.

Theorem 2.3

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_k = (x_k, y_k)$, let P_j, P_{j+1} be two fixed adjacent vertices of Π, let $(\alpha, \beta) \in Int\, \Pi$ and $1 \le q \le \infty$. Assume that $\overline{A} = \{A_1, \dots, A_N\}$ is a Banach N-tuple, that B is a Banach space and that T is a linear operator $T : \overline{A} \longrightarrow B$.

If $T : A_k \longrightarrow B$ is weakly compact for all $1 \le k \le N$ with $k \ne j, j+1$, then $T : \overline{A}_{(\alpha,\beta),q;K} \longrightarrow B$ is also weakly compact.

Proof. It suffices to give the proof for $q = \infty$ because, according to definition of K-spaces, we have $\overline{A}_{(\alpha,\beta),q_1;K} \hookrightarrow \overline{A}_{(\alpha,\beta),q_2;K}$ provided that $q_1 \le q_2$.

If, say the exceptional vertices are P_1 and P_2 we claim that

(3) For each $r \in \mathbb{N}$ there are positive constants $C_{3,r}, \dots, C_{N,r}$ such that for any $a \in \overline{A}_{(\alpha,\beta),\infty;K}$ we have

$$\inf \left\{ r\|a_1\|_{A_1} + r\|a_2\|_{A_2} + \sum_{j=3}^{N} C_{j,r}^{-1} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j \right\} \le \|a\|_{(\alpha,\beta),\infty;K}.$$

Indeed, let $px + qy = s$ be the equation of the line through P_1 and P_2 (see Figure 2.1). Clearly $px_1 + qy_1 = s$ and $px_2 + qy_2 = s$. Moreover $p\alpha + q\beta < s$ for $(\alpha, \beta) \in \text{Int}\, \Pi$. Thus

$$p(x_1 - \alpha) + q(y_1 - \beta) > 0 \quad \text{and} \quad p(x_2 - \alpha) + q(y_2 - \beta) > 0.$$

We can then find $(m, n) \in \mathbb{Z}^2$ so that

$$2^{m(x_i - \alpha) + n(y_i - \beta)} \ge r \quad \text{for } i = 1, 2.$$

The choice

$$C_{j,r} = 2^{m(\alpha - x_j) + n(\beta - y_j)}, \quad j = 3, \dots, N$$

gives now (3).

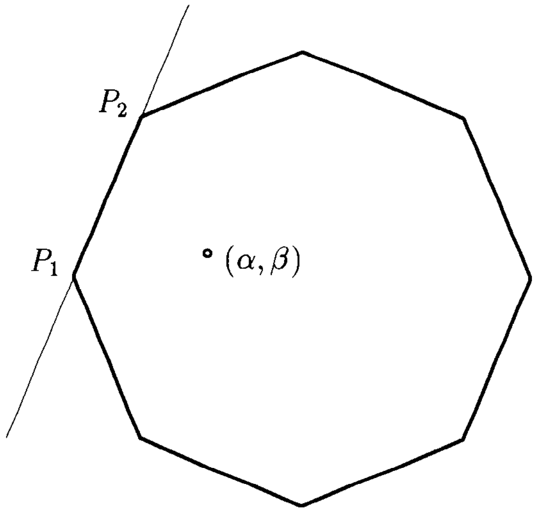

Figure 2.1

As a direct consequence of (3) we get that

$$V \subset \frac{2}{r}T(U_{A_1}) + \frac{2}{r}T(U_{A_2}) + \sum_{j=3}^{N} 2C_{j,r}T(U_{A_j})$$

where $V = T\big(U_{\overline{A}_{(\alpha,\beta),\infty;K}}\big)$. So we can construct maps

$$f_{i,r} : V \longrightarrow \frac{2}{r}\; T(U_{A_i}), \quad i = 1, 2$$

and

$$g_{j,r} : V \longrightarrow 2C_{j,r}T(U_{A_j}), \quad j = 3, \ldots, N$$

such that for any $a \in V$ it holds

$$\sum_{i=1}^{2} f_{i,r}(a) + \sum_{j=3}^{N} g_{j,r}(a) = a\,. \tag{4}$$

Let $\widetilde{T} : \ell_1(V) \longrightarrow B$ be the operator assigning to each summable family $(\lambda_v)_{v\in V}$ the sum $\widetilde{T}(\lambda_v) = \sum_{v\in V} \lambda_v v$. Since $\widetilde{T}\big(U_{\ell_1(V)}\big) = T\big(U_{\overline{A}_{(\alpha,\beta),\infty;K}}\big)$ we see that in order to prove weak compactness of T it is enough to show that $\widetilde{T}$ is weakly compact. With this aim, let $R_{i,r}, S_{j,r} : \ell_1(V) \longrightarrow B$ be the operators defined by

$$R_{i,r}(\lambda_v) = \sum_{v \in V} \lambda_v f_{i,r}(v), \quad i = 1, 2$$
$$S_{j,r}(\lambda_v) = \sum_{v \in V} \lambda_v g_{j,r}(v), \quad j = 3, \dots, N.$$

By (4) we have

$$\widetilde{T} = \sum_{i=1}^{2} R_{i,r} + \sum_{j=3}^{N} S_{j,r}.$$

Besides

$$\sum_{i=1}^{2} \|R_{i,r}\| \leq \frac{\left(\|T\|_{A_1,B} + \|T\|_{A_2,B}\right)}{r} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Hence $\widetilde{T}$ is the limit (in the operator norm) of the sequence $\left(\sum_{j=3}^{N} S_{j,r}\right)_{r=1}^{\infty}$, and each operator $S_{j,r}$ is weakly compact because

$$S_{j,r}\left(U_{\ell_1(V)}\right) \subset 2C_{j,r}T\left(U_{A_j}\right).$$

Consequently, $\widetilde{T}$ is also weakly compact.

The proof is complete. $\square$

Our next result refers to J-spaces. We denote by $\ell_\infty(D)$ the collection of all bounded families of scalars with D as indexed set.

Theorem 2.4

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_k = (x_k, y_k)$, let P_j, P_{j+1} be two fixed adjacent vertices of Π, let $(\alpha, \beta) \in Int\, \Pi$ and $1 \leq q \leq \infty$. Assume that $\overline{B} = \{B_1, \dots, B_N\}$ is a Banach N-tuple, that A is a Banach space and that T is a linear operator $T : A \longrightarrow \overline{B}$.

If $T : A \longrightarrow B_k$ is weakly compact for all $1 \leq k \leq N$ with $k \neq j, j+1$, then

$$T : A \longrightarrow \overline{B}_{(\alpha,\beta),q;J}$$

is also weakly compact.

Proof. Since

$$\overline{B}_{(\alpha,\beta),q_1;J} \hookrightarrow \overline{B}_{(\alpha,\beta),q_2;J} \quad \text{if} \quad q_1 \le q_2$$

we may assume that $q = 1$.

If, say, the exceptional vertices are P_1 and P_2, we have this time

(5) For each $r \in \mathbb{N}$ there are positive constants $C_{3,r}, \ldots, C_{N,r}$ such that for any $a \in A$ it holds

$$\|Ta\|_{(\alpha,\beta),1;J} \le \max\left\{\frac{1}{r}\|Ta\|_{B_1}, \frac{1}{r}\|Ta\|_{B_2}, C_{3,r}\|Ta\|_{B_3}, \ldots, C_{N,r}\|Ta\|_{B_N}\right\}.$$

Indeed, arguing as in (3), we can find $(m,n) \in \mathbb{Z}^2$ such that

$$2^{m(x_i-\alpha)+n(y_i-\beta)} \le \frac{1}{r} \quad \text{for} \quad i = 1, 2.$$

Thus

$$\begin{aligned}\|Ta\|_{(\alpha,\beta),1;J} \le & 2^{-\alpha m-\beta n} J(2^m, 2^n; Ta)\\ \le & \max\left\{\frac{1}{r}\|Ta\|_{B_1}, \frac{1}{r}\|Ta\|_{B_2}, C_{3,r}\|Ta\|_{B_3}, \ldots, C_{N,r}\|Ta\|_{B_N}\right\}\end{aligned}$$

where we have put $C_{j,r} = 2^{m(x_j-\alpha)+n(y_j-\beta)}$, $j = 3, \ldots, N$.

We claim that

$$\begin{aligned} W &= T^*\big(U_{\overline{B}^*_{(\alpha,\beta),1;J}}\big)\\ &\subset \frac{2}{r}T^*\big(U_{B_1^*}\big) + \frac{2}{r}T^*\big(U_{B_2^*}\big) + 2C_{3,r}T^*\big(U_{B_3^*}\big) + \cdots + 2C_{N,r}T^*\big(U_{B_N^*}\big). \end{aligned} \tag{6}$$

Here T^* is the adjoint operator of T.

To establish (6) take any $f \in W$. Then $f = gT$ with

$$|g(z)| \le \|z\|_{(\alpha,\beta),1;J} \le \max\left\{\frac{1}{r}\|z\|_{B_1}, \frac{1}{r}\|z\|_{B_2}, C_{3,r}\|z\|_{B_3}, \ldots, C_{N,r}\|z\|_{B_N}\right\}$$

for all $z \in T(A)$. Let $\overline{D} = \{D_1, \ldots, D_N\}$ be the N-tuple given by

$$D_j = \begin{cases} \left(T(A), \dfrac{1}{r}\|\cdot\|_{B_j}\right) & \text{if} \quad j = 1, 2\\ \big(T(A), C_{j,r}\|\cdot\|_{B_j}\big) & \text{if} \quad j = 3, \ldots, N\,. \end{cases}$$

We have that

$$g \in \Delta\left(\overline{D}\right)^* = D_1^* + \ldots + D_N^*.$$

Whence there are $h_j \in D_j^*$ with $\|h_j\| \le 2$ such that

$$g(z) = \sum_{j=1}^{N} h_j(z), \quad z \in T(A).$$

Applying Hahn-Banach theorem, we can extend h_j to a bounded functional $g_j \in B_j^*$ with

$$\|g_j\| = \begin{cases} \dfrac{1}{r}\|h_j\| \le \dfrac{2}{r} & \text{for } \ j = 1, 2 \\ C_{j,r}\|h_j\| \le 2C_{j,r} & \text{for } \ j = 3, \dots, N. \end{cases}$$

Therefore

$$f = gT \in \frac{2}{r}\, T^*\big(U_{B_1^*}\big) + \frac{2}{r}\, T^*\big(U_{B_2^*}\big) + 2C_{3,r}T^*\big(U_{B_3^*}\big) + \dots + 2C_{N,r}T^*\big(U_{B_N^*}\big).$$

This gives (6).

We can now construct maps

$$\begin{aligned} \varphi_{i,r} &: W \longrightarrow \frac{2}{r}\, T^*\left(U_{B_i^*}\right), \quad i = 1, 2 \\ \psi_{j,r} &: W \longrightarrow 2\, C_{j,r}T^*\left(U_{B_j^*}\right), \quad j = 3, \dots, N \end{aligned}$$

such that for any $f \in W$ it holds

$$\sum_{i=1}^{2} \varphi_{i,r}(f) + \sum_{j=3}^{N} \psi_{j,r}(f) = f. \tag{7}$$

Let B be the closure of $T(A)$ in $\overline{B}_{(\alpha,\beta),1;J}$, and let $J : B \longrightarrow \ell_\infty(W)$ be the isometric embedding given by $J\,(Ta) = \big(f(a)\big)_{f\in W}$, $a \in A$. Write $\widehat{T} = JT : A \longrightarrow \ell_\infty(W)$. In order to check that $\widehat{T}$ is weakly compact define operators $R_{i,r}, S_{j,r} : A \longrightarrow \ell_\infty(W)$ by

$$\begin{aligned} R_{i,r}(a) &= \ (< a, \varphi_{i,r}(f) >)_{f\in W}\,, \quad i = 1, 2 \\ S_{j,r}(a) &= \ (< a, \psi_{j,r}(f) >)_{f\in W}\,, \quad j = 3, \dots, N\,. \end{aligned}$$

It follows from

$$\|S_{j,r}(a)\|_{\ell_\infty(W)} \le 2C_{j,r}\|Ta\|_{B_j}, \quad j = 3, \dots, N$$

and weak compactness of $T : A \longrightarrow B_j$ that each operator $S_{j,r}$ is weakly compact for $3 \leq j \leq N$. On the other hand, (7) gives that

$$\widehat{T} - \sum_{j=3}^{N} S_{j,r} = \sum_{i=1}^{2} R_{i,r}$$

and

$$\sum_{i=1}^{2} \|R_{i,r}\| \leq \frac{2}{r}\left(\|T\|_{A,B_1} + \|T\|_{A,B_2}\right) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Whence $\widehat{T}$ is the limit of a sequence of weakly compact operators, and it is therefore weakly compact.

Finally, weak compactness of $T : A \longrightarrow \overline{B}_{(\alpha,\beta),q;J}$ follows from equality $\widehat{T} = JT$ taking into account that J is an isometric embedding. $\square$

Remark 2.5. Theorems 2.3 and 2.4 are best possible. That is to say, they fail if the weakly compactness condition is not fulfilled at three of the vertices of Π or if the exceptional vertices are not adjacent.

Indeed, Examples 2.1 and 2.2 prove the first statement. Let us check the second statement. Let Π be the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$, let (α,β) be the center of Π and $q = \infty$. Take the 4-tuple $\overline{A} = \{A_1,\ldots,A_4\}$ where

$$A_j = \begin{cases} \ell_2 & \text{for} \quad j = 2,3 \\ \ell_\infty & \text{for} \quad j = 1,4, \end{cases}$$

take $B = \ell_\infty$ and choose T as the identity operator $T(\xi_n) = (\xi_n)$. Then all except for two restrictions of T are weakly compact but they do not correspond to adjacent vertices. We can determine $\overline{A}_{(\alpha,\beta),\infty;K}$ by using Corollary 1.6/(a) or [7], Example 1.25. It turns out that $\overline{A}_{(\alpha,\beta),\infty;K} = \ell_\infty$. The interpolated operator coincides then with the identity of ℓ_∞ and so it fails to be weakly compact.

A similar example can be shown for for the J-case.

We pass now to the general case of non-degenerated N-tuples. We shall work with vector valued sequence spaces modeled on the sum $\Sigma(\overline{B})$, and on the intersection $\Delta(\overline{A})$. Let

$$F_{m,n} = \left(\Sigma(\overline{B}), K(2^m,2^n;\cdot)\right), \quad G_{m,n} = \left(\Delta(\overline{A}), J(2^m,2^n;\cdot)\right), \quad (m,n) \in \mathbb{Z}^2,$$

put

$$\ell_q(2^{-\alpha m-\beta n}F_{m,n}) = \Big\{(b_{m,n})_{(m,n)\in\mathbb{Z}^2} : \quad b_{m,n} \in \Sigma(\overline{B}),$$
$$\|(b_{m,n})\|_{\ell_q} = \Big(\sum_{(m,n)\in\mathbb{Z}^2} \big(2^{-\alpha m-\beta n}K(2^m,2^n;b_{m,n})\big)^q \Big)^{1/q} < \infty \Big\}$$

and define $\ell_q(2^{-\alpha m-\beta n}G_{m,n})$ similarly. Consider also the metric injection $J : \overline{B}_{(\alpha,\beta),q;K} \longrightarrow \ell_q(2^{-\alpha m-\beta n}F_{m,n})$ defined by $J(b) = (\cdots, b, b, b, \cdots)$, and the surjection $Q : \ell_q(2^{-\alpha m-\beta n}G_{m,n}) \longrightarrow \overline{A}_{(\alpha,\beta),q;J}$ given by $Q(u_{m,n}) = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n}$.

As it was pointed out in [9], any bounded operator S between vector valued ℓ_q-spaces $(1 < q < \infty)$ is weakly compact provided all its components (regarded S as a matrix) are weakly compact. This fact was called in [9] "the Σ_q-property" of the ideal of weakly compact operators.

Theorem 2.6

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon, let P_j, P_{j+1} be two fixed adjacent vertices of Π, let $(\alpha,\beta) \in Int\,\Pi$ and $1 < q < \infty$. Assume that $\overline{A} = \{A_1,\dots,A_N\}$ and $\overline{B} = \{B_1,\dots,B_N\}$ are Banach N-tuples, and let $T : \overline{A} \longrightarrow \overline{B}$ be a linear operator such that $T : A_k \longrightarrow B_k$ is weakly compact for all $1 \le k \le N$ with $k \ne j, j+1$. Then

(a) $$T : \overline{A}_{(\alpha,\beta),q;K} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$$

and

(b) $$T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \overline{B}_{(\alpha,\beta),q;J}$$

are also weakly compact.

Proof. According to Theorem 2.3, each component $P_{r,s}JT$ of the operator $JT : \overline{A}_{(\alpha,\beta),q;K} \longrightarrow \ell_q(2^{-\alpha m-\beta n}F_{m,n})$ is weakly compact. So, by Σ_q-property, JT is weakly compact, and therefore $T : \overline{A}_{(\alpha,\beta),q;K} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$ is also weakly compact.

The proof of (b) is similar but using this time Theorem 2.4. □

Remark 2.7. Under certain density conditions on the N-tuples $\overline{A}$ and $\overline{B}$, Theorem 2.6/(a) has been independently obtained by Carro and Nikolova [3] by means of a different approach.

Remark 2.8. Assumption $1 < q < \infty$ is essential for Theorem 2.6 as the following example shows:

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$ and consider the N-tuple of scalar weighted sequence spaces over $\mathbb{Z}^2$ given by

$$\overline{\ell}_2 = \{\ell_2(2^{-mx_1-ny_1}), \ell_2(2^{-mx_2-ny_2}), \dots, \ell_2(2^{-mx_N-ny_N})\}.$$

All spaces of $\overline{\ell}_2$ are reflexive. However, according to [6], Theorem 3.1, we have that

$$(\overline{\ell}_2)_{(\alpha,\beta),1;K} = (\overline{\ell}_2)_{(\alpha,\beta),1;J} = \ell_1(2^{-\alpha m-\beta n})$$

and

$$(\overline{\ell}_2)_{(\alpha,\beta),\infty;K} = (\overline{\ell}_2)_{(\alpha,\beta),\infty;J} = \ell_\infty(2^{-\alpha m-\beta n})$$

which are not reflexive spaces.

Remark 2.9. Techniques used in Theorems 2.3, 2.4 and 2.6 also work if we replace the ideal of weakly compact operators for any other surjective closed operator ideal (resp. injective closed operator ideal) satisfying the Σ_q-condition (see [9] for definitions of these concepts). Examples of such ideals are Banach-Sacks operators, Rosenthal operators and decomposing operators.

As a direct consequence of Theorem 2.6 we have the following result on reflexivity of K- and J-spaces.

Corollary 2.10

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_k = (x_k, y_k)$, let P_j, P_{j+1} be two fixed adjacent vertices of Π and let $(\alpha,\beta) \in Int\ \Pi$. If $\overline{A} = \{A_1, \dots, A_N\}$ is a Banach N-tuple with all spaces A_k being reflexive for $1 \le k \le N$ with $k \ne j, j+1$, then the spaces $\overline{A}_{(\alpha,\beta),q;K}$ and $\overline{A}_{(\alpha,\beta),q;J}$ are reflexive provided that $1 < q < \infty$.

Our last result refers to operators acting from a J-space into a K-space. In that case we can give a complete characterization for weak compactness of the interpolated operator.

Theorem 2.11

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, let $(\alpha,\beta) \in Int\ \Pi$ and $1 < q < \infty$. Assume that $\overline{A} = \{A_1, \dots, A_N\}$ and $\overline{B} = \{B_1, \dots, B_N\}$ are Banach N-tuples, and that $T : \overline{A} \longrightarrow \overline{B}$.

Then a necessary and sufficient condition for $T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$ to be weakly compact is the weak compactness of $T : \Delta(\overline{A}) \longrightarrow \Sigma(\overline{B})$.

Proof. The scheme

$$\Delta(\overline{A}) \hookrightarrow \overline{A}_{(\alpha,\beta),q;J} \xrightarrow{T} \overline{B}_{(\alpha,\beta),q;K} \hookrightarrow \Sigma(\overline{B})$$

shows that weak compactness of $T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$ implies that $T : \Delta(\overline{A}) \longrightarrow \Sigma(\overline{B})$ is weakly compact.

Let us check that the condition is sufficient. Write $\widehat{T} = JTQ$. Thus $\widehat{T}$ acts from $\ell_q\big(2^{-\alpha m-\beta n}G_{m,n}\big)$ into $\ell_q\big(2^{-\alpha m-\beta n}F_{m,n}\big)$, and its components are $P_{r,s}\widehat{T}R_{j,k}$. Since $P_{r,s}\widehat{T}R_{j,k} : 2^{-\alpha j-\beta k}G_{j,k} \longrightarrow 2^{-\alpha r-\beta s}F_{r,s}$ coincides with the operator T acting from $2^{-\alpha j-\beta k}G_{j,k}$ into $2^{-\alpha r-\beta s}F_{r,s}$, our assumption on T yields that each component $P_{r,s}\widehat{T}R_{j,k}$ of $\widehat{T}$ is weakly compact. It follows then from Σ_q-property that $\widehat{T}$ is weakly compact. But $\widehat{T} = JTQ$ being J a metric injection and Q a surjection. Therefore we conclude that $T : \overline{A}_{(\alpha,\beta),q;J} \longrightarrow \overline{B}_{(\alpha,\beta),q;K}$ is also weakly compact.

Acknowledgments. The authors would like to thank María J. Carro and Tomás Schonbek for some helpful discussions.

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